Probability

Days Six–Seven

\(\sigma\)-Rings, \(\sigma\)-Algebras and Monotone Classes

Today we extend several notions introduced in Days One–Two. These extensions are aimed at adding more “flexibility.”

**Terminology.** Let \(X\) be some non-empty set, and let \(\mathcal{C}\) be a collection of subsets of \(X\). We introduce the following two technical conditions:

(\(\sigma\)) Whenever \((A_n)_{n=1}^\infty\) is a sequence of sets in \(\mathcal{C}\), it follows that the union \(\bigcup_{n=1}^\infty A_n\) again belongs to \(\mathcal{C}\).

(\(\delta\)) Whenever \((A_n)_{n=1}^\infty\) is a sequence of sets in \(\mathcal{C}\), it follows that the intersection \(\bigcap_{n=1}^\infty A_n\) again belongs to \(\mathcal{C}\).

**Exercise 1.** Let \(X\) be some non-empty set, and let \(\mathcal{C} \subset \mathcal{P}(X)\) be a collection, which is complemented in \(X\), in the sense that

\[
\bullet \quad A \in \mathcal{C} \Rightarrow X \setminus A \in \mathcal{C}.
\]

Prove that \(\mathcal{C}\) has property (\(\sigma\)), if and only if \(\mathcal{C}\) has property (\(\delta\)).

**Definition.** Let \(X\) be some non-empty set. A ring \(\mathcal{R}\) of sets in \(X\), which has property (\(\sigma\)), is called a \(\sigma\)-ring. An algebra \(\mathcal{A}\) of sets in \(X\), which has property (\(\sigma\)), is called a \(\sigma\)-algebra.

**Remark 1.** If \(\mathcal{R}\) is a \(\sigma\)-ring, then \(\mathcal{R}\) has property (\(\delta\)). Indeed, if one starts with some sequence \((A_n)_{n=1}^\infty \subset \mathcal{R}\), and if we form the sets \(B_n = A_1 \setminus A_n\), then \(B_n \in \mathcal{R}\), \(\forall n \geq 1\), so the union \(B = \bigcup_{n=1}^\infty B_n\) again belongs to \(\mathcal{R}\). Then \(\mathcal{R}\) will also contain

\[
A_1 \setminus B = A_1 \setminus \left[ \bigcap_{n=1}^\infty (A_1 \setminus A_n) \right] = \bigcap_{n=1}^\infty A_n.
\]

**Exercise 2.** (“Working” definition for \(\sigma\)-rings.) Let \(X\) be some non-empty set, and let \(\mathcal{R}\) be some non-empty collection of subsets of \(X\). Prove that the following are equivalent:

(i) \(\mathcal{R}\) is a \(\sigma\)-ring of sets in \(X\).

(ii) \(\mathcal{R}\) has property (\(\sigma\)) and:

\[
\bullet \quad A, B \in \mathcal{R} \Rightarrow A \setminus B \in \mathcal{R},
\]

**Exercise 3.** (“Working” definition for \(\sigma\)-algebras.) Let \(X\) be some non-empty set, and let \(\mathcal{A}\) be some non-empty collection of subsets of \(X\). Prove that the following are equivalent:
(i) $\mathcal{A}$ is a $\sigma$-algebra of sets in $X$.

(ii) $\mathcal{A}$ has property $(\sigma)$ and is complemented in $X$.

(iii) $\mathcal{A}$ has property $(\delta)$ and is complemented in $X$.

**Fact 1.** Given a family $(\mathcal{A}_i)_{i \in I}$ of $\sigma$-rings (or $\sigma$-algebras) of sets in $X$, the intersection $\bigcap_{i \in I} \mathcal{A}_i$ is again a $\sigma$-ring (or $\sigma$-algebra) of sets in $X$.

