Locally Convex Vector Spaces III:
The Metric Point of View

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

Warning! Some proofs are based on Exercises from previous lectures. The reader is urged to solve all Exercises from CW I-III and LCVS I-II. (See Remarks 1-6 below.)

Convention. Throughout this note \( K \) will be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and all vector spaces are over \( K \).

A. Locally convex topologies defined by seminorms

In this sub-section we outline a method of constructing locally convex topologies using seminorms.

**Notation.** Given a vector space \( X \) and a seminorm \( p \) on \( X \), we define the two unit \( p \)-balls as the sets:

\[
B_X(p) = \{ x \in X : p(x) < 1 \},
\]
\[
\overline{B}_X(p) = \{ x \in X : p(x) \leq 1 \}.
\]

(When there is no danger of confusion, the subscript \( X \) will be omitted from the notation.)

**Remarks 1-7.** Most of the properties listed below are essentially contained in CW II (see the brief references provided)

1. The sets \( B(p) \) and \( \overline{B}(p) \) are convex and balanced. The set \( B(p) \) is openly absorbing, and its Minkowski functional is: \( q_{B(p)} = p \). (Exercise 8 from CW II.)

2. If \( A \subset X \) is convex and openly absorbing, then \( A \) can be recovered from its Minkowski functional as:

\[
A = \{ x \in X : q_A(x) < 1 \}.
\]

In particular, if \( A \) is also balanced, then \( q_A \) is a seminorm, and one has the equality \( A = B(q_A) \). (Proposition 2 from CW II).

3. A sufficient condition for a subset \( 0 \in A \subset X \) to be openly absorbing is that \( A \) is open with respect to some linear topology on \( X \). (Exercise 6 from CW II.)

4. For two seminorms \( p \) and \( q \), the following are equivalent:

(i) \( p(x) \leq q(x), \forall x \in X \);
(ii) \( B(p) \supset B(q) \).
The implication \( (i) \Rightarrow (ii) \) is trivial. For the implication \( (ii) \Rightarrow (i) \), observe that for every \( x \in \mathcal{X} \) and every \( \varepsilon > 0 \), one has \( q\left(\frac{1}{q(x)+\varepsilon} x\right) = \frac{q(x)}{q(x)+\varepsilon} < 1 \), so if \( (ii) \) holds then this implies also \( p\left(\frac{1}{q(x)+\varepsilon} x\right) \leq 1 \), i.e. \( p(x) \leq q(x) + \varepsilon \), \( \forall \varepsilon > 0 \), hence \( (i) \) follows.

5. If \( p \) is a seminorm on \( \mathcal{X} \) then

\[
\varepsilon \mathcal{B}(p) = \{ x \in \mathcal{X} : p(x) < \varepsilon \},
\]

\[
\varepsilon \mathcal{B}(p) = \{ x \in \mathcal{X} : p(x) \leq \varepsilon \}.
\]

Clearly one has \( x \in \varepsilon \mathcal{B}(p) \iff \varepsilon^{-1} x \in \mathcal{B}(p) \iff p(\varepsilon^{-1} x) < 1 \iff p(x) < \varepsilon \), and similar equivalences with \( \leq \) in place of \(<\).

6. If \( p \) is a seminorm on \( \mathcal{X} \) then

\[
|p(x) - p(y)| \leq p(x - y), \ \forall x, y \in \mathcal{X}.
\]

Use \( p(x - y) = p(y - x) \), and the triangle inequalities \( p(y) + p(x - y) \geq p(x) \) and \( p(x) + p(y - x) \geq p(y) \).

7. If \( \mathcal{X} \) is equipped with a linear topology \( \Sigma \), and \( p \) is a seminorm on \( \mathcal{X} \), then the following are equivalent:

(i) \( p \) is continuous;

(ii) \( p \) is continuous at \( 0 \);

(iii) \( \mathcal{B}(p) \) is a neighborhood of \( 0 \);

(iii') \( \mathcal{B}(p) \) is a neighborhood of \( 0 \).

First of all, the implications \( (i) \Rightarrow (iii) \Rightarrow (iii') \) are trivial. For the implication \( (iii') \Rightarrow (ii) \), we must show that, assuming \( (iii') \), for every \( \varepsilon > 0 \), the set \( \mathcal{V} = \{ x \in \mathcal{X} : p(x) < \varepsilon \} \) is a neighborhood of \( 0 \). But this is trivial, since \( \mathcal{V} \supseteq \varepsilon \mathcal{B}(p) \). Finally, for the implication \( (ii) \Rightarrow (i) \), we notice that if \( p \) is continuous at \( 0 \), and \( x_\lambda \to x \), then \( (x_\lambda - x) \to 0 \), and then the inequalities \( 0 \leq |p(x_\lambda) - p(x)| \leq p(x_\lambda - x) \) will clearly force \( p(x_\lambda) \to p(x) \).

The starting observation in this section is contained in the following result.

**Proposition 1.** Let \( \mathcal{X} \) be a locally convex vector space.

(i) For every balanced convex neighborhood \( \mathcal{A} \) of \( 0 \), the Minkowski functional \( q_\mathcal{A} \) is a continuous seminorm on \( \mathcal{X} \).

(ii) The correspondence

\[
\left\{ \mathcal{A} \subset \mathcal{X} \mid \mathcal{A} \text{ open, convex, balanced, } \mathcal{A} \ni 0 \right\} \ni \mathcal{A} \longmapsto q_\mathcal{A} \in \left\{ q : \mathcal{X} \to [0, \infty) \mid q \text{ continuous seminorm on } \mathcal{X} \right\}
\]

is bijective. Its inverse is the map:
\[ \left\{ q : \mathcal{X} \to [0, \infty) \mid q \text{ continuous seminorm on } \mathcal{X} \right\} \ni p \longmapsto \mathcal{B}(p) \in \left\{ \mathcal{A} \subset \mathcal{X} \mid \mathcal{A} \text{ open, convex, balanced, } \mathcal{A} \ni 0 \right\}. \] (2)

**Proof.** (i). Fix some balanced convex neighborhood \( \mathcal{A} \) of 0. We already know (CW II) that \( q_{\mathcal{A}} \) is a seminorm on \( \mathcal{A} \), and \( \mathcal{A} = \mathcal{B}(q_{\mathcal{A}}) \). By Remark 7, \( q_{\mathcal{A}} \) is continuous.

