ON SPECTRA OF LÜDERS OPERATIONS

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ABSTRACT. Given an $n$-tuple $A = (A_1, \ldots, A_n)$ consisting of positive operators on a Hilbert space $\mathcal{H}$, satisfying $\sum_{j=1}^{n} A_j = I$, we discuss the equation $\Lambda_A(X) = I - X$, where $\Lambda_A$ is the Lüders operation:

$$\Lambda_A : \mathcal{B}(\mathcal{H}) \ni X \mapsto \sum_{j=1}^{n} \frac{A_j^{1/2} X A_j^{1/2}}{2} \in \mathcal{B}(\mathcal{H})$$

and we identify two fairly general cases when the only solution is the “expected” one $X = \frac{1}{2} I$, namely

(a) when $A$ consists of commuting elements in a unital Banach algebra (in this case $\mathcal{B}(\mathcal{H})$ is replaced by a unital Banach algebra, and the positivity condition is stated in terms of spectra), or
(b) when the von Neumann algebra $\mathfrak{M} = \mathfrak{A}''$ is finite, and, when viewed as a von Neumann sub-algebra of $\mathcal{M} = (\mathfrak{A} \cup \{X, X^*\})''$, there exists a faithful conditional expectation of $\mathcal{M}$ onto $\mathfrak{M}$.

1. Introduction

Given an $n$-tuple $A = (A_1, \ldots, A_n)$ consisting of positive operators on a Hilbert space $\mathcal{H}$, satisfying the identity $\sum_{k=1}^{n} A_k = I$, where $I$ denotes the identity operator, the Lüders operation associated with $A$ is the map $\Lambda_A : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, defined by

$$\Lambda_A(B) = \sum_{k=1}^{n} \frac{A_k^{1/2} B A_k^{1/2}}{2}, \quad B \in \mathcal{B}(\mathcal{H}).$$

Using the terminology from [1] the Lüders operation is a special case of quantum operations, constructed using arbitrary $n$-tuples $T = (T_1, \ldots, T_n)$ satisfying $\sum_{k=1}^{n} T_k T_k^* \leq I$, as the maps $\Phi_T$ defined by

$$\Phi_T(B) = \sum_{k=1}^{n} T_k B T_k^*, \quad B \in \mathcal{B}(\mathcal{H}).$$

The maps (1) have been extensively investigated by Arias, Gheondea and Gudder in [1], where the focus was the study of the fixed point subspace, i.e. the space

$$\mathcal{B}(\mathcal{H})^{\Phi_T} = \{B \in \mathcal{B}(\mathcal{H}) : \Phi_T(B) = B\}.$$ 

The main motivation for [1] was an older result of Gudder and this author (see [3]), where it was shown that, in the case $n = 2$, that is, when one uses the system $\mathcal{A} = (A, I - A)$, with $0 \leq A \leq I$, then the fixed point space $\mathcal{B}(\mathcal{H})^{\Lambda_A}$, of the Lüders operation, coincides with the commutant of $A$, i.e. the set $\{A\}' = \{B \in \mathcal{B}(\mathcal{H}) : AB = BA\}$. One of the results from [1] states that, if $\mathcal{H}$ is finite dimensional, then the fixed point space of an arbitrary Lüders operation $\Lambda_A$ on $\mathcal{H}$ again coincides with the commutant of $\mathcal{A}$, i.e. $\mathcal{A}' = \{B \in \mathcal{B}(\mathcal{H}) : A_k B = B A_k, \forall k\}$. (A similar
conclusion holds for quantum operations as well.) In [1] it has been conjectured that the equality

$$B(\mathcal{H})^{A} = A'$$

does hold if $A'$ is injective. As shown in [1], injectivity is a necessary condition for (2). The test case for this conjecture is when $A$ is abelian, and even in this case there is no proof available.

Another problem of interest, discussed in [2], was the operator equation

$$\Lambda (A(B)) = I - B,$$

where $A = (A_1, \ldots, A_n)$ is, as before, an $n$-tuple of positive operators satisfying $\sum_{k=1}^{n} A_k = I$. For future reference, we will call such an $A$ a Lüders system on $\mathcal{H}$. Not surprisingly, it can be easily proven that, if $A$ is a Lüders pair (i.e. $n = 2$), then the equation (3) has only one solution: $B = \frac{1}{2}I$. Gudder [2] proved this in the finite dimensional case, and Wang, Du and Zuo [6] generalized it to the infinite dimensional case. It is worth pointing out that in both proofs, the fact that $A$ is abelian (forced by the fact that $A$ has two elements $A$ and $I - A$) plays an important role. One may then easily conjecture that the same conclusion holds for arbitrary abelian Lüders systems. As it turns out, if the abelian version of the above mentioned conjecture holds (i.e. $A$ abelian $\Rightarrow B(\mathcal{H})^{A} = A'$), then the equation (3) indeed has only the “expected” solution, if $A$ is abelian. To see this, all one needs to do is to notice the fact that (3) implies that $B$ is a fixed point of the operation $\Sigma = \Lambda (A) \circ \Lambda (A)$, and $\Sigma = \Lambda (A(2))$ – the Lüders operation associated with the system $A^{(2)} = (A_j A_k)_{j,k=1}^{n}$ (consisting of $n^2$ elements).

