4. Compactness

**Definition.** Let $X$ be a topological space $X$. A subset $K \subset X$ is said to be compact set in $X$, if it has the finite open cover property:

$(f.o.c)$ Whenever $\{D_i\}_{i \in I}$ is a collection of open sets such that $K \subset \bigcup_{i \in I} D_i$, there exists a finite sub-collection $D_{i_1}, \ldots, D_{i_n}$ such that $K \subset D_{i_1} \cup \cdots \cup D_{i_n}$.

An equivalent description is the finite intersection property:

$(f.i.p.)$ If $\{F_i\}_{i \in I}$ is a collection of closed sets such that for any finite sub-collection $F_{i_1}, \ldots, F_{i_n}$ we have $K \cap F_{i_1} \cap \cdots F_{i_n} \neq \emptyset$, it follows that $K \cap \left( \bigcap_{i \in I} F_i \right) \neq \emptyset$.

A topological space $(X, T)$ is called compact if $X$ itself is a compact set.

**Remark 4.1.** Suppose $(X, T)$ is a topological space, and $K$ is a subset of $X$. Equip $K$ with the induced topology $T|_K$. Then it is straightforward from the definition that the following are equivalent:

- $K$ is compact, as a subset in $(X, T)$;
- $(K, T|_K)$ is a compact space, that is, $K$ is compact as a subset in $(K, T|_K)$.

The following three results, whose proofs are immediate from the definition, give methods of constructing compact sets.

**Proposition 4.1.** A finite union of compact sets is compact.

**Proposition 4.2.** Suppose $(X, T)$ is a topological space and $K \subset X$ is a compact set. Then for every closed set $F \subset X$, the intersection $F \cap K$ is again compact.

**Proposition 4.3.** Suppose $(X, T)$ and $(Y, S)$ are topological spaces, $f : X \to Y$ is a continuous map, and $K \subset X$ is a compact set. Then $f(K)$ is compact.

The following results discuss compactness in Hausdorff spaces.

**Proposition 4.4.** Suppose $(X, T)$ is topological Hausdorff space.

(i) Any compact set $K \subset X$ is closed.

(ii) If $K$ is a compact set, then a subset $F \subset K$ is compact, if and only if $F$ is closed (in $X$).

**Proof.** (i) The key step is contained in the following

**Claim:** For every $x \in X \setminus K$, there exists some open set $D_x$ with $x \in D_x \subset X \setminus K$.

Fix $x \in X \setminus K$. For every $y \in K$, using the Hausdorff property, we can find two open sets $U_y$ and $V_y$ with $U_y \ni x$, $V_y \ni y$, and $U_y \cap V_y = \emptyset$. Since we obviously have $K \subset \bigcup_{y \in K} V_y$, by compactness, there exist points $y_1, \ldots, y_n \in K$, such that $K \subset V_{y_1} \cup \cdots \cup V_{y_n}$. The claim immediately follows if we then define $D_x = U_{y_1} \cap \cdots \cap U_{y_n}$.

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Using the Claim we now see that we can write the complement of $K$ as a union of open sets:

$$X \setminus K = \bigcup_{x \in X \setminus K} D_x,$$

so $X \setminus K$ is open, which means that $K$ is indeed closed. (ii). If $F$ is closed, then $F$ is compact by Proposition 4.2. Conversely, if $F$ is compact, then by (i) $F$ is closed.

**Corollary 4.1.** Let $X$ be a compact Hausdorff space, let $Y$ be a topological space, and let $f : X \to Y$ be a continuous map, which is bijective. Then $f$ is a homeomorphism, i.e. the inverse map $f^{-1} : Y \to X$ is continuous.