(ii). For simplicity, let \( \mathfrak{B} \) denote the source set of (1), and let \( \mathfrak{P} \) denote the collection of all continuous seminorms.

First of all, for every \( p \in \mathfrak{P} \), the unit ball \( \mathcal{B}(p) \) is convex and openly absorbing (by Remark 1). Furthermore, \( \mathcal{B}(p) \) is also open, by continuity, so the correspondence (2) indeed takes values in \( \mathfrak{B} \).

Secondly, by Remarks 1 and 2, we also know that

\[ q_{\mathcal{B}(p)} = p, \forall p \in \mathfrak{P}; \]
\[ \mathcal{B}(q_{\mathcal{A}}) = \mathcal{A}, \forall \mathcal{A} \in \mathfrak{C}, \]

which clearly show that (1) and (2) are inverses of each other. \( \square \)

**Theorem-Definition 1.** Let \( \mathcal{X} \) be a vector space, and let \( \mathfrak{P} \) be a non-empty collection of seminorms on \( \mathcal{X} \). If we define, for every \( \varepsilon > 0 \) and \( p \in \mathfrak{P} \) the set

\[ \mathcal{U}_{p,\varepsilon} = \{ x \in \mathcal{X} : p(x) < \varepsilon \}, \]

there exists a unique locally convex topology \( \mathfrak{T} \) on \( \mathcal{X} \), such that the collection

\[ \mathcal{U} = \{ \mathcal{U}_{p,\varepsilon} : p \in \mathfrak{P}, \varepsilon > 0 \} \]

constitutes a fundamental system of \( \mathfrak{T} \)-neighborhoods of 0. Moreover:

(i) all \( p \in \mathfrak{P} \) are \( \mathfrak{T} \)-continuous;

(ii) \( \mathfrak{T} \) is the weakest among all locally convex topologies on \( \mathcal{X} \) which make all \( p \in \mathfrak{P} \) are continuous.

The topology \( \mathfrak{T} \) is referred to as the **locally convex topology defined by** \( \mathfrak{P} \), as is denoted by \( \mathfrak{T}(\mathfrak{P}) \). If \( \mathfrak{P} \) is a singleton \( \{ p \} \), we denote this topology simply by \( \mathfrak{T}(p) \).

**Proof.** Consider the collection \( \mathfrak{C} = \{ \mathcal{B}(p) : p \in \mathfrak{P} \} \). Since all sets in \( \mathfrak{C} \) are convex, balanced, and (openly) absorbing, by Theorem 1 from LCVS I, there exists a unique locally convex topology \( \mathfrak{T} \), for which the collection

\[ \mathcal{U} = \{ \varepsilon \mathcal{B}(p) : \varepsilon > 0, p \in \mathfrak{P} \} \] (3)

is a fundamental system of \( \mathfrak{T} \)-neighborhoods of 0.

To prove (i) we simply notice that, since every \( p \in \mathfrak{P} \) can be written (by Remark 1) as the Minkowski functional of its unit ball \( p = q_{\mathcal{B}(p)} \), and \( \mathcal{B}(p) \) is by construction a convex balanced \( \mathfrak{T} \)-neighborhood of 0, the continuity of \( p \) follows from Proposition 1(i).

To prove (ii), fix another locally convex topology \( \mathfrak{T}' \) such that all \( p \in \mathfrak{P} \) are \( \mathfrak{T}' \)-continuous, and let us show that \( \mathfrak{T}' \supset \mathfrak{T} \). Since both \( \mathfrak{T} \) and \( \mathfrak{T}' \) are linear, all we need to show is that:

\footnote{By Remark 5, \( \varepsilon \mathcal{B}(p) = \mathcal{U}_{p,\varepsilon} \).}
(\*) every \( \mathcal{I} \)-neighborhood of 0 is also a \( \mathcal{I}' \)-neighborhood of 0.

Since \( \mathcal{I} \) has \( \mathcal{U} \) as a fundamental system of \( \mathcal{I} \)-neighborhoods for 0, the above condition is equivalent to the condition that every set in \( \mathcal{U} \) is a \( \mathcal{I}' \)-neighborhood of 0. Using (3) and the fact that dilations are homeomorphisms in linear topologies we now see that (\*) is in fact equivalent to the following.

(\**) For every \( p \in \mathcal{P} \), the unit ball \( \mathcal{B}(p) \) is a \( \mathcal{I}' \)-neighborhood of 0.

But (\**) is clearly true, since all \( p \in \mathcal{P} \) are \( \mathcal{I}' \)-continuous (so the \( \mathcal{B}(p) \)'s are in fact open neighborhoods of 0.) \( \square \)

**Remark 8.** With note notations as above, if we consider, for every \( p \in \mathcal{P}, \varepsilon > 0 \), the set

\[ W_{pe} = \{ x \in \mathcal{X} : p(x) \leq \varepsilon \} = \varepsilon \mathcal{B}(p), \]

then the collection

\[ \mathcal{W} = \{ W_{pe} : p \in \mathcal{P}, \varepsilon > 0 \} \]

also constitutes a fundamental system of \( \mathcal{I} \)-neighborhoods of 0. This is pretty obvious from the inclusions \( U_{pe} \subset W_{pe} \subset U_{p,2\varepsilon} \).

**Remark 9.** Every locally convex topology on a vector space \( \mathcal{X} \) can be constructed as \( \mathcal{I}(\mathcal{P}) \), for a suitably chosen collection \( \mathcal{P} \) of seminorms on \( \mathcal{X} \). More precisely, if \( \mathcal{G} \) is a locally convex topology, we can simply take \( \mathcal{P} \) to be the collection of all \( \mathcal{G} \)-continuous seminorms. On the one hand, by Theorem-Definition 1, since all \( p \in \mathcal{P} \) are \( \mathcal{G} \)-continuous, we automatically have the inclusion \( \mathcal{G} \supset \mathcal{I}(\mathcal{P}) \). On the other hand,

(\*) every \( \mathcal{G} \)-neighborhood of 0 is also a \( \mathcal{I}(\mathcal{P}) \)-neighborhood of 0,

so we also have the inclusion \( \mathcal{G} \subset \mathcal{I}(\mathcal{P}) \). Property (\*) can be proven as follows. Start with some \( \mathcal{G} \)-neighborhood \( \mathcal{V} \) of 0, and choose a balanced open convex set \( \mathcal{A} \subset \mathcal{V} \). By Proposition 1(i), the Minkowski functional \( q_A \) is \( \mathcal{G} \)-continuous, hence \( q_A \) belongs to \( \mathcal{P} \). By Remark 2, \( \mathcal{A} = \mathcal{B}(q_A) \), so by the definition of \( \mathcal{I}(\mathcal{P}) \), it follows that \( \mathcal{A} \) is a \( \mathcal{I}(\mathcal{P}) \)-neighborhood of 0, and so will be \( \mathcal{V} \supset \mathcal{A} \).

