Locally Convex Vector Spaces V: 
(Abstract) Duality Theory

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

This section contains an important conceptual discussion on duality, which in a special case encodes the interplay between a locally convex space and its (topological) dual. Subsection D is optional.

A. Dual Pairings

Convention. Throughout this note \( \mathbb{K} \) will be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and all vector spaces are over \( \mathbb{K} \).

Definitions. Suppose one has two vector spaces \( \mathcal{X} \) and \( \mathcal{Y} \). A dual pairing of \( \mathcal{X} \) with \( \mathcal{Y} \) is a bilinear\(^1\) map \( \Phi : \mathcal{X} \times \mathcal{Y} \to \mathbb{K} \), which is non-degenerate, in the following sense:

- If \( x \in \mathcal{X} \) is such that \( \Phi(x, y) = 0 \), \( \forall \ y \in \mathcal{Y} \), then \( x = 0 \).
- If \( y \in \mathcal{Y} \) is such that \( \Phi(x, y) = 0 \), \( \forall \ x \in \mathcal{X} \), then \( y = 0 \).

Equivalently, \( \Phi \) can be represented by two\(^2\) linear injective maps

\[
\begin{align*}
# : \mathcal{X} &\ni x \longmapsto x^# \in \mathcal{Y}', \\
# : \mathcal{Y} &\ni y \longmapsto y^# \in \mathcal{X}',
\end{align*}
\]

satisfying the equality

\[
x^#(y) = y^#(x), \quad \forall \ x \in \mathcal{X}, \ y \in \mathcal{Y}.
\]

Example 1. Given an arbitrary vector space \( \mathcal{X} \), and a linear subspace \( \mathcal{F} \subset \mathcal{X}' \), which separates the points of \( \mathcal{X} \) (i.e. if \( x_1 \neq x_2 \), there exists \( f \in \mathcal{F} \) with \( f(x_1) \neq f(x_2) \)), then the map \( \Phi : \mathcal{X} \times \mathcal{F} \ni (x, f) \longmapsto f(x) \in \mathbb{K} \) is a dual pairing.

Example 2. We can specialize the above example to the following situation. Assume \( \mathcal{X} \) is a locally convex topological vector space. In particular the topology \( \mathcal{T} \), which \( \mathcal{X} \) is equipped with, is Hausdorff, and the dual space \( \mathcal{X}^* = (\mathcal{X}, \mathcal{T})^* \) separates the points in \( \mathcal{X} \). Then the map \( \Phi : \mathcal{X} \times \mathcal{X}^* \ni (x, \phi) \longmapsto \phi(x) \in \mathbb{K} \) is a dual pairing.

B. The Weak Dual Topologies

\(^1\) This means that \( \Phi \) is linear in each variable.

\(^2\) Although the notation is abusive, it is very convenient!
Definitions. Assume now a dual pairing is given, defined by the maps (1) and (2) satisfying (3). If we endow \( Y' \) with the weak* topology, we can apply the pull-back construction (see LCVS II) using the map (1) to obtain a topology, denoted by \( w^# \) on \( X \). Specifically, this is the \textit{weakest topology which makes the map} \( \#: X \rightarrow (Y', w^*) \) \textit{continuous}. Since the weak* topology on \( Y' \) is Hausdorff locally convex, and the map \( \# \) is linear and injective, the topology \( w^# \) on \( X \) is Hausdorff locally convex.

Similarly, one also constructs a Hausdorff locally convex topology \( w^# \) on \( Y' \), as the \textit{weakest} topology that makes the map \( \#: Y \rightarrow (X', w^*) \) \textit{continuous}.

The two topologies constructed above are called the \textit{weak topologies associated with the dual pairing}.

Remarks 1-2. Use the notations as above.

1. Convergence in \((X, w^#)\) and in \((Y, w^#)\) can be characterized as follows:
   
   (i) \( x_\lambda \xrightarrow{w^#} x \) (in \( X \)), if and only if one of the following equivalent conditions holds:
   
   \[
   x_\lambda^#(y) \rightarrow x^#(y), \quad \forall y \in Y, \\
   y^#(x_\lambda) \rightarrow y^#(x), \quad \forall y \in Y.
   \]
   
   (ii) \( y_\lambda \xrightarrow{w^#} y \) (in \( X \)), if and only if one of the following equivalent conditions holds:
   
   \[
   y_\lambda^#(x) \rightarrow y^#(x), \quad \forall x \in X, \\
   x^#(y_\lambda) \rightarrow x^#(y), \quad \forall x \in X.
   \]

2. In terms of seminorms:
   
   (i) the topology \( w^# \) on \( X \) is defined by the family \( \mathfrak{P}_Y^X = \{p_y\}_{y \in Y} \), given by
   
   \[
   p_y(x) = |y^#(x)| = |x^#(y)|, \quad x \in X, \ y \in Y;
   \]

   (ii) the topology \( w^# \) on \( Y \) is defined by the family \( \mathfrak{P}_X^Y = \{p_x\}_{x \in X} \), given by
   
   \[
   p_x(y) = |x^#(y)| = |y^#(x)|, \quad x \in X, \ y \in Y;
   \]

Theorem 1. Use the notations above.

(i) For any \( x \in X \), the linear functional \( x^# : Y \rightarrow K \) is continuous in the \( w^# \) topology.

(i') If we equip the space \( Y^* = (Y, w^#)^* \) with the weak* topology, the map

\[
\#: (X, w^#) \ni x \longmapsto x^# \in (Y^*, w^*)
\] (4)

is a topological linear isomorphism.

(ii) For any \( y \in Y \), the linear functional \( y^# : X \rightarrow K \) is continuous in the \( w^# \) topology.
(ii') If we equip the space $\mathcal{X}^* = (\mathcal{X}, w^\#)^*$ with the weak* topology, the map

$$
# : (\mathcal{Y}, w^\#) \ni y \longmapsto y^\# \in (\mathcal{X}^*, w^*)
$$

is a topological linear isomorphism.

**Proof.** By symmetry, it suffices to prove only (i) and (i'). Property (i) is quite trivial, by Remark 1.

To prove (ii) we first remark that (4) is injective, by definition.

