6. Metric spaces

In this section we review the basic facts about metric spaces.

Definitions. A metric on a non-empty set $X$ is a map

$$d : X \times X \rightarrow [0, \infty)$$

with the following properties:

1. If $x, y \in X$ are points with $d(x, y) = 0$, then $x = y$;
2. $d(x, y) = d(y, x)$, for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

A metric space is a pair $(X, d)$, where $X$ is a set, and $d$ is a metric on $X$.

Given a metric space $(X, d)$, and a set $A \subset X$, one defines its diameter to be the “number”

$$\text{diam}_d(A) = \sup \{d(x, y) : x, y \in A\} \in [0, \infty).$$

(When there is no danger of confusion, the subscript will be omitted.)

Notations. If $(X, d)$ is a metric space, then for any point $x \in X$ and any $r > 0$, we define the open and closed balls:

$$B_r(x) = \{y \in X : d(x, y) < r\},$$
$$\overline{B}_r(x) = \{y \in X : d(x, y) \leq r\}.$$  

Definition. Suppose $(X, d)$ is a metric space. Then $X$ carries a natural topology constructed as follows. We say that a set $D \subset X$ is open, if it has the property:

- for every $x \in D$, there exists some $r_x > 0$, such that $B_{r_x}(x) \subset D$.

One can prove that the collection

$$T_d = \{D \subset X : D \text{ open}\}$$

is indeed a topology, i.e. we have

- $\emptyset$ and $X$ are open;
- if $(D_i)_{i \in I}$ is a family of open sets, then $\bigcup_{i \in I} D_i$ is again open;
- if $D_1$ and $D_2$ are open, then $D_1 \cap D_2$ is again open.

The topology thus constructed is called the metric topology.

Remark 6.1. Let $(X, d)$ be a metric space. Then for every $p \in X$, and for every $r > 0$, the set $B_r(p)$ is open, and the set $\overline{B}_r(p)$ is closed.

If we start with some $x \in B_r(p)$, an if we define $r_x = r - d(x, p)$, then for every $y \in B_{r_x}(x)$ we will have

$$d(y, p) \leq d(y, x) + d(x, p) < r_x + d(x, p) = r,$$

so $y$ belongs to $B_r(p)$. This means that $B_{r_x}(x) \subset B_r(p)$. Since this is true for all $x \in B_r(p)$, it follows that $B_r(p)$ is indeed open.

To prove that $\overline{B}_r(p)$ is closed, we need to show that its complement

$$X \setminus \overline{B}_r(p) = \{x \in X : d(x, p) > r\}$$
is open. If we start with some \( x \in X \setminus \mathbb{B}_r(p) \), an if we define \( \rho_x = d(p, x) - r \), then for every \( y \in \mathbb{B}_r(x) \) we will have

\[
d(y, p) \geq d(p, x) - d(y, x) > d(p, x) - \rho_x = r,
\]

so \( y \) belongs to \( X \setminus \mathbb{B}_r(p) \). This means that \( \mathbb{B}_{\rho_x}(x) \subset X \setminus \mathbb{B}_r(p) \). Since this is true for all \( x \in X \setminus \mathbb{B}_r(p) \), it follows that \( X \setminus \mathbb{B}_r(p) \) is indeed open.

**Remark 6.2.** The metric topology on a metric space \((X, d)\) is Hausdorff. Indeed, if we start with two points \( x, y \in X \), with \( x \neq y \), then if we choose \( r \) to be a real number, with

\[
0 < r < \frac{d(x, y)}{2},
\]

then we have \( \mathbb{B}_r(x) \cap \mathbb{B}_r(y) = \emptyset \). (Otherwise, if we have a point \( z \in \mathbb{B}_r(x) \cap \mathbb{B}_r(y) \), we would have \( 2r < d(x, y) \leq d(x, z) + d(y, z) < 2r \), which is impossible.)

**Remark 6.3.** Let \((X, d)\) be a metric space, and let \( M \) be a subset of \( X \). Then \( d|_{M \times M} \) is a metric on \( M \), and the metric topology on \( M \) defined by this metric is precisely the **induced topology** from \( X \). This means that a set \( A \subset M \) is open in \( M \) if and only if there exists some open set \( D \subset X \) with \( A = M \cap D \).

The metric space framework is particularly convenient because one can use **sequential convergence**.

**Definition.** Let \((X, d)\) be a metric space. For a point \( x \in X \), we say that a sequence \((x_n)_{n \geq 1} \subset X \) is **convergent to** \( x \), if \( \lim_{n \to \infty} d(x_n, x) = 0 \).

**Exercise 1.** Prove that, given \( x \in X \) and as sequence \((x_n)_{n=1}^\infty \subset X \), the above definition is equivalent to the fact that \((x_n)_{n \in \mathbb{N}} \) converges to \( x \), as a net, with respect to the metric topology. (See Section 2).

**Remark 6.4.** Let \((X, d)\) is a metric space, and if the sequence \((x_n)_{n \geq 1} \subset X \) is convergent to some point \( x \in X \), then

\[
\lim_{n \to \infty} d(x_n, y) = d(x, y), \quad \forall y \in X.
\]

This is an immediate consequence of the inequalities

\[
d(x, y) - d(x_n, x) \leq d(x_n, y) \leq d(x, y) + d(x_n, x).
\]

Among other things, the equality (1) gives the fact that \((x_n)_{n \geq 1} \) cannot be convergent to any other point \( y \neq x \). (This can be also obtained using Remark 6.2 and Exercise 1.) Therefore, if \((x_n)_{n \geq 1} \) is convergent to some \( x \), then \( x \) is uniquely determined, and will be denoted by \( \lim_{n \to \infty} x_n \).

Sequential convergence is useful for characterizing closure.

**Proposition 6.1.** Let \((X, d)\) be a metric space, and let \( A \subset X \) be a non-empty subset. For a point \( x \in X \), the following are equivalent:

(i) \( x \) belongs to the closure \( \overline{A} \) of \( A \);

(ii) there exists some sequence \((x_n)_{n \geq 1} \subset A \), with \( \lim_{n \to \infty} x_n = x \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume \( x \in \overline{A} \). This means that

(*) For every open set \( D \subset X \) with \( D \ni x \), the intersection \( D \cap A \) is non-empty.
We use this property for the open sets $B_{1/n}(x)$, $n = 1, 2, \ldots$. So, for every integer $n \geq 1$, we can find a point $x_n \in B_{1/n}(x) \cap A$. This way we have built a sequence $(x_n)_{n \geq 1} \subset A$, such that

$$d(x_n, x) < \frac{1}{n}, \quad \forall n \geq 1.$$ 

It is clear that this gives $x = \lim_{n \to \infty} x_n$.

(ii) $\Rightarrow$ (i). Assume $x$ satisfies property (ii). Fix $(x_n)_{n \geq 1} \subset A$ to be a sequence with $\lim_{n \to \infty} x_n = x$. We need to prove property (*). Start with some arbitrary open set $D \subset X$, with $x \in D$. Let $\varepsilon > 0$ be chosen such that $B_{\varepsilon}(x) \subset D$. Since $\lim_{n \to \infty} d(x_n, x) = 0$, there exists some $n_\varepsilon$ such that $d(x_{n_\varepsilon}, x) < \varepsilon$. It is now clear that $x_{n_\varepsilon} \in B_{\varepsilon}(x) \cap A \subset D \cap A$, so the intersection $D \cap A$ is indeed non-empty. □

Continuity can be characterized using sequential convergence, as follows.