**Remark 10.** If a locally convex space \( \mathcal{X} \) has its topology defined by a family \( \mathcal{P} \) of seminorms, then for a net \( (x_\lambda)_{\lambda \in \Lambda} \) in \( \mathcal{X} \) and some \( x \in \mathcal{X} \), the following conditions are equivalent:

(i) \( x_\lambda \to x \);

(ii) \( p(x_\lambda - x) \to 0 \), \( \forall p \in \mathcal{P} \).

Of course, the implication (i) \( \Rightarrow \) (ii) is trivial. Conversely, if condition (ii) holds, then for every \( p \in \mathcal{P} \) and every \( \varepsilon > 0 \), there exists \( \lambda_{p,\varepsilon} \in \Lambda \), such that \( p(x_\lambda - x) < \varepsilon \), \( \forall \lambda \succ \lambda_{p,\varepsilon} \). Equivalently, using the notations from Theorem-Definition 1, it follows that for every \( \mathcal{U} \in \mathcal{U} \), there exists \( \lambda_{\mathcal{U}} \) such that

\[ x_\lambda \in \mathcal{U} + x, \quad \forall, \lambda \succ \lambda_{\mathcal{U}}, \]

and since by construction the collection \( \{ \mathcal{U} + x : \mathcal{U} \in \mathcal{U} \} \) is a fundamental system of neighborhoods for \( x \), it follows that \( (x_\lambda) \) indeed converges to \( x \).
Remark 11. Given a family $\mathfrak{P}$ of seminorms on $\mathcal{X}$, the locally convex topology $\mathfrak{T}(\mathfrak{P})$ is Hausdorff, if and only if the “null set” $\mathcal{N}(\mathfrak{P}) = \{ x \in \mathcal{X} : p(x) = 0, \forall p \in \mathfrak{P} \}$ is equal to the singleton set $\{0\}$. This follows from the obvious equality $\bigcap_{\varepsilon > 0} \varepsilon \mathcal{B}(p) = \{ x \in \mathcal{X} : p(x) = 0 \}$, which implies: $\mathcal{N}(\mathfrak{P}) = \bigcap_{p \in \mathfrak{P}} \varepsilon \mathcal{B}(p)$, and then the desired equivalence is a consequence of Remark 1 from LCVS I. It should be noted that, without any additional assumptions on $\mathfrak{P}$ the null set $\mathcal{N}(\mathfrak{P})$ is a $\mathfrak{T}(\mathfrak{P})$-closed linear subspace, namely the closure $\overline{\{0\}}$ of 0.

**Comment.** The whole point of Theorem-Definition 1 (and of Theorem 1 from LCVS I) is that, in many instances, the collection $\mathfrak{P}$ (or $\mathfrak{C}$ from the Theorem) can be chosen to be (very) small. Therefore the following question is important: Given two collections $\mathfrak{P}$ and $\mathfrak{Q}$ of seminorms on $\mathcal{X}$, when does one have the inclusion $\mathfrak{T}(\mathfrak{Q}) \subset \mathfrak{T}(\mathfrak{P})$? Of course, by Theorem-Definition 1, the following conditions are equivalent:

- $\mathfrak{T}(\mathfrak{Q}) \subset \mathfrak{T}(\mathfrak{P})$;
- every $q \in \mathfrak{Q}$ is $\mathfrak{T}(\mathfrak{P})$-continuous.

What we want, however, is to give a characterization which is intrinsic to $\mathfrak{P}$ and $\mathfrak{Q}$. One such formulation is contained in the following result.

**Proposition 2.** Let $\mathcal{X}$ be a vector space and let $\mathfrak{T} = \mathfrak{T}(\mathfrak{P})$ be the locally convex topology defined by a family $\mathfrak{P}$ of seminorms on $\mathcal{X}$. For a seminorm $q$ on $\mathcal{X}$, the following are equivalent:

(i) $q$ is $\mathfrak{T}$-continuous;

(ii) there exists $p_1, \ldots, p_n \in \mathfrak{P}$ and $t_1, \ldots, t_n \geq 0$, such that:

$$q(x) \leq t_1p_1(x) + \cdots + t_np_n(x), \quad \forall x \in \mathcal{X}. \quad \text{(4)}$$

**Proof.** $(i) \Rightarrow (ii)$. Assume $p$ is continuous, so that the unit ball $\mathcal{B}(p)$ is an open convex balanced $\mathfrak{T}$- neighborhood of 0. Using Theorem-Definition 1 (and Remark 2), there exists $p_1, \ldots, p_n \in \mathfrak{P}$, and $\varepsilon_1, \ldots, \varepsilon_n > 0$, such that

$$\varepsilon_1 \mathcal{B}(p_1) \cap \cdots \cap \varepsilon_n \mathcal{B}(p_n) \subset \mathcal{B}(q).$$

Consider the seminorm $p = \varepsilon_1^{-1}p_1 + \cdots + \varepsilon_n^{-1}p_n$. The proof of $(ii)$ will be finished once we show that $q(x) \leq p(x), \forall x \in \mathcal{X}$. By Remark 4 this inequality is equivalent to the inclusion $\mathcal{B}(p) \subset \mathcal{B}(q)$. Start then with some $x \in \mathcal{B}(p)$, i.e.

$$\varepsilon_1^{-1}p_1(x) + \cdots + \varepsilon_n^{-1}p_n(x) < 1.$$ 

In particular, for every $k = 1, \ldots, n$ we have the inequality $\varepsilon_k^{-1}p_k(x) < 1$, which is equivalent to $p_k(x) < \varepsilon_k$, i.e. $x \in \varepsilon_k \mathcal{B}(p_k)$. In other words, it follows that $x \in \varepsilon_1 \mathcal{B}(p_1) \cap \cdots \cap \varepsilon_n \mathcal{B}(p_n) \subset \mathcal{B}(q)$, and we are done.