**Proof.** What we need to prove is the fact that, whenever $D \subset X$ is open, it follows that its preimage under $f^{-1}$, that is, the set $f(D) \subset Y$, is again open. To get this fact we take complements. Consider the set $X \setminus D$, which is closed in $X$. Since $X$ is compact, by Proposition 4.2 it follows that $X \setminus D$ is compact. By Proposition 4.3, the set $f(X \setminus D)$ is compact in $Y$, and finally by Proposition 4.4 it follows that $f(X \setminus D)$ is closed. Now we are done, because the fact that $f$ is bijective gives $Y \setminus f(D) = f(X \setminus D)$, and the fact that $Y \setminus f(D)$ is closed means that $f(D)$ is indeed open in $Y$. □

**Proposition 4.5.** Every compact Hausdorff space is normal.

**Proof.** Let $X$ be a compact Hausdorff space. Let $A, B \subset X$ be two closed sets with $A \cap B = \emptyset$. We need to find two open sets $U, V \subset X$, with $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. We start with the following

**Particular case:** Assume $B$ is a singleton, $B = \{b\}$.

The proof follows line by line the first part of the proof of part (i) from Proposition 4.4. For every $a \in A$ we find open sets $U_a$ and $V_a$, such that $U_a \ni a$, $V_a \ni b$, and $U_a \cap V_a = \emptyset$. Using Proposition 4.4 we know that $A$ is compact, and since we clearly have $A \subset \bigcup_{a \in A} U_a$, there exist $a_1, \ldots, a_n \in A$, such that $U_{a_1} \cup \cdots \cup U_{a_n} \supset A$. Then we are done by taking $U = U_{a_1} \cup \cdots \cup U_{a_n}$, and $V = V_{a_1} \cap \cdots \cap V_{a_n}$.

Having proven the above particular case, we proceed now with the general case. For every $b \in B$, we use the particular case to find two open sets $U_b$ and $V_b$, with $U_b \supset A$, $V_b \ni b$, and $U_b \cap V_b = \emptyset$. Arguing as above, the set $B$ is compact, and we have $B \subset \bigcup_{b \in B} V_b$, so there exist $b_1, \ldots, b_n \in B$, such that $V_{b_1} \cup \cdots \cup V_{b_n} \supset B$. Then we are done by taking $U = U_{b_1} \cap \cdots \cap U_{b_n}$, and $V = V_{b_1} \cup \cdots \cup V_{b_n}$.

**Examples 4.1.** A. Any closed interval of the form $[a, b]$ is a compact set in $\mathbb{R}$. To prove this, we start with an arbitrary collection $\mathcal{D}$ of open subsets of $\mathbb{R}$, with $[a, b] \subset \bigcup_{D \in \mathcal{D}} D$, and we wish to find $D_1, \ldots, D_n \in \mathcal{D}$, with $D_1 \cup \cdots \cup D_n \supset [a, b]$. To get this we consider the set

$$M = \{ t \in (a, b) : \text{ there exist } D_1, \ldots, D_n \in \mathcal{D} \text{ with } D_1 \cup \cdots \cup D_n \supset [a, t] \},$$

so that all we need is the fact that $b \in M$. We first show that the number $s = \sup M$ also belongs to $M$. On the one hand, there exists some open set $D \in \mathcal{D}$ with $D \ni s$, so there exists $\varepsilon > 0$ such that $D \supset (s - \varepsilon, s + \varepsilon)$. On the other hand, there exists $t \in M$ with $s - \varepsilon < t$, so there exist $D_1, \ldots, D_n \supset [a, t] \supset [a, s - \varepsilon]$. Since we clearly have

$$D \cup D_1 \cup \cdots \cup D_n \supset [a, s - \varepsilon] \cup (s - \varepsilon, s + \varepsilon) = [a, s + \varepsilon],$$

we have $s \in M$. □
it follows that indeed \( s \) belongs to \( M \). Finally, by the same argument as above, it follows that the set \( M \) will also contain the intersection \( [s, s + \varepsilon] \cap [a, b] \), which clearly forces \( s = b \).

**B.** The above result has a famous generalization - the Bolzano-Weierstrass Theorem - which states that a set \( X \subset \mathbb{R} \) is compact, if and only if

(i) \( X \) is closed, and

(ii) \( X \) is bounded, in the sense that there exists some \( C \geq 0 \) such that

\[
|x| \leq C, \quad \forall x \in X.
\]

The fact that every compact set \( X \subset \mathbb{R} \) is closed and bounded is clear (use the finite open cover property with \( \bigcup_{n=1}^{\infty} (-n, n) = \mathbb{R} \supset X \)). Conversely, if \( X \) is closed and bounded, then \( X \) is a closed subset of some interval of the form \([-C, C]\), which is compact by A, so \( X \) itself is compact.

The Bolzano-Weierstrass Theorem has the following important consequence.

