Locally Convex Vector Spaces V: Linear Continuous Maps and Topological Duals

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

In this section we take a closer look at continuity for linear maps from one locally convex space into another.

A. Basic Definitions and Notations

Conventions & Notations. Throughout this note $\mathbb{K}$ will be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and all vector spaces are over $\mathbb{K}$. As a matter of terminology, given a vector space $\mathcal{X}$, the term *functional on* $\mathcal{X}$ designates a map $\phi : \mathcal{X} \to \mathbb{K}$.

Given two $\mathbb{K}$-vector spaces $\mathcal{X}$ and $\mathcal{Y}$, we denote by $\textsc{Lin}_{\mathbb{K}}(\mathcal{X}, \mathcal{Y})$ the space of all $\mathbb{K}$-linear maps $\mathcal{X} \to \mathcal{Y}$. (When there is no danger of confusion, the field $\mathbb{K}$ is omitted from the notation.)

Given $T \in \textsc{Lin}(\mathcal{X}, \mathcal{Y})$ and $x \in \mathcal{X}$, the vector $T(x)$ will be denoted simply by Tx. Given $S \in \textsc{Lin}(\mathcal{Y}, \mathcal{Z})$, the composition $S \circ T \in \textsc{Lin}(\mathcal{X}, \mathcal{Z})$ will be simply denoted by ST.

Suppose now $\mathfrak{T}$ is a linear topology on $\mathcal{X}$ and $\mathfrak{S}$ is a linear topology on $\mathcal{Y}$. We denote by $\mathcal{L}_{\mathbb{K}}^{\mathfrak{T},\mathfrak{S}}(\mathcal{X}, \mathcal{Y})$ the space of all $\mathbb{K}$-linear operators, which are *continuous* with respect to the given topologies. When there is no danger of confusion, the subscripts and superscripts will be removed from the notation. In the case when $(\mathcal{X}, \mathfrak{T}) = (\mathcal{Y}, \mathfrak{S})$, the above space will simply be denoted by $\mathcal{L}_{\mathbb{K}}^{\mathfrak{T}}(\mathcal{X})$, or simply $\mathcal{L}(\mathcal{X})$.

A central problem in Functional Analysis is to *identify* topological vector spaces. For this purpose we introduce the following terminology.

Definition. Suppose $\mathcal{X}_1$ and $\mathcal{X}_2$ are vector spaces, equipped with linear topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. A linear map $T : \mathcal{X}_1 \to \mathcal{X}_2$ is said to be a *topological linear isomorphism*, if T is bijective (hence a linear isomorphism), and both $T : (\mathcal{X}_1, \mathfrak{T}_1) \to (\mathcal{X}_2, \mathfrak{T}_2)$ and $T^{-1} : (\mathcal{X}_2, \mathfrak{T}_2) \to (\mathcal{X}_1, \mathfrak{T}_1)$ are continuous.

B. Linear Continuous Functionals, Separation, and Closed Convex Hulls

In the study of linear continuous maps, an important special case is always given great attention, namely when the target space is the ground field $\mathbb{K}$. If $\mathcal{X}$ is a vector space, the space $\textsc{Lin}_{\mathbb{K}}(\mathcal{X}, \mathbb{K})$, which is referred to as the *algebraic dual of* $\mathcal{X}$, will be denoted simply by $\mathcal{X}'$.

In the case when $\mathcal{X}$ is endowed with a linear topology $\mathfrak{T}$, and we endow $\mathbb{K}$, with the natural topology $\mathfrak{T}_{\mathbb{K}}$, the space $\mathcal{L}_{\mathbb{K}}^{\mathfrak{T},\mathfrak{T}_{\mathbb{K}}}(\mathcal{X}, \mathbb{K})$, is referred to as the *(topological)* $\mathcal{T}$*-dual space of* $\mathcal{X}$, and is denoted simply by $(\mathcal{X}, \mathfrak{T})^*$, or just $\mathcal{X}^*$, when there is no danger of confusion.

The first result in this section is a collection of characterizations of linear continuous functionals, most of which were already presented in earlier sections.

Proposition 1. *Let $\mathcal{X}$ be vector space, equipped with a linear topology $\mathfrak{T}$.. For a linear functional $\phi : \mathcal{X} \to \mathbb{K}$, the following are equivalent:*

(i) *ϕ is $\mathfrak{T}$-continuous*[1]*;*

(i') *the functional* $\text{Re}\,\phi : \mathcal{X} \to \mathbb{R}$ *is $\mathfrak{T}$-continuous;*

(ii) *ϕ is $\mathfrak{T}$-continuous at* 0*;*

(ii') *the functional* $\text{Re}\,\phi : \mathcal{X} \to \mathbb{R}$ *is $\mathfrak{T}$-continuous at* 0*;*

(iii) $\text{Ker}\,\phi$ *is $\mathfrak{T}$-closed;*

(iii') Ker *$Re\,\phi$ is $\mathfrak{T}$-closed;*

(iv) *there exists $s > 0$, such that the set*

$$\mathcal{U}_\phi(s) = \{x \in \mathcal{X} : |\phi(x)| < s\}$$

is a $\mathfrak{T}$-neighborhood of 0*;*

(iv') *there exists $s > 0$, such that the set*

$$\mathcal{X}_\phi^-(s) = \{x \in \mathcal{X} : \text{Re}\,\phi(x) < s\}$$

is a $\mathfrak{T}$-neighborhood of 0*;*

(v) *for every $s > 0$ the set $\mathcal{U}_\phi(s)$, defined in* (iv)*, is a $\mathfrak{T}$-neighborhood of* 0*;*

(v') *for every $s > 0$ the set $\mathcal{X}_\phi^-(s)$, defined in* (iv')*, is a $\mathfrak{T}$-neighborhood of* 0*;*

(vi) $|\phi|$ *is $\mathfrak{T}$-continuous.*

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and $(i') \Leftrightarrow (ii') \Leftrightarrow (iii')$ are shown in Lemma 2 from TVS III. The equivalence $(i) \Leftrightarrow (i') \Leftrightarrow (iv') \Leftrightarrow (v')$ is discussed in Lemm a 1 from CW III. The implications $(i) \Rightarrow (vi) \Rightarrow (v) \Rightarrow (ii) \Rightarrow (iv)$ are trivial. Finally, the implication $(iv) \Rightarrow (v)$ is easy to prove, since by linearity, for $s, t > 0$, the conditions $|\phi(x)| < s$ and $|\phi(ts^{-1}x)| < t$ are equivalent, which shows that – using the notations from (iv) – one has

$$\mathcal{U}_\phi(t) = ts^{-1}\mathcal{U}_\phi(s), \ \ \forall\, s, t > 0.$$

In particular, if $\mathcal{U}_\phi(s)$ is a neighborhood of 0, then so is $\mathcal{U}_\phi(t)$, for every $t > 0$. □

[1] It is understood that $\mathbb{K}$ is equipped with its natural topology.

Remark 1. Given a vector space $\mathcal{X}$ and a linear functional $\phi : \mathcal{X} \to \mathbb{K}$, the map $|\phi| : \mathcal{X} \ni x \longmapsto |\phi(x)| \in [0, \infty)$ defines a *seminorm* on $\mathcal{X}$.

One important feature of topological duals in the locally convex Hausdorff case is described by the following result.

Proposition 2. *If $\mathcal{X}$ is a locally convex topological vector space, then $\mathcal{X}^*$ separates the points of $\mathcal{X}$, in the following sense: for any $x, y \in \mathcal{X}$, such that $x \neq y$, there exists $\phi \in \mathcal{X}^*$, such that $\phi(x) \neq \phi(y)$.*

Proof. Since $\mathcal{X}$ is locally convex and Hausdorff, there exists some open convex set $\mathcal{A} \ni y$, such that $x \notin \mathcal{A}$. The existence of ϕ then follows from the "Easy" Hahn-Banach Separation Theorem (from CW III). $\square$

Continuing our discussion on topological duals, we now take a closer look at an important class of convex sets.

Definition. Suppose $\mathcal{X}$ is a vector space, equipped with a linear topology $\mathfrak{T}$. A subset $\mathcal{H} \subset \mathcal{X}$ is said to be a *$\mathfrak{T}$-closed half-space in $\mathcal{X}$*, if there exist $\phi \in (\mathcal{X}, \mathfrak{T})^*$ and $s \in \mathbb{R}$, such that

$$\mathcal{H} = \{x \in \mathcal{X} : \operatorname{Re} \phi(x) \leq s\}.$$

Note that, by continuity, $\mathcal{H}$ is indeed closed. Note also that, by linearity, $\mathcal{H}$ is also convex.

Using this terminology, one has the following important result.