**Proposition 6.2.** Let $X$ and $Y$ be metric spaces, and let $f : X \to Y$ be a function. For a point $p \in X$, the following are equivalent:

(i) $f$ is continuous at $p$;

(ii) for every $\varepsilon > 0$, there exists some $\delta_\varepsilon > 0$ such that

$$d(f(x), f(p)) < \varepsilon, \quad \text{for all } x \in X \text{ with } d(x, p) < \delta_\varepsilon.$$ 

(iii) if $(x_n)_{n \geq 1} \subset X$ is a sequence with $\lim_{n \to \infty} x_n = p$, then $\lim_{n \to \infty} f(x_n) = f(p)$.

**Proof.** (i) $\Rightarrow$ (ii). The condition that $f$ is continuous at $p$ means

(*) for every open set $D \subset Y$, with $D \ni f(p)$, there exists some open set $E \subset X$, with $p \in E \subset f^{-1}(D)$.

Assume $f$ is continuous at $p$. For every $\varepsilon > 0$, we consider the open ball $B^Y_{\varepsilon}(f(p))$. Using (*), there exists some open set $E \subset X$, with $E \ni p$, and $f(E) \subset B^Y_{\varepsilon}(f(p))$. In particular, there exists $\delta > 0$, such that $B^X_\delta(p) \subset E$, so now we have

$$f(B^X_{\delta}(p)) \subset B^Y_{\varepsilon}(f(p)),$$

which clearly gives (ii).

(ii) $\Rightarrow$ (iii). Assume $f$ satisfies (ii), and start with some sequence $(x_n)_{n \geq 1} \subset X$, which converges to $p$. For every $\varepsilon > 0$, we choose $\delta_\varepsilon > 0$ as in (ii), and using the fact that $\lim_{n \to \infty} x_n = p$, we can also choose some $N_\varepsilon$ such that

$$d(x_n, p) < \delta_\varepsilon, \quad \forall n \geq N_\varepsilon.$$ 

Using (ii) this will give

$$d(f(x_n), f(p)) < \varepsilon, \quad \forall n \geq N_\varepsilon.$$ 

In other words, we get the fact that

$$\lim_{n \to \infty} (f(x_n), f(p)) = 0,$$

which means that we indeed have $\lim_{n \to \infty} f(x_n) = f(p)$.

(iii) $\Rightarrow$ (i). Assume $f$ satisfies (iii), but $f$ is not continuous at $p$. By (*) this means that there exists some open set $D_0 \subset Y$ with $D_0 \ni f(p)$, such that

(*') for every open set $E \subset X$ with $E \ni p$, we have $f(E) \notin D_0$. 

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It is clear that any other open set $D$, with $f(p) \in D \subset D_0$, will again satisfy property $(\star')$. Fix then some $r > 0$, such that $B^Y_r(f(p)) \subset D_0$. Using condition $(\star')$ it follows that for every integer $n \geq 1$, we have

$$f(B^X_{1/n}(p)) \not\subset B^Y_r(f(p)).$$

This means that, for every integer $n \geq 1$, we can find a point $x_n \in X$ such that $d(x_n, p) < \frac{1}{n}$ and $d(f(x_n), f(p)) \geq r$.

It is then clear that the sequence $(x_n)_{n \geq 1} \subset X$ is convergent to $p$, but the sequence $(f(x_n))_{n \geq 1} \subset Y$ is not convergent to $f(p)$. This will contradict (iii). □

**Exercise 2.** Let $(X, d)$ be a metric space, and let $A \subset X$ be some non-empty subset. Prove the equality $\text{diam}(A) = \text{diam}(\overline{A})$.

Sequential convergence can also be used for characterizing compactness, but we are going to formulate the appropriate result a little later (see Theorem ??).

**Definition.** Let $(X, d)$ be a metric space. For a point $x \in X$ and a non-empty subset $A \subset X$, one defines the distance from $x$ to $A$ as the number

$$d(x, A) = \inf \{ d(x, a) : a \in A \}.$$ 

**Exercise 3.** Let $(X, d)$ be a metric space, and let $A \subset X$ be a non-empty subset.

(i) For a point $x \in X$, prove that the equality $d(x, A) = 0$ is equivalent to the fact that $x \in \overline{A}$.

(ii) Prove the inequality

$$|d(x, A) - d(y, A)| \leq d(x, y), \ \forall x, y \in X.$$

Using (ii) conclude that the map $X \ni x \mapsto d(x, A) \in [0, \infty)$ is continuous.

**Proposition 6.3.** Let $(X, d)$ be a metric space. When equipped with the metric topology, $X$ is normal.

**Proof.** Let $A$ and $B$ be closed subsets of $X$ with $A \cap B = \emptyset$. We need to find open sets $U, V \subset X$, with $U \supset A$, $V \supset B$, and $U \cap V = \emptyset$. We are going to use a converse of Urysohn Lemma. More explicitly, let us define the function $f : X \to [0, 1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \ x \in X.$$ 

Notice that by Exercise 3, both the numerator and denominator are continuous, and the denominator never vanishes. So $f$ is indeed continuous. It is obvious that $f|_A = 0$ and $f|_B = 1$, so if we take the open sets $U = f^{-1}((-\infty, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, \infty))$, we clearly get the desired result. □

An important feature in the study of metric spaces is covered by the following.

**Definitions.** Let $(X, d)$ be a metric space.

A. We say that a sequence $(x_n)_{n \geq 1} \subset X$ is a Cauchy sequence, if:
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(C) for every $\varepsilon > 0$, there exists some integer $N_\varepsilon \geq 1$ such that
$$d(x_m, x_n) < \varepsilon, \ \forall m, n \geq N_\varepsilon.$$ It is obvious that every convergent sequence is Cauchy.

B. We say that $(X, d)$ is complete, if every Cauchy sequence is convergent.

The following result gives some equivalent characterizations.

**Proposition 6.4.** Let $(X, d)$ be a metric space. The following are equivalent.

(i) $(X, d)$ is complete.

(ii) Every sequence $(x_n)_{n \geq 1} \subset X$, with
$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty,$$

is convergent.

(iii) Every Cauchy sequence has a convergent subsequence.

**Proof.** (i) $\Rightarrow$ (ii). Assume $X$ is complete. Let $(x_n)_{n \geq 1} \subset X$ be a sequence with property (2). To prove (ii) it suffices to show that $(x_n)_{n \geq 1}$ is Cauchy. For every $N \geq 1$ we define
$$R_N = \sum_{n=N}^{\infty} d(x_{n+1}, x_n).$$

Using (2) we get $\lim_{N \to \infty} R_N = 0$, so for every $\varepsilon > 0$ there exists some $N(\varepsilon)$ with $R_N(\varepsilon) < \varepsilon$. Notice also that the sequence $(R_N)_{N \geq 1}$ is decreasing. If $m > n \geq N(\varepsilon)$, then
$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{\infty} d(x_{k+1}, x_k) = R_n \leq R_N(\varepsilon) < \varepsilon,$$

so $(x_n)_{n \geq 1}$ is indeed Cauchy.