The purpose of this note is to show directly that, the equation (3) has only one solution (the “expected” one), in the cases of interest suggested in [1], thus offering two generalizations of the results of Gudder, Wang, Du and Zuo. Our approach will employ two results (Theorems 1 and 2 below) concerning the spectrum of certain general operations involving (spectrally) positive elements.

2. Results

The first result provides the needed spectral estimate in the abelian case.

**Theorem 1.** Let $\mathfrak{A}$ be a unital Banach algebra, and let $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be a $2n$-tuple of commuting elements in $\mathfrak{A}$. If the spectra of all $x$’s and all $y$’s are contained in $[0, \infty)$, then the linear operator $\Theta : \mathfrak{A} \to \mathfrak{A}$, defined by

$$\Theta (a) = \sum_{k=1}^{n} x_k a y_k, \ a \in \mathfrak{A},$$

is continuous and has its spectrum contained in $[0, \infty)$.

**Proof.** Continuity is pretty obvious. Define, for any $b \in \mathfrak{A}$, the linear operators $L_b, R_b : \mathfrak{A} \to \mathfrak{A}$ by $L_b a = ba$ and $R_b a = ab, \forall a \in \mathfrak{A}$. It is well known that, for any $b \in \mathfrak{A}$, the linear operators $L_b$ and $R_b$ are both continuous (in fact one has $\|L_b\| = \|R_b\| = \|b\|$), and, as elements in the Banach algebra $\mathcal{L}(\mathfrak{A})$ of all linear continuous operators on $\mathfrak{A}$, their spectra are given by

$$\text{Spec}_{\mathcal{L}(\mathfrak{A})}(L_b) = \text{Spec}_{\mathcal{L}(\mathfrak{A})}(R_b) = \text{Spec}_{\mathfrak{A}}(b).$$

In particular, all spectra $\text{Spec}_{\mathcal{L}(\mathfrak{A})}(L_{x_j}), \text{Spec}_{\mathcal{L}(\mathfrak{A})}(R_{y_j}), j = 1, \ldots, n$ are contained in $[0, \infty)$. 
Consider then sub-algebra $\mathcal{M}$ generated by the set
$$S = \{I, L_{x_1}, \ldots, L_{x_n}, R_{y_1}, \ldots, R_{y_n}\},$$
which is a unital abelian sub-algebra of $L(\mathfrak{A})$. The fact that $\mathcal{M}$ is abelian follows easily from two simple observations:

(i) $R_b L_c = L_c R_b, \forall b, c \in \mathfrak{A}$;
(ii) if $b, c \in \mathfrak{A}$ commute, then $R_b$ and $R_c$ commute, and likewise, $L_b$ and $L_c$ commute.

If we define now the enhanced set
$$\mathcal{E} = \mathcal{M} \cup \bigcup_{T \in \mathcal{M}} \{ (T - \lambda I)^{-1} : \lambda \in \mathbb{C} \setminus \text{Spec}_{L(\mathfrak{A})}(T) \},$$
then $\mathcal{E}$ is again abelian, and so is the Banach sub-algebra $\mathcal{B} \subset L(\mathfrak{A})$, generated by $\mathcal{E}$. By construction, we have the spectral permanence equalities

$$\text{Spec}_{\mathcal{B}}(T) = \text{Spec}_{L(\mathfrak{A})}(T), \forall T \in \mathcal{M}. \quad (4)$$

Consider now the Gelfand transform $\mathcal{B} \ni T \mapsto \hat{T} \in C(\Gamma_{\mathcal{B}})$, defined by $\hat{T}(\gamma) = \gamma(T), \forall T \in \mathcal{B}, \gamma \in \Gamma_{\mathcal{B}}$, where $\Gamma_{\mathcal{B}}$ denotes character space. It is well-known that, for every $T \in \mathcal{B}$ one has the equality

$$\text{Spec}_{\mathcal{B}}(T) = \text{Range } \hat{T}. \quad (5)$$

In particular, if consider the elements $X_j = L_{x_j}, Y_j = R_{y_j} \in \mathcal{M}, j = 1, \ldots, n$, so that we have $\Theta = \sum_{k=1}^{n} X_j Y_j \in \mathcal{M}$, then using (4) we obtain

$$\text{Spec}_{L(\mathfrak{A})}(\Theta) = \text{Spec}_{\mathcal{B}}(\Theta) = \text{Range } \left( \sum_{j=1}^{n} \hat{X}_j \hat{Y}_j \right). \quad (6)$$