Theorem 1 (Closed Convex Hull Theorem) *Suppose $\mathcal{X}$ is a locally convex topological vector space. For any non-empty subset $\mathcal{S} \subset \mathcal{X}$, the closure $\overline{\text{CONV}(\mathcal{S})}$ of the convex hull of $\mathcal{S}$ is equal to the intersection of all closed half-spaces that contain $\mathcal{S}$.*

Proof. Let $\mathcal{C}$ denote the convex hull $\text{CONV}(\mathcal{S})$ and let $\mathcal{C}'$ denote the intersection of all closed half-spaces that contain $\mathcal{S}$. On the one hand, since all closed half-spaces are convex, it is clear that $\mathcal{C}'$ is a convex set that contains $\mathcal{S}$, thus $\mathcal{C}'$ contains $\mathcal{C}$. On the other hand, since $\mathcal{C}'$ is also closed, being an intersection of closed sets, it follows that $\mathcal{C}' \supset \overline{\mathcal{C}}$. To finish the proof, we argue by contradiction, assuming the existence of some $v \in \mathcal{C}'$, which does not belong to $\overline{\mathcal{C}}$. By local convexity, there exists some open convex set $\mathcal{A} \ni v$, such that $\mathcal{A} \cap \mathcal{C} = \varnothing$. By the Hahn-Banach separation Theorem (se CW III), there exists $\phi \in \mathcal{X}^*$ and $s \in \mathbb{R}$, such that

$$\operatorname{Re} \phi(c) \geq s > \operatorname{Re} \phi(a), \ \ \forall\, a \in \mathcal{C},\ a \in \mathcal{A}.$$

In particular, if we define $\mathcal{H} = \{x \in \mathcal{X} : \operatorname{Re} \phi(s) \geq s\}$, then[2] $\mathcal{H}$ is a closed-half-space which contains $\mathcal{C}$, but does not contain v. This forces $v \notin \mathcal{C}'$, which is impossible. $\square$

Corollary 1. *If $\mathcal{X}$ is a locally convex topological vector space, then for any linear subspace $\mathcal{Y}$, its closure is:*

$$\overline{\mathcal{Y}} = \bigcap_{\substack{\phi \in \mathcal{X}^* \\ \phi\big|_{\mathcal{Y}} = 0}} \operatorname{Ker} \phi. \tag{1}$$

[2] We can also write $\mathcal{H} = \{x \in \mathcal{X} : \operatorname{Re}(-\phi)(x) \leq -s\}$.

Proof. Denote the right-hand side of (1) by $\mathcal{Z}$. Clearly $\mathcal{Z}$ is a closed linear subspace (since it is an intersection of closed linear subspaces), which contains $\mathcal{Y}$ (since $\phi\big|_{\mathcal{Y}} = 0 \Leftrightarrow \mathcal{Y} \subset \operatorname{Ker}\phi$), therefore $\mathcal{Z}$ clearly contains $\overline{\mathcal{Y}}$. To prove the other inclusion, we must prove the implication $x \notin \overline{\mathcal{Y}} \Rightarrow x \notin \mathcal{X}$. Start with $x \notin \overline{\mathcal{Y}}$. By Theorem 1, there exists a closed half-space $\mathcal{H}$ that contains $\mathcal{Y}$, but not x. Write $\mathcal{H} = \{x \in \mathcal{X} \,:\, \operatorname{Re}\phi(x) \leq s\}$, for some $\phi \in \mathcal{X}^*$ and $s \in \mathbb{R}$, so that the conditions $\mathcal{Y} \subset \mathcal{H} \not\ni x$ read:

$$\operatorname{Re}\phi(y) \leq s < \operatorname{Re}\phi(x), \;\; \forall\, y \in \mathcal{Y}. \tag{2}$$

As it turns out, the first inequality in (2) in fact implies $\phi\big|_{\mathcal{Y}} = 0$. (Indeed, if this were not true, one could find some $y \in \mathcal{Y}$ with $\phi(y) \neq 0$, so the vector $y_1 = \frac{s+1}{\phi(y)} y \in \mathcal{Y}$ would then have $\phi(y_1) = s+1$, which is impossible.) Going back to (2) we now have $s \geq 0$, so $x \notin \operatorname{Ker}\phi$. Since $\phi\big|_{\mathcal{Y}} = 0$, this forces $x \notin \mathcal{Z}$. □

Comment. Theorem 1 raises an interesting problem. Assume $\mathfrak{T}$ and $\mathfrak{T}'$ are two different Hausdorff topologies on $\mathcal{X}$. Then obviously one has

$$\{\mathcal{A} \subset \mathcal{X} \,:\, \mathcal{A} \text{ convex, } \mathfrak{T}\text{-open}\} \neq \{\mathcal{A} \subset \mathcal{X} \,:\, \mathcal{A} \text{ convex, } \mathfrak{T}'\text{-open}\}.$$

Despite the above inequality, is it possible to have the equality

$$\{\mathcal{A} \subset \mathcal{X} \,:\, \mathcal{A} \text{ convex, } \mathfrak{T}\text{-}\textit{closed}\} = \{\mathcal{A} \subset \mathcal{X} \,:\, \mathcal{A} \text{ convex, } \mathfrak{T}'\text{-}\textit{closed}\}?$$

Surprisingly the answer is affirmative, as indicated by the following result. (See also the example that follows, in which we actually have $\mathfrak{T}' \subsetneq \mathfrak{T}$.)

Proposition 3. *Given two locally convex Hausdorff topologies $\mathfrak{T}$ and $\mathfrak{T}'$ on $\mathcal{X}$, the following statements are equivalent.*

(i) *For any convex set $\mathcal{C}$, the conditions*

- *$\mathcal{C}$ is $\mathfrak{T}$-closed, and*
- *$\mathcal{C}$ is $\mathfrak{T}'$-closed*

are equivalent.

(ii) *For any linear functional $\phi : \mathcal{X} \to \mathbb{K}$, the conditions*

- *ϕ is $\mathfrak{T}$-continuous, and*
- *ϕ is $\mathfrak{T}'$-continuous*

are equivalent, that is, one has the equality: $(\mathcal{X}, \mathfrak{T})^* = (\mathcal{X}, \mathfrak{T}')^*$.

(iii) *For any non-empty subset $\mathcal{S} \subset \mathcal{X}$, one has the equality*

$$\overline{\operatorname{conv}\mathcal{S}}^{\mathfrak{T}} = \overline{\operatorname{conv}\mathcal{S}}^{\mathfrak{T}'} \tag{3}$$

between $\mathfrak{T}$- and $\mathfrak{T}'$-closures of $\operatorname{conv}\mathcal{S}$.

Proof. $(i) \Rightarrow (ii)$. Assume condition (i). To prove condition (ii) start with some linear functional ϕ and we simply observe that, since $\operatorname{Ker}\phi$ is obviously convex (in fact a linear subspace), by (i) one has the equivalence

$$\operatorname{Ker}\phi\ \mathfrak{T}\text{-closed} \Leftrightarrow \operatorname{Ker}\phi\ \mathfrak{T}'\text{-closed}$$

and then everything follows from Proposition 1.

$(ii) \Rightarrow (iii)$. Assume $(\mathcal{X},\mathfrak{T})^* = (\mathcal{X},\mathfrak{T}')^*$. In particular, for a linear functional ϕ and an $s \in \mathbb{R}$, when we consider the half-space $\mathcal{H} = \{x \in \mathcal{X} : \operatorname{Re}\phi(x) \leq s\}$, the conditions

- $\mathcal{H}$ is $\mathfrak{T}$-closed, and
- $\mathcal{H}$ is $\mathfrak{T}'$-closed

are equivalent. Therefore, for any $\mathcal{S} \subset \mathcal{X}$, one has the equality

$$\{\mathcal{H} \subset \mathcal{X},\ \mathfrak{T}\text{-closed half-space},\ \mathcal{H} \supset \mathcal{S}\} = \{\mathcal{H} \subset \mathcal{X},\ \mathfrak{T}'\text{-closed half-space},\ \mathcal{H} \supset \mathcal{S}\}, \qquad (4)$$

and then (3) follows from Theorem 1.

The implication $(iii) \Rightarrow (i)$ is trivial. $\square$

Example 1. Suppose $(\mathcal{X},\mathfrak{T})$ is a locally convex topological vector space. Consider the collection $\mathfrak{P}_{\mathrm{w}} = \{p_\phi : \phi \in \mathcal{X}^*\}$ of seminorms, defined by $p_\phi(x) = |\phi(x)|$. Consider the locally convex topology $\mathrm{w}_{\mathfrak{T}}$, defined by $\mathfrak{P}_{\mathrm{w}}$. Since $\mathfrak{X}^*$ separates the points, this topology is Hausdorff. Remark that, by construction, the condition $x_\lambda \xrightarrow{\mathrm{w}_{\mathfrak{T}}} x$ is equivalent to the condition that $p_\phi(x_\lambda - x) \to 0$, $\forall\, \phi \in \mathcal{X}^*$, which in turn is equivalent to:

$$\phi(x_\lambda) \to \phi(x),\ \ \forall\, \phi \in \mathcal{X}^*.$$

Therefore, $\mathrm{w}_{\mathfrak{T}}$ is *the weakest topology on $\mathcal{X}$, with respect to which all $\phi \in \mathcal{X}^*$ are continuous.* In particular, one has the inclusion $\mathrm{w}_{\mathfrak{T}} \subset \mathfrak{T}$, but also the equality of topological duals: $(\mathcal{X},\mathrm{w}_{\mathfrak{T}})^* = (\mathcal{X},\mathfrak{T})^*$. In particular, by Proposition 3, it follows that, for every $\mathcal{S} \subset \mathcal{X}$, the $\mathfrak{T}$- and the $\mathrm{w}_{\mathfrak{T}}$-closure of $\mathrm{CONV}\,\mathcal{S}$ coincide: $\overline{\mathrm{CONV}\,\mathcal{S}}^{\mathfrak{T}} = \overline{\mathrm{CONV}\,\mathcal{S}}^{\mathrm{w}_{\mathfrak{T}}}$.