Next we show that (4) is surjective. Suppose $\phi : \mathcal{Y} \to \mathbb{K}$ is linear and $w^\#$-continuous, and let us prove that there exists (a unique) $x \in \mathcal{X}$, such that $\phi = x^\#$. Using Remark 2, the discussion from LCVS IV (Remark 3), there exist $x_1, \ldots, x_n \in \mathcal{X}$ and $t_1, \ldots, t_n > 0$, such that

$$
|\phi(y)| \leq t_1|x_1^\#(y)| + \cdots + t_n|x_n^\#(y)|, \quad \forall y \in \mathcal{Y}.
$$

Using Exercise ?? from HB, (applied to the seminorms $p_j(y) = t_j|x_j^\#(y)|$, $j = 1, \ldots, n$), there exist linear functionals $\phi_1, \ldots, \phi_n : \mathcal{Y} \to \mathbb{K}$, such that $\phi = \phi_1 + \cdots + \phi_n$, and

$$
|\phi_j(y)| \leq t_j|x_j^\#(y)|, \quad \forall y \in \mathcal{Y}, \quad j = 1, \ldots, n.
$$

Fix for the moment $j$, and consider the linear functional $\psi_j = t_jx_j^# : \mathcal{Y} \to \mathbb{K}$. Since $|\phi_j(y)| \leq |\psi_j(y)|$, $\forall y \in \mathcal{Y}$, we must have the inclusion $\text{Ker} \psi_j \subset \text{Ker} \phi_j$, so $\mathcal{X}/\text{Ker} \phi_j$ is a quotient of $\mathcal{X}/\text{Ker} \psi_j$. Since $\psi_j$ is linear, we must have

$$
1 \geq \dim(\text{Ker} \psi_j) \geq \dim(\text{Ker} \phi_j) \geq 0,
$$

so either

1. $\dim(\text{Ker} \psi_j) = \dim(\text{Ker} \phi_j) \leq 1$, in which case $\text{Ker} \phi_j = \text{Ker} \psi_j$, or
2. $1 = \dim(\text{Ker} \psi_j) > \dim(\text{Ker} \phi_j) = 0$, in which case $\text{Ker} \phi_j = \mathcal{X}$.

In either case we get $\phi_j = \alpha_j \psi_j$, for some $\alpha_j \in \mathbb{K}$ (in case (ii), $\alpha_j = 0$). So now we have $\phi_j = \alpha_j t_j x_j^#$, $j = 1, \ldots, n$, so

$$
\phi(y) = \alpha_1 t_1 x_1^#(y) + \cdots + \alpha_1 t_1 x_1^#(y) = (\alpha_1 t_1 x_1 + \cdots + \alpha_n t_n x_n)^#(y), \quad \forall y \in \mathcal{Y},
$$

i.e. $\phi = (\alpha_1 t_1 x_1 + \cdots + \alpha_n t_n x_n)^#$.

Having proven that (4) is a linear isomorphism, its continuity, as well as the continuity of its inverse, are tautologic by Remark 1, since the condition $x^\# \xrightarrow{w^*} x^\#$ (in $\mathcal{Y}^*$) is equivalent to

$$
x^\#_{\lambda}(y) \to x^\#(y), \quad \forall y \in \mathcal{Y},
$$

which in turn is equivalent to $x_{\lambda} \xrightarrow{w^*} x$ (in $\mathcal{X}$).

We now re-visit the two examples from sub-section A.

**Example 1.** Given an arbitrary vector space $\mathcal{X}$, and a linear subspace $\mathcal{F} \subset \mathcal{X}'$, which separates the points of $\mathcal{X}$, then, as pointed out in sub-section A, we have is a dual pairing
between $\mathcal{X}$ and $\mathcal{X}$. The weak topology $\mathcal{w}^#$ on $\mathcal{X}$ is referred to as the weak $\mathcal{F}$-topology, and is denoted by $\mathcal{w}^F$. The other weak topology $\mathcal{w}^#$ on $\mathcal{F}$ is simply the restriction of the weak* topology (from $\mathcal{X}''$) to $\mathcal{F}$.

**Example 2.** If $(\mathcal{X}, \mathcal{T})$ is a locally convex topological vector space (thus $\mathcal{T}$ is Hausdorff), then, as discussed in Example 2 from sub-section A, we have a dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$. The topology $\mathcal{w}^#$ on $\mathcal{X}$ is simply the weak topology $\mathcal{w}_\mathcal{T}$, implemented by $\mathcal{T}$, which was introduced in LCVS IV. In particular (see LCVS IV), for every set $\mathcal{S} \subset \mathcal{X}$, one has the equalities

$$\text{conv}^\mathcal{T} \mathcal{S} = \text{conv}^\mathcal{w}_\mathcal{T} \mathcal{S} = \text{conv}^\mathcal{w}^\# \mathcal{S}.$$  \hspace{1cm} (6)

As before, the other topology $\mathcal{w}^#$ on $\mathcal{X}^*$ is simply the weak* topology. As a consequence of Theorem 1, we also have a natural topological linear isomorphism $\mathcal{#} : (\mathcal{X}, \mathcal{w}_\mathcal{T}) \ni x \mapsto x^\# \in (\mathcal{X}^*, \mathcal{w}^*)$, defined by $x^\#(\phi) = \phi(x), \forall x \in \mathcal{X}, \phi \in \mathcal{X}^*$.

**Exercise 1*. Equip the direct sum $\mathcal{X} = \bigoplus_I \mathbb{K}$ with the locally convex sum topology $\mathcal{T}$. Show that the weak topology $\mathcal{w}^#$ is strictly weaker than $\mathcal{T}$. Conclude that there does not exist locally convex spaces $\mathcal{Y}$, such that $(\mathcal{X}, \mathcal{T})$ is topologically linearly isomorphic to $(\mathcal{Y}^*, \mathcal{w}^*)$.

**Exercise 2.** Let $I$ be some non-empty set. Consider the spaces $\mathcal{X}_{\text{prod}} = \prod_I \mathbb{K}$ and $\mathcal{X}_{\text{sum}} = \bigoplus_I \mathbb{K}$.

(i) Prove that the map $\Phi : \mathcal{X}_{\text{prod}} \times \mathcal{X}_{\text{sum}} \rightarrow \mathbb{K}$, defined by$^3$

$$\Phi(x, y) = \sum_{i \in I} x_i y_i, \ \forall x = (x_i)_{i \in I} \in \mathcal{X}_{\text{prod}}, \ y = (y_i)_{i \in I} \in \mathcal{X}_{\text{sum}},$$

establishes a dual pairing.

(ii) Prove that the topology $\mathcal{w}^#$ on $\mathcal{X}_{\text{prod}}$ coincides with the product topology $\mathcal{T}_{\text{prod}}$.

(iii) Prove that the topology $\mathcal{w}^#$ on $\mathcal{X}_{\text{sum}}$ is strictly weaker than the locally convex sum topology. Therefore we only have an algebraic linear isomorphism $\mathcal{X}_{\text{sum}} \simeq (\mathcal{X}_{\text{prod}}, \mathcal{T}_{\text{prod}})^*$.