**Exercise 4.** Prove Fact 1.

Using Fact 1, we see that given an arbitrary collection $\mathcal{C} \subset \mathcal{P}(X)$, we can consider the families

$$
\Gamma = \{ \mathcal{R} : \mathcal{R} \text{ $\sigma$-ring of sets in } X, \text{ with } \mathcal{R} \supset \mathcal{C} \},
$$

$$
\Theta = \{ \mathcal{A} : \mathcal{A} \text{ $\sigma$-algebra of sets in } X, \text{ with } \mathcal{A} \supset \mathcal{C} \},
$$

which are both *non-empty*, since they contain for example $\mathcal{P}(X)$. The intersection

$$
\mathbf{S}(\mathcal{C}) = \bigcap_{\mathcal{R} \in \Gamma} \mathcal{R},
$$

which is a $\sigma$-ring, is referred to as the *$\sigma$-ring on $X$ generated by $\mathcal{C}$*. The intersection

$$
\mathbf{\Sigma}(\mathcal{C}) = \bigcap_{\mathcal{A} \in \Theta} \mathcal{A},
$$

which is a $\sigma$-algebra, is referred to as the *algebra on $X$ generated by $\mathcal{C}$*. The features of these collections are as follows.

**Fact 2.** Let $\mathcal{C}$ be some non-empty collection of subsets of $X$.

(i) $\mathbf{S}(\mathcal{C})$ is the smallest $\sigma$-ring that contains $\mathcal{C}$, that is, whenever $\mathcal{R}$ is a $\sigma$-ring that contains $\mathcal{C}$, it follows that $\mathcal{R}$ also contains $\mathbf{S}(\mathcal{C})$.

(ii) $\mathbf{\Sigma}(\mathcal{C})$ is the smallest $\sigma$-algebra that contains $\mathcal{C}$, that is, whenever $\mathcal{A}$ is a $\sigma$-algebra that contains $\mathcal{C}$, it follows that $\mathcal{A}$ also contains $\mathbf{\Sigma}(\mathcal{C})$.

**Exercise 5.** Prove Fact 2.

Unfortunately, there is no easy way of constructing the $\sigma$-ring (or $\sigma$-algebra) generated by an arbitrary collection. One way to circumvent this problem is by introducing the following two refined versions of properties $(\sigma)$ and $(\delta)$, which refer to a collection $\mathcal{C}$ of subsets of $X$:

$(\sigma_1)$ Whenever $(\mathcal{A}_n)_{n=1}^\infty$ is a sequence of sets in $\mathcal{C}$, which is increasing, that is $A_1 \subset A_2 \subset \ldots$, it follows that the union $\bigcup_{n=1}^\infty A_n$ also belongs to $\mathcal{C}$. 

2
Whenever \((A_n)_{n=1}^\infty\) is a sequence of sets in \(C\), which is decreasing, that is \(A_1 \supset A_2 \supset \ldots\), it follows that the intersection \(\bigcap_{n=1}^\infty A_n\) also belongs to \(C\).

We also introduce a third technical condition, which is a refinement of property \((\sigma)\)

\((\sigma_0)\) Whenever \((A_n)_{n=1}^\infty\) is a sequence of disjoint sets in \(C\), it follows that the union \(\bigcup_{n=1}^\infty A_n\) again belongs to \(C\).

With this terminology one has the following result.

**Proposition 1.** For a ring \(R\) of sets in \(X\), the following are equivalent:

(i) \(R\) is a \(\sigma\)-ring;

(ii) \(R\) has property \((\sigma_0)\).

(iii) \(R\) has property \((\sigma_\uparrow)\).

**Proof.** The implication \((i) \Rightarrow (ii)\) is immediate, since one clearly has the implication \((\sigma) \Rightarrow (\sigma_0)\).

To prove the implication \((\sigma_0) \Rightarrow (\sigma_\uparrow)\), we assume condition \((\sigma_0)\), we start with some increasing sequence \(A_1 \subset A_2 \subset \ldots\) in \(R\), and we show that the union \(A = \bigcup_{n=1}^\infty A_n\) again belongs to \(R\). To get this fact, using \((\sigma_0)\), all we need is to write \(A\) as a union of a sequence of disjoint sets in \(R\). If we define the sets \(D_1 = A_1\) and

\[
D_n = A_n \setminus (A_1 \cup \cdots \cup A_{n-1}), \quad \forall n \geq 2,
\]

then on the one hand all the \(D\)'s are in \(R\) (which is a ring). On the other hand it is obvious that all the \(D\)'s are disjoint, and \(A = \bigcup_{n=1}^\infty D_n\).