$(ii) \Rightarrow (i)$. Assume $q$ satisfies $(4)$, and let us show that $q$ is $\mathfrak{T}$-continuous. We use Remark 7, so we start with a net $(x_\lambda)_{\lambda \in \Lambda}$ in $\mathcal{X}$ which converges to 0, and let us prove that

$$q(x_\lambda) \to 0. \quad \text{(5)}$$
With \( p_1, \ldots, p_n \) as in (4), we know (by continuity) that \( p_k(x_\lambda) \to 0, \forall k = 1, \ldots, n \). The desired conclusion (5) then follows immediately from the inequalities

\[
0 \leq q(x_\lambda) \leq t_1 p_1(x_\lambda) + \cdots + t_n p_n(x_\lambda). \quad \square
\]

Comments. A. Using an intrinsic terminology, one could say that a seminorm \( q \) is dominated by the seminorm family \( \mathcal{P} \), if it satisfies condition (ii) from Proposition 2. With this language, the above result simply says that \( q \) is \( \mathcal{I}(\mathcal{P}) \)-continuous, if and only if \( q \) is dominated by \( \mathcal{P} \).

B. Suppose one starts with a collection \( \mathcal{P} \) of seminorms on \( X \). It is obvious that, for any sub-collection \( \mathcal{P}_0 \subset \mathcal{P} \), one has the inclusion \( \mathcal{I}(\mathcal{P}_0) \subset \mathcal{I}(\mathcal{P}) \). If one wants then to replace \( \mathcal{P} \) with \( \mathcal{P}_0 \), without changing the locally convex topologies they define, a necessary and sufficient condition is: every \( p \in \mathcal{P} \) is dominated by \( \mathcal{P}_0 \). This observation is our main tool in producing small families of seminorms that define a given locally convex topology.

Example 1. If \( \mathcal{P} = \{p_1, \ldots, p_n\} \) is a finite collection of seminorms on \( X \), then there exists one seminorm \( q \), so that \( \mathcal{I}(\mathcal{P}) = \mathcal{I}(q) \), namely \( q = p_1 + \cdots + p_n \). Obviously \( q \) is \( \mathcal{I}(\mathcal{P}) \)-continuous. Conversely, since \( p_k \leq q \), it follows that all \( p_k, k = 1, \ldots, n \) are \( \mathcal{I}(q) \) continuous.

Comment. Going back to Theorem-Definition 1, it is useful to ask when the collection \( \mathcal{U} \) introduced there is in fact a basic \( \mathcal{I}(\mathcal{P}) \)-neighborhood system for 0. As before, we want an intrinsic characterization. For this purpose, we introduce the following pre-order\(^2\) relation relation on the set of all seminorms:

\[
p \succ q \iff \exists t > 0, \text{ such that } tp \geq q.
\]

Using this pre-order relation one can also introduce an equivalence relation defined by:

\[
p \sim q \iff p \succ q \text{ and } q \succ p.
\]

Using this terminology, one has the following result.

Proposition 3. Using the notations from Theorem-Definition 1, the collection \( \mathcal{U} \) is a basic system of \( \mathcal{I}(\mathcal{P}) \)-neighborhoods of 0, if and only if \( \mathcal{P} \) is directed, in the sense that:

(D) for any \( p_1, p_2 \in \mathcal{P} \), there exists \( p \in \mathcal{P} \) with \( p \succ p_1 \) and \( p \succ p_2 \).

Proof. We already know from Theorem-Definition 1 that \( \mathcal{U} = \{\varepsilon \mathcal{B}(p) : p \in \mathcal{P}, \varepsilon > 0\} \) is a fundamental system of neighborhood of 0. This means that the condition that \( \mathcal{U} \) is basic is equivalent to the condition that \( \mathcal{U} \) is a filter (i.e. for any \( U_1, U_2 \in \mathcal{U} \), there exists \( U \in \mathcal{U} \) with \( U \subset U_1 \cap U_2 \)).

Suppose \( \mathcal{U} \) is a filter and let us show that \( \mathcal{P} \) is directed. Start with two seminorms \( p_1, p_2 \in \mathcal{P} \). In particular the unit balls \( \mathcal{B}(p_1) \) and \( \mathcal{B}(p_2) \) both belong to \( \mathcal{U} \), so there exist \( p \in \mathcal{P} \) and \( \varepsilon > 0 \), such that

\[
\varepsilon \mathcal{B}(p) \subset \mathcal{B}(p_1) \cap \mathcal{B}(p_2)
\]

(6)

Since, for any \( x \in X \) one has the equivalences

\[
x \in \varepsilon \mathcal{B}(p) \iff p(x) < \varepsilon \iff \varepsilon^{-1} p(x) \iff x \in \mathcal{B}(\varepsilon^{-1} p),
\]

\(^2\) What is missing from the definition of an honest order relation is: \( (p \succ q \text{ and } q \succ p) \Rightarrow p = q \).
we see that \((6)\) forces the inclusions \(B(\varepsilon^{-1} p) \subset B(p_1)\) and \(B(\varepsilon^{-1} p) \subset B(p_2)\), so using Remark 4, we get \(p_1, p_2 \leq \varepsilon^{-1} p\), thus \(p > p_1, p_2\).

Conversely, assume \(\mathfrak{P}\) is directed, and let us prove that \(\mathfrak{U}\) is a filter. Start with two sets \(\mathcal{U}_1, \mathcal{U}_2 \in \mathfrak{U}\), represented as \(\mathcal{U}_1 = \varepsilon_1 B(p_1)\) and \(\mathcal{U}_2 = \varepsilon_2 B(p_2)\), with \(p_1, p_2 \in \mathfrak{P}\) and \(\varepsilon_1, \varepsilon_2 > 0\).

Use the directedness of \(\mathfrak{P}\) to produce some \(p \in \mathfrak{P}\) and some \(t > 0\), such that \(p_1, p_2 \leq tp\), and define \(\varepsilon = \min\{t^{-1}\varepsilon_1, t^{-1}\varepsilon_2\}\). To finish the proof, we will show that

\[
\varepsilon B(p) \subset \varepsilon_1 B(p_1) \cap \varepsilon_2 B(p_2).
\]

Start now with some \(x \in \varepsilon B(p)\). We have, of course the inequality \(p(x) < \varepsilon \leq t^{-1}\varepsilon_k\), which implies \(tp(x) < \varepsilon_k\), \(k = 1, 2\). By the choice of \(p\) this yields \(p_k(x) < \varepsilon_k\), \(k = 1, 2\), so \(x\) indeed belongs to the intersection \(\varepsilon_1 B(p_1) \cap \varepsilon_2 B(p_2)\).