(ii) $\Rightarrow$ (iii). Start with some Cauchy sequence $(y_k)_{k \geq 1}$. For every $n \geq 1$ choose an integer $N(n) \geq 1$ such that
$$d(y_k, y_\ell) < \frac{1}{2^n}, \ \forall k, \ell \geq N(n).$$

Start with some arbitrary $k_1 \geq N(1)$ and define recursively an entire sequence $(k_n)_{n \geq 1}$ of integers, by
$$k_{n+1} = \max\{k_n + 1, N(n + 1), \ n \geq 1\}.$$ Clearly we have $k_1 < k_2 < \ldots$, and since we have
$$k_{n+1} > k_n \geq N(n), \ \forall n \geq 1,$$

using (3), we get
$$d(y_{k_{n+1}}, y_{k_n}) < \frac{1}{2^n}, \ \forall n \geq 1.$$ So if we define the subsequence $x_n = y_{k_n}, \ n \geq 1$, we will have
$$\sum_{n=1}^{\infty} d(x_{n+1}, x_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1,$$

so the subsequence $(x_n)_{n \geq 1}$ satisfies condition (2). By (ii) the subsequence $(x_n)_{n \geq 1}$ is convergent.
(iii) ⇒ (i). Assume condition (iii) holds. Start with some Cauchy sequence \((x_n)_{n \geq 1}\). For every integer \(n \geq 1\) we put

\[
S_n = \sup_{\ell, m \geq n} d(x_\ell, x_m).
\]

Since \((x_n)_{n \geq 1}\) is Cauchy, we have

\[
\lim_{n \to \infty} S_n = 0.
\]

Using the assumption, we can find a subsequence \((x_{k_m})_{m \geq 1}\) (defined by an increasing sequence of integers \(1 \leq k_1 < k_2 < \ldots\)) which is convergent to some point \(x\). We are going to prove that the entire sequence \((x_n)_{n \geq 1}\) is convergent to \(x\). Fix for the moment \(n \geq 1\). For every \(m \geq n\), we have \(k_m \geq m \geq n\), so we have

\[
S_n \geq d(x_n, x_{k_m}), \quad \forall \ m \geq n.
\]

By Remark 3.4, we also know that

\[
\lim_{m \to \infty} d(x_n, x_{k_m}) = d(x_n, x),
\]

so if we take \(\lim_{m \to \infty}\) in (5) we will get

\[
d(x_n, x) \leq S_n.
\]

Since this estimate holds for arbitrary \(n \geq 1\), using (4) we immediately get the fact that \((x_n)_{n \geq 1}\) is indeed convergent to \(x\). □

**Proposition 6.5.** Suppose \((X, d)\) is a complete metric space, and \(Y\) is a subset of \(X\). The following are equivalent:

(i) \(Y\) is complete, when equipped with the metric from \(X\);

(ii) \(Y\) is closed in \(X\), in the metric topology.

**Proof.** (i) ⇒ (ii). Assume \(Y\) is complete, and let us prove that \(Y\) is closed. Start with a point \(x \in Y\). Then there exists a sequence \((y_n)_{n \geq 1} \subset Y\) with \(\lim_{n \to \infty} y_n = x\). Notice that \((y_n)_{n \geq 1}\) is Cauchy in \(Y\), so by assumption, \((y_n)_{n \geq 1}\) is convergent to some point in \(Y\). This will then clearly force \(x \in Y\).

(ii) ⇒ (i). Assume \(Y\) is closed, and let us prove that \(Y\) is complete. Start with a Cauchy sequence \((y_n)_{n \geq 1} \subset Y\). Since \(X\) is complete, the sequence \((y_n)_{n \geq 1}\) is convergent to some point \(x \in X\). Since \(Y\) is closed, this forces \(x \in Y\). □

The following discussion gives a method of constructing complete metric spaces.

**Definitions.** Let \((X, d)\) be a metric space. We define

\[
\text{cs}(X, d) = \{ \mathbf{x} = (x_n)_{n \geq 1} : \mathbf{x} \text{ Cauchy sequence in } X\}.
\]

We say that two Cauchy sequences \(\mathbf{x} = (x_n)_{n \geq 1}\) and \(\mathbf{y} = (y_n)_{n \geq 1}\) in \(X\) are equivalent, if

\[
\lim_{n \to \infty} d(x_n, y_n) = 0.
\]

In this case we write \(\mathbf{x} \sim \mathbf{y}\). (It is fairly obvious that \(\sim\) is indeed an equivalence relation.) We define the quotient space

\[
\overline{X} = \text{cs}(X, d)/\sim.
\]

For an element \(\mathbf{x} \in \text{cs}(X, d)\), we denote its equivalence class by \(\overline{x}\).

Finally, for a point \(x \in X\), we define \(\langle x \rangle \in \overline{X}\), to be the equivalence class of the constant sequence \(x\) (which is obviously Cauchy).
Remark 6.5. Let \((X, d)\) be a metric space. If \(x = (x_n)_{n \geq 1}\) and \(y = (y_n)_{n \geq 1}\) are Cauchy sequences in \(X\), then the sequence of real numbers \((d(x_n, y_n))_{n \geq 1}\) is convergent. Indeed, for any \(m, n\) we have
\[
|d(x_m, y_m) - d(x_n, y_n)| \leq |d(x_m, y_m) - d(x_n, y_n)| + |d(x_n, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n).
\]
We can then define
\[
\delta(x, y) = \lim_{n \to \infty} d(x_n, y_n).
\]

Proposition 6.6. Let \((X, d)\) be a metric space.
A. The map \(\delta: \text{cs}(X, d) \times \text{cs}(X, d) \to [0, \infty)\) has the following properties:
(i) \(\delta(x, y) = \delta(y, x)\), \(\forall x, y \in \text{cs}(X, d)\);
(ii) \(\delta(x, y) \leq \delta(x, z) + \delta(z, y)\), \(\forall x, y, z \in \text{cs}(X, d)\);
(iii) \(\delta(x, y) = 0 \iff x \sim y\);
(iv) If \(x, x', y, y' \in \text{cs}(X, d)\) are such that \(x \sim x'\) and \(y \sim y'\), then \(\delta(y, x) = \delta(x', y')\).
B. The map \(\bar{d}: \bar{X} \times \bar{X} \to [0, \infty)\), correctly defined by
\[
\bar{d}(\bar{x}, \bar{y}) = \delta(x, y), \quad \forall x, y \in \text{cs}(X, d),
\]
is a metric on \(\bar{X}\).
C. The map \(X \ni x \mapsto (x) \in \bar{X}\) is isometric, in the sense that
\[
\bar{d}(x), (y) = d(x, y), \quad \forall x, y \in X.
\]

Proof. A. Properties (i), (ii) and (iii) are obvious. To prove property (iv) let \(x = (x_n)_{n \geq 1}\), \(x' = (x'_n)_{n \geq 1}\), \(y = (y_n)_{n \geq 1}\), and \(y' = (y'_n)_{n \geq 1}\). The inequality
\[
d(x'_n, y'_n) \leq d(x'_n, x_n) + d(x_n, y_n) + d(y'_n, y_n),
\]
combined with \(\lim_{n \to \infty} d(x'_n, x_n) = \lim_{n \to \infty} d(y_n, y'_n) = 0\) immediately gives
\[
\delta(x', y') = \lim_{n \to \infty} d(x'_n, y'_n) \leq \lim_{n \to \infty} d(x_n, y_n) = \delta(x, y).
\]
By symmetry we also have \(\delta(x, y) \leq \delta(x', y')\), and we are done.
B. This is immediate from A.
C. Obvious, from the definition. \(\square\)

Proposition 6.7. Let \((X, d)\) be a metric space.
(i) For any Cauchy sequence \(x = (x_n)_{n \geq 1}\) in \(X\), one has
\[
\lim_{n \to \infty} (x_n) = \bar{x}, \text{ in } \bar{X}.
\]
(ii) The metric space \((\bar{X}, \bar{d})\) is complete.

Proof. (i). For every \(n \geq 1\), we have
\[
\bar{d}(x_n, x) = \lim_{m \to \infty} d(x_n, x_m).
\]
Now if we start with some \(\varepsilon > 0\), and we choose \(N_\varepsilon\) such that
\[
d(x_n, x_m) < \varepsilon, \quad \forall m, n \geq N_\varepsilon,
\]
then (6) shows that
\[
\bar{d}(x_n, x) \leq \varepsilon, \quad \forall n \geq N_\varepsilon,
\]
so we indeed have
\[
\lim_{n \to \infty} \tilde{d}(\langle x_n \rangle, \bar{x}) = 0.
\]

(ii). Let \((p_k)_{k \geq 1}\) be a Cauchy sequence in \(\tilde{X}\). Using (i), we can choose, for each \(k \geq 1\), an element \(x_k \in X\), such that
\[
\tilde{d}(\langle x_k \rangle, p_k) \leq \frac{1}{2^k}.
\]

Claim 1: The sequence \(x = (x_k)_{k \geq 1}\) is Cauchy in \(X\).