To prove the implication \((iii) \Rightarrow (i)\), we assume \(R\) has property \((\sigma_\uparrow)\), we fix some arbitrary sequence \((A_n)_{n=1}^\infty\) in \(R\), and we show that the union \(A = \bigcup_{n=1}^\infty A_n\) again belongs to \(R\). Using the fact that \(R\) is a ring, it follows that for every \(n \geq 1\), the finite union \(B_n = \bigcup_{k=1}^n A_k\) belongs to \(R\). Since we have \(B_1 \subset B_2 \subset \ldots\), with \(\bigcup_{n=1}^\infty B_n = A\), by property \((\sigma_\uparrow)\), it follows that \(A\) indeed belongs to \(R\). \(\square\)

**Definition.** A collection \(M\) of subsets in \(X\) is called a **monotone class**, if it has both properties \((\sigma_\uparrow)\) and \((\delta_\downarrow)\).

With this terminology, Proposition 1 has the following consequence:

**Corollary 1.** For a ring \(R\) of sets in \(X\), the following are equivalent:

(i) \(R\) is a \(\sigma\)-ring;

(ii) \(R\) is a monotone class. \(\square\)
Fact 3. Given a family \( \{M_i\}_{i \in I} \) of monotone classes in \( X \), the intersection \( \bigcap_{i \in I} M_i \) is again a monotone class in \( X \).


Using Fact 3, we see that given an arbitrary collection \( C \subseteq \mathcal{P}(X) \), we can consider the family
\[
M = \{ M : M \text{ monotone class in } X, \text{ with } M \supset C \},
\]
which is non-empty, since it contains for example \( \mathcal{P}(X) \). The intersection
\[
M(C) = \bigcap_{M \in M} M,
\]
which is a monotone class, is referred to as the monotone class in \( X \) generated by \( C \). The main feature is follows.

Fact 4. Let \( C \) be a collection of subsets in \( X \). The collection \( M(C) \) is the smallest monotone class in \( X \) that contains \( C \), that is, whenever \( M \) is a monotone class in \( X \) that contains \( C \), it follows that \( M \) also contains \( M(C) \).


Using this terminology, has the following important result.

Monotone Class Theorem. Let \( C \) be a collection of sets in \( X \), and let \( R(C) \) and \( A(C) \) be the ring and algebra generated by \( C \). The \( \sigma \)-ring \( S(C) \) and the \( \sigma \)-algebra \( \Sigma(C) \) generated by \( C \), are given as: \( S(C) = M(R(C)) \) and \( \Sigma(C) = M(A(C)) \).

Proof. Let us first observe that since \( S(C) \) is a ring that contains \( C \), it follows that \( S(C) \) contains the ring \( R(C) \) generated by \( C \). Secondly, since \( S(C) \) is also a monotone class, which contains \( R(C) \), it follows that \( S(C) \) contains the monotone class generated by \( R(C) \), that is, one has the inclusion
\[
S(C) \supset M(R(C)).
\]
To prove the other inclusion, since we clearly have the inclusion \( M(R(C)) \supset C \), we see that all we need to show is the fact that \( M(R(C)) \) is a \( \sigma \)-ring. Since \( M(R(C)) \) is a monotone class, by Proposition 1, it suffices to show that \( M(R(C)) \) is just a ring. Denote for simplicity \( R(C) \) by \( R \), so what we need to prove (see the “working definition” from Days One–Two) is the fact that, for any two sets \( A, B \in M(R) \), the sets \( A \setminus B \) and \( A \cup B \) again belong to \( M(R) \). We analyze first the following
Particular Case: Assume $A \in \mathcal{R}$.

To prove that $A \setminus B, A \cup B \in M(\mathcal{R}), \forall B \in M(\mathcal{R})$, we argue indirectly, as follows. We fix $A \in \mathcal{R}$, and we define the collection

$$\mathcal{B} = \{ B \subset X : A \setminus B, A \cup B \in M(\mathcal{R}) \},$$

so that we need to show the inclusion

$$M(\mathcal{R}) \subset \mathcal{B},$$

and we prove this inclusion by showing that $\mathcal{B}$ is a monotone class, which contains $\mathcal{R}$. The fact that $\mathcal{B}$ contains $\mathcal{R}$ is trivial, since one has the implications

$$A, B \in \mathcal{R} \Rightarrow A \setminus B, A \cup B \in \mathcal{R}.$$

To show that $\mathcal{B}$ is a monotone class, we must check properties $(\sigma \uparrow)$ and $(\delta \downarrow)$.