**Proposition 4.6.** Let \( X \) be a compact space, and let \( f : X \rightarrow \mathbb{R} \) be a continuous function. Then \( f \) attains its maximum and minimum values, in the sense that there exist points \( x_1, x_2 \in X \), such that

\[
f(x_1) \leq f(x) \leq f(x_2), \quad \forall x \in X.
\]

**Proof.** Consider the set \( K = f(X) \) which is compact in \( \mathbb{R} \). Since \( K \) is bounded, the quantities \( t_1 = \inf K \) and \( t_2 = \sup K \) are finite. Since \( K \) is closed, both \( t_1 \) and \( t_2 \) belong to \( K \), so there exist \( x_1, x_2 \in X \) such that \( t_1 = f(x_1) \) and \( t_2 = f(x_2) \).

The following is a useful technical result, which deals with the notion of uniform convergence.

**Theorem 4.1 (Dini).** Let \( X \) be a compact space, let \( (f_n)_{n=1}^{\infty} \) be a sequence of continuous functions \( f_n : X \rightarrow \mathbb{R} \). Assume the sequence \( (f_n)_{n=1}^{\infty} \) is monotone, in the sense that either

(i) \( f_1 \leq f_2 \leq \ldots \), or

(ii) \( f_1 \geq f_2 \geq \ldots \).

Assume \( f : X \rightarrow \mathbb{R} \) is another continuous function, with the property that the sequence \( (f_n)_{n=1}^{\infty} \) converges pointwise to \( f \), that is:

\[
\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad \forall x \in X.
\]

Then the sequence \( (f_n)_{n=1}^{\infty} \) is converges uniformly to \( f \), in the sense that

\[
\lim_{n \rightarrow \infty} \left[ \max_{x \in X} |f_n(x) - f(x)| \right] = 0.
\]

**Proof.** Replacing \( f_n \) with \( -f_n \), we can assume that \( f = 0 \), which means that we have \( \lim_{n \rightarrow \infty} f_n(x) = 0, \forall x \in X \). Replacing (if necessary) \( f_n \) with \(-f_n \), we can also assume that the sequence \( (f_n)_{n=1}^{\infty} \) satisfies (i). In particular, each \( f_n \) is non-negative.

For each \( n \geq 1 \), let us define \( s_n = \max_{x \in X} f_n(x) \), so that what we need to prove is the fact that \( \lim_{n \rightarrow \infty} s_n = 0 \). Using (i) it is clear that we have the inequalities

\[
s_1 \geq s_2 \geq \cdots \geq 0,
\]

so the desired result is equivalent to showing that

\[
\inf \{s_n : n \geq 1 \} = 0.
\]
4. Compactness

We are going to prove this equality by contradiction. Assume there exists some \( \varepsilon > 0 \), such that

\[ s_n \geq \varepsilon, \forall n \geq 1. \]

For each integer \( n \geq 1 \), let us define the set

\[ F_n = \{ x \in X : f_n(x) \geq \varepsilon \}, \]

so that (use Proposition 4.6) \( F_n \neq \emptyset \). Since \( F_n = f_n^{-1}([\varepsilon, \infty)) \), it is clear that \( F_n \) is closed, for every \( n \geq 1 \). Using \((\downarrow)\) it is obvious that we have the inclusions

\[ F_1 \supset F_2 \supset \ldots, \]

so using the finite intersection property, it follows that

\[ \bigcap_{n=1}^{\infty} F_n \neq \emptyset. \]

But this leads to a contradiction, because if we pick an element \( x \in \bigcap_{n=1}^{\infty} F_n \), then we will have \( f_n(x) \geq \varepsilon, \forall n \geq 1 \), and then the equality \( \lim_{n \to \infty} f_n(x) = 0 \) is impossible. \( \square \)

Comment. The reader is strongly encouraged to try to work the two exercises below, before reading Theorems 4.2 and 4.3. Strictly speaking these results do not require the more sophisticated machinery employed in these two theorems.

Exercise \( 1 \). Let \( X \) be a compact space. Equip the product space \( X \times \mathbb{R} \) with the product topology (here \( \mathbb{R} \) is equipped with the standard topology). Prove that a set of the form \( K = X \times [a, b] \) is compact in \( X \times \mathbb{R} \).