**C. Polars and the Bipolar Theorem**

As we have already seen in Example 2, the closure of convex hulls depends only on the interaction between the ambient space and its (topological) dual. Therefore, it is expected that the operation of taking closed convex hulls to admit an “abstract” characterization, within the framework of dual pairs.

**Definitions.** Suppose a dual pairing between two vector spaces $\mathcal{X}$ and $\mathcal{Y}$ is given, defined by the maps (1) and (2) satisfying (3). For a non-empty set $\mathcal{A} \subset \mathcal{X}$, we define its **polar** (in $\mathcal{Y}$) to be the set:

$$\mathcal{A}^\circ = \{ y \in \mathcal{Y} : \text{Re} y^\#(a) \leq 1, \ \forall a \in \mathcal{A} \}.$$

$^3$ The sum is clearly finite, since $y_i \neq 0$ only for finitely many $i \in I$. 


Similarly, for a non-empty set \( B \subset Y \), we define its polar (in \( Y \)) to be the set:

\[
B^\circ = \{ x \in X : \Re x^\#(b) \leq 1, \ \forall b \in B \}.
\]

**Exercise 3.** Prove that, using the notations as above, for two non-empty sets \( A \subset B \subset \mathcal{X} \), one has the reverse inclusion \( A^\circ \supset B^\circ \).

**Proposition 2.** With the notations as above, if \( A \subset \mathcal{X} \) is non-empty, then \( A^\circ \) is convex, \( w^\# \)-closed in \( \mathcal{Y} \), and contains 0. Likewise, if \( B \subset \mathcal{Y} \) is non-empty, then \( B^\circ \) is convex, \( w^\# \)-closed in \( \mathcal{X} \), and contains 0.

**Proof.** By symmetry, we only need to prove the first statement. The fact that \( A^\circ \) contains 0 is trivial. To prove convexity, start with \( y, z \in A^\circ \) and some \( t \in [0, 1] \), and let us show that \( ty + (1-t)z \in A^\circ \). This is, however, trivial, since by the linearity of the map (5) we have

\[
\Re[(ty + (1-t)z)^\#(a)] = \Re[ty^\#(a) + (1-t)z^\#(a)] =
\]

\[
t[\Re y^\#(a)] + (1-t)[\Re z^\#(a)] \leq 1, \ \forall a \in A.
\]

Finally, to prove that \( A^\circ \) is \( w^\# \)-closed, we start with some net \( (y_\lambda) \) in \( A^\circ \), which is \( w^\# \)-convergent to some \( y \in \mathcal{Y} \), and we show that \( y \in A^\circ \). This is again trivial, since the condition \( y_\lambda \xrightarrow{w^\#} y \) implies \( y_\lambda^\#(a) \to y^\#(a) \), \( \forall a \in A \), so the inequalities \( \Re y_\lambda^\#(a) \leq 1, \ \forall \lambda \), will force \( \Re y^\#(a) \leq 1 \).

**Theorem 2** (Bipolar Theorem). Use the notations and hypotheses from the above definition. Let \( w^\# \) be the weak topology on \( \mathcal{X} \) associated with the dual pairing. For any subset \( A \subset \mathcal{X} \), the \( w^\# \)-closure of the convex hull of \( A \cup \{0\} \) is given by the equality

\[
\overline{\text{conv}(A \cup \{0\})}^{w^\#} = (A^\circ)^\circ.
\]  

**Proof.** Using Proposition 2, we already know that the bi-polar \((A^\circ)^\circ\) is \( w^\# \)-closed (in \( \mathcal{X} \)), convex, and contains 0. Note also that \((A^\circ)^\circ\) contains \( A \). Indeed, if \( a \in A \), then for every \( y \in A^\circ \) one has \( a^\#(y) = y^\#(a) \), so we now have

\[
\Re a^\#(y) = \Re y^\#(a) \leq 1, \ \forall y \in A^\circ.
\]

(The second inequality comes from the very definition of \( A^\circ \).)

Since \((A^\circ)^\circ\) is \( w^\# \)-closed, convex, and it contains \( A \cup \{0\} \), we have the inclusion

\[
\overline{\text{conv}(A \cup \{0\})}^{w^\#} \subset (A^\circ)^\circ.
\]

To prove the other inclusion, we start with some element \( x \in \mathcal{X} \) that does not belong to \( \overline{\text{conv}(A \cup \{0\})}^{w^\#} \), and we show that \( x \not\in (A^\circ)^\circ \). First of all, by Theorem 1 from LCVS IV, we know there exist \( \phi \in (\mathcal{X}, w^\#)^* \) and \( s \in \mathbb{R} \), such that

\[
\Re \phi(a) \leq s < \Re \phi(x), \ \forall a \in A \cup \{0\}.
\]

In particular (using \( a = 0 \)), it follows that \( \Re \phi(x) > s \geq 0 \), so if we take \( t = (\Re \phi(x) + s)/2 \), and we define \( \psi : \mathcal{X} \to \mathbb{K} \) is still linear and \( w^\# \)-continuous, but now satisfies

\[
\Re \psi(a) < 1 < \Re \psi(x), \ \forall a \in A \cup \{0\}.
\]
By Theorem 1, we know that there exists (a unique) \( y \in \mathcal{Y} \), such that \( \psi = y^\# \), so the above inequalities now read:

\[
\Re y^\#(a) < 1 < \Re y^\#(x), \quad \forall a \in \mathcal{A} \cup \{0\}.
\]

In particular, from the first inequality we can conclude that \( y \in \mathcal{A}^\circ \), and then the second inequality shows that \( x \notin (\mathcal{A}^\circ)^\circ \).

Comment. Many textbooks use a notion that is slightly different from ours, by defining (use the notations as above), for non-empty \( \mathcal{A} \subset \mathcal{X} \) and \( \mathcal{B} \subset \mathcal{Y} \), the sets

\[
\mathcal{A}^\ominus = \{ y \in \mathcal{Y} : |y^\#(a)| \leq 1, \ \forall a \in \mathcal{A} \},
\]

\[
\mathcal{B}^\ominus = \{ x \in \mathcal{X} : |x^\#(b)| \leq 1, \ \forall b \in \mathcal{B} \}.
\]

Definition. With the above notations, the set \( \mathcal{A}^\ominus \) is called the absolute polar of \( \mathcal{A} \) (in \( \mathcal{Y} \)), and the set \( \mathcal{B}^\ominus \) is called the absolute polar of \( \mathcal{B} \) (in \( \mathcal{X} \)).