Using again (4), we know that $\text{Spec}_{\mathcal{B}}(X_j), \text{Spec}_{\mathcal{B}}(Y_j) \subset [0, \infty), j = 1, \ldots, n$, which by (5) means that the continuous functions $\hat{X}_1, \ldots, \hat{X}_n, \hat{Y}_1, \ldots, \hat{Y}_n \in C(\Gamma_{\mathcal{B}})$ all have their ranges in $[0, \infty)$. It now follows immediately that the function $\sum_{j=1}^{n} \hat{X}_j \hat{Y}_j \in C(\Gamma_{\mathcal{B}})$ also has its range in $[0, \infty)$, so going back to (6), we get the desired inclusion $\text{Spec}_{L(\mathfrak{A})}(\Theta) \subset [0, \infty)$. ☐

**Corollary 1.** Assume $\mathfrak{A}$ is a unital Banach algebra, and $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{A}$ are commuting elements, each having its spectrum contained in $[0, \infty]$, such that $\sum_{j=1}^{n} x_j y_j = 1$. For any $\lambda \in \mathbb{C}$ and any $\mu \in (0, \infty)$, the equation

$$\sum_{j=1}^{n} x_j z y_j = \lambda(1 + \mu)1 - \mu z, \quad z \in \mathfrak{A} \quad (7)$$

has only one solution: $z = \lambda 1$.

**Proof.** By Theorem 1, the linear continuous map
$$\Theta : \mathfrak{A} \ni a \mapsto \sum_{j=1}^{n} x_j a y_j \in \mathfrak{A}.$$

has its spectrum contained in $[0, \infty)$. In particular, for any $\mu > 0$ the linear continuous map $\Theta + \mu \text{Id} : \mathfrak{A} \to \mathfrak{A}$ is invertible, thus the only solution of the equation

$$\Theta(a) = -\mu a \quad (8)$$

is $a = \mu 1$. \hfill \square
is the trivial one: $a = 0$.

Suppose now $z$ is a solution of (7). Since $\Theta(1) = 1$, if we subtract $\Theta(\lambda 1) = \lambda 1$ from both sides of (7), we get

$$\Theta(z - \lambda 1) = \mu(\lambda 1 - z),$$

which means that $a = z - \lambda 1$ is a solution of (8), so we indeed have $z = \lambda 1$. □

**Corollary 2.** If $\mathcal{H}$ is a Hilbert space, and $A = (A_1, \ldots, A_n)$ is a Lüders system on $\mathcal{H}$, consisting of commuting operators, then the equation

$$\Lambda_A(B) = I - B, \ B \in \mathcal{B}(\mathcal{H}),$$

has exactly one solution: $B = \frac{1}{2}I$.

**Proof.** Immediate from Corollary 1, applied to $\mathfrak{A} = \mathcal{B}(\mathcal{H})$ with $x_j = y_j = A_j^{1/2}$, $\lambda = \frac{1}{2}$ and $\mu = 1$. □

We now turn our attention to the second special case, for which we introduce the following terminology.

**Definition.** Given a von Neumann algebra $\mathcal{M}$, we call a von Neumann sub-algebra $\mathfrak{A} \subset \mathcal{M}$ faithfully injected in $\mathcal{M}$, if

(i) $\mathfrak{A}$ contains the unit of $\mathcal{M}$, and
(ii) there exists a faithful conditional expectation $E$ of $\mathcal{M}$ onto $\mathfrak{A}$.

Recall that, given unital C*-algebras $\mathfrak{A} \subset \mathcal{M}$ with common unit, the condition that a linear map $E : \mathcal{M} \to \mathcal{M}$ is a conditional expectation of $\mathcal{M}$ onto $\mathfrak{A}$ means that $E$ is a contractive idempotent (i.e. $\|E\| = 1$ and $E \circ E = E$) with Range $E = \mathfrak{A}$. According to Tomiyama’s Theorem ([5]), such a map $E$ is completely positive and also satisfies the condition

$$E(ab) = aE(b), \ \forall a, b \in \mathfrak{A}, x \in \mathcal{M}.$$  

The condition that $E$ is faithful means that, whenever $x \in \mathcal{M}$ is a positive element such that $E(x) = 0$, it follows that $x = 0$.