To check property $(\sigma \uparrow)$ we start with an increasing sequence $B_1 \subset B_2 \subset \ldots$ in $\mathcal{B}$ and we show that the union $B = \bigcup_{n=1}^{\infty} B_n$ again belongs to $\mathcal{B}$. This is however clear, since one has

- $A \setminus B = \bigcap_{n=1}^{\infty} (A \setminus B_n)$, with $(A \setminus B_n)_{n=1}^{\infty}$ a decreasing sequence in the monotone class $M(\mathcal{R})$,
- $A \cup B = \bigcup_{n=1}^{\infty} (A \cup B_n)$, with $(A \cup B_n)_{n=1}^{\infty}$ an increasing sequence in the monotone class $M(\mathcal{R})$,

so $A \setminus B$ and $A \cup B$ both belong to $M(\mathcal{R})$.

To check property $(\delta \downarrow)$ we start with a decreasing sequence $B_1 \supset B_2 \supset \ldots$ in $\mathcal{B}$ and we show that the intersection $B = \bigcap_{n=1}^{\infty} B_n$ again belongs to $\mathcal{B}$. This is however clear, since one has

- $A \setminus B = \bigcup_{n=1}^{\infty} (A \setminus B_n)$, with $(A \setminus B_n)_{n=1}^{\infty}$ an increasing sequence in the monotone class $M(\mathcal{R})$,
- $A \cup B = \bigcap_{n=1}^{\infty} (A \cup B_n)$, with $(A \cup B_n)_{n=1}^{\infty}$ a decreasing sequence in the monotone class $M(\mathcal{R})$,

so $A \setminus B$ and $A \cup B$ both belong to $M(\mathcal{R})$.

Having proven the particular case, we proceed with the general case. Fix $B \in M(\mathcal{R})$ and let us show that

$$A \setminus B, A \cup B \in M(\mathcal{R}), \forall A \in M(\mathcal{R}).$$

We argue again indirectly, by defining

$$\mathcal{A} = \{ A \subset X : A \setminus B, A \cup B \in M(\mathcal{R}) \},$$

so that we need to show that $\mathcal{A}$ is a monotone class, which contains $\mathcal{R}$. The fact that $\mathcal{A}$ contains $\mathcal{R}$ follows from the particular case above. The fact that $\mathcal{A}$ is a monotone class is proven exactly as above.
The other equality $\Sigma(\mathcal{C}) = M(A(\mathcal{C}))$ is immediate. \qed

The following "estimate" result can often be employed to test whether a set does not belong to the $\sigma$-ring generated by a collection.

**Proposition 2.** Let $\mathcal{C}$ be a collection of sets in $X$. If a set $A$ belongs to the $\sigma$-ring $S(\mathcal{C})$ generated by $\mathcal{C}$, then there exists a sequence $(B_n)_{n=1}^{\infty} \subset \mathcal{C}$, such that $A \subset \bigcup_{n=1}^{\infty} B_n$.

**Proof.** Define the collection

$$U = \{ A \subset X : \text{there exists } (B_n)_{n=1}^{\infty} \subset \mathcal{C}, \text{ such that } A \subset \bigcup_{n=1}^{\infty} B_n \},$$

so that what we need to prove is the inclusion

$$U \supset S(\mathcal{C}).$$

We prove this inclusion by shown that $U$ is a $\sigma$-ring, which contains $\mathcal{C}$. The inclusion $U \supset \mathcal{C}$ is trivial. To check that $U$ is a $\sigma$-ring, we must show that

- $(\sigma)$ whenever $(A_n)_{n=1}^{\infty}$ is a sequence in $U$, it follows that the union $\bigcup_{n=1}^{\infty} A_n$ again belongs to $U$;
- $(\ast) A, B \in U \Rightarrow A \setminus B \in U$.

Property $(\ast)$ is trivial, since whenever $A$ belongs to $U$, it follows that $U$ contains all subsets of $A$. Property $(\sigma)$ is also pretty clear, since if one takes for each $n \in \mathbb{N}$ a countable collection in $\mathcal{C}$ whose union covers $A_n$, then putting all these collections together yields another countable collection in $\mathcal{C}$ whose union covers $\bigcup_{n=1}^{\infty} A_n$. \qed

**Exercise 8.** Prove that a finite collection $\mathcal{C}$ of subsets of $X$ is always a monotone class.

**Exercise 9.** Prove that a finite ring (or algebra) in $X$ is a $\sigma$-ring (or $\sigma$-algebra).