The result below deal with the “absolute” version of Proposition 2 and the relationship between the absolute and the “honest” polars.

Proposition 3 Use the notations as above. If \( \mathcal{A} \subset \mathcal{X} \) is a non-empty subset, then:

(i) The absolute polar \( \mathcal{A}^\ominus \) is \( w^\# \)-closed, convex, and balanced (hence \( \mathcal{A}^\ominus \ni 0 \));

(ii) \( \mathcal{A}^\ominus = [\text{bal } \mathcal{A}]^\circ \).

(Recall that \( \text{bal } \mathcal{A} \) denotes the balanced hull of \( \mathcal{A} \), defined as \( \bigcup_{|\alpha| \leq 1} \alpha \mathcal{A} \).)

Proof. (ii). Suppose first \( y \in \mathcal{A}^\ominus \), and let us prove that \( y \in [\text{bal } \mathcal{A}]^\circ \), i.e. \( \Re y^\#(\alpha a) \), for all \( a \in \mathcal{A} \), and every \( \alpha \in \mathbb{K} \) with \( |\alpha| \leq 1 \). This is, however, obvious, since

\[
\Re y^\#(\alpha a) \leq |y^\#(\alpha a)| = |\alpha| \cdot |y^\#(a)| \leq |\alpha| \leq 1.
\]

Conversely, if \( y \in [\text{bal } \mathcal{A}]^\circ \), then we can choose, for every \( a \in \mathcal{A} \), a scalar \( \alpha_a \in \mathbb{K} \) such that \( |\alpha_a| = 1 \) and \( \alpha_a y^\#(a) = |y^\#(a)|(\in \mathbb{R}) \). In particular, for every \( a \in \mathcal{A} \), we have

\[
|y^\#(a)| = \alpha_a y^\#(a) = \Re[\alpha_a y^\#(a)] = \Re y^\#(\alpha_a a) \leq 1
\]

(the last inequality follows from the assumption that \( y \) is in \( [\text{bal } \mathcal{A}]^\circ \)), so \( y \) indeed belongs to \( \mathcal{A}^\ominus \).

(i). By part (ii) and Proposition 2, the absolute polar \( \mathcal{A}^\ominus \) is \( w^\# \)-closed, convex (and contains 0). If \( \alpha \in \mathbb{K} \) is such that \( |\alpha| \leq 1 \), then for every \( y \in \mathcal{A}^\ominus \) we have:

\[
|(\alpha y)^\#(a)| = |\alpha| \cdot |y^\#(a)| \leq |\alpha| \leq 1,
\]

which means that \( \alpha y \) is also in \( \mathcal{A}^\ominus \).

The “absolute” version of Theorem 2 is the following.

Theorem 3 (Absolute Bipolar Theorem) Use the notations and hypotheses from the preceding definition. Let \( w^\# \) be the weak topology on \( \mathcal{X} \) associated with the dual pairing. For any non-empty subset \( \mathcal{A} \subset \mathcal{X} \), one has the equality

\[
\overline{\text{conv}(\text{bal } \mathcal{A})}^{w^\#} = (\mathcal{A}^\circ)^\ominus.
\]
Proof. By Proposition 3, we know that

\[ \mathcal{A}^\circ = [\text{BAL} \mathcal{A}]^\circ. \]  

(9)

Since \( \mathcal{A}^\circ \) is already balanced, again by Proposition 3, we get the equality \( (\mathcal{A}^\circ)^\circ = (\mathcal{A}^\circ)^\circ \), so going back to (9) we now have

\( (\mathcal{A}^\circ)^\circ = ([\text{BAL} \mathcal{A}]^\circ)^\circ. \)

The desired equality (8) now follows from the one above, and Theorem 2 (applied to BAL \( \mathcal{A} \)).

Besides (absolute) polars, there is a third construction associated with dual pairs, described in the following group of Exercises.

Exercises 3-5. Use notations as above. For non-empty \( \mathcal{A} \subset \mathcal{X} \) and \( \mathcal{B} \subset \mathcal{Y} \), let:

\[ \mathcal{A}^{\text{ann}} = \{ y \in \mathcal{Y} : y^\#(a) = 0, \ \forall \ a \in \mathcal{A} \}, \]

\[ \mathcal{B}^{\text{ann}} = \{ x \in \mathcal{X} : x^\#(b)y^\#(a) = 0, \ \forall \ b \in \mathcal{B} \}. \]

3. Prove the equalities\(^4\)

\[ \mathcal{A}^{\text{ann}} = (\text{SPAN} \mathcal{A})^\circ = (\text{SPAN} \mathcal{A})^{\text{ann}} = (\text{SPAN} \mathcal{A})^\circ. \]

4. Prove that \( \mathcal{A}^{\text{ann}} \) is a \( \text{w}^\# \)-closed linear subspace in \( \mathcal{Y} \).

5. Prove that \((\mathcal{A}^{\text{ann}})^{\text{ann}} = \text{SPAN} \mathcal{A}^{\text{w}^\#} \).

Definition. The set \( \mathcal{A}^{\text{ann}} \) is called\(^5\) the annihilator of \( \mathcal{A} \) (in \( \mathcal{Y} \)), and the set \( \mathcal{B}^{\text{ann}} \) is called the annihilator of \( \mathcal{B} \) (in \( \mathcal{X} \)).

Comment. The (Absolute) Bipolar Theorem has numerous application in Functional Analysis, especially in the setting described in Example 2. Specifically, if \( (\mathcal{X}, \mathcal{T}) \) is a locally convex topological vector space, we can from a dual pairing with its topological dual \( \mathcal{X}^* = (\mathcal{X}, \mathcal{T})^* \), and then by Theorems 3 and 4, it follows that, for every \( \mathcal{A} \subset \mathcal{X} \), one has the equalities

\[ \text{conv}(\mathcal{A} \cup \{0\})^\mathcal{T} = (\mathcal{A}^\circ)^\circ; \]

\[ \text{conv}(\text{BAL} \mathcal{A})^\mathcal{T} = (\mathcal{A}^\circ)^\circ. \]

(10) \hspace{1cm} (11)

Some subtle applications are illustrated in sub-section D.

Exercise 6. Let \( \text{op} \) be any one of the three operations: “\( \circ \)” (polar), “\( \square \)” (absolute polar), or “\( \text{ann} \)” (annihilator). Prove that, given a dual pairing between \( \mathcal{X} \) and \( \mathcal{Y} \), for a non-empty set \( \mathcal{A} \subset \mathcal{X} \) one has the equality \( [(\mathcal{A}^\text{op})^{\text{op}}]^\text{op} = \mathcal{A}^\text{op} \).
D*. Other natural topologies on dual pairs

This optional sub-section introduces the reader to several new, but equally natural topologies associated with a dual pairing. Our discussion concludes with a result of Mackey (Theorem 5 below), which is a perfect illustration of the power of the ”Duality Machine” constructed throughout this section. With few exceptions, most results are formulated as Exercises.