With this terminology, one has the following result.

**Theorem 2.** Assume $\mathcal{M}$ is a von Neumann algebra, and $\mathfrak{A}$ is a finite faithfully injected von Neumann sub-algebra of $\mathcal{M}$. If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathfrak{A}$ are positive elements, then the linear operator

$$\Theta : \mathcal{M} \ni x \mapsto \sum_{j=1}^{n} a_j x b_j \in \mathcal{M}$$

has no negative eigenvalues.

**Proof.** Fix a faithful conditional expectation $E$ of $\mathcal{M}$ onto $\mathfrak{A}$. Let $\mathfrak{Z}$ denote the center of the (finite) von Neumann algebra $\mathfrak{A}$, and let

$$(9) \quad \mathfrak{A} \ni a \mapsto a^\mathfrak{Z} \in \mathfrak{Z}$$

denote the center valued trace. It is well known (see for instance [4] with references therein) that (9) is a faithful conditional expectation of $\mathfrak{A}$ onto $\mathfrak{Z}$, so the map

$$(10) \quad \mathcal{M} \ni x \mapsto E(x)^{\mathfrak{Z}} \in \mathfrak{Z}$$
is now a faithful conditional expectation of $\mathfrak{M}$ onto $\mathfrak{Z}$. Identify, using the Gelfand transform, $\mathfrak{Z} \cong C(\Gamma_3)$, and define, for every $\gamma \in \Gamma_3$ the state $\phi_\gamma : \mathfrak{M} \to \mathbb{C}$ by

$$\phi_\gamma(x) = \gamma(\langle x | x \rangle).$$

Denote, for every $\gamma \in \Gamma_3$, the Hilbert space separate-completion of $\mathfrak{M}$ with respect to the scalar product

$$\langle x | y \rangle_\gamma = \phi_\gamma(x^* y),$$

by $L^2_\gamma(\mathfrak{M})$, and denote by $T_\gamma : \mathfrak{M} \to L^2_\gamma(\mathfrak{M})$ the canonical map.

**Claim 1.** For any $a, b \in \mathfrak{A}$ and $\gamma \in \Gamma_3$ there exists a unique linear continuous $W_{\gamma}^{ab} \in \mathcal{B}(L^2_\gamma(\mathfrak{M}))$, such that $W_{\gamma}^{ab}T_\gamma x = T_\gamma(axb), \forall x \in \mathfrak{M}$.

Start of by observing that, for any $x \in \mathfrak{M}$, using the inequality $x^* a^* ax \leq \|a\|^2 x^* x$, and the properties of conditional expectations, we have:

$$0 \leq E((axb)^*(axb)) = E(b^* x^* a^* axb) \leq \|a\|^2 E(b^* x^* x) = \|a\|^2 \|b\|^2 E(x^* x),$$

Applying the center-valued trace we also have

$$\|b^* E(x^* x)\|^2 = (E(x^* x)^{1/2}bb^* E(x^* x)^{1/2})^2 \leq (\|b\|^2 E(x^* x))^2 = \|b\|^2 |E(x^* x)|^2,$$

so going back to (12) we now have

$$0 \leq E((axb)^*(axb)) \leq \|a\|^2 \|b\|^2 |E(x^* x)|^2.$$ 

so if we apply $\gamma$, we now get

$$0 \leq \phi_\gamma((axb)^*(axb)) \leq \|a\|^2 \|b\|^2 \phi_\gamma(x^* x).$$

Not only that the above inequality proves the implication $T_\gamma x = 0 \Rightarrow T_\gamma(axb) = 0$, which means that one has a correctly defined linear operator

$$V_{\gamma}^{ab} : \text{Range } T_\gamma \ni T_\gamma x \longmapsto T_\gamma(axb) \in \text{Range } T_\gamma,$$

but it also proves the inequality

$$\|V_{\gamma}^{ab}v\|_{L^2_\gamma(\mathfrak{M})} \leq \|a\| \cdot \|b\| \cdot \|v\|_{L^2_\gamma(\mathfrak{M})}, \forall v \in \text{Range } T_\gamma,$$

so $V_{\gamma}^{ab}$ indeed extends to a linear continuous operator $W_{\gamma}^{ab}$ on $L^2_\gamma(\mathfrak{M})$. The uniqueness is pretty obvious.

**Claim 2.** Use the notations as above. For any $\gamma \in \Gamma_3$ one has the following implications:

(i) If $a, b \in \mathfrak{A}$ are self-adjoint, then so is $W_{\gamma}^{ab}$.