Indeed, for \(k \geq \ell \geq 1\) we have
\[
d(x_k, x_\ell) = \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \tilde{d}(\langle x_k \rangle, p_k) + \tilde{d}(p_k, p_\ell) + \tilde{d}(\langle x_\ell \rangle, \langle x_k \rangle) \leq \tilde{d}(p_k, p_\ell) + \frac{1}{2^k}.
\]

This clearly gives
\[
\lim_{n \to \infty} \left[ \sup_{k, \ell \geq N} d(x_k, x_\ell) \right] \leq \lim_{n \to \infty} \left[ \sup_{k, \ell \geq N} \tilde{d}(p_k, p_\ell) \right] = 0,
\]
so \(x = (x_k)_{k \geq 1}\) is indeed Cauchy.

The proof of (ii) will then be finished, once we prove:

Claim 2: We have \(\lim_{n \to \infty} p_k = \bar{x}\) in \(\tilde{X}\).

To see this, we observe that, for \(\ell \geq k \geq 1\) we have the inequality
\[
(7) \quad \tilde{d}(p_k, \langle x_\ell \rangle) \leq \tilde{d}(\langle x_k \rangle, \langle x_\ell \rangle) \leq \frac{1}{2^k} + d(x_k, x_\ell).
\]

If we now start with some \(\varepsilon > 0\), and we choose \(N_\varepsilon\) such that
\[
d(x_k, x_\ell) < \varepsilon, \quad \forall k, \ell \geq N_\varepsilon,
\]
then (7) gives
\[
\tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall \ell \geq k \geq N_\varepsilon.
\]

If we keep \(k \geq N_\varepsilon\) fixed and take \(\lim_{\ell \to \infty}\), using (i) we get
\[
\tilde{d}(p_k, \bar{x}) = \lim_{\ell \to \infty} \tilde{d}(p_k, \langle x_\ell \rangle) \leq \frac{1}{2^k} + \varepsilon, \quad \forall k \geq N_\varepsilon.
\]

The above estimate clearly proves that
\[
\lim_{k \to \infty} \tilde{d}(p_k, \bar{x}) = 0,
\]
so the sequence \((p_k)_{k \geq 1}\) is convergent (to \(\bar{x}\)).

\[\square\]

Definition. The metric space \((\tilde{X}, \tilde{d})\) is called the completion of \((X, d)\).

The completion has a certain universality property. In order to formulate this property we need the following

Definition. Let \((X, d)\) and \((Y, \rho)\) be metric spaces. A map \(f : X \to Y\) is said to be a Lipschitz function, if there exists some constant \(C \geq 0\), such that
\[
\rho(f(x), f(x')) \leq C \cdot d(x, x'), \quad \forall x, x' \in X.
\]
Such a constant \(C\) is then called a Lipschitz constant for \(f\).
§6. Metric spaces

Proposition 6.8. Let $(X, d)$ be a metric space, and let $(\bar{X}, \bar{d})$ be its completion. If $(Y, \rho)$ is a complete metric space, and $f : X \to Y$ is a Lipschitz function with Lipschitz constant $C \geq 0$, then there exists a unique continuous function $\bar{f} : \bar{X} \to Y$, such that

$$\bar{f}((x)) = f(x), \ \forall x \in X.$$ 

Moreover, $\bar{f}$ is Lipschitz, with Lipschitz constant $C$.

Proof. Start with some Cauchy sequence $x = (x_n)_{n \geq 1}$ in $X$. Using the inequality

$$\rho(f(x_m), f(x_n)) \leq C \cdot d(x_m, x_n), \ \forall m, n \geq 1,$$

it is obvious that $(f(x_n))_{n \geq 1}$ is a Cauchy sequence in $Y$. Since $Y$ is complete, this sequence is convergent. Define,

$$\phi(x) = \lim_{n \to \infty} f(x_n).$$

This way we have constructed a map $\phi : \text{cs}(X, d) \to Y$.

Claim: If $x \sim x'$, then $\phi(x) = \phi(x')$.

Indeed, if $x = (x_n)_{n \geq 1}$ and $x' = (x'_{n})_{n \geq 1}$, then the Lipschitz property will give

$$\rho(f(x_n), f(x'_n)) \leq C \cdot d(x_n, x'_n), \ \forall n \geq 1,$$

and using the fact that $\lim_{n \to \infty} d(x_n, x'_n) = 0$, we get $\lim_{n \to \infty} \rho(f(x_n), f(x'_n)) = 0$. This clearly forces

$$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(x'_n).$$

Having proven the claim, we now see that we have a correctly defined map $\bar{f} : \bar{X} \to Y$, with the property that

$$\bar{f}(x) = \phi(x), \ \forall x \in \text{cs}(X, d).$$

The equality

$$\bar{f}((x)) = f(x), \ \forall x \in X$$

is trivially satisfied.

Let us check now that $\bar{f}$ is Lipschitz, with Lipschitz constant $C$. Start with two points $p, p' \in \bar{X}$, represented as $p = \bar{x}$ and $p' = \bar{x}'$, for two Cauchy sequences $x = (x_n)_{n \geq 1}$ and $x' = (x'_{n})_{n \geq 1}$ in $X$. Using the definition, we have

$$\bar{f}(p) = \lim_{n \to \infty} f(x_n) \text{ and } \bar{f}(p') = \lim_{n \to \infty} f(x'_{n}).$$

This will give

$$\rho(f(p), f(p')) = \lim_{n \to \infty} \rho(f(x_n), f(x'_{n})).$$

Notice however that

$$\rho(f(x_n), f(x'_{n})) \leq C \cdot d(x_n, x'_n), \ \forall n \geq 1,$$

so taking the limit yields

$$\rho(f(p), f(p')) = \lim_{n \to \infty} \rho(f(x_n), f(x'_n)) \leq C \cdot \lim_{n \to \infty} d(x_n, x'_n) = C \cdot \bar{d}(p, p').$$

Finally, let us show that $\bar{f}$ is unique. Let $F : \bar{X} \to Y$ be another continuous function with $F((x)) = f(x)$, for all $x \in X$. Start with an arbitrary point $p \in \bar{X}$. Let $x = (x_n)_{n \geq 1}$ be a Cauchy sequence converging to $p$. Since $X$ is complete, this sequence is convergent. Define

$$\phi(x) = \lim_{n \to \infty} f(x_n).$$

This way we have constructed a map $\phi : \text{cs}(X, d) \to Y$. Then

$$\phi(p) = \lim_{n \to \infty} f(x_n) = F((x)) = f(x).$$

Therefore, $\phi(p) = F(p)$, so $\phi = F$. This proves that $\bar{f}$ is unique.
\( \tilde{X} \), represented as \( p = x \), for some Cauchy sequence \( x = (x_n)_{n \geq 1} \) in \( X \). Since \( \lim_{n \to \infty} x_n = p \) in \( \tilde{X} \), by continuity we have

\[
F(p) = \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} f(x_n) = \phi(x) = \tilde{f}(p).
\]

\[\square\]

**Corollary 6.1.** Let \((X,d)\) be a metric space, let \((Y,\rho)\) be a complete metric space, and let \( f : X \to Y \) be an isometric map, that is

\[
\rho(f(x), f(x')) = d(x, x'), \ \forall x, x' \in X.
\]

Then the map \( \tilde{f} : \tilde{X} \to Y \), given by the above result, is isometric and \( \tilde{f}(\tilde{X}) = \overline{f(X)} \) - the closure of \( f(X) \) in \( Y \).