Definition. Suppose a dual pairing between \( X \) and \( Y \) is given, defined by the maps (1) and (2) satisfying (3). A locally convex topology \( T \) on \( X \) is said to be compatible with the (given) dual pairing, if one has the equality \((X, T)^* = (X, w^#)^*\).

Exercises 7-9. Suppose a dual pairing between \( X \) and \( Y \) is given, defined by the maps (1) and (2) satisfying (3), and a locally convex topology \( T \) on \( X \) is given.

7. Prove that the following conditions are equivalent:

   (i) \((X, w^#)^* \subset (X, T)^*\);
   (ii) for every \( y \in Y \), the linear functional \( y^# : X \to K \) is \( T \)-continuous;
   (iii) \( T \) is stronger than \( w^# \), i.e. one has the inclusion \( T \supset w^# \).

8. Assume \( T \) satisfies one of the conditions (i)-(iii) from the preceding Exercise. Consider, by condition (ii), the linear map \( # : Y \ni y \mapsto y^# \in (X, T)^*(\subset X') \). Prove that the following are equivalent:

   (i) \( T \) is compatible with the dual pairing;
   (ii) the map \( # \) is surjective;
   (ii') the map \( # \) is a linear isomorphism;
   (iii) when we equip \( Y \) with the \( w^# \)-topology, and the dual space \((X, T)^* \) with the weak* topology, the map \( # \) is a topological linear isomorphism.

9. Prove that the \( w^# \)-topology on \( X \) is the weakest among all topologies compatible with the dual pairing.

Exercise 10. Start with a locally convex topological vector space \((X, T)\), and we use the dual pairing between \( X \) and \( Y = (X, T)^* \). Prove that both \( T \) and \( w_T \) (the weak topology implemented by \( T \)), are compatible with this dual pairing.

Comment. Since for a dual pairing between \( X \) and \( Y \), there always exists a smallest locally convex topology on \( X \), which is compatible with the pairing (namely \( w^# \)), it is natural to ask whether there exists a strongest one. Remarkably (see Theorem 4 below), this question has an affirmative answer.

In preparation for our approach to the above mentioned problem, we start off by introducing the following
**Notations.** Assume a dual pairing between $\mathcal{X}$ and $\mathcal{Y}$ is given, defined by the maps (1) and (2) satisfying (3). Define the following collections of sets:

- $\mathcal{B}^\#_X = \{ \mathcal{A} \subset \mathcal{X} : \mathcal{A} \text{ non-empty } w^\#\text{-bounded} \}$;
- $\tilde{\mathcal{B}}^\#_X = \{ \mathcal{A} \subset \mathcal{X} : \mathcal{A} \text{ non-empty convex, balanced, } w^\#\text{-bounded, and } w^\#\text{-closed} \}$;
- $\mathcal{K}^\#_X = \{ \mathcal{A} \subset \mathcal{X} : \text{ non-empty convex, balanced, and } w^\#\text{-compact} \}$.

(Similarly, one defines the collections $\mathcal{B}^\#_Y$, $\tilde{\mathcal{B}}^\#_Y$, and $\mathcal{K}^\#_Y$.)

Note that we have the inclusions: $\mathcal{K}^\#_X \subset \tilde{\mathcal{B}}^\#_X \subset \mathcal{B}^\#_X$.

**Remarks 3-4.** Use the notations as above.

3. Since the $w^\#$-topology on $\mathcal{X}$ is defined by the family $\{p_y\}_{y \in \mathcal{Y}}$, given by $p_y(x) = |y^#(x)| = |x^#(y)|$, by Exercise 1 from LCVS III, it follows that a non-empty set $\mathcal{A} \subset \mathcal{X}$ is $w^\#$-bounded, if and only if,

$$\sup_{a \in \mathcal{A}} |a^#(y)| < \infty, \quad \forall y \in \mathcal{Y}. \quad (12)$$

Likewise, a non-empty set $\mathcal{B} \subset \mathcal{Y}$ is $w^\#$-bounded, if and only if,

$$\sup_{b \in \mathcal{B}} |b^#(x)| < \infty, \quad \forall x \in \mathcal{X}. \quad (13)$$

4. If $\mathcal{A} \subset \mathcal{X}$ is $w^\#$-bounded, then so is the closure $\overline{\text{conv}(\text{bal } \mathcal{A})}^{\#}$, which now is a member of $\tilde{\mathcal{B}}^\#_X$. This follows again from Exercise 1 from LCVS III.

**Exercises 11-15.** Use the notations as above.

11. Prove that, if $\mathcal{M}$ is any one of the collections $\mathcal{B}^\#_X$, $\tilde{\mathcal{B}}^\#_X$, and $\mathcal{K}^\#_X$, then $\mathcal{M}$ is **directed**, i.e. for any $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{M}$, there exists $\mathcal{A} \in \mathcal{M}$, such that $\mathcal{A} \supset \mathcal{A}_1 \cup \mathcal{A}_2$.

12. Show that, if $\mathcal{B} \in \mathcal{B}^\#_Y$, then its absolute polar $\mathcal{B}^\circ$ is **absorbing** (in $\mathcal{X}$) and furthermore, its associated Minkowski functional $q_{\mathcal{B}^\circ}$ is given by

$$q_{\mathcal{B}^\circ}(x) = \sup_{b \in \mathcal{B}} |b^#(x)|, \quad \forall x \in \mathcal{X}.$$

13. Assume $\mathcal{B} \in \tilde{\mathcal{B}}^\#_Y$, and consider the linear subspace $\mathcal{Z} = \text{span } \mathcal{B} \subset \mathcal{Y}$.

   (i) Prove that $\mathcal{Z} = \bigcup_{t \in [0, \infty)} t\mathcal{B} = \bigcup_{n=1}^{\infty} n\mathcal{B}$, so in particular, $\mathcal{B}$ is **absorbing in** $\mathcal{B}$.

   (ii) On $\mathcal{Z}$ consider the Minkowski functional $q_{\mathcal{B}}$ associated with $\mathcal{B}$. Prove that $q_{\mathcal{B}}$ is a **norm** on $\mathcal{Z}$.