(ii) If $a, b \in \mathfrak{A}$ are positive, then so is $W_{\gamma}^{ab}$.

To prove (i), we use the scalar product on $L^2_\gamma(\mathfrak{M})$, which comes from (11), so we must show the identity

$$\langle x | ayb \rangle_\gamma = \langle axb | y \rangle_\gamma, \forall x, y \in \mathfrak{M}.$$ 

The right-hand side is

$$\langle axb | y \rangle_\gamma = \gamma(E((axb)^* y)^2).$$

Using the properties of conditional expectations and of the center-valued trace we have

$$E((axb)^* y)^2 = E(bx^* ay)^2 = (bE(x^* ay))^2 = (E(x^* ay)b)^2 = E(x^* ayb)^2,$$
so if we apply $\gamma$, then using (14) we now get
\[ \langle axb | y \rangle_\gamma = \gamma(\langle x^*ayb \rangle)^3 = \langle x | ayb \rangle_\gamma, \]
and (13) is proven.

(ii). This is quite trivial, because we can write $axb = a_1^{1/2}(a_1^{1/2}xb^{1/2})b^{1/2}$, so if we consider the self-adjoint operator $X = W_\gamma^{a_1^{1/2},b^{1/2}} \in B(L_2^2(M))$, then clearly $W_\gamma^{ab} = X^2$, so $W_\gamma^{ab}$ is indeed positive.

Having proven Claim 2, let us define, for every $\gamma \in \Gamma$, the operator
\[ \Theta_\gamma = \sum_{j=1}^n W_\gamma^{a_j,b_j} \in B(L_2^2(M)), \]
so that
\[ \Theta_\gamma(T_\gamma x) = T_\gamma \Theta_\gamma, \]
and $\Theta_\gamma$ is positive.

Suppose now $\lambda < 0$ is a negative scalar, and $x \in M$ is such that $\Theta_\gamma x = \lambda x$, and let us prove that $x = 0$. Using (A) it follows that
\[ \Theta_\gamma(T_\gamma x) = T_\gamma(\lambda x) = \lambda(T_\gamma x), \quad \forall \gamma \in \Gamma. \]

By condition (B) this forces $T_\gamma x = 0$, in $L_2^2(M)$, that is, $\phi_\gamma(x^*x) = 0$, which by definition means
\[ \gamma(\langle x^*x \rangle_\gamma^3) = 0 \quad \forall \gamma \in \Gamma. \]

The above condition forces, of course
\[ E(x^*x)^2 = 0 \quad \text{(in } Z), \]
which by the faithfulness of both $E$ and the central-valued trace forces $x^*x = 0$, i.e. $x = 0$. \hfill \Box

**Corollary 3.** Assume $A$ and $M$ are as in Theorem 2, and $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ are positive elements, such that $\sum_{j=1}^n a_jb_j = 1$. For any $\lambda \in \mathbb{C}$ and any $\mu \in (0, \infty)$, the equation
\[ \sum_{j=1}^n a_jz b_j = \lambda(1 + \mu)1 - \mu z, \quad z \in M \]
has only one solution: $z = \lambda 1$.

**Proof.** By Theorem 2, the linear continuous map
\[ \Theta : M \ni x \mapsto \sum_{j=1}^n a_jxb_j \in A. \]
has no negative eigenvalues. In particular, for any $\mu > 0$ the linear continuous map $\Theta + \mu \text{Id} : M \rightarrow M$ is one-to-one, thus the only solution of the equation
\[ \Theta(x) = -\mu x, \quad x \in M, \]
is the trivial one: $x = 0$. The rest of the proof is the same as in Corollary 1. \hfill \Box

**Corollary 4.** If $H$ is a finite dimensional Hilbert space, and $A = (A_1, \ldots, A_n)$ is a Lüders system on $H$, then the equation
\[ \Lambda_A(B) = I - B, \quad B \in B(H), \]
has exactly one solution: $B = \frac{1}{2} I$.

**Proof.** Immediate from Corollary 3, applied to the (finite dimensional) von Neumann algebras $A = M = B(H)$ with $a_j = b_j = A_j^{1/2}$, $\lambda = \frac{1}{2}$ and $\mu = 1$. \hfill \Box
Comment. The above results use positivity in an essential way. For arbitrary quantum operations, such as (1), one cannot prevent the appearance of negative eigenvalues. For example, if one considers the simplest possible quantum operation
\[ \Phi(X) = UXU^*, \]
with \( U \in \mathcal{B}(\mathcal{H}) \) unitary, then the equation \( \Phi(X) = -X \) could have many non-trivial solutions. For instance 
\[ U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \] and 
\[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

References


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