Hint: Start with an arbitrary collection \( D \) of open subsets of \( X \times \mathbb{R} \), with \( X \times [a, b] \subset \bigcup_{D \in D} D \), and prove there exist \( D_1, \ldots, D_n \in D \) with \( D_1 \cup \cdots \cup D_n \supset X \times [a, b] \). Argue as above, by considering the set

\[ M = \{ t \in [a, b] : \text{there exist } D_1, \ldots, D_n \in D \text{ with } D_1 \cup \cdots \cup D_n \supset X \times [a, t] \}, \]

and prove that \( b \in M \), by showing first that \( \sup M \in M \).

Exercise \( 2 \). Let \( n \geq 1 \) be an integer, and equip \( \mathbb{R}^n \) with the product topology, where \( \mathbb{R} \) is equipped with the standard topology.

A. Prove that "closed boxes" of the form \( B = [a_1, b_1] \times \cdots \times [a_n, b_n] \) are compact in \( \mathbb{R}^n \).

B. Consider the coordinate maps \( \pi_k : \mathbb{R}^n \to \mathbb{R}, \ k = 1, \ldots, n \). Declare a set \( X \subset \mathbb{R}^n \) bounded, if all sets \( \pi_k(X) \subset \mathbb{R}, \ k = 1, \ldots, n \), are bounded. Prove the general Bolzano-Weierstrass Theorem: a set \( X \subset \mathbb{R}^n \) is compact, if and only if \( X \) is closed and bounded.

C. Prove that a set \( X \subset \mathbb{C} \) is compact, if and only if \( X \) is closed and bounded, in the sense that there exists some constant \( C \geq 0 \), such that

\[ |\zeta| \leq C, \ \forall \zeta \in X. \]

Hints: A. Use induction and the preceding Exercise.

B. Argue exactly as in Example 4.1.B.

C. Analyze the two notions of boundedness (the coordinate definition given in B, and the absolute value definition given in C) and show that they are equivalent, under the canonical homeomorphism \( \mathbb{R}^2 \cong \mathbb{C} \).

Besides the two equivalent conditions (F.O.C) and (F.I.P.), there are some other useful characterizations of compactness, listed in the following.
THEOREM 4.2. Let \((X, T)\) be a topological space. The following are equivalent:

(i) \(X\) is compact.

(ii) (Alexander sub-base Theorem) There exists a sub-base \(S\) with the finite open cover property:

(s) For any collection \(\{S_i \mid i \in I\} \subset S\) with \(X = \bigcup_{i \in I} S_i\), there exists a finite sub-collection \(\{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}\) (for some finite sequence of indices \(i_1, i_2, \ldots, i_n \in I\)) such that \(X = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_n}\).

(iii) Every ultrafilter in \(X\) is convergent.

(iv) Every net in \(X\) has a convergent subnet.

PROOF. (i) \(\Rightarrow\) (ii). This is obvious. (In fact any sub-base has the open cover property.)

(ii) \(\Rightarrow\) (iii). Let \(\mathcal{U}\) be an ultrafilter on \(X\). Assume \(\mathcal{U}\) is not convergent to any point \(x \in X\). By Proposition 3.2 it follows that, for each \(x \in X\), one can find a set \(S_x \in S\) with \(S_x \ni x\), but such that \(S_x \not\in \mathcal{U}\). Using property (s), one can find a finite collection of points \(x_1, \ldots, x_n \in X\), such that

\[
S_{x_1} \cup \cdots \cup S_{x_n} = X.
\]

Since \(S_{x_p} \not\in \mathcal{U}\), it means that \(X \setminus S_{x_p}\) belongs to \(\mathcal{U}\), for every \(p = 1, \ldots, n\). Then, using (1), we get

\[
\mathcal{U} \ni (X \setminus S_{x_1}) \cap \cdots \cap (X \setminus S_{x_n}) = \emptyset,
\]

which is impossible.