   (iii) Let $\mathcal{T}^\text{norm}_{\mathcal{B}}$ denote the locally convex topology on $\mathcal{Z}$, defined by the (semi)norm $q_{\mathcal{B}}$. Prove that $\mathcal{T}^\text{norm}_{\mathcal{B}}$ is **stronger** than the induced $w^\#$-topology (from $\mathcal{Y}$). In other words, if $(z_\lambda)$ is a net in $\mathcal{Z}$, such that $q_{\mathcal{B}}(z_\lambda) \to 0$, then $z_\lambda \overset{w^\#}{\to} 0$ (in $\mathcal{Y}$).

---

6 By (i) this makes sense, since $\mathcal{B}$ is absorbing in $\mathcal{Z}$. 
14. Assume \( \mathcal{B} \in \mathfrak{B}\). Using the notations from the preceding Exercise, define the metric \( d_{\mathcal{B}} \) on \( \mathcal{Z} \) by \( d_{\mathcal{B}}(z_1, z_2) = q_{\mathcal{B}}(z_1 - z_2) \). Prove that the following are equivalent:

(i) \( (\mathcal{Z}, d_{\mathcal{B}}) \) is a complete metric space (equivalently, \( (\mathcal{Z}, T^{\text{norm}}) \) is a Frechet space);
(ii) \( (\mathcal{B}, d_{\mathcal{B}}) \) is a complete metric space.

A set \( \mathcal{B} \in \mathfrak{B} \), satisfying one of the above conditions, is said to be self-complete.

15. Prove that, if \( \mathcal{B} \in \mathfrak{B} \), then \( \mathcal{B} \) is self-complete.

We now continue with an interesting and very useful technical result. The preceding three Exercises are essential in the proof.

**Theorem 3** (Strong Boundedness). Suppose a dual pairing between \( X \) and \( Y \) is given, defined by the maps (1) and (2) satisfying (3). If \( \mathcal{B} \in \mathfrak{B} \) is self-complete and \( \mathcal{A} \in \mathfrak{B} \), then: \( \sup_{b \in \mathcal{B}} |a^\#(b)| < \infty \).

**Proof.** Since \( \mathcal{B} \in \mathfrak{B} \) is self-complete, we can apply the results from Exercises 12-14. Let us denote the Minkowski functional \( q_{\mathcal{B}} \) (on \( \mathcal{Z} = \text{span} \mathcal{B} \subset Y \)) simply by \( \| \cdot \|_{\mathcal{B}} \) (after all, we know it defines a norm), so collecting the facts from Exercises 12-14, we know that:

(A) the norm topology \( T^{\text{norm}} \) on \( \mathcal{Z} \) is stronger than the induced \( w^\# \)-topology (from \( Y \));

(B) \( (\mathcal{Z}, T^{\text{norm}}) \) is Frechet.

Consider now the linear maps \( \theta_a = a^\#|_\mathcal{Z} : \mathcal{Z} \to K, a \in \mathcal{A} \). Since all these maps are continuous with respect to the induced \( w^\# \)-topology (from \( Y \)), by (A) it follows that all \( \theta_a \), \( a \in \mathcal{A} \) are also \( T \)-continuous. Since \( \mathcal{A} \subset X \) is \( w^\# \)-bounded, by Remark 3 we have (12), so in particular (specializing to \( y \in \mathcal{Z} \)), we now have: \( \sup_{a \in \mathcal{A}} |\theta_a(z)| < \infty, \forall z \in \mathcal{Z} \). Since \( (\mathcal{Z}, T) \) is Frechet, by the Equi-Continuity Principle (see TVS IV), it follows that the collection \( \Theta = \{\theta_a\}_{a \in \mathcal{A}} \) is equi-continuous. In particular, there exists a \( T \)-neighborhood of 0 in \( \mathcal{Z} \), such that

\[
|\theta_a(z)| \leq 1, \quad \forall z \in \mathcal{U}, \ a \in \mathcal{A}. \tag{14}
\]

Choose \( r > 0 \), such that \( \mathcal{U} \supset \{z \in \mathcal{Z} : \|z\|_{\mathcal{B}} \leq r\} \). With this choice of \( r \), let us now observe that if we start with arbitrary \( a \in \mathcal{A} \) and \( b \in \mathcal{B} \), then \( \|b\|_{\mathcal{B}} \leq 1 \), so \( rb \in \mathcal{U} \), which by (14) yields \( 1 \geq |\theta_a(rb)| = r|a^\#(b)| \), thus proving the desire conclusion, explicitly: \( \sup_{b \in \mathcal{B}} |a^\#(b)| \leq 1/r \).

**Corollary 1.** Given \( X \) and \( Y \) in a dual pairing as above, if \( \mathcal{B} \in \mathfrak{B} \) and \( \mathcal{A} \in \mathfrak{B} \), then:

\[
\sup_{b \in \mathcal{B}} |a^\#(b)| < \infty.
\]

**Proof.** Immediate from Theorem 3 and Exercise 15.

**Notations.** Suppose, as before, a dual pairing between \( X \) and \( Y \) is given. By Exercise 12 we know that, if \( \mathcal{B} \in \mathfrak{B} \), the functional \( X \ni x \mapsto \sup_{b \in \mathcal{B}} |b^\#(x)| \in [0, \infty) \) defines a seminorm, namely the Minkowski functional \( q_{\mathcal{B}} \) associated with the absolute polar of \( \mathcal{B} \). (We know that \( \mathcal{B}^c \) is already convex and balanced. By Exercise 12, \( \mathcal{B}^c \) is also absorbing in
Given a non-empty collection \( \mathcal{M} \subset \mathcal{B}_Y^# \), we consider the family \( \Omega_{2\mathcal{M}} = \{ q_\mathcal{M} : \mathcal{B} \in \mathcal{M} \} \) of seminorms on \( \mathcal{X} \), and we denote by \( \mathcal{T}_{2\mathcal{M}} \) the locally convex topology on \( \mathcal{X} \), defined by \( \Omega_{2\mathcal{M}} \). In terms of convergence, the topology \( \mathcal{T}_{2\mathcal{M}} \) is characterized as follows. For a net \( (x_\lambda) \) in \( \mathcal{X} \), the condition \( x_\lambda \xrightarrow{\mathcal{T}_{2\mathcal{M}}} 0 \) is equivalent to the condition: \( \left[ \sup_{b \in \mathcal{M}} |b^#(x_\lambda)| \right] \to 0, \forall \mathcal{B} \in \mathcal{M} \).

Likewise, for a collection \( \mathcal{M} \subset \mathcal{B}_x^# \), the corresponding topology \( \mathcal{T}_{2\mathcal{M}} \) on \( \mathcal{Y} \) is defined.