Definition. The topology $\mathrm{w}_{\mathfrak{T}}$, constructed above, is called the *weak topology implemented by $\mathfrak{T}$.*

Example 2. Let $\mathcal{X}$ be an infinite dimensional vector space. Let $\mathfrak{T}$ be the strongest locally convex topology on $\mathcal{X}$ (see Exercise 1 from LCVS I). We know that $\mathfrak{T}$ is defined in such a way that *every balanced convex absorbing set $\mathcal{A}$ is a $\mathfrak{T}$-neighborhood of* 0. By construction, *all* linear functionals $\phi \in \mathcal{X}'$ are $\mathfrak{T}$-continuous. The weak topology $\mathrm{w}_{\mathfrak{T}}$ is then the locally convex topology described in Exercise 2 from LCVS I, so we know (see Exercise 3 from LCVS I) that $\mathrm{w}_{\mathfrak{T}} \subsetneq \mathfrak{T}$.

Comment. One general reason why the inclusion $\mathrm{w}_{\mathfrak{T}} \subset \mathfrak{T}$ is strict in many instances, is the following. If $(\mathcal{X},\mathfrak{T})$ is an infinite dimensional locally convex topological vector space, the weak topology $\mathrm{w}_{\mathfrak{T}}$ has a peculiar property: *every $\mathrm{w}_{\mathfrak{T}}$-neighborhood of* 0 *contains a closed infinite dimensional linear subspace.* Indeed, if we start with some neighborhood $\mathcal{V}$, then using the notations from Example 1, there exist $\phi_1,\dots,\phi_n \in \mathcal{X}^*$ and $\varepsilon_1,\dots,\varepsilon_n > 0$,

such that $\mathcal{V} \supset \varepsilon_1 \mathcal{B}(p_{\phi_1}) \cap \dots \cap \varepsilon_n \mathcal{B}(p_{\phi_n})$, so $\mathcal{V}$ will clearly contain the closed subspace $(\operatorname{Ker}\phi_1) \cap \dots \cap (\operatorname{Ker}\phi_n)$.

C. Application: Compact Convex Sets

In this sub-section we discuss an important application of the Closed Convex Hull Theorem. In preparation for Theorem 2 below, we introduce the following terminology.

Definitions. Let $\mathcal{X}$ be a vector space, and let $\mathcal{C} \subset \mathcal{X}$ be a non-empty convex subset. A subset $\mathcal{S} \subset \mathcal{C}$ is said to be *extremal in* $\mathcal{C}$, if

(E) *whenever $x, y \in \mathcal{C}$ are such that $tx + (1-x)y \in \mathcal{S}$, for some $t \in (0,1)$, it follows that both x and y belong to $\mathcal{S}$.*

A point $x \in \mathcal{C}$ is called an *extremal point in* $\mathcal{C}$, is the singleton set $\mathcal{S} = \{x\}$ is an extremal subset in $\mathcal{C}$.

Lemma 1. *Suppose $\mathcal{X}$ is a locally convex topological vector space, $\mathcal{C} \subset \mathcal{X}$ is a non-empty compact convex subset, and $\mathcal{M} \subset \mathcal{C}$ is non-empty, compact, convex and extremal in $\mathcal{C}$. Suppose $\phi \in \mathcal{X}^*$, and define the quantity $m = \sup_{x \in \mathcal{M}} \operatorname{Re}\phi(x)$. Then the set*

$$\mathcal{S} = \{x \in \mathcal{C} : \operatorname{Re}\phi(x) = m\}$$

is non-empty, compact, convex, and extremal in $\mathcal{C}$.

Proof. First of all, since $\mathcal{M}$ is compact, the maximum of (the continuous function) $\operatorname{Re}\phi : \mathcal{M} \to \mathbb{R}$ is attained, i.e. $\mathcal{S}$ is non-empty.

Secondly, since ϕ is continuous, the set $\mathcal{S}$ is closed (in $\mathcal{M}$), thus compact.

Thirdly, the convexity of $\mathcal{S}$ is quite obvious, from the linearity of ϕ.

Finally, to prove extremality, start with $x, y \in \mathcal{C}$ such that $tx + (1-t)y \in \mathcal{S}$, for some $t \in (0,1)$, and let us prove that both x and y in fact belong to $\mathcal{S}$. First of all, since $\mathcal{S} \subset \mathcal{M}$, and $\mathcal{M}$ is extremal in $\mathcal{C}$, it follows that both x and y belong to $\mathcal{M}$. Argue now by contradiction, assuming that either $x \notin \mathcal{S}$ or $y \notin \mathcal{S}$. Then at least one of the inequalities $\operatorname{Re}\phi(x) \leq m$ and $\operatorname{Re}\phi(y) \leq m$ would be strict, which (use the inequalities $t > 0$ and $1 - t > 0$) forces

$$m = tm + (1-t) > t\operatorname{Re}\phi(x) + (1-t)\operatorname{Re}\phi(y) = \operatorname{Re}\phi(tx + (1-t)y) = m$$

(last equality is due to the fact that $tx+(1-t)y$ belongs to $\mathcal{S}$), which is clearly impossible. □

Theorem 2 (Krein-Milman). *Suppose $\mathcal{X}$ is a locally convex topological vector space and $\mathcal{C} \subset \mathcal{X}$ is a non-empty compact convex subset.*

(i) *Every non-empty compact convex set $\mathcal{A}$, which is an extremal subset of $\mathcal{C}$, contains at least on extreme point of $\mathcal{C}$. In particular*[3]*, the set*

$$\operatorname{ext}\mathcal{C} = \{x \in \mathcal{C} : x \text{ extremal point in } \mathcal{C}\}$$

is non-empty.

[3] $\mathcal{C}$ is obviously extremal in itself.

(ii) $\overline{\text{CONV}(\operatorname{ext}\mathcal{C})} = \mathcal{C}$.

Proof. (i). Fix a non-empty compact convex extremal subset $\mathcal{A}$ of $\mathcal{C}$. Consider the collection $\mathfrak{E}$ of all non-empty compact convex subsets of $\mathcal{A}$, which are extremal subsets of $\mathcal{C}$. Obviously $\mathfrak{E}$ is non-empty, since $\mathcal{A} \in \mathfrak{E}$. Equip $\mathfrak{E}$ with the order relation

$$\mathcal{E}_1 \succeq \mathcal{E}_2 \Leftrightarrow \mathcal{E}_1 \subset \mathcal{E}_2.$$

We claim that $\mathfrak{E}$ *possesses at least one maximal element.* To prove this, we apply Zorn Lemma, so it suffices to show that any totally ordered sub-collection $\mathfrak{E}' = \{\mathcal{E}_i : i \in I\}$ admits an upper bound in $\mathfrak{E}$. Indeed, if we take $\mathcal{E} = \bigcap_{i \in I} \mathcal{E}_i$, then obviously $\mathcal{E}$ is compact and convex. Since for every finite set of indices $F \subset I$, there is one j such that $\mathcal{E}_j \subset \mathcal{E}_i$, $\forall\, i \in F$, thus we have $\bigcap_{i \in F} \mathcal{E}_i \supset \mathcal{E}_j \neq \varnothing$, by compactness and the *Finite Intersection Property*, it follows that $\mathcal{E}$ is also non-empty. Finally, the extremality of $\mathcal{E}$ is quite clear, since if we start with two points $x, y \in \mathcal{C}$ and some $t \in (0,1)$, such that $tx + (1-t)y \in \mathcal{E}$, then $tx + (1-t)y \in \mathcal{E}_i$, $\forall\, i \in I$, so by the extremality of each $\mathcal{E}_i$, we must have $x, y \in \mathcal{E}_i$, $\forall\, i \in I$.

Fix now some maximal element $\mathcal{M} \in (\mathfrak{E}, \succeq)$, and let us prove that $\mathcal{M}$ is a singleton. Argue by contradiction, assuming $\mathcal{M}$ contains at least two points $a \neq b$. Choose then a linear continuous functional $\phi \in \mathcal{X}^*$, such that $\phi(a) \neq \phi(b)$. Multiplying, if necessary, ϕ by -1 or $\pm i$ (in the complex case), we can assume $\operatorname{Re}\phi(a) < \operatorname{Re}\phi(b)$. In particular, if we define $m = \max_{x \in \mathcal{M}} \operatorname{Re}\phi(x)$, we have the strict inequality $\operatorname{Re}\phi(a) < m$, so the set $\mathcal{S} = \{x \in \mathcal{M} : \operatorname{Re}\phi(x) = m\}$ does not contain a, which forces $\mathcal{S} \subsetneq \mathcal{M}$. We also know by Lemma 1 that $\mathcal{S}$ is non-empty, compact, convex, and extremal in $\mathcal{C}$, and this clearly contradicts the maximality of $\mathcal{M}$ in $(\mathfrak{E}, \succeq)$.