(iii) \(\Rightarrow\) (iv). Assume condition (iii) holds, and let \((x_\lambda)_{\lambda \in \Lambda}\) be some net in \(X\). Denote by \(\Phi\) the collection of all finite subsets of \(\Lambda\) (including the empty set), and define, for each \(F \in \Phi\), the sets

\[
\Lambda_F = \{\lambda \in \Lambda : \lambda \succ \mu, \forall \mu u \in F\},
\]

\[
G_F = \{x_\lambda : \lambda \in \Lambda_F\}
\]

(\(use\ the\ convention\ \Lambda_\emptyset = \Lambda\). Since we obviously have \(\Lambda_{F_1} \cap \Lambda_{F_2} = \Lambda_{F_1 \cup F_2}\), \(\forall F_1, F_2 \in \Phi\), we get the equalities

\[
G_{F_1} \cap G_{F_2} = G_{F_1 \cup F_2}, \forall F_1, F_2 \in \Phi,
\]

which prove the fact that the collection \(\mathcal{G} = \{G_F\}_{F \in \Phi}\) is a filter in \(X\). Let \(\mathcal{U}\) be then some ultrafilter with \(\mathcal{U} \supseteq \mathcal{G}\). By the hypothesis (iii), the ultrafilter \(\mathcal{U}\) is convergent to some point \(x \in X\), which means that \(\mathcal{U}\) contains the collection \(\mathcal{N}_x\) of all neighborhoods of \(x\). In particular, we get the fact that

\[
N \cap G_F \neq \emptyset, \forall N \in \mathcal{N}_x, F \in \Phi.
\]

We are now in position to define a subnet of \((x_\lambda)_{\lambda \in \Lambda}\), which will prove to be convergent to \(x\). Consider the set

\[
\Sigma = \{(N, \lambda) \in \mathcal{N}_x \times \Lambda : x_\lambda \in N\},
\]

equipped with the ordering

\[
(N_1, \lambda_1) \succ (N_2, \lambda_2) \iff \begin{cases} N_1 \subset N_2 \\ \lambda_1 \succ \lambda_2 \end{cases}
\]

Let us remark that \(\Sigma\) is a directed set. Indeed, if we start with two elements \(\sigma_1 = (N_1, \lambda_1)\) and \(\sigma_2 = (N_2, \lambda_2)\) in \(\Sigma\), then using (2) with \(N = N_1 \cap N_2\), and \(F = \{\lambda_1, \lambda_2\}\), we get the existence of some \(\lambda \in \Lambda\) with \(\lambda \succ \lambda_1, \lambda \succ \lambda_2\), and such
that \( x_\lambda \in N \). This means that the pair \( \sigma = (N, \lambda) \) belongs to \( \Sigma \), and it also satisfies \( \sigma \succ \sigma_1, \sigma \succ \sigma_2 \).

Consider also the map

\[ \phi : \Sigma \ni (N, \lambda) \mapsto \lambda \in \Lambda. \]

Again using (2), it follows that \( \phi \) is a directed map.

Define then the net \( (y_\sigma)_{\sigma \in \Sigma} \) by \( y_\sigma = x_{\phi(\sigma)}, \forall \sigma \in \Sigma \). By construction, \( (y_\sigma)_{\sigma \in \Sigma} \) is a subnet of \( (x_\lambda)_{\lambda \in \Lambda} \). We show now that \( (y_\sigma)_{\sigma \in \Sigma} \) is convergent to \( x \). Start with some arbitrary neighborhood \( N \) of \( x \). Use (2) with \( F = \emptyset \), to find some \( \lambda_N \in \Lambda \), such that \( x_{\lambda_N} \in N \). Put \( \sigma_N = (N, \lambda_N) \). If \( \sigma = (V, \lambda) \in \Sigma \) is such that \( \sigma \succ \sigma_N \), then in particular we have \( x_\lambda \in M \subset N, \) i.e. \( y_\sigma \in N \). In other words we have

\[ y_\sigma \in N, \ \forall \sigma \succ \sigma_N, \]

thus proving that \( (y_\sigma)_{\sigma \in \Sigma} \) is indeed convergent to \( x \).