**Comment.** Many “natural” topologies associated with dual pairs are constructed according to the recipe given above. For instance, if one lets \( \mathcal{M}_0 \) be the collection of all singletons in \( \mathcal{Y} \), the associated topology \( \mathcal{T}_{2\mathcal{M}_0} \) is precisely the \( \mathcal{W}^# \)-topology on \( \mathcal{X} \). Below we explore two other possible choices of \( \mathcal{M}(\subset \mathcal{B}_Y^#) \), such that \( \mathcal{M} \supset \mathcal{M}_0 \). (This is sufficient condition for \( \mathcal{T}_{2\mathcal{M}} \) to be Hausdorff.)

**Definitions.** Use the notations as above.

A. If \( \mathcal{M} = \mathcal{B}_Y^# \) (the collection of all non-empty \( \mathcal{W}^* \)-bounded sets in \( \mathcal{Y} \)), the associated topology \( \mathcal{T}_{2\mathcal{M}} \) is referred to as the **strong topology (on \( \mathcal{X} \)) associated with the dual pairing**, and is denoted by \( \mathcal{S}^# \). Similarly, the strong topology \( \mathcal{S}^# \) on \( \mathcal{Y} \) can be constructed.

B. If \( \mathcal{M} = \mathcal{K}_Y^# \) (the collection of all non-empty convex, balanced, \( \mathcal{W}^# \)-compact sets in \( \mathcal{Y} \)), the associated topology \( \mathcal{T}_{2\mathcal{M}} \) is referred to as the **Mackey topology (on \( \mathcal{X} \)) associated with the dual pairing**, and is denoted by \( \mathcal{M}^# \). Similarly, the Mackey topology \( \mathcal{M}^# \) on \( \mathcal{Y} \) can be constructed.

**Remark 5.** Since, for two collections \( \mathcal{M} \subset \mathcal{M}'(\subset \mathcal{B}_Y^#) \) we have the inclusions \( \Omega_{2\mathcal{M}} \subset \Omega_{2\mathcal{M}'} \), it follows that the associated topologies satisfy the same inclusion \( \mathcal{T}_{2\mathcal{M}} \subset \mathcal{T}_{2\mathcal{M}'} \). If we apply this observation to (let \( \mathcal{M}_0 \) be as in the preceding Comment) the inclusions \( \mathcal{M}_0 \subset \mathcal{K}_Y^# \subset \mathcal{B}_Y^# \), we get the inclusions \( \mathcal{W}^# \subset \mathcal{M}^# \subset \mathcal{S}^# \).

**Comment.** Theorem 3 above simply states that all self-complete sets \( \mathcal{B} \in \mathcal{B}_Y^# \) are \( \mathcal{S}^# \)-bounded, thus justifying its naming. Likewise, Corollary 1 above states that all \( \mathcal{W}^# \)-compact convex sets are \( \mathcal{S}^# \)-bounded.

**Exercises 16-17.** Use the above notations and definitions.

16. Prove that, if we consider the collection \( \mathcal{M} = \mathcal{B}_Y^# \setminus \mathcal{B}_x^# \) of all non-empty, convex, balanced, \( \mathcal{W}^# \)-bounded and \( \mathcal{W}^# \)-closed subsets in \( \mathcal{Y} \), then the associated topology \( \mathcal{T}_{2\mathcal{M}} \) coincides with the strong topology \( \mathcal{S}^# \).

17. Prove that, if \( \mathcal{M} \) is either one of the collections \( \mathcal{B}_Y^#, \mathcal{B}_x^#, \text{ or } \mathcal{K}_Y^#, \) then

   (i) the collection \( \mathcal{B} = \{ B^\alpha : B \in \mathcal{M} \} \) constitutes a basic system of neighborhoods of 0 in \( (\mathcal{X}, \mathcal{T}_{2\mathcal{M}}) \);

   (ii) the collection of seminorms \( \Omega_{2\mathcal{M}} \) is directed.

**Comment.** The exercise below shows that, in general, the strong topology is **not** always compatible with the dual pair. The Mackey topology, however, is! (See Theorem 4 below.)

**Exercise 18.** Let \( \mathcal{X} = \bigoplus_N \mathbb{K} \) be paired with itself, using the bilinear form \( \Phi : \mathcal{X} \times \mathcal{X} \to \mathbb{K} \), defined by

\[
\Phi(x, y) = \sum_{n=1}^{\infty} x_n y_n, \quad \forall x = (x_n)_{n=1}^{\infty}, \quad y = (y_n)_{n=1}^{\infty} \in \mathcal{X}.
\]
Denote by \( \pi_n : \mathcal{X} \to \mathbb{K}, n \in \mathbb{N} \), the coordinate maps.

(i) Prove that a non-empty set \( \mathcal{A} \subset \mathcal{X} \), the following are equivalent:

- \( \mathcal{A} \) is \( \text{w}^\# \)-compact;
- \( \mathcal{A} \) is \( \text{w}^\# \)-bounded, and there exists \( m \in \mathbb{N} \) such that \( \pi_n(a) = 0, \forall a \in \mathcal{A}, n > m. \)

(ii) Prove that, for a non-empty set \( \mathcal{A} \subset \mathcal{X} \), the following are equivalent:

- \( \mathcal{A} \) is \( \text{w}^\# \)-bounded;
- \( \sup_{a \in \mathcal{A}} |\pi_n(a)| < \infty, \forall n \in \mathbb{N}. \)
- \( \mathcal{A} \) is \( \text{m}^\# \)-bounded;

(iii) Show that \( \text{m}^\# \subset \text{s}^\# \), by exhibiting an \( \text{m}^\# \)-bounded set, which is not \( \text{s}^\# \)-bounded.

(iv*) Identify the \( \text{w}^\# \), \( \text{s}^\# \), and \( \text{s}^\# \)-topologies.

**Theorem 4** (Mackey-Arens). Use the notations as above. The Mackey topology \( \text{m}^\# \) (on \( \mathcal{X} \)) is the strongest among all locally convex topologies that are compatible with the (given) dual pairing between \( \mathcal{X} \) and \( \mathcal{Y} \).