Example 2. Given an arbitrary collection \(\mathfrak{P}\) of seminorms, we can always find a directed one, which defines the same topology. Define \(\mathfrak{Q} = \{\sum_{p \in \mathfrak{P}} p : \mathfrak{F} \subset \mathfrak{P}\text{ finite}\}\). Clearly \(\mathfrak{Q}\) is directed, it contains \(\mathfrak{P}\), hence \(\mathbb{T}(\mathfrak{P}) \subset \mathbb{T}(\mathfrak{Q})\), and furthermore, every \(q \in \mathfrak{Q}\) is dominated by \(\mathfrak{P}\), so we also have the other inclusion \(\mathbb{T}(\mathfrak{Q}) \subset \mathbb{T}(\mathfrak{P})\). Let us also remark that, if \(\mathfrak{P}\) is infinite, then \(\text{card } \mathfrak{Q} = \text{card } \mathfrak{P}\). Of course, when \(\mathfrak{P}\) is finite, \(\mathfrak{Q}\) can be replaced by a singleton, as discussed in Example 1.

Exercises 1-3 In the following three problems \((X, \mathfrak{T})\) is a locally convex space.

1. Assume \(\mathfrak{T}\) is defined by a collection \(\mathfrak{P}\) of seminorms.

   (i) Prove that a set \(S \subset X\) is bounded, if and only if

   \[
   \sup_{x \in S} p(x) < \infty, \quad \forall p \in \mathfrak{P}.
   \]

   (ii) Prove that, if \(S \subset X\) is bounded, then the closure \(\overline{\text{conv}(\text{bal } S)}^{\mathfrak{T}}\) is also bounded.

2. Let \(Y \subset X\) be a linear subspace and let \(\pi : X \to X/Y\) denote the quotient map.

   (i) Show that, if \(p\) is a seminorm on \(X\), the functional \(\hat{p} : X/Y \to [0, \infty)\), defined by

   \[
   \hat{p}(v) = \inf\{p(x) : x \in X, \pi(x) = v\}, \quad v \in X/Y
   \]

   is a seminorm on the quotient space \(X/Y\).

   (ii) Show that, if \(p\) is continuous, then \(\hat{p}\) is continuous with respect to the quotient topology.

   (iii) Show that, if the topology \(\mathfrak{T}\) (on \(X\)) is defined by a directed family \(\mathfrak{P}\) of seminorms (on \(X\)), then the quotient topology (on \(X/Y\)) is defined by the family \(\hat{\mathfrak{P}} = \{\hat{p} : p \in \mathfrak{P}\}\).

Exercises 3-5 In the following three problems we start with a family \((X_i)_{i \in I}\) of locally convex vector spaces. We denote the product space \(\prod_{i \in I} X_i\) by \(Y\) and we equip it with the product topology. We denote the direct sum \(\bigoplus_{i \in I} X_i\) by \(X\) and we equip it with the locally convex sum topology.
3. Suppose for each \( i \in I \) the topology on \( X_i \) is defined by a family \( \Omega_i \) of seminorms (on \( X_i \)). Prove that all the maps in the set

\[
\Omega = \bigcup_{i \in I} \{q \circ \pi_i : q \in \Omega_i\}
\]

are continuous seminorms on the product space \( Y \), and furthermore, the product topology is \( T(\Omega) \).

4. Denote, for each \( i \in I \), by \( P_i \) the collection of all continuous seminorms on \( X_i \), and let \( P = \prod_{i \in I} P_i \). For each \( p = (p_i)_{i \in I} \in P \), define \( \tilde{p} : X \to [0, \infty) \) by

\[
\tilde{p}(x) = \sum_{i \in I} p_i(x_i), \quad \forall x = (x_i)_{i \in I} \in X.
\]

(By the definition of the direct sum, \( x_i \neq 0 \) only for finitely many \( i \)'s, so the above sum has in fact only finitely many non-zero terms.) Show that

(i) for every \( p \in P \) the map \( \tilde{p} \) is a seminorm on \( X \);

(ii) the collection \( \tilde{P} = \{\tilde{p} : p \in P\} \) defines the locally convex sum topology;

(iii) \( P \) is directed.

5*. Assume all \( X_i, i \in I \), are Hausdorff. Use the notations from Exercise ?? from LCVS II. Prove that a subset \( S \subset X \) is bounded, if and only if there exists a finite set \( F \subset I \), such that \( S \subset X(F) \), and \( S \) is bounded in \( X(F) \). (HINT: Argue by contradiction, assuming that, for every finite set \( F \subset I \), there is some \( x = (x_i)_{i \in I} \in S \) and some \( j \in I \setminus F \), such that \( x_j \neq 0 \). Reduce the statement to the case when \( I = \mathbb{N} \), as follows. Using the above assumption, there exists a countable infinite subset \( J \subset I \), such that, for every \( j \in J \), there exists \( x = (x_i)_{i \in I} \in S \) with that \( x_j \neq 0 \). Use the restriction map \( R : \bigoplus_{i \in I} X_i \to \bigoplus_{j \in J} X_j \) and replace \( S \) with \( R(S) \). Assuming now \( I = \mathbb{N} \), construct, for each \( k \in \mathbb{N} \), a continuous seminorm \( p_k \) on \( X_k \), such that \( \sup\{p_k(x_k) : x = (x_n)_{n \in \mathbb{N}} \in S\} \geq k \). If one considers the system \( p = (p_n)_{n \in \mathbb{N}} \in P \), then \( \sup\{\tilde{p}(x) : x \in S\} = \infty \), thus a contradiction is reached.)

Exercises 6-9. Let \( I \) be some non-empty set. Consider the spaces \( X = \bigoplus_{i \in I} \mathbb{K} \) and \( Y = \prod_{i \in I} \mathbb{K}(= \mathbb{K}^I) \).