**Proof.** To show that \( \tilde{f}(\tilde{X}) = \overline{f(X)} \), start with some arbitrary point \( y \in \overline{f(X)} \). Then there exists a sequence \((x_n)_{n \geq 1} \subset X\), with \( \lim_{n \to \infty} f(x_n) = y \). Since \((f(x_n))_{n \geq 1}\) is Cauchy in \( Y \), and

\[
d(x_m, x_n) = \rho(f(x_m), f(x_n)), \ \forall m, n \geq 1,
\]

it follows that the sequence \( x = (x_n)_{n \geq 1} \) is cauchy in \( X \). We then have

\[
y = \lim_{n \to \infty} f(x_n) = \tilde{f}(\tilde{x}).
\]

Finally, we show that \( \tilde{f} \) is isometric. Start with two points \( p, q \in \tilde{X} \), represented as \( p = \tilde{x} \) and \( q = \tilde{z} \), for some Cauchy sequences \( x = (x_n)_{n \geq 1} \) and \( z = (z_n)_{n \geq 1} \) in \( X \). Then by construction we have

\[
\rho(\tilde{f}(p), \tilde{f}(q)) = \lim_{n \to \infty} \rho(\tilde{f}(x_n), \tilde{f}(z_n)) = \lim_{n \to \infty} \rho(f(x_n), f(z_n)) =
\]

\[
= \lim_{n \to \infty} d(x_n, z_n) = \tilde{d}(\tilde{x}, \tilde{z}) = d(p, q).
\]

\[\square\]

**Corollary 6.2.** If \((X,d)\) is a complete metric space, and \( \tilde{X} \) is its completion, then the map \( \iota : X \ni x \mapsto \langle x \rangle \in \tilde{X} \) is bijective.

**Proof.** Apply the previous result to the map \( \text{Id} : X \to X \), to get a bijective (isometric) map \( \tilde{\text{Id}} : \tilde{X} \to X \). Since the map \( \tilde{\text{Id}} \) is obviously a left inverse for \( \iota \), it follows that \( \iota \) itself is bijective. \[\square\]

Since metric spaces have “nice” features (for example the fact that one has various topological properties in terms of sequences), it is natural to address the following question: *Given a topological Hausdorff space \( X \), when does there exist a metric \( d \) on \( X \), such that the given topology coincides with the metric topology defined by \( d \)?* A topological Hausdorff space with the above property is said to be *metrizable*. It is difficult to give non-trivial necessary and sufficient conditions for metrizability. One instance in which this is possible is the compact case (see the comment following Lemma 6.1 below). Here is a useful result, which is an example of a sufficient condition for metrizability.

**Proposition 6.9 (Metrizability of Countable Products).** Let \((X_i, d_i)_{i \in I}\) be a countable family of metric spaces. Then the product space \( X = \prod_{i \in I} X_i \), equipped with the product topology, is metrizable.
PROOF. Denote by $\mathcal{T}$ the product topology on $X$. What we need is a metric $d$ on $X$, such that the maps

$$\text{Id} : (X, d) \to (X, \mathcal{T}) \quad \text{and} \quad \text{Id} : (X, \mathcal{T}) \to (X, d)$$

are continuous. (Here the notation $(X, d)$ signifies that $X$ is equipped with the metric topology defined by $d$.) For each $i \in I$, let $\pi_i : X \to X_i$ denote the projection onto the $i$th coordinate.

**Case I:** Assume $I$ is finite. In this case we define the metric $d$ on $X$ as follows. If $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ are elements in $X$, we put

$$d(x, y) = \max_{i \in I} d_i(x_i, y_i).$$

The continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the fact that all maps

$$\pi_i : (X, d) \to (X_i, d_i), \quad i \in I$$

are continuous. This is obvious, because by construction we have

$$d_i(\pi_i(x), \pi_i(y)) \leq d(x, y), \quad \forall x, y \in X.$$

Conversely, to prove the continuity of $\text{Id} : (X, \mathcal{T}) \to (X, d)$, we are going to prove that every $d$-open set is open in the product topology. It suffices to prove this only for open balls. Fix then $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ and $r > 0$, and consider the open ball $B_r(x)$. If we define, for each $i \in I$, the open ball $B_{i}^{X_i}(x_i)$, then it is obvious that

$$B_{i}^{X_i}(x_i) = \bigcap_{i \in I} \pi_i^{-1}(B_{r}(x_i)),$$

and since $\pi_i$ are all continuous, this proves that $B_r(x)$ is indeed open in the product topology.

**Case II:** Assume $I$ is infinite. In this case we identify $I = \mathbb{N}$. For every $n \in \mathbb{N}$ we define a new metric $\delta_n$ on $X_n$, as follows. If

$$\sup_{p, q \in X_n} d_n(p, q) \leq 1,$$

we put $\delta_n = d_n$. Otherwise, we define

$$\delta_n(p, p) = \frac{d_n(p, q)}{1 + d_n(p, q)}, \quad \forall p, q \in X_n.$$

It is not hard to see that the metric topology defined by $\delta_n$ coincides with the one defined by $d_n$. The advantage is that $\delta_n$ takes values in $[0, 1]$. We define the metric $d : X \times X \to [0, \infty)$, as follows. If $x = (x_n)_{n \in \mathbb{N}}$ and $y = (y_n)_{n \in \mathbb{N}}$ are elements in $\prod_{n \in \mathbb{N}} X_n$, we define

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} = \sum_{n=1}^{\infty} \frac{\delta_n(x_n, y_n)}{2^n}.$$ 

Due to the fact that $\delta_n$ takes values in $[0, 1]$, the above series is convergent, and it obviously defines a metric on $X$.

As above, the continuity of the map $\text{Id} : (X, d) \to (X, \mathcal{T})$ is equivalent to the continuity of all the maps $\pi_n : (X, d) \to (X_n, d_n)$, or equivalently for $\pi_n : (X, d) \to (X_n, \delta_n)$, $n \in \mathbb{N}$. But this is an immediate consequence of the (obvious) inequalities

$$\delta_n(\pi_n(x), \pi_n(y)) \leq 2^n \cdot d(x, y), \quad \forall x, y \in X.$$
As before, in order to prove the continuity of the other map \( \text{Id} : (X, T) \to (X, d) \), we start with some \( d \)-open set \( D \), and we show that \( D \) is open in the product topology. Since \( D \) is a union of \( d \)-open balls, we need to prove that for any \( x \in X \) and any \( r \geq 0 \), the open ball \( B_r(x) \), in \( (X, d) \), is a neighborhood of \( x \) in the product topology. Fix \( x = (x_n)_{n \in \mathbb{N}} = \prod_{n \in \mathbb{N}} X_n \), as well as \( r > 0 \). Choose some integer \( N \geq 1 \), such that
\[
\sum_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{r}{2}
\]
and define, for each \( k \in \{1, 2, \ldots, N\} \) the set
\[
D_k = \{ y = (y_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : \delta_n(x_k, y_k) < \frac{r}{2} \},
\]
It is clear that \( D_k \) is open in the product topology, for each \( k = 1, 2, \ldots, N \). (This is a consequence of the fact that \( D_k = \pi_k^{-1}(B_{r/2}(x_k)) \), where \( B_{r/2}(x_k) \) is the \( \delta_k \)-open ball in \( X_k \)). Then the set \( D = D_1 \cap D_2 \cap \cdots \cap D_N \) is also open in the product topology. Obviously we have \( x \in D \). We now prove that \( D \subset B_r(x) \). Start with some arbitrary \( y \in D \), say \( y = (y_n)_{n \in \mathbb{N}} \). On the one hand, we have
\[
\delta_k(x_k, y_k) < \frac{r}{2}, \quad \forall k \in \{1, 2, \ldots, N\},
\]
so we get
\[
\sum_{n=1}^{N} \frac{1}{2^n} \delta_n(x_n, y_n) < \frac{r}{2} \sum_{n=1}^{N} \frac{1}{2^n} < \frac{r}{2}.
\]
On the other hand, since \( \delta_n \) takes values in \( [0, 1) \), we also have
\[
\sum_{n=N+1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} < \frac{r}{2},
\]
so we get
\[
d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \delta_n(x_n, y_n) < r,
\]
thus proving that \( y \) indeed belongs to \( B_r(x) \).