\((iv) \Rightarrow (i)\). Assume (iv), and let us prove that \( X \) has the finite intersection property (F.I.P.). Let \( \{F_i\}_{i \in I} \) is a collection of closed sets such that for any finite sub-collection \( F_i, \ldots, F_n \) we have \( F_1 \cap \ldots \cap F_n \neq \emptyset \), and let us prove that \( \bigcap_{i \in I} F_i \neq \emptyset \). Denote by \( \Omega \) the collection of all finite non-empty subsets of \( I \), and define, for each \( J \in \Omega \), the non-empty closed set \( F_J = \bigcap_{i \in J} F_i \). With this notation, what we have to prove is the fact that \( \bigcap_{J \in \Omega} F_J \neq \emptyset \). The advantage here is the fact that \( \Omega \) is directed (by inclusion). Since \( F_J \neq \emptyset \), for each \( J \in \Omega \), we can choose an element \( x_J \in F_J \). This way (use the Axiom of Choice) we can construct a net \( (x_J)_{J \in \Omega} \). Using (iv) there exists a subnet \( (y_\sigma)_{\sigma \in \Sigma} \subset (x_J)_{J \in \Omega} \) which is convergent to some point \( x \in X \). In order to finish the proof, it suffices to show that \( x \in F_J, \ \forall J \in \Omega \). As above, denote by \( N_x \) the collection of all neighborhoods of \( x \). Since each \( F_J \) is closed, all we need to prove is the fact that

\[ N \cap F_J \neq \emptyset, \ \forall N \in N_x, J \in \Omega. \]

Fix \( N \in N_x \) and \( J \in \Omega \). On the one hand, since \( \phi : \Sigma \to \Omega \) is a directed map, there exists \( \sigma_1 \in \Sigma \), such that \( \phi(\sigma) \succ J, \forall \sigma \succ \sigma_1 \). On the other hand, since \( (y_\sigma)_{\sigma \in \Sigma} \) is convergent to \( x \), there exists some \( \sigma_2 \in \Sigma \), such that \( y_\sigma \in N, \forall \sigma \succ \sigma_2 \). If we choose \( \sigma \in \Sigma \) such that \( \sigma \succ \sigma_1 \) and \( \sigma \succ \sigma_2 \), then on the one hand we have \( y_\sigma = x_{\phi(\sigma)} \in F_{\phi(\sigma)} \subset F_J \) (here we use the fact that, for \( J, J_1 \in \Omega \), one has \( J_1 \succ J \Rightarrow F_{J_1} \subset F_J \)), and on the other hand we also have \( y_\sigma \in N \). This means precisely that \( y_\sigma \in N \cap F_J \). \( \Box \)

An interesting application of the above result is the following:

**Theorem 4.3** (Tihonov). Suppose one has a family \( (X_i, \mathcal{T}_i)_{i \in I} \) of compact topological spaces. Then the product space \( \prod_{i \in I} X_i \) is compact in the product topology.

**Proof.** We are going to use the ultrafilter characterization (iii) from the preceding Theorem. Let \( \mathcal{U} \) be an ultrafilter on \( X = \prod_{i \in I} X_i \). Denote by \( \pi_i : X \to X_i, \) \( i \in I \) the coordinate maps. Since each \( X_i \) is compact, it follows that, for every \( i \in I \), the ultrafilter \( \pi_{i*}(\mathcal{U}) \) (in \( X_i \)) is convergent to some point \( x_i \in X_i \). If we form the element \( x = (x_i)_{i \in I} \in X \), this means that \( \pi_{i*}(\mathcal{U}) \) is convergent to \( \pi_i(x) \), for every \( i \in I \). Then, by the ultrafilter characterization of the product topology (see section 3) it follows that \( \mathcal{U} \) is convergent to \( x \). \( \Box \)
Comment. Another interesting application of Theorem 4.2 is the following construction. Suppose \((X, \mathcal{T})\) is a compact Hausdorff space, and \((x_i)_{i \in I} \subset X\) is an arbitrary family of elements. (Here \(I\) is an arbitrary set.) Suppose \(\mathcal{U}\) is an ultrafilter on \(I\). If we regard the family \((x_i)_{i \in I}\) simply as a function \(f : I \to X\), then we can construct the ultrafilter \(f^*(\mathcal{U})\) on \(X\). More explicitly

\[
f^*(\mathcal{U}) = \{ W \subset X : \text{the set}\{i \in I : x_i \in W\} \text{ belongs to } \mathcal{U}\}.
\]

Since \(X\) is compact Hausdorff, the ultrafilter \(f^*(\mathcal{U})\) is convergent to some unique point \(x \in X\). This point is denoted by \(\lim_{\mathcal{U}} x_i\).