(ii). Consider the set $\mathcal{C}_0 = \overline{\text{CONV}(\operatorname{ext}\mathcal{C})}$. Obviously, since $\mathcal{C}$ is convex, closed (in fact compact), and it contains $\operatorname{ext}\mathcal{C}$, we have the inclusion $\mathcal{C} \supset \mathcal{C}_0$. To prove that in fact we have the equality $\mathcal{C} = \mathcal{C}_0$, we argue by contradiction, assuming the existence of some $a \in \mathcal{C}$ which does not belong to $\mathcal{C}_0$. By the Closed Convex Hull Theorem, there exists $\phi \in \mathcal{X}^*$ and some $s \in \mathbb{R}$, such that

$$\operatorname{Re}\phi(a) > s \geq \operatorname{Re}\phi(x), \ \ \forall\, x \in \operatorname{ext}\mathcal{C}. \tag{5}$$

Define, as in the proof of part (i), the quantity $m = \max_{x \in \mathcal{C}} \phi(x)$, and the set

$$\mathcal{S} = \{x \in \mathcal{C} : \phi(x) = m\},$$

so that $m > s$, hence by (5) we have

$$\mathcal{S} \cap \operatorname{ext}\mathcal{C} = \varnothing. \tag{6}$$

By Lemma 1, $\mathcal{S}$ is non-empty, compact, convex, and extremal in $\mathcal{C}$, so by (i) $\mathcal{S}$ must contain an extreme point of $\mathcal{C}$, which is impossible by (6). $\square$

In preparation for the next result (Theorem 3 below), the reader must solve the following:

Exercise 1. Let $\mathcal{X}$ be a locally convex topological vector space. Suppose $\mathcal{C}_1, \dots, \mathcal{C}_n \subset \mathcal{X}$ are non-empty compact convex subsets, and let us consider the set $\mathcal{C} = \text{CONV}(\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_n)$. Prove that

(i) $\mathcal{C}$ is compact;

(ii) $\operatorname{ext}\mathcal{C} \subset \bigcup_{i=1}^{n} \operatorname{ext}\mathcal{C}_i$.

(Hint: Use induction, so everything is reduced to the case $n = 2$.)

Theorem 3 (Milman). *Suppose $\mathcal{X}$ is a locally convex topological vector space and $\mathcal{C} \subset \mathcal{X}$ is a non-empty compact convex subset. If a subset $\mathcal{S} \subset \mathcal{X}$ is such that*

$$\overline{\operatorname{conv}\mathcal{S}} = \mathcal{C},$$

then all extreme points of $\mathcal{C}$ belong to $\overline{\mathcal{S}}$.

Proof. Throughout the entire proof we can assume that $\mathbb{K} = \mathbb{R}$, that is, we are going to regard $\mathcal{X}$ as a real locally convex topological vector space. (Convexity is, after all, dependent only on the real vector space structure.)

Replacing $\mathcal{S}$ with its closure $\overline{\mathcal{S}}$, we can also assume that $\mathcal{S}$ is closed, hence compact (being a subset of $\mathcal{C}$). Fix some extreme point $c \in \operatorname{ext}\mathcal{C}$, and let us prove that c belongs to $\mathcal{S}$.

Claim: *For every finite set $F \subset \mathcal{X}^*$ and every $\varepsilon > 0$, there exists some point $s \in \mathcal{S}$, such that: $|\phi(s - c)| \leq \varepsilon$, $\forall\, \phi \in F$.*

Replacing F with $F \cup (-1)F$, suffices to prove the existence of $s \in \mathcal{S}$, such that

$$\phi(s - c) \leq \varepsilon, \ \ \forall\, \phi \in F. \tag{7}$$

List now $F = \{\phi_1, \dots, \phi_n\}$, and define the sequence $(\mathcal{C}_k)_{k=0}^n$, $(\mathcal{D}_k)_{k=0}^n$ recursively by $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{D}_0 = \varnothing$, and

$$\begin{aligned} \mathcal{C}_k &= \{x \in \mathcal{C}_{k-1} \,:\, \phi_k(x - c) \leq \varepsilon\}, \\ \mathcal{D}_k &= \{x \in \mathcal{C}_{k-1} \,:\, \phi_k(x - c) \geq \varepsilon\}. \end{aligned}$$

Notice that, by construction, the $\mathcal{C}_k$'s and the $\mathcal{D}_k$'s are convex and closed, hence compact. Since we have $\mathcal{C}_{k-1} = \mathcal{C}_k \cup \mathcal{D}_k$ we also have the equalities

$$\mathcal{C} = \mathcal{C}_k \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_k, \ \ \forall\, k = 1, \dots, n. \tag{8}$$

With these notations, we see that in order to prove the Claim, it suffices to show that $\mathcal{S} \cap \mathcal{C}_n \neq \varnothing$. Argue by contradiction, assuming that $\mathcal{S}$ is disjoint from $\mathcal{C}_n$, which by (8), would force $\mathcal{S}$ to be contained in the union $\mathcal{M} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n$. In particular (by the above Exercise), since $\mathcal{M}$ is a union of compact convex sets, its convex hull $\operatorname{conv}\mathcal{M}$ is compact, thus closed. Since $\mathcal{S} \subset \mathcal{M} \subset \mathcal{C}$, it follows that

$$\mathcal{C} = \overline{\operatorname{conv}\,\mathcal{S}} \subset \operatorname{conv}\mathcal{M} \subset \mathcal{C},$$

so now we have $\mathcal{C} = \operatorname{conv}(\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_n)$. By the above Exercise, the point $c \in \operatorname{ext}\mathcal{C}$ must belong to some $\mathcal{D}_j$, which is clearly impossible, by the definition of the $\mathcal{D}$'s.

Having proven the above Claim, it is now clear, that we can construct a net (s_λ) in $\mathcal{S}$, which converges to c in the *weak topology* $\mathrm{w}_{\mathfrak{T}}$, implemented by the given topology $\mathfrak{T}$ on $\mathcal{X}$. Since $\mathcal{S}$ is $\mathfrak{T}$-compact, and $\mathrm{w}_{\mathfrak{T}}$ is weaker than $\mathfrak{T}$, it follows that $\mathcal{S}$ is also $\mathrm{w}_{\mathfrak{T}}$-compact, hence also $\mathrm{w}_{\mathfrak{T}}$-closed. Now we are done, because the condition $s_\lambda \xrightarrow{\mathrm{w}_{\mathfrak{T}}} c$ will force $c \in \mathcal{S}$. $\square$

D. Linear Continuous Maps on Locally Convex Spaces

In the case of linear continuous maps between locally convex spaces, one can use the metric point of view introduced in LCVS III, in conjunction with the following result.

Theorem 4. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are locally convex spaces, whose topologies are defined by two families of seminorms: $\mathfrak{P}$ on $\mathcal{X}$ and $\mathfrak{Q}$ on $\mathcal{Y}$. For a linear map $T : \mathcal{X} \to \mathcal{Y}$, the following are equivalent.*

(i) *T is continuous.*

(ii) *T is continuous at 0.*

(iii) *For every $q \in \mathfrak{Q}$, there exist $p_1, \dots, p_n \in \mathfrak{P}$, such that:*

$$\sup \big\{ q(Tx) \, : \, x \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n) \big\} < \infty.$$

(Recall that, for a seminorm p, the notation $\mathcal{B}(p)$ identifies the unit ball $\{x \in \mathcal{X} : p(x) < 1\}$.)

(iv) *For every $q \in \mathfrak{Q}$, there exist $p_1, \dots, p_n \in \mathfrak{P}$, and $t_1, \dots, t_n \geq 0$, such that:*

$$q(Tx) \leq t_1 p_1(x) + \cdots + t_n p_n(x), \ \ \forall \, x \in \mathcal{X}. \tag{9}$$

Proof. The implication $(i) \Leftrightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$. Assume T is continuous at 0 and fix a seminorm $q \in \mathfrak{Q}$. Since q is continuous, the unit ball $\mathcal{B}(q) = \{y \in \mathcal{Y} : q(y) < 1\}$ is a (open) neighborhood of 0 in $\mathcal{Y}$. In particular, since T is continuous at 0, the preimage $\mathcal{A} = T^{-1}\mathcal{B}(q)$, which is simply given by

$$\mathcal{A} = \{x \in \mathcal{X} \, : \, q(Tx) < 1\}$$

is a neighborhood of 0 in $\mathcal{X}$, so by construction (see LCVS III) there exist $p_1, \dots, p_n \in \mathfrak{P}$ and $\varepsilon_1, \dots, \varepsilon_n > 0$, such that

$$\varepsilon_1 \mathcal{B}(p_1) \cap \cdots \cap \varepsilon_n \mathcal{B}(p_n) \subset \mathcal{A}.$$

Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. Suppose now $x \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n)$. Then

$$\varepsilon x \in \varepsilon \mathcal{B}(p_1) \cap \cdots \cap \varepsilon \mathcal{B}(p_n) \subset \varepsilon_1 \mathcal{B}(p_1) \cap \cdots \cap \varepsilon_n \mathcal{B}(p_n) \subset \mathcal{A},$$

so (by linearity) we get $q(\varepsilon Tx) = q(T(\varepsilon x)) < 1$. In other words, we have

$$\sup \big\{ q(Tx) \, : \, x \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n) \big\} \leq \varepsilon^{-1} < \infty,$$

thus proving (iii).