6. For every \( i \in I \), let \( p_i : Y \to [0, \infty) \) be the map

\[
q_i(x) = |x_i|, \quad \forall x = (x_i)_{i \in I} \in Y.
\]

Show that \( q_i \) are seminorms on \( Y \).

7. Show that the collection \( \Omega = \{q_i\}_{i \in I} \) defines the product topology \( T_{\text{prod}} \) on \( Y \).

8. Show that \( X \) is dense in \( Y \) in the product topology.

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9. Define, for every $I$-tuple $t = (t_i)_{i \in I} \in [0, \infty)^I$, the map $p_t : X \to [0, \infty)$ by

$$p_t(x) = \sum_{i \in I} t_i|x_i|, \quad \forall x = (x_i)_{i \in I} \in X.$$  

(As in Exercise 5, only finitely many terms in the above sum are non-zero.) Show that:

(i) $p_t$ is a seminorm on $X$, for every $t \in [0, \infty)^I$;
(ii) the collection $\mathcal{P} = \{p_t : t \in [0, \infty)^I\}$ defines the locally convex sum topology $\mathcal{T}_{\text{sum}}$ on $X$;
(iii) $\mathcal{P}$ is directed.

We conclude this sub-section with a discussion of completeness for locally convex case. These exercise rely on the material covered in TVS IV.

**Exercises 10-12.** Assume $(X, \mathcal{T})$ is a locally convex topological vector space.

10. Suppose $p$ is a continuous seminorm on $X$.

(i) Prove that $p$ is uniformly continuous (see TVS for the definition), when regarded as a map $p : X \to \mathbb{R}$.
(ii) Consider the map $\tilde{p} : \tilde{X} \to \mathbb{R}$ constructed in Exercise 14 from TVS IV (upon identifying the completion $\tilde{\mathbb{R}}$ with $\mathbb{R}$). Show that $\tilde{p}$ is a continuous seminorm on $\tilde{X}$.

Assume the topology on $X$ is defined by a family $\mathcal{P}$ of seminorms.

11. Show that, for a net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$, the following are equivalent:

(i) $(x_\lambda)_{\lambda \in \Lambda}$ is Cauchy;
(ii) for every $p \in \mathcal{P}$, and every $\varepsilon > 0$, there exists $\lambda_{p,\varepsilon} \in \Lambda$, such that $p(x_\lambda - x_\mu), \forall \lambda, \mu > \lambda_{p,\varepsilon}$.

12. With the notations as in Exercise 10, show that, the locally convex topology on $\tilde{X}$, defined by the collection $\tilde{\mathcal{P}} = \{\tilde{p}\}_{p \in \mathcal{P}}$, coincides with $\tilde{\mathcal{T}}$. In particular $\tilde{\mathcal{T}}$ is a locally convex topology.

**B. Metrizability**

The special case, when a Hausdorff locally convex topology is defined by a countable collection of seminorms, is particularly interesting in the light of the following result.

**Theorem 2** (Metrizability Theorem). For a locally convex topological vector space $(X, \mathcal{T})$, the following conditions are equivalent:

(ii) The topology $\mathcal{T}$ is metrizable, i.e. there exists a metric $d$ on $X$, so that $\mathcal{T}$ coincides with the metric topology defined by $d$.  

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(ii) The topology $\mathcal{T}$ is first countable$^3$.

(iii) There exists a countable collection $\mathfrak{P}$ of seminorms which defines $\mathcal{T}$.

Moreover, in the case when one of the above conditions is satisfied, the metric $d$ from (i) can be chosen to be translation invariant, i.e.

$$d(x + z, y + z) = d(x, y), \ \forall \ x, y, z \in \mathcal{X}.$$

**Proof.** We already know from TVS IV B that (i) $\iff$ (ii), so we only need to prove the equivalence (ii) $\iff$ (iii).

(ii) $\implies$ (iii). Assume 0 has a countable basic $\mathcal{T}$-neighborhood system $\mathfrak{A} = \{\mathcal{A}_n\}_{n=1}^\infty$. Without any loss of generality$^4$ we can assume that all $\mathcal{A}_n$’s are open, convex, and balanced. Consider then the Minkowski functionals $P = \{q_{\mathcal{A}_n}\}_{n=1}^\infty$. By Proposition 1, every seminorm in $P$ is $\mathcal{T}$-continuous, so by Theorem-Definition 1 we have the inclusion $\mathcal{T} \supset \mathcal{T}(P)$. To prove the other inclusion, we must show that (see Remark 9 and the Comment preceding Proposition 2): all $\mathcal{T}$-continuous seminorms are dominated by $P$. Start with some $\mathcal{T}$-continuous seminorm $p$. The unit ball $B(p) = \{x \in \mathcal{X} : p(x) < 1\}$ is then an open convex balanced neighborhood of 0, so by construction there is some $n \in \mathbb{N}$ such that $\mathcal{A}_n \subset B(p)$. Of course (by Proposition 1), $\mathcal{A}_n$ coincides with the unit ball $B(q_{\mathcal{A}_n})$, and then (by Remark 4) the inclusion $B(q_{\mathcal{A}_n}) \subset B(p)$ yields the inequality $p \leq q_{\mathcal{A}_n}$.

(iii) $\implies$ (ii). This is quite trivial, since given a family $\mathfrak{P}$ of seminorms that define the topology, the collection $\mathfrak{V} = \{1_nB(p) : p \in \mathfrak{P}, n \in \mathbb{N}\}$ clearly constitutes a fundamental system of neighborhoods of 0. Of course, if $\mathfrak{P}$ is countable, then so is $\mathfrak{V}$. \qed

The two exercises below provide an explicit construction of a metric $d$ with the properties discussed in the Metrizability Theorem.

**Exercises 13-14.** Suppose $(\mathcal{X}, \mathcal{T})$ is metrizable locally convex topological space, whose topology is defined by the family of seminorms $\mathfrak{P} = \{p_n\}_{n=1}^\infty$.

13. Prove that, for every $n$, the map $d_n : \mathcal{X} \times \mathcal{X} \to [0, 1)$ defined by:

$$d_n(x, y) = \frac{p_n(x - y)}{1 + p_n(x - y)}, \ \forall \ x, y \in \mathcal{X},$$

is a translation invariant contractive semi-metric$^5$, in the sense that: Note that:

(i) $d_n(x, y) = d_n(y, x) \geq 0, \ \forall \ x, y \in \mathcal{X}$;

(ii) $d_n(x, y) \leq d_n(x, z) + d_n(z, y), \ \forall \ x, y, z \in \mathcal{X}$;

---

$^3$ This means that every point $x \in \mathcal{X}$ has a countable fundamental (or basic) $\mathcal{T}$-neighborhood system. Because of translation invariance, it suffices to check this condition only at $x = 0$.