**Exercise 4.** With the notations from the proof of Theorem 6.10, prove that if each \( (X_i, d_i) \) is complete, then so is \( (X, d) \).

**Exercise 5.** Let \( (X, d) \) be a metric space. Prove that, when we equip \( X \times X \) with the product topology, the map \( d : X \times X \to [0, \infty) \) is continuous.

We conclude with a discussion on compactness, and its characterization in terms of sequential convergence. The following terminology will be useful.

**Definition.** Let \( (X, d) \) be a metric space. For a number \( \varepsilon > 0 \), we say that a subset \( A \subset X \) is \( \varepsilon \)-discrete in \( X \), if \( d(a, b) \geq \varepsilon \), for all \( a, b \in A \) with \( a \neq b \).

With this terminology one has the following interesting result.

**Lemma 6.1.** For a metric space \( (X, d) \), the following conditions are equivalent.

(i) For every \( \varepsilon > 0 \), all \( \varepsilon \)-discrete subsets of \( X \) are finite;

(ii) For every \( r > 0 \), there exists a finite set \( F \subset X \), such that \( X = \bigcup_{p \in F} B_r(p) \).
(iii) For every \( \delta > 0 \), there exists an integer \( n \geq 1 \), and sets \( A_1, \ldots, A_n \subset X \) with \( A_1 \cup \cdots \cup A_n = X \), and \( \text{diam}(A_k) \leq \delta, \forall k = 1, \ldots, n \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume condition (i) holds, and let us prove (ii) by contradiction. Assume there exists some \( r > 0 \) such that, for every finite set \( F \subset X \), one has a strict inclusion

\[
\bigcup_{x \in F} B_r(p) \subseteq X.
\]

Start with some arbitrary point \( a_1 \in X \), and construct recursively a sequence \( (a_n)_{n \geq 1} \subset X \), by choosing

\[
a_{n+1} \in X \setminus \big[ B_\varepsilon(a_1) \cup \cdots \cup B_\varepsilon(a_n) \big], \quad \forall n \geq 1.
\]

This will then force

\[
d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1,
\]

so \( A = \{ a_n : n \in \mathbb{N} \} \) will be an infinite \( \varepsilon \)-discrete set, thus contradicting (i).

(ii) \( \Rightarrow \) (iii). This is trivial, since

\[
\text{diam}(B_r(p)) \leq 2r, \quad \forall r > 0, \ p \in X.
\]

(iii) \( \Rightarrow \) (i). Assume (iii), and let us prove (i). Fix \( \varepsilon > 0 \), and let \( A \subset X \) be some \( \varepsilon \)-discrete subset. Use condition (iii) to find some integer \( n \geq 1 \), and sets \( A_1, \ldots, A_n \subset X \) with \( A_1 \cup \cdots \cup A_n = X \) and \( \text{diam}(A_k) < \varepsilon, \forall k = 1, \ldots, n \). Remark that, for every \( a \in A \), there exists some \( \iota(a) \in \{ 1, \ldots, n \} \), such that \( a \in A_{\iota(a)} \). This way we have constructed a function \( \iota : A \ni a \mapsto \iota(a) \in \{ 1, \ldots, n \} \). Remark that \( \iota \) is injective. Indeed, if \( a, b \in A \) are such that \( \iota(a) = \iota(b) = k \), then \( a, b \in A_k \), so we have

\[
d(a, b) \leq \text{diam}(A_k) < \varepsilon,
\]

and then the \( \varepsilon \)-discreteness of \( A \) will force \( a = b \). Having constructed an injective map \( \iota : A \to \{ 1, \ldots, n \} \), it follows that \( A \) itself is finite. \( \square \)

**Remarks 6.6.** A. If a metric space \((X, d)\) satisfies the above equivalent conditions, then \( X \) is second countable, i.e. there is a countable base for the topology. To see this, we use condition (ii) to find, for every integer \( n \geq 1 \), a finite set \( F_n \subset X \), such that

\[
X = \bigcup_{x \in F_n} B_{\frac{1}{n}}(x).
\]

Then the countable collection

\[
\mathcal{V} = \bigcup_{n=1}^{\infty} \{ B_{\frac{1}{n}}(x) \}_{x \in F_n}
\]

is a base for the topology. To prove this fact, we need to show that if \( D \subset X \) is some open set \( D \), and \( p \in D \), then there exists \( V \in \mathcal{V} \) with \( p \in V \subset D \). First we choose \( r > 0 \) such that \( B_r(p) \subset D \), then we take an integer \( n > 2/r \) and we pick a point \( x \in F_n \) with \( p \in B_{\frac{1}{n}}(x) \). For every \( z \in B_{\frac{1}{n}}(x) \), we have

\[
d(z, p) \leq d(z, x) + d(x, p) < \frac{2}{n} < r,
\]

which means that if we take \( V = B_{\frac{1}{n}}(x) \in \mathcal{V} \), we have the \( p \in V \subset B_r(p) \subset D \).

B. If \((X, d)\) is a metric space which is compact in the metric topology, then it obviously satisfies condition (ii) in Lemma 6.1. In particular, \( X \) is second countable.
Definition. We say that a topological space $X$ is sequentially compact, if every sequence in $X$ has a convergent subsequence.

With this terminology, one has the following result.

Lemma 6.2. Suppose $X$ is a topological space which is second countable, i.e. it has a countable base for the topology. The following are equivalent:

(i) $X$ is compact;

(ii) $X$ is sequentially compact.

Proof. Fix a countable base $\mathcal{V} = \{V_k\}_{k=1}^{\infty}$ for the topology.

(i) $\Rightarrow$ (ii). Assume $X$ is compact. Start with some sequence $(x_n)_{n=1}^{\infty} \subset X$, and let us indicate how to construct a convergent subsequence. For every $n \geq 1$, we define the closed set

$$T_n = \overline{\{x_k : k > n\}}.$$

It is obvious that the family of closed sets $(T_n)_{n \geq 1}$ has the finite intersection property, i.e. for every finite set $F$ of indices, we have

$$\bigcap_{n \in F} T_n \neq \emptyset.$$

(This follows from the fact that the $T_n$'s form a decreasing sequence of sets.) By compactness, it follows that

$$\bigcap_{n \geq 1} T_n \neq \emptyset.$$

Take a point $x \in \bigcap_{n \geq 1} T_n$. Consider the set $K_x = \{k \in \mathbb{N} : \forall \mathcal{V} \ni x\}$, and let us list it as $K_x = \{k_1 < k_2 < \ldots \}$. It may happen that $K_x$ is finite, say $\{k_1 < \cdots < k_p\}$. In that case we define $k_n = k_p, \forall n > p$.

Since $\mathcal{V}$ is a base for the topology, it follows that the collection $\mathcal{V}_x = \{V_{k_n}\}_{n=1}^{\infty}$ is a basic system of neighborhoods of $x$. Define, for each integer $m \geq 1$, the set $W_m = V_{k_1} \cap \cdots \cap V_{k_m}$, so that the system $\{W_m\}_{m=1}^{\infty}$ is again a basic system of neighborhoods of $x$, but with the additional property that $W_1 \supset W_2 \supset \ldots$.