$(iii) \Rightarrow (iv)$. Assume condition (iii). Fix $q \in \mathfrak{Q}$, as well as $p_1, \dots, p_n \in \mathfrak{P}$ as in (iii), and let $t = \sup \big\{ q(Tx) \, : \, x \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n) \big\}$. To prove (iv), we now show that

$$q(Tx) \leq t p_1(x) + \cdots + t p_n(x), \ \ \forall \, x \in \mathcal{X}. \tag{10}$$

Start with some arbitrary $x \in \mathcal{X}$, and then for every $\varepsilon > 0$, the vector

$$x_\varepsilon = [p_1(x) + \cdots + p_n(x) + \varepsilon]^{-1} x$$

satisfies

$$p_k(x_\varepsilon) = \frac{p_k(x)}{p_1(x) + \cdots + p_n(x) + \varepsilon} < 1, \ \ \forall\, k = 1, \dots, n,$$

so $x_\varepsilon \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n)$ By condition (ii) it follows that $q(Tx_\varepsilon) \leq t$, and then by linearity we also get

$$q(Tx) \leq t[p_1(x) + \cdots + p_n(x) + \varepsilon].$$

Since the above inequality holds for all $\varepsilon > 0$, it forces the desired inequality (10)

$(iv) \Rightarrow (i)$. Assume condition (iv), and let us prove that T is continuous. Start with some net (x_λ) in $\mathcal{X}$, which converges to some $x \in \mathcal{X}$, and let us show that the net (Tx_λ) converges to Tx, which means:

$$q(Tx_\lambda - Tx) \to 0, \ \ \forall\, q \in \mathfrak{Q}. \tag{11}$$

Fix $q \in \mathfrak{P}$ and use condition (iv) to produce $p_1, \dots, p_n \in \mathfrak{P}$ and $t_1, \dots, t_n \geq 0$, satisfying (9). One the one hand, by linearity, we have $Tx_\lambda - Tx = T(x_\lambda - x)$, so (9) yields

$$0 \leq q(Tx_\lambda - Tx) \leq t_1 p_1(x_\lambda - x) + \cdots + t_n p_n(x_\lambda - x). \tag{12}$$

On the other hand, since $x_\lambda \to x$, we also know that $p(x_\lambda - x) \to 0$, $\forall\, p \in \mathfrak{P}$, so (12) clearly forces $q(Tx_\lambda - Tx) \to 0$. $\square$

Comment. In the above Theorem, the equivalence $(i) \Leftrightarrow (ii)$ holds in much greater generality (see TVS I).

Remark 2. If in addition to the hypotheses in Theorem 2, we assume that $\mathfrak{P}$ is *directed*, then the four conditions are also equivalent to each of the following:

(iii') *For every $q \in \mathfrak{Q}$, there exists $p \in \mathfrak{P}$, such that:*

$$\sup\big\{ q(Tx) \,:\, x \in \mathcal{B}(p) \big\} < \infty.$$

(iv') *For every $q \in \mathfrak{Q}$, there exist $p \in \mathfrak{P}$, and $t \geq 0$, such that:*

$$q(Tx) \leq t p(x), \ \ \forall\, x \in \mathcal{X}. \tag{13}$$

(First of all, since $\mathfrak{P}$ is directed, for any $p_1, \dots, p_n \in \mathfrak{P}$, there exist $p \in \mathfrak{P}$ and $s > 0$, such that $p_1 + \cdots + p_n \leq sp$. In particular, condition (iv) from Theorem 2 clearly implies condition (iv') above. In turn, the inequality (13) clearly yields

$$\sup\{q(Tx) \,:\, x \in \mathcal{B}(p)\} \leq t < \infty,$$

so we also have the implication $(iv') \Rightarrow (iii')$. Finally condition (iii') trivially implies condition (iii) from Theorem 2.)

Remarks 3-4. If we specialize to the case when $\mathcal{Y} = \mathbb{K}$, then the only seminorm needed for $\mathbb{K}$ is $q(y) = |y|$. Therefore, in this case Theorem 2 and Remark 2 have the following statements.

3. For a linear functional $\phi : \mathcal{X} \to \mathbb{K}$, the following are equivalent.

(i) *ϕ is continuous.*

(ii) *ϕ is continuous at* 0.

(iii) *There exist $p_1, \dots, p_n \in \mathfrak{P}$, such that:*

$$\sup \big\{ |\phi(x)| \,:\, x \in \mathcal{B}(p_1) \cap \cdots \cap \mathcal{B}(p_n) \big\} < \infty.$$

(iv) *There exist $p_1, \dots, p_n \in \mathfrak{P}$, and $t_1, \dots, t_n \geq 0$, such that:*

$$|\phi(x)| \leq t_1 p_1(x) + \cdots + t_n p_n(x), \;\; \forall\, x \in \mathcal{X}.$$

4. If $\mathfrak{P}$ is *directed*, then the above conditions are also equivalent to each of the following:

(iii') *There exists $p \in \mathfrak{P}$, such that:*

$$\sup \big\{ |\phi(x)| \,:\, x \in \mathcal{B}(p) \big\} < \infty.$$

(iv') *There exist $p \in \mathfrak{P}$, and $t \geq 0$, such that:*

$$|\phi(x)| \leq t p(x), \;\; \forall\, x \in \mathcal{X}.$$

E. The Natural Topology on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$

All spaces $\textsc{Lin}(\mathcal{X}, \mathcal{Y})$, $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}'$, and $\mathcal{X}^*$ are vector spaces, so it is legitimate to ask whether they carry some natural linear topologies. As it turns out, to build such topologies all one needs is a linear topology on the target space $\mathcal{Y}$, as indicated in the following construction.

Definition. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces, and $\mathfrak{T}$ is a linear topology on $\mathcal{Y}$. Define, for every $x \in \mathcal{X}$ the evaluation map

$$E_x : \textsc{Lin}(\mathcal{X}, \mathcal{Y}) \ni T \longmapsto Tx \in \mathcal{Y}.$$

We can now consider the system $\boldsymbol{E} = (E_x)_{x \in \mathcal{X}}$ and apply the *joint $\boldsymbol{E}$-pull-back construction* from TVS II, to produce a topology denoted by $\mathfrak{T}_{\text{so}}$ on $\textsc{Lin}(\mathcal{X}, \mathcal{Y})$. This topology is the *weakest topology on $\textsc{Lin}(\mathcal{X}, \mathcal{Y})$, which makes all the evaluation maps E_x, $x \in \mathcal{X}$, continuous.* Since all E_x's are linear, $\mathfrak{T}_{\text{so}}$ is a linear topology. In terms of convergence, for a net (T_λ) and some T in $\textsc{Lin}(\mathcal{X}, \mathcal{Y})$, the condition $T_\lambda \xrightarrow{\mathfrak{T}_{\text{so}}} T$ is equivalent to:

$$T_\lambda x \xrightarrow{\mathfrak{T}} Tx \text{ (in } \mathcal{Y}), \, \forall\, x \in \mathcal{X}.$$

The topology $\mathfrak{T}_{\text{so}}$ is referred to[4] as the *strong operator topology induced by* $\mathfrak{T}$. Note that, if $\mathfrak{T}$ is Hausdorff, then so is $\mathfrak{T}_{\text{so}}$ (see TVS II).

[4] There is nothing "strong" about this topology. We use this (unfortunate) terminology for consistency reasons, in order to keep it in line with Hilbert space Operator Theory. It would be more natural to call it the *topology of point-wise convergence.*

When we specialize to the case when $\mathcal{Y} = \mathbb{K}$ (with its natural topology), so we work with the algebraic dual space $\mathcal{X}'$, this topology is simply denoted by W^*, and is referred to as the *weak* (or weak dual) topology*. As noticed above, the weak* topology is always Hausdorff.

Convention. Assuming $\mathcal{X}$ is also equipped with a linear topology, we use the same notations and terminology, when the above topologies are restricted to the spaces of linear continuous maps on $\mathcal{X}$. Thus $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ – which is a linear subspace in $\text{Lin}(\mathcal{X}, \mathcal{Y})$ – and $\mathcal{X}^*$ – which is a linear subspace in $\mathcal{X}'$ – will carry the induced topologies, which will be denoted again by $\mathfrak{T}_{\text{so}}$ and W^*, respectively.

Remark 5. If $\mathcal{Y}$ is a *complete* topological vector space and $\mathcal{X}$ is an arbitrary vector space, then $(\text{Lin}(\mathcal{X}, \mathcal{Y}), \mathfrak{T}_{\text{so}})$ is also complete. Indeed, if we start with a net $(T_\lambda)_{\lambda \in \Lambda}$ in $\text{Lin}(\mathcal{X}, \mathcal{Y})$, which is $\mathfrak{T}_{\text{so}}$-Cauchy, then for every $x \in \mathcal{X}$, the net $(T_\lambda x)_{\lambda \in \Lambda}$ is Cauchy in $\mathcal{Y}$, so it has a limit Tx. It is pretty obvious that T is linear, and $T_\lambda \xrightarrow{\mathfrak{T}_{so}} T$.