**Proof.** The first step is to prove that \( \text{m}^\# \) is compatible with the dual pairing. By Remark 5, we already know that \( \text{m}^\# \supset \text{w}^\# \), so we have the inclusion \( (\mathcal{X}, \text{w}^\#)^* \subset (\mathcal{X}, \text{m}^\#)^* \). Therefore, by Exercise 8, in order to prove that \( \text{m}^\# \) is compatible with the dual pairing, it suffices to show that any linear \( \text{m}^\# \)-continuous functional \( \phi : \mathcal{X} \to \mathbb{K} \) can be represented as \( \phi = y^\# \), for some \( y \in \mathcal{Y} \). Using Exercise 17 (ii) and the results from LCVS III D, we know that there exist \( \mathcal{B} \in \mathcal{K}_\mathcal{Y}^\# \) and \( t > 0 \), such that \( |\phi(x)| \leq tq_{\mathcal{K}^\#}(x), \forall x \in \mathcal{X} \). In particular the linear functional \( \psi = \frac{1}{t} \phi \), which is still \( \text{m}^\# \)-continuous, now satisfies:

\[
|\psi(x)| \leq q_{\mathcal{B}^\#}(x), \forall x \in \mathcal{X}. \tag{15}
\]

Consider now the dual pairing between \( \mathcal{X} \) and its algebraic dual \( \mathcal{X}' \), and denote by \( \text{w}^\#_0 \) the corresponding weak dual topologies. Since the linear map

\[
T : (\mathcal{Y}, \text{w}^\#) \ni y \mapsto y^\# \in (\mathcal{X}', \text{w}^\#_0)
\]

is clearly continuous, it follows that \( T(\mathcal{B}) \) is \( \text{w}^\#_0 \)-compact in \( \mathcal{X}' \). Since \( T(\mathcal{B}) \) is balanced, by the Absolute Bipolar Theorem, we have the equality

\[
T(\mathcal{B}) = [T(\mathcal{B})^\square]^\square. \tag{16}
\]

(The inner absolute polar is taken in \( \mathcal{X} \), the outer absolute polar is taken in \( \mathcal{X}' \).) By construction, the absolute polar \( T(\mathcal{B})^\square \), in \( \mathcal{X} \), is given by:

\[
T(\mathcal{B})^\square = \{ x \in \mathcal{X} : |y^\#(x)| \leq 1, \forall y \in \mathcal{B} \} = \mathcal{B}^\square,
\]

where the second absolute polar is computed relative to the dual pairing between \( \mathcal{X} \) and \( \mathcal{Y} \). In particular, by condition (15) we have \( |\psi(x)| \leq 1, \forall x \in \mathcal{B}^\square = T(\mathcal{B})^\square \), so \( \psi \) belongs to
But now we are done, since (16) forces \( \psi \) to belong to \( T(B) \), which means that there exists some \( b \in B \), such that \( \psi = b^\# \), and then \( \phi = t\psi = (tb)^\# \).

To finish the proof, we start with an arbitrary locally convex topology \( \mathcal{X} \) on \( X \), which is compatible with the dual pairing between \( X \) and \( Y \), and we show that \( \mathcal{X} \) is weaker than the Mackey topology \( m^\# \). Fix some \( \mathcal{X} \)-neighborhood \( V \) of 0, and let us show that \( V \) is also a \( m^\# \)-neighborhood of 0. Using local convexity (see LCVS I), there exists another \( \mathcal{X} \)-neighborhood \( W \) of 0, which is also closed, convex, and balanced, such that \( W \subset V \), and then it suffices to show that \( W \) is a \( m^\# \)-neighborhood of 0. By the Alaoglu-Bourbaki Theorem, the set

\[
C = \{ \phi \in (X, \mathcal{X})^\ast : |\phi(x)| \leq 1, \ \forall x \in W \}
\]

is compact in the dual space \((X, \mathcal{X})^\ast\) in the weak* topology. It is trivial that \( C \) is also convex and balanced. By Remarks 3-4, we know that the correspondence

\[
L : (Y, w^\#) \ni y \mapsto y^\# \in ((X, \mathcal{X})^\ast, w^\ast)
\]

is a topological linear isomorphism, so the set \( B = L^{-1}(C) \subset Y \) is \( w^\# \)-compact, convex, and balanced. Notice now that, by construction,

\[
B = \{ y \in Y : y^\# \in C \} = \{ y \in Y : |y^\#(x)| \leq 1, \ \forall x \in W \} = W^{\text{cl}}.
\]  

(That is, \( B \) is the absolute polar of \( W \) in \( Y \)). Since \( W \) is \( \mathcal{X} \)-closed and convex, and \( \mathcal{X} \) is compatible with the dual pair, it follows that \( W \) is also \( w^\# \)-closed in \( X \), and then using (17) and the Absolute Bipolar Theorem, we obtain the equality \( W = B^{\text{cl}} \), so by Exercise 17, \( W \) is indeed an \( m^\# \)-neighborhood of 0.

The following result is a beautiful illustration of the power of the “Duality Machine.” Although it might be possible to prove it in different ways, the proof presented here is the most elegant one.

**Theorem 5 (Mackey)** Suppose \( \mathcal{X} \) and \( \mathcal{X}' \) are two locally convex Hausdorff topologies on a vector space \( X \), such that \((X, \mathcal{X})^\ast = (X, \mathcal{X}')^\ast\). Then for every subset \( A \subset X \), the following conditions are equivalent:

(i) \( A \) is \( \mathcal{X} \)-bounded;

(ii) \( A \) is \( \mathcal{X}' \)-bounded.

**Proof.** Denote the topological dual space simply by \( Y \), and put it in the standard dual pairing with \( X \), so that both \( \mathcal{X} \) and \( \mathcal{X}' \) are compatible with this dual pairing. In particular, by the Mackey-Arens, we have the inclusions

\[
w^\# \subset \mathcal{X}, \ \mathcal{X}' \subset m^\#,
\]

so it suffices to prove that, for \( A \subset X \), the following are equivalent:

(i') \( A \) is \( w^\# \)-bounded;

(ii') \( A \) is \( m^\# \)-bounded.
Since $\mathcal{W}^\# \subset \mathcal{M}^\#$, the implication $(ii') \Rightarrow (i')$ is trivial. Assume now $\mathcal{A}$ is $\mathcal{W}^\#$-bounded, and let us prove that it is also $\mathcal{M}^\#$-bounded. Using the definition of the Mackey topology – as the locally convex topology defined by the Minkowski seminorms $q_B$, $B \in \mathcal{K}_Y^\#$ – and Exercise 1 from LCVS III, this amounts to showing that $\sup_{a \in \mathcal{A}} q_B(a) < \infty$, $\forall B \in \mathcal{K}_Y^\#$. By Exercise 12, this is equivalent to the condition $\sup_{a \in \mathcal{A}} \left[ \sup_{b \in B} |b^\#(a)| \right] < \infty$, $\forall B \in \mathcal{K}_Y^\#$, which is immediate from Corollary 1.