The key feature of $x$ is the given by the Followings

Claim 1: For every $\ell, m \geq 1$, there exists some integer $N(\ell, m) > \ell$ such that $x_{N(\ell, m)} \in W_m$.

This is a consequence of the fact that, for every $\ell \geq 1$, the point $x$ belongs to the closure $\{x_N : N > \ell\}$, so using the fact that $W_m$ is a neighborhood of $x$, we have

$$W_m \cap \{x_N : N > \ell\} \neq \emptyset.$$

Using Claim 1, we define a sequence $(\ell_n)_{n \geq 1}$ of integers, recursively by

$$\ell_n = N(\ell_{n-1}, n), \forall n \geq 1.$$

(The initial term $\ell_0$ is chosen arbitrarily.) We have, by construction, $\ell_0 < \ell_1 < \ell_2 < \ldots$, and

$$x_{\ell_n} \in W_n, \forall n \geq 1,$$

so $(x_{\ell_n})_{n=1}^{\infty}$ is indeed a subsequence of $(x_n)_{n=1}^{\infty}$, which is obviously convergent to $x$.

(ii) $\Rightarrow$ (i). Assume $X$ is sequentially compact, and let us prove that $X$ is compact. Start with an arbitrary Start with a collection $(D_i)_{i \in I}$ of open sets, with $\bigcup_{i \in I} D_i = X$. We need to find a finite set of indices $I_0 \subset I$, such that $\bigcup_{i \in I_0} D_i = X$.

First we show that:
Claim 2: There exists a countable set of indices $I_1 \subset I$, such that

$$\bigcup_{i \in I_1} D_i = X.$$  

For each $i \in I$, we define the set

$$M(i) = \{k \in \mathbb{N} : V_k \subset D_i\}.$$  

Since $\mathcal{V} = \{V_k\}_{k=1}^{\infty}$ is a base for the topology, we have

$$D_i = \bigcup_{k \in M(i)} V_k, \forall i \in I.$$  

Consider then the union $M = \bigcup_{i \in I} M(i)$, which is countable, being a subset of the integers. We clearly have

$$\bigcup_{k \in M} V_k = \bigcup_{i \in I} \bigcup_{k \in M(i)} V_k = \bigcup_{i \in I} D_i = X.$$  

For every $k \in M$ we choose an $i_k \in I$, such that $k \in M_{i_k}$. Then if we take

$$I_1 = \{i_k : k \in M\},$$  

then $I_1$ is obviously countable, and since we clearly have $W_k \subset D_{i_k}$, we get

$$X = \bigcup_{k \in M} V_k \subset \bigcup_{k \in M} D_{i_k} = \bigcup_{i \in I_1} D_i,$$

so the Claim is proven.

Let us list the countable set $I_1$ as

$$I_1 = \{i_k : k \geq 1\}.$$  

(Of course, if $I_1$ is already finite, there is nothing to prove. So we will assume that $I_1$ is infinite.) In order to finish the proof, we must find some $k$, such that $D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k} = X$. Assume no such $k$ can be found, which means that

$$D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k} \subset X, \forall k \geq 1.$$  

In other words, if we define for each $k \geq 1$, the closed set

$$A_k = X \setminus (D_{i_1} \cup D_{i_2} \cup \cdots \cup D_{i_k})$$

we have

$$A_k \neq \emptyset, \forall k \geq 1.$$  

For each $k \geq 1$ we choose a point $x_k \in A_k$. This way we have constructed a sequence $(x_k)_{k \geq 1} \subset X$, so using property (i) we can find a convergent subsequence. This means that we have a sequence of integers

$$1 \leq k_1 < k_2 < \ldots$$

and a point $x \in X$, such that $\lim_{n \to \infty} x_{k_n} = x$. Notice that, since

$$k_n \geq n, \forall n \geq 1,$$

and since the sequence $(A_k)_{k \geq 1}$ is decreasing, we get the fact that, for each $m \geq 1$, we have

$$x_{k_n} \in A_m, \forall n \geq m.$$
Since \( A_m \) is closed, this forces \( x \in A_m \), for all \( m \geq 1 \). But this is clearly impossible, since

\[
\bigcap_{m \geq 1} A_m = X \setminus \left( \bigcup_{m \geq 1} (D_{i_1} \cup \cdots \cup D_{i_m}) \right) = X \setminus \left( \bigcup_{i \in I_1} D_i \right) = \emptyset. \quad \Box
\]

**COMMENT.** The reader may think that Lemma 6.2 is somehow out of context here, since it deals with arbitrary topological spaces. It turns out that any second countable compact Hausdorff space is metrizable. (This is precisely the statement of Urysohn Metrizability Theorem, which is discussed in II.6.)

Going back to the problem of characterizing compactness in metric spaces, we have the following fundamental result.

**Theorem 6.1.** Let \((X, d)\) be a metric space. The following are equivalent:

(i) \( X \) is compact in the metric topology;

(ii) \( X \) is sequentially compact in the metric topology;

(iii) \((X, d)\) is complete, and for every \( \varepsilon > 0 \), all \( \varepsilon \)-discrete subsets of \( X \) are finite.

**Proof.** (i) \( \Rightarrow \) (ii). This implication is immediate from Remark 6.6.B and Lemma 6.2.

(ii) \( \Rightarrow \) (iii). Assume condition (ii) holds. By Remark Proposition 6.4, it is obvious that \((X, d)\) is complete. Fix \( \varepsilon > 0 \), as well as an arbitrary \( \varepsilon \)-discrete subset \( A \subset X \). We prove that \( A \) is finite by contradiction. Assume \( A \) is infinite. Then there exists a sequence \((a_n)_{n \geq 1} \subset A\), such that \( a_m \neq a_n \), for all \( m \neq n \), so

\[d(a_m, a_n) \geq \varepsilon, \quad \forall m > n \geq 1.\]

It is clear that no subsequence of \((a_n)_{n \geq 1}\) is Cauchy, which means that \((a_n)_{n \geq 1}\) does not have any convergent subsequence, thus contradicting (ii).

(iii) \( \Rightarrow \) (i). To get this implication, it suffices to prove (iii) \( \Rightarrow \) (ii). Indeed, by Remark 6.6.A, \( X \) is second countable, so by Lemma 6.2, sequential compactness of \( X \) will imply compactness.

Assume then \((X, d)\) has property (iii), and let us prove that \( X \) is sequentially compact. Start with an arbitrary sequence \((x_n)_{n=1}^{\infty} \subset X\), and let us indicate how one can construct a convergent subsequence. The key step in this construction is contained in the following.

**Claim:** For any infinite subset \( A \subset X \), and any \( \varepsilon > 0 \), then there exists an infinite subset \( B \subset A \), with \( \text{diam}(B) \leq \varepsilon \).

Indeed, by Lemma 6.1, we know that there exists an integer \( n \geq 1 \) and sets \( A_1, \ldots, A_n \subset X \), with \( A_1 \cup \cdots \cup A_n \), and \( \text{diam}(A_k) \leq \varepsilon, \forall k = 1, \ldots, n \). If we put \( B_k = A \cap A_k \), then we still have \( \text{diam}(B_k) \leq \varepsilon, \forall k = 1, \ldots, n \), as well as \( A = B_1 \cup \cdots \cup B_n \). Since \( A \) is infinite, one of the \( B \)'s is infinite.