In particular (if we use $\mathcal{Y} = \mathbb{K}$), the algebraic dual $\mathcal{X}'$ is complete, when equipped with the W^*-topology.

Remark 6. If the topology $\mathfrak{T}$ (on $\mathcal{Y}$) is *locally convex*, then the topology $\mathfrak{T}_{\text{so}}$ on $\text{Lin}(\mathcal{X}, \mathcal{Y})$ and on $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is also locally convex. This follows from the discussion in LCVS II. Furthermore, if $\mathfrak{T}$ is defined by a family $\mathfrak{Q}$ of seminorms on $\mathcal{Y}$, then the locally convex topology $\mathfrak{T}_{\text{so}}$ is defined by the family of seminorms

$$\mathfrak{P} = \{q \circ E_x \,:\, q \in \mathfrak{Q},\, x \in \mathcal{X}\}.$$

In particular, the weak* topology on $\mathcal{X}'$ (or $\mathcal{X}^*$) is always locally convex, being defined by the collection $\mathfrak{P} = \{p_x \,:\, x \in \mathcal{X}\}$, where

$$p_x(\phi) = |\phi(x)|, \;\; \forall\, \phi \in \mathcal{X}',\, x \in \mathcal{X}.$$

Exercises 2-3. Suppose $\mathcal{X}_i$, $i \in I$ are vector spaces. For any vector space $\mathcal{Y}$, let

$$\Pi : \prod_{i \in I} \text{Lin}(\mathcal{Y}, \mathcal{X}_i) \to \text{Lin}(\mathcal{Y}, \prod_{i \in I} \mathcal{X}_i)$$
$$\Sigma : \prod_{i \in I} \text{Lin}(\mathcal{X}_i, \mathcal{Y}) \to \text{Lin}(\bigoplus_{i \in I} \mathcal{X}_i, \mathcal{Y})$$

be defined as follows:

$$\Pi(T)y = (T_i y)_{i \in I}, \;\; \forall\, T = (T_i)_{i \in I} \in \prod_{i \in I} \text{Lin}(\mathcal{Y}, \mathcal{X}_i),\, y \in \mathcal{Y}; \tag{14}$$

$$\Sigma(S)x = \sum_{i \in I} S_i x_i, \;\; \forall\, S = (S_i)_{i \in I} \in \prod_{i \in I} \text{Lin}(\mathcal{X}_i, \mathcal{Y}),\, x = (x_i)_{i \in I} \in \bigoplus_{i \in I} \mathcal{X}_i. \tag{15}$$

(Note that in (15) the sum is *finite*.)

2. Prove that Π and Σ are linear isomorphisms.

3. Assume $\mathcal{Y}$ and $\mathcal{X}_i$, $i \in I$, are locally convex spaces. Equip all spaces $\mathcal{L}(\mathcal{X}_i, \mathcal{Y})$ and $\mathcal{L}(\mathcal{Y}, \mathcal{X}_i)$, $i \in I$, with the strong operator topologies. Let $\mathcal{X}_{\text{prod}} = \prod_{i \in I} \mathcal{X}_i$ be equipped with the product topology, and let $\mathcal{X}_{\text{sum}} = \bigoplus_{i \in I} \mathcal{X}_i$ be equipped with the locally convex sum topology. Prove the following.

(i) If $T = (T_i)_{i\in I} \in \prod_{i\in I} \mathcal{L}(\mathcal{Y}, \mathcal{X}_i)$, then $\Pi(T) : \mathcal{Y} \to \mathcal{X}_{\text{prod}}$ is continuous. When we equip the product space $\prod_{i\in I} \mathcal{L}(\mathcal{Y}, \mathcal{X}_i)$ with the product topology, and the space $\mathcal{L}(\mathcal{Y}, \mathcal{X}_{\text{prod}})$ with the strong operator topology, then the map

$$\Pi_c : \prod_{i\in I} \mathcal{L}(\mathcal{Y}, \mathcal{X}_i) \ni T \longmapsto \Pi(T) \in \mathcal{L}(\mathcal{Y}, \mathcal{X}_{\text{prod}})$$

is a *topological* linear isomorphism.

(ii) If $S = (S_i)_{i\in I} \in \prod_{i\in I} \mathcal{L}(\mathcal{X}_i, \mathcal{Y})$, then $\Sigma(S) : \mathcal{X}_{\text{sum}} \to \mathcal{Y}$ is continuous. When we equip the product space $\prod_{i\in I} \mathcal{L}(\mathcal{X}_i, \mathcal{Y})$ with the product topology, and the space $\mathcal{L}(\mathcal{X}_{\text{sum}}, \mathcal{Y})$ with the strong operator topology, then the map

$$\Sigma_c : \prod_{i\in I} \mathcal{L}(\mathcal{X}_i, \mathcal{Y}) \ni S \longmapsto \Sigma(S) \in \mathcal{L}(\mathcal{X}_{\text{sum}}, \mathcal{Y})$$

is a *topological* linear isomorphism.

Comment. Assume $\mathcal{X}_i$, $i \in I$, are locally convex. If we apply Exercise 3 with $\mathcal{Y} = \mathbb{K}$, we obtain the existence of a natural *topological* linear isomorphism

$$\Sigma_c : \prod_{i\in I} (\mathcal{X}_i)^* \xrightarrow{\sim} \Big(\bigoplus_{i\in I} \mathcal{X}_i\Big)^*. \tag{16}$$

(It is understood that on $\mathcal{X}_{\text{sum}} = \bigoplus_{i\in I} \mathcal{X}_i$ one uses the locally convex sum topology, on all $(\mathcal{X}_i)^*$ and $\mathcal{X}_{\text{sum}}$ one uses the weak* topology, and on $\prod_{i\in I}(\mathcal{X}_i)^*$ one uses the product topology.)

This is sometimes quoted as a basic rule by stating: *"the dual of the sum is the product of the duals."*

F. Alaoglu-Bourbaki Compactness Theorem

The result we are about to prove concerns *compactness* in the weak* topology. Since the most popular applications come from a special case (for *normed* vector space version), we will discuss them elsewhere.

Theorem 5 (Alaoglu-Bourbaki). *Assume the vector space $\mathcal{X}$ is equipped with a linear topology. For any neighborhood $\mathcal{V}$ of 0 in $\mathcal{X}$, and any $\rho > 0$, the set*

$$\mathcal{K}_{\mathcal{V},\rho} = \{\phi \in \mathcal{X}^* \,:\, \sup_{x\in\mathcal{V}} |\phi(x)| \leq \rho\}$$

is compact in $(\mathcal{X}^*, \text{w}^*)$.

Proof. Consider the set $T = \{\alpha \in \mathbb{K} \,:\, |\alpha| \leq \rho\}$, and define the map

$$\Theta : \mathcal{K}_{\mathcal{V},\rho} \ni \phi \longmapsto \big(\phi(x)\big)_{x\in\mathcal{V}} \in \prod_{\mathcal{V}} T.$$

Denote the range $\Theta(\mathcal{K}_{\mathcal{V},\rho})$ simply by $\mathcal{R}$.

CLAIM: *When we equip $\mathcal{K}_{\mathcal{V},\rho}$ with the w^*-topology and we equip the set $\mathcal{R} \subset \prod_{\mathcal{V}} T$ with induced product topology, the map $\Theta : \mathcal{K}_{\mathcal{V},\rho} \to \mathcal{R}$ is a homeomorphism..*

Remark first that Θ is *injective*. Indeed, if $\phi, \psi \in \mathcal{K}_{\mathcal{V},\rho}$ are such that $\Theta(\phi) = \Theta(\psi)$, then $\phi(x) = \psi(x)$, $\forall\, x \in \mathcal{V}$, so since $\mathcal{V}$ is *absorbing*[5], by linearity, one also has $\phi(x) = \psi(x)$,

[5] That is, for every $x \in \mathcal{X}$, there exists $t > 0$ such that $tx \in \mathcal{V}$

$\forall\, x \in \mathcal{X}$, i.e. $\phi = \psi$. To prove the fact that Θ is a homeomorphism, we must show that for a net (ϕ_λ) in $\mathcal{K}_{\mathcal{V},\rho}$ and some $\phi \in \mathcal{K}_{\mathcal{V},\rho}$, the conditions

- $\phi_\lambda \xrightarrow{\text{w}^*} \phi$, and
- $\Theta(\phi_\lambda) \to \Theta(\phi)$ (in $\prod_{\mathcal{V}} T$ with respect to the product topology)

are equivalent. Explicitly this means that the conditions

$$\phi_\lambda(x) \to \phi(x),\ \forall\, x \in \mathcal{X}, \tag{17}$$

$$\phi_\lambda(x) \to \phi(x),\ \forall\, x \in \mathcal{V}, \tag{18}$$

ought to be equivalent. This is, however, trivial, again using linearity and the fact that $\mathcal{V}$ is absorbing.