$^4$ If each $\mathcal{A}_n$ is replaced by another neighborhood $\mathcal{A}_n'$ of 0, with $\mathcal{A}_n' \subset \mathcal{A}_n$, then $2' = \{\mathcal{A}_n'\}_{n=1}^\infty$ is again a basic system of neighborhoods.

$^5$ The only condition that is missing from the definition of a metric is the equivalence $d_n(x, y) = 0 \iff x = y$. 

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(iii) \( d_n(x, y) = d_n(x + z, y + z), \forall x, y, z \in \mathcal{X}; \)
(iv) if \( |\alpha| \leq 1, \) then \( d_n(\alpha x, \alpha y) \leq d_n(x, y), \forall x, y \in \mathcal{X}. \)

(HINT: Consider the function \( f(t) = \frac{t}{1+t}, t \in [0, \infty] \) – with the convention \( f(\infty) = 1, \) and show that \( f(s + t) \leq f(s) + f(t), \forall s, t \in [0, \infty]. \) Note also that \( f : [0, \infty] \to [0, 1] \) is a strictly increasing homeomorphism.)

14. Use the notations as above, and define the map \( d : \mathcal{X} \times \mathcal{X} \to [0, 1], \) by
\[
d(x, y) = \sum_{n=1}^{\infty} \frac{d_n(x, y)}{2^n}, \quad x, y \in \mathcal{X}.
\]

(i) Prove that \( d \) satisfies properies (i)-(iv) from the preceding exercise.

(ii) Prove that \( d \) is a translation invariant metric on \( \mathcal{X}, \) whose metric topology coincides with \( \mathcal{T}. \)

The next two exercises are somehow related to the previous two. As we shall see, besides the metric constructed in Exercise 13-14, there is another natural “candidate,” which fails, however, due to some very subtle obstructions.

**Exercises 15-16.** Suppose \((\mathcal{X}, \mathcal{T})\) is a metrizable locally convex topological space. Therefore its topology can be defined by a countable family \( \mathfrak{P} = \{p_n\}_{n=1}^{\infty} \) of seminorms. Define, for every \( n \in \mathbb{N} \) the seminorm \( q_n = p_1 + \cdots + p_n. \) (Obviously the collection \( \mathfrak{Q} = \{q_n\}_{n=1}^{\infty} \) defines the same locally convex topology.) Define the map \( \delta : \mathcal{X} \to \mathcal{X} \to [0, 1] \) by
\[
\delta(x, y) = \lim_{n \to \infty} \frac{q_n(x - y)}{1 + q_n(x - y)}.
\]

15. Prove that:

(i) \( \delta \) is a translation invariant metric on \( \mathcal{X}; \)

(ii) the metric topology \( \mathcal{T}_\delta \) is stronger than \( \mathcal{T}; \)

(iii) the addition \( \mathcal{X} \times \mathcal{X} \ni (x, y) \mapsto x + y \in \mathcal{X} \) is \( \mathcal{T}_\delta \)-continuous.

16. Give an example in which the multiplication \( \mathbb{K} \times \mathcal{X} \ni (\alpha, y) \mapsto \alpha x \in \mathcal{X} \) is not \( \mathcal{T}_\delta \)-continuous. Conclude that, in general, \((\mathcal{X}, \mathcal{T}_\delta)\) may fail to be a topological vector space.

Metrizability appears naturally in the presence of bounded sets, as indicated in the following two exercises.

**Exercises 17-18** Assume \((\mathcal{X}, \mathcal{T})\) is a locally convex topological vector space (in particular \( \mathcal{T} \) is Hausdorff).

17. Prove that the following are equivalent:

(i) There exists a bounded neighborhood of 0;

(ii) There exists some neighborhood \( \mathcal{V} \) of 0, such that \( \{\varepsilon \mathcal{V}\}_{\varepsilon > 0} \) is a basic neighborhood system for 0.
(iii) There exists one seminorm \( p \), so that \( \mathcal{T} = \mathcal{F}(p) \). Moreover, any seminorm with this property is a norm, i.e. one has the implication: \( p(x) = 0 \Rightarrow x = 0 \).

In particular, \( (\mathcal{X}, \mathcal{T}) \) is metrizable. Prove that \( d(x, y) = p(x - y) \) defines a translation invariant metric, whose metric topology coincides with \( \mathcal{T} \).

18\circledast Suppose \( \mathcal{B} \subset \mathcal{X} \) is non-empty, convex and balanced.

(i) Prove that linear span \( \mathcal{Z} = \text{span} \mathcal{B} \) can also be represented as: \( \mathcal{Z} = \bigcup_{t \in [0, \infty)} t\mathcal{B} = \bigcup_{n=1}^{\infty} n\mathcal{B} \). In particular, \( \mathcal{B} \) is absorbing in \( \mathcal{Z} \), so we can consider its associated Minkowski functional \( q_\mathcal{B} \) on \( \mathcal{Z} \), defined by

\[ q_\mathcal{B}(z) = \inf\{t > 0 : z \in t\mathcal{B}\}. \]

(ii) Prove that, if \( \mathcal{B} \) is bounded, then \( q_\mathcal{B} \) is a norm on \( \mathcal{Z} \), which from now on we will denote by \( \| \cdot \|_\mathcal{B} \).

(iii) Prove that the locally convex topology \( \mathcal{T}^\text{norm}_\mathcal{B} \) on \( \mathcal{Z} \), defined by \( \| \cdot \|_\mathcal{B} \) is stronger than the induced topology \( \mathcal{T} \big| _\mathcal{Z} \). In other words, for a net \( (z_\lambda)_{\lambda \in \Lambda} \subset \mathcal{Z} \), one has the implication \( \|z_\lambda\|_\mathcal{B} \to 0 \Rightarrow z_\lambda \xrightarrow{\mathcal{T}} 0 \).

C. Frechet spaces

As we have already seen in TVS IV, the metrizability problem is often associated with the completeness problem.