Having proven the Claim, we now proceed with the construction of a convergent subsequence of \((x_n)_{n=1}^{\infty}\). Of course, if \((x_n)_{n=1}^{\infty}\) has a constant subsequence, there is nothing to prove. We can assume (after passing to a subsequence) that in fact we have \( x_m \neq x_n, \forall m > n \geq 1 \). This gives the fact that, for any infinite set of indices \( M \subset \mathbb{N} \), the set \( \{x_m : m \in M\} \) is infinite. Use the Claim, to find a sequence \((M_k)_{k=1}^{\infty}\), of infinite subsets of \( \mathbb{N} \), with

\[ (*) \quad M_1 \supset M_2 \supset \ldots; \]

\[ (**) \quad \text{diam}(\{x_m : m \in M_k\}) \leq 2^{-k}, \forall k \geq 1. \]
Using the fact that the sets $M_k$, $k \geq 1$ are all infinite, one can construct (recursively) a sequence of integers $m_1 < m_2 < \ldots$, such that $m_k \in M_k$, $\forall \, k \geq 1$. Remark now that for $k > \ell \geq 1$, using (*) one has $m_k, m_\ell \in M_{\ell}$, so by (***) we get
\[ d(x_{m_k}, x_{m_\ell}) \leq \frac{1}{2\ell}. \]
This proves that the subsequence $(x_{m_k})_{k=1}^\infty$ is Cauchy, hence convergent. \hfill \Box

**Corollary 6.3.** Let $(X, d)$ be a metric space.

A. For a subset $K \subset X$, the following are equivalent:

(i) $K$ is compact;

(ii) every sequence in $K$ has a subsequence which is convergent to a point in $K$.

B. For a subset $A \subset X$, the following are equivalent:

(i) the closure $\overline{A}$ is compact in $X$;

(ii) every sequence in $A$ has a subsequence which is convergent (to a point in $X$, not necessarily in $A$).

Moreover, if $(X, d)$ is complete, then the above conditions are also equivalent to:

(iii) for each $\varepsilon > 0$, all $\varepsilon$-discrete subsets of $A$ are finite;

(iv) for every $r > 0$, there exists a finite subset $F \subset X$, such that $A \subset \bigcup_{p \in F} B_r(p)$;

(v) for every $\delta > 0$, there exists an integer $n \geq 1$, and sets $B_1, \ldots, B_n \subset X$ with $A \subset B_1 \cup \cdots \cup B_n$, and $\text{diam}(B_k) \leq \delta$, $\forall \, k = 1, \ldots, n$.

**Proof.** A. Obvious, since the topology on $K$ is defined by the metric $d|_{K \times K}$.

B. The implication (i) $\Rightarrow$ (ii) is clear.

To prove the implication (ii) $\Rightarrow$ (i), we start by assuming that $A$ has property (ii), and we prove that every sequence in $\overline{A}$ has a subsequence, which is convergent to some point in $\overline{A}$. (By part A this will prove that $\overline{A}$ is compact.) Start with some sequence $(x_n)_{n=1}^\infty \subset \overline{A}$, and let us construct a subsequence, which converges to some point in $\overline{A}$. For each $n \geq 1$, we use the fact that $x_n \in \overline{A}$, to choose some $a_n \in A$, such that $d(a_n, x_n) < \frac{1}{n}$. Use condition (iv) to produce a subsequence $(a_{k_n})_{n=1}^\infty$, which converges to some point $x \in X$. Notice that
\[ d(x_{k_n}, x) \leq d(x_{k_n}, a_{k_n}) + d(a_{k_n}, x) \leq \frac{1}{k_n} + d(a_{k_n}, x), \quad \forall \, n \geq 1. \]
Since $\lim_{n \to \infty} k_n = \infty$, this proves that $\lim_{n \to \infty} d(x_{k_n}, x) = 0$, i.e. the subsequence $(x_{k_n})_{n=1}^\infty$ converges to $x$. Since $\overline{A}$ is closed, and $x_{k_n} \in \overline{A}$, $\forall \, n \geq 1$, it follows that $x$ itself belongs to $\overline{A}$.

To prove the other equivalent conditions, we continue with the extra assumption that $(X, d)$ is complete. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are clear.

(v) $\Rightarrow$ (i). Assume $A$ has property (v). We first observe that $\overline{A}$ is complete, being a closed subset of $X$, therefore, using Theorem 6.1 and Lemma 6.1, we see that, in order to prove that $\overline{A}$ is compact, it suffices to prove the following improved version of (v):

(v') for every $\delta > 0$, there exists an integer $n \geq 1$, and sets $A_1, \ldots, A_n \subset X$ with $\overline{A} = A_1 \cup \cdots \cup A_n$, and $\text{diam}(A_k) \leq \delta$, $\forall \, k = 1, \ldots, n$.

But this is immediate, since if one takes the sets $B_1, \ldots, B_n$ as in (v), then by Exercise 2 we also have $\text{diam}(B_k) \leq \varepsilon$, $\forall \, k = 1, \ldots, n$. Since $\overline{B_1 \cup \cdots \cup B_n}$ is closed,
it follows that \( A \subset B_1 \cup \cdots \cup B_n \), and then \((v')\) follows by taking \( A_k = A \cap B_k \), \( \forall k = 1, \ldots, n \).

**Corollary 6.4.** Let \( X \) and \( Y \) be metric spaces, and let \( f : X \to Y \) be a continuous map. If \( X \) is compact, then \( f \) is uniformly continuous, that is,

- for every \( \varepsilon > 0 \), there exists some \( \delta_\varepsilon > 0 \), such that \( d(f(x), f(x')) < \varepsilon \), for all \( x, x' \in X \) with \( d(x, x') < \delta_\varepsilon \).

**Proof.** Suppose \( f \) is not uniformly continuous, so there exists some \( \varepsilon_0 > 0 \), with the property that for any \( \delta > 0 \) there exists \( x, x' \in X \), with \( d(x, x') < \delta \), but \( d(f(x), f(x')) \geq \varepsilon_0 \). In particular, one can construct two sequences \((x_n)_{n \geq 1}\) and \((x'_n)_{n \geq 1}\) with

\[
d(x_n, x'_n) < \frac{1}{n} \quad \text{and} \quad d(f(x_n), f(x'_n)) \geq \varepsilon_0, \quad \forall n \geq 1.\tag{8}
\]

Using compactness, we can find a subsequence \((x_{n_k})_{k \geq 1}\) of \((x_n)_{n \geq 1}\) which converges to some point \( p \). On the one hand, we have

\[
d(p, x'_{n_k}) \leq d(p, x_{n_k}) + d(x_{n_k}, x'_{n_k}) < d(p, x_{n_k}) + \frac{1}{n_k}, \quad \forall k \geq 1,
\]

which proves that

\[
\lim_{k \to \infty} x'_{n_k} = p.\tag{9}
\]

On the other hand, using (8) we also have

\[
\varepsilon_0 \leq d(f(x_{n_k}), f(x'_{n_k})) \leq d(f(p), f(x_{n_k})) + d(f(p), f(x'_{n_k})),
\]

which leads to a contradiction, because the equalities

\[
\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} x'_{n_k} = p,
\]

together with the continuity of \( f \), will force

\[
\lim_{k \to \infty} d(f(p), f(x_{n_k})) = \lim_{k \to \infty} d(f(p), f(x'_{n_k})) = 0.
\]

**Remark 6.7.** Let \( X \) be a metric space. Then any compact subset \( K \subset X \) is **closed** (this is a consequence of the fact that \( X \) is Hausdorff) and **bounded**, in the sense that \( \text{diam}(K) < \infty \).

In general however the converse is not true, i.e. there are metric spaces in which closed bounded sets may fail to be compact.

**Exercise 6.** Construct an example of a complete metric space \((X, d)\), with \( \text{diam}(X) < \infty \), but such that \( X \) is not compact.

**Exercise 7.** Start with a metric space \( X \), and let \((x_n)_{n \geq 1} \subset X\) be a sequence which is convergent to some point \( x \). Prove that the set

\[
K = \{x\} \cup \{x_n : n \geq 1\}
\]

is compact in \( X \).