Remark now that, since T is compact, the product space $\prod_{\mathcal{V}} T$ is also compact in the product topology, by Tihonov's Theorem. Therefore, using the Claim, all we need to do is to show that $\mathcal{R}$ is *closed* in the product topology. Suppose (ϕ_λ) is a net in $\mathcal{K}_{\mathcal{V},\rho}$, so that $\Theta(\phi_\lambda) \to \Gamma = (\Gamma_x)_{x\in\mathcal{V}} \in \prod_{x\in\mathcal{V}} T$ in the product topology, and let us prove the existence of (a unique) $\phi \in \mathcal{K}_{\mathcal{V},\rho}$, such that $\Theta(\phi) = \Gamma$. Treating Γ as a function $\Gamma : \mathcal{V} \to \mathbb{K}$, we know that, by the definition of the product topology, we have

$$\Gamma(x) = \lim_\lambda \phi_\lambda(x),\ \ \forall\, x \in \mathcal{V}. \tag{19}$$

Remark now that:

($*$) *if $x \in \mathcal{X}$ and $\alpha, \beta \in \mathbb{K}$ are such that αx and βx both belong to $\mathcal{V}$, then $\beta\Gamma(\alpha x) = \alpha\Gamma(\beta x)$.*

This follows immediately from (19), which yields: $\beta\Gamma(\alpha x) = \lim_\lambda \phi_\lambda(\alpha\beta x)$.

Using ($*$) we can now define $\phi : \mathcal{X} \to \mathbb{K}$ as follows. Start with some $x \in \mathcal{X}$, choose some $t > 0$ such that $tx \in \mathcal{V}$ (use here the fact that $\mathcal{V}$ is absorbing), and let $\phi(x) = t^{-1}\Gamma(tx)$. The point of ($*$) is that the value $t^{-1}\Gamma(tx)$ *is independent of the choice of* t. By construction we have

$$\phi(x) = \Gamma(x),\ \ \forall\, x \in \mathcal{V}, \tag{20}$$

Property ($*$) also yields:

$$\phi(\alpha x) = \alpha\phi(x),\ \ \forall\, x \in \mathcal{X},\, \alpha \in \mathbb{K}.$$

To prove linearity, we start with $x, y \in \mathcal{X}$ and – using the fact that $\mathcal{V}$ is a neighborhood of 0 – we choose $t > 0$, such that tx, ty and $tx + ty$ all belong to $\mathcal{V}$. Then

$$\begin{aligned}\phi(x+y) = t^{-1}\Gamma(tx+ty) = t^{-1}\lim_\lambda \phi_\lambda(tx+ty) = t^{-1}\lim_\lambda[\phi_\lambda(tx) + \phi_\lambda(ty)] = \\ = t^{-1}[\lim_\lambda \phi_\lambda(tx) + \lim_\lambda \phi_\lambda(ty)] = t^{-1}[\Gamma(tx) + \Gamma(ty)] = \phi(x) + \phi(y).\end{aligned}$$

Now we have a linear functional $\phi : \mathcal{X} \to \mathbb{K}$, satisfying

$$\lim_\lambda \phi_\lambda(x) = \phi(x),\ \ \forall\, x \in \mathcal{V}. \tag{21}$$

Using the definition of $\mathcal{K}_{\mathcal{V},\rho}$, the limit (21) yields the inequality

$$|\phi(x)| \leq \rho, \ \ \forall x \in \mathcal{V}. \tag{22}$$

On the one hand, this shows that the set

$$\mathcal{U}_\phi(2\rho) = \{x \in \mathcal{X} \, : \, |\phi(x)| < 2\rho\}$$

contains $\mathcal{V}$, so $\mathcal{U}_\phi(2\rho)$ is a neighborhood of 0, and then by Proposition 1 it follows that ϕ is indeed continuous. On the other hand, (22) now tells us that ϕ in fact belongs to $\mathcal{K}_{\mathcal{V},\rho}$, and then (19) forces $\Gamma = \Theta(\phi)$. $\square$

Remark 7. If $\mathcal{Y}$ is any linear subspace, such that $\mathcal{X}^* \subset \mathcal{Y} \subset \mathcal{X}'$, then[6] the set $\mathcal{K}_{\mathcal{V},\rho}$ is also compact in $\mathcal{Y}$, in the induced w*-topology. In particular, $\mathcal{K}_{\mathcal{V},\rho}$ is also compact in $(\mathcal{X}', \mathrm{w}^*)$.

G. The (Topological) Transpose

Definitions. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces, and $S \in \mathrm{Lin}(\mathcal{X}, \mathcal{Y})$, Then for any vector space $\mathcal{Z}$, one has a linear map

$$S^{\mathrm{transp}_{\mathcal{Z}}} : \mathrm{Lin}(\mathcal{Y}, \mathcal{Z}) \ni L \longmapsto LS \in \mathrm{Lin}(\mathcal{X}, \mathcal{Z}).$$

The linear map $S^{\mathrm{transp}_{\mathcal{Z}}}$ is referred to as the *algebraic $\mathcal{Z}$-transpose of S.*

Assuming now $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are all equipped with linear topologies, and S is linear and continuous, it is obvious that if $L \in \mathrm{Lin}(\mathcal{Y}, \mathcal{Z})$ is continuous, then so is LS. Therefore, by restricting $S^{\mathrm{transp}_{\mathcal{Z}}}$ to $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$, we obtain a linear map

$$S^{*_{\mathcal{Z}}} : \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \ni L \longmapsto LS \in \mathcal{L}(\mathcal{X}, \mathcal{Z}),$$

which is referred to as the *topological $\mathcal{Z}$-transpose of S.* (When there is no danger of confusion, $S^{*_{\mathcal{Z}}}$ is simply denoted by S^*.

In particular, when $\mathcal{Z} = \mathbb{K}$, we speak of the *topological dual transpose* $S^* : \mathcal{Y}^* \to \mathcal{X}^*$.

One key feature of the transpose operation is *functoriality*, which states that, if $\mathcal{X} \xrightarrow{T} \mathcal{M} \xrightarrow{S} \mathcal{Y}$ are linear and continuous, then the composition

$$\mathcal{L}(\mathcal{Y}, \mathcal{Z}) \xrightarrow{S^*} \mathcal{L}(\mathcal{M}, \mathcal{Z}) \xrightarrow{T^*} \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

is $(ST)^*$, i.e. one has the identity

$$(ST)^* = T^* S^*.$$

Another key feature is *continuity*, as described in the following result.

Proposition 4. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be equipped with linear topologies, and let $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. If we equip the spaces $\mathcal{L}(\mathcal{X}, \mathcal{Z})$ and $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$ with the strong operator topologies, then the linear map*

$$S^* : (\mathcal{L}(\mathcal{Y}, \mathcal{Z}), \mathfrak{T}_{\text{s-o}}) \to (\mathcal{L}(\mathcal{X}, \mathcal{Z}), \mathfrak{T}_{\text{s-o}})$$

is continuous.

In particular, when we equip the topological dual spaces $\mathcal{X}^*$ and $\mathcal{Y}^*$ with their weak topologies, the linear map $S^* : (\mathcal{Y}^*, \mathrm{w}^*) \to (\mathcal{X}^*, \mathrm{w}^*)$ is continuous.*

[6] This follows from a well know fact in Topology: *if Y is a topological space, and Z is a subset in Y, then every $K \subset Z$, which is compact in Z, relative to the induced topology, is also compact in Y.*

Proof. Suppose $L_\lambda \xrightarrow{\mathfrak{T}_{\rm so}} L$ in $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$, and let us prove that $S^*(L_\lambda) \xrightarrow{\mathfrak{T}_{\rm so}} S^*(L)$ in $\mathcal{L}(\mathcal{X}, \mathcal{Z})$, which means that

$$S^*(L_\lambda)x \to S^*(L)x \text{ (in } \mathcal{Z}), \forall\, x \in \mathcal{X}.$$

The above condition is, however, trivial, since $S^*(L_\lambda)x = L_\lambda(Sx)$ and $S^*(L)x = L(Sx)$, and we know (by the condition $L_\lambda \xrightarrow{\mathfrak{T}_{\rm so}} L$) that: $L_\lambda y \to Ly$ (in $\mathcal{Z}$), $\forall\, y \in \mathcal{Y}$. □

Concerning the functoriality of the transpose operation, one has certain interesting connections, illustrated in the following two Exercises.

Exercise 4*. Suppose $\mathcal{X}$ is equipped with a linear topology, and $\mathcal{Y}$ is a locally convex topological vector space. Prove that, for $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the following are equivalent:

(i) T has *dense range in* $\mathcal{Y}$;

(ii) the topological dual transpose $S^* : \mathcal{Y}^* \to \mathcal{X}^*$ is *injective*;

(iii) for any vector space $\mathcal{Z}$, equipped with a linear topology, the map $S^* : \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \to \mathcal{L}(\mathcal{X}, \mathcal{Z})$ is injective.

(Hint: The implications $(i) \Rightarrow (iii) \Rightarrow (ii)$ are trivial. When proving $(ii) \Rightarrow (i)$, let $\mathcal{M}$ be the closure of Range S, and argue – using Corollary 1 – that, if $\mathcal{M} \subsetneq \mathcal{Y}$, then there exists $y \in \mathcal{Y}$ and $\phi \in \mathcal{Y}^*$, such that $\phi\big|_{\mathcal{M}} = 0$, but $\phi(y) \neq 0$.)

Exercise 5*. Let $(\mathcal{Y}, \mathfrak{T})$ be a locally convex topological vector space, and let $\mathcal{X} \subset \mathcal{Y}$ be a linear subspace. Equip $\mathcal{X}$ with the induced topology $\mathfrak{T}\big|_{\mathcal{X}}$, so that the inclusion map $J : \mathcal{X} \hookrightarrow \mathcal{Y}$ is continuous. Prove that the topological dual transpose $J^* : \mathcal{Y}^* \to \mathcal{X}^*$ is *surjective*. (Hint: Let $\mathfrak{P}$ be the directed collection of all $\mathfrak{T}$-continuous seminorms on $\mathcal{Y}$, so that the induced topology $\mathfrak{T}\big|_{\mathcal{X}}$ is defined by the collection $\mathfrak{P}\big|_{\mathcal{X}}$ consisting of the restrictions to $\mathcal{X}$ of all the seminorms in $\mathfrak{P}$. Argue that, if $\phi \in \mathcal{X}^*$, there exists $p \in \mathfrak{P}$, such that $|\phi(x) \leq p(x)$, $\forall\, x \in \mathcal{X}$. Use the Hahn-Banach Theorem to produce $\psi \in \mathcal{Y}^*$, such that $\psi\big|_{\mathcal{X}} = \phi$.)

Comment. When comparing the above two Exercises, one notices that in Exercise 4, there is no counterpart of property (iii) from Exercise 4. In other words, using the notations and assumptions as in Exercise 5, one cannot derive the surjectivity of

$$J^* : \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \to \mathcal{L}(\mathcal{X}, \mathcal{Z}), \tag{23}$$

for *arbitrary* (locally convex) topological vector spaces $\mathcal{Z}$. It is quite trivial to see that the surjectivity of (23) may even fail when $\mathcal{Z} = \mathcal{X}$. After all, the condition that

$$J^* : \mathcal{L}(\mathcal{Y}, \mathcal{X}) \to \mathcal{L}(\mathcal{X}, \mathcal{X}) \tag{24}$$

is surjective, is equivalent to the existence of some $P \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, satisfying

$$Px = x, \;\; \forall\, x \in \mathcal{X}.$$

A linear map $P : \mathcal{Y} \to \mathcal{X}$, satisfying the above condition, is called a *projection of* $\mathcal{Y}$ *onto* $\mathcal{X}$. Using this language, the surjectivity of (23), for *any* $\mathcal{Z}$, is equivalent to the existence

of a continuous projection P of $\mathcal{Y}$ ont $\mathcal{X}$. (Indeed, if such a P exists, then $J^*(LP) = L$, $\forall\, L \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$.)

This discussion then legitimizes the formulation of the so-called PROJECTION PROBLEM: *Given a topological vector space $\mathcal{Y}$, and a linear subspace $\mathcal{X} \subsetneq \mathcal{Y}$, decide if a continuous projection of $\mathcal{Y}$ onto $\mathcal{X}$ exists.* An equivalent of describing such projection is to think them as operators $Q \in \mathcal{L}(\mathcal{Y})$, satisfying[7] $Q^2 = Q$ and $\operatorname{Range} Q = \mathcal{X}$. It is trivial to see that any $Q \in \mathcal{L}(\mathcal{Y})$, satisfying $Q^2 = Q$, is forced to have *closed range*, therefore the Projection Problem is only meaningful when $\mathcal{X}$ is a closed linear subspace. The two problems below show examples when the Projection Problem has an affirmative answer. (A negative example will be discussed in a different chapter.)

Exercise 6*. Assume $\mathcal{Y}$ is a locally convex topological vector space, and $\mathcal{X} \subset \mathcal{Y}$ is a *finite dimensional* linear subspace. Show that a continuous projection of $\mathcal{Y}$ onto $\mathcal{X}$ exists. (HINT: Fix a topological linear isomorphism $S : \mathcal{X} \xrightarrow{\sim} \mathbb{K}^n$, and consider the coordinate maps $\pi_j : \mathbb{K}^n \to \mathbb{K}$, $j = 1, \dots, n$. Use Exercise 5 to obtain the existence of $\phi_1, \dots \phi_n \in \mathcal{Y}^*$, such that $(\pi_j \circ S)(x) = \pi_j(x)$, $\forall\, x \in \mathcal{X}$, $j = 1, \dots, n$. Form the linear continuous operator $\Phi = (\phi_1, \dots, \phi_n) : \mathcal{Y} \to \mathbb{K}^n$, and let $P = S^{-1}\Phi$.)

Exercise 7*. Assume $\mathcal{Y}$ is a locally convex topological vector space, and $\mathcal{X} \subset \mathcal{Y}$ is a closed linear subspace of *finite co-dimension*, i.e. the quotient space $\mathcal{Y}/\mathcal{X}$ is finite dimensional. Show that a continuous projection of $\mathcal{Y}$ onto $\mathcal{X}$ exists. (HINT: Let $\Pi : \mathcal{Y} \to \mathcal{Y}/\mathcal{X}$ denote the quotient map. Fix some linear basis $\{v_1, \dots, v_n\}$ for $\mathcal{Y}/\mathcal{X}$, let $y_1, \dots, y_n \in \mathcal{Y}$ be such that $\Pi y_j = v_j$, $\forall\, j = 1, \dots, n$, and let $S : \mathcal{Y}/\mathcal{X} \to \mathcal{Y}$ be the unique linear map satisfying $Sv_j = y_j$, $\forall\, j$. Argue that $Q : \mathcal{Y} \to \mathcal{Y}$, defined by $Qy = y - S\Pi y$, is the desired projection.)

Exercise 8*. Suppose $\mathcal{Y}$ is a locally convex topological vector space, and $\mathcal{X} \subset \mathcal{Y}$ is a closed linear subspace. Using the notation from Exercise 5, we already know that $J^* : \mathcal{Y}^* \to \mathcal{X}^*$ is surjective and continuous, when both $\mathcal{X}^*$ and $\mathcal{Y}^*$ are equipped with their respective W* topologies. Prove that J^* is also open, so the W* topology on $\mathcal{X}^*$ is the *quotient* topology on $\mathcal{Y}^*/\operatorname{Ker} J^*$. (HINT: Suffices to show that, if $\mathcal{A}$ is a basic open set in $\mathcal{Y}^*$, which can be written as $\mathcal{A} = \{\phi \in \mathcal{Y}^* : \max_{y \in \mathcal{F}} |\phi(y)| < 1\}$, for some finite subset $\mathcal{F} \subset \mathcal{Y}$, then $J^*\mathcal{A}$ is open in $\mathcal{X}^*$. Argue first that, if we consider the subspace $\mathcal{Z} = \mathcal{X} + \operatorname{Span} \mathcal{F}$, and the inclusion $K : \mathcal{Z} \hookrightarrow \mathcal{Y}$, then $K^*\mathcal{A}$ is open in $\mathcal{Z}$, in a very obvious way. This shows that we can replace $\mathcal{Y}$ with $\mathcal{Z}$, so we can take advantage of Exercise 7.)

Exercise 9. In Exercise 8, why do we need $\mathcal{X}$ to be *closed* in $\mathcal{Y}$? Find an example, where $\mathcal{X} \subsetneq \mathcal{Y}$ is *dense*, such that J^* is not open.

Exercise 10. Let $\mathcal{X}$ be a locally convex topological vector space, and let $\mathcal{Y} \subset \mathcal{X}$ be a closed linear subspace. Equip the quotient space $\mathcal{X}/\mathcal{Y}$ with the quotient topology, and let $\Pi : \mathcal{X} \to \mathcal{X}/\mathcal{Y}$ denote the quotient map. From Exercise 5 we already know that $\Pi^* : \mathcal{L}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}) \to \mathcal{L}(\mathcal{X}, \mathcal{Z})$ is injective, for any topological vector space $\mathcal{Z}$. Prove that

(i) $\operatorname{Range} \Pi^* = \{T \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \,:\, T\big|_{\mathcal{Y}} = 0\}$;

(ii) When we equip $\operatorname{Range} \Pi^*$ with the induced so topology, the map

$$\Pi^* : (\mathcal{L}(\mathcal{X}/\mathcal{Y}, \mathcal{Z}), \mathfrak{T}_{\text{so}}) \to (\operatorname{Range} \Pi^*, \mathfrak{T}_{\text{so}})$$

[7] Using the multiplicative notation, Q^2 stands for $Q \circ Q$.

is a topological linear isomorphism.

Exercise 11. Suppose $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are vector spaces equipped with linear topologies, and let $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Prove that the left composition operator

$$\Lambda_S : (\mathcal{L}(\mathcal{Z}, \mathcal{X}), \mathfrak{T}_{\text{so}}) \ni L \longmapsto SL \in (\mathcal{L}(\mathcal{Z}, \mathcal{Y}), \mathfrak{T}_{\text{so}})$$

is continuous.