**Definition.** A locally convex topological space \( \mathcal{X} \) is called a Frechet space, if it is an \( (F) \)-space, that is, \( \mathcal{X} \) is metrizable and complete.

**Remarks 12-14.** Based on the results from TVS IV and LCVS II, the following statements yield three general methods of constructing Frechet spaces:

12. If \( \mathcal{X} \) is a Frechet space, then: a linear subspace \( \mathcal{Y} \subset \mathcal{X} \), equipped with the induced topology is a Frechet space, if and only if \( \mathcal{Y} \) is closed in \( \mathcal{X} \). (See Remark 1' from TVS IV and Example 1 from LCVS II.)

13. If \( (\mathcal{X}_j)_{j \in J} \) is a family of metrizable locally convex topological vector spaces, then: the product space \( \prod_{j \in J} \mathcal{X}_j \), equipped with the product topology, is an Frechet space, if and only if all \( \mathcal{X}_j \)'s are Frechet spaces, and the index set \( J \) is countable. (Remark 2' from TVS IV and Example 2 from LCVS II.)

In particular, all finite dimensional topological vector spaces are Frechet spaces.

14. If \( \mathcal{X} \) is a Frechet space, and \( \mathcal{Y} \subset \mathcal{X} \) is a closed linear subspace, then the quotient space \( \mathcal{X}/\mathcal{Y} \), equipped with the quotient topology, is a Frechet space. (See Remark 3' from TVS IV and Proposition 1 from LCVS II.)
One other method of constructing normed Frechet spaces\(^6\) is based on Exercise 18, in connection with which we introduce the following terminology.

**Definition.** Suppose \(X\) is a vector space, and \(B\) is a non-empty, convex, balanced subset, so that by Exercise 18 (i), \(B\) is absorbing in \(Z = \text{span}\ B\). In particular the Minkowski functional \(q_B\) is defined on \(Z\). Define \(d_B : Z \times Z \to [0, \infty)\), by \(d_B(z, z') = q_B(z - z')\).

We say that \(B\) is **self-complete**, if it satisfies the following two conditions.

(i) \(q_B\) is a norm on \(Z\), or equivalently, \(d_B\) is a metric on \(Z\). In this case, \(q_B\) will be denoted by \(\|\cdot\|_B\), and will be referred to as the \(B\)-norm.

(ii) \((B, d_B)\) is a complete metric space.

In this case, the locally convex topology on \(Z\), defined by it, will be denoted by \(\mathfrak{T}_B^{\text{norm}}\), and will be referred to as the \(B\)-norm topology.

**Remarks 15-16.** Use the notations as above, and assume that \(B \subset X\) is non-empty, convex and balanced.

15. Condition (i) above is equivalent to: \(\bigcap_{t>0} tB = \{0\}\) (see CW II). By Exercise 18 (ii), this can be achieved, for instance, if \(B\) is bounded relative to some Hausdorff linear topology on \(X\).

16. Assuming now condition (i), we see that condition (ii) is equivalent to either one of the following conditions:

(iii) \((Z, d_B)\) is a complete metric space, and \(B\) is closed in \(Z\), relative to the metric topology.

(iii') \((Z, d_B)\) is a complete metric space, and 

\[
B = \{z \in Z : \|z\|_B \leq 1\}. 
\]

(7)

In particular, \((Z, \mathfrak{T}_B^{\text{norm}})\) is a Frechet space.

It is obvious that \((iii') \Rightarrow (iii) \Rightarrow (ii)\), so we only need to justify the implication \((ii) \Rightarrow (iii')\). This follows from the well known features of the Minkowski functional (see CW II), which using the above notations yield the inclusions:

\[
\{z \in Z : \|z\|_B < 1\} \subset B \subset \{z \in Z : \|z\|_B \leq 1\}. 
\]

If we start with some element \(z \in Z\), such that \(\|z\|_B\), then clearly the sequence \((z_n)\) given by \(z_n = (1 - \frac{1}{n})z\) belongs to \(B\) and is convergent in norm to \(z\), so by the completeness of \(B\), it follows that \(z\) belongs to \(B\), thus proving (7). To prove that \((Z, d_B)\) is complete, start with some sequence \((z_n)_{n=1}^\infty \subset Z\) is Cauchy, relative to \(d_B\), and notice\(^7\) that there exists some \(t > 0\), such that \(\|z_n\|_B \leq t\), \(\forall n\). Obviously the sequence \((\frac{1}{n}z_n)_{n=1}^\infty\) is Cauchy in \((B, d_B)\), thus convergent to some \(b \in B\), and then \((z_n)\) will clearly converge to \(tb\) in \((Z, d_B)\).

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\(^6\) Later on, this terminology will be replaced: normed Frechet spaces will be called Banach spaces

\(^7\) Since \(\|z_n\|_B - \|z_m\|_B \leq \|z_m - z_n\|_B = d_B(z_m, z_n)\), we see that the sequence \((\|z_n\|_B)_{n=1}^\infty \subset [0, \infty)\) is Cauchy in \(\mathbb{R}\), thus bounded.
Exercise 19. Let $(\mathcal{X}, \mathfrak{T})$ be a locally convex topological vector space. Prove that, if $\mathcal{B}$ is a non-empty, convex, balanced, and $\mathfrak{T}$-compact, then $\mathcal{B}$ is self-complete.

Comment. In connection with the preceding exercise, the reader is warned that on the space $\mathcal{Z} = \text{span} \mathcal{B}$ we now have two topologies: (i) the $\mathcal{B}$-norm topology $\mathfrak{T}_\mathcal{B}^\text{norm}$, and (ii) the induced topology $\mathfrak{T}|_{\mathcal{Z}}$, which, in general, is strictly weaker than the norm topology. One way to see this is by observing that $\mathcal{B}$ is compact in $(\mathcal{Z}, \mathfrak{T}|_{\mathcal{Z}})$, but cannot be compact in $(\mathcal{Z}, \mathfrak{T}_\mathcal{B}^\text{norm})$, unless $\mathcal{Z}$ is finite dimensional. (See TVS III.)

Exercise 20*. Consider the space $\mathcal{X} = \bigoplus_{i \in I} \mathbb{K}$, equipped with the locally convex sum topology $\mathfrak{T}_\text{sum}$. (See Exercise 6-9.) Prove that, if $I$ is infinite, then $(\mathcal{X}, \mathfrak{T}_\text{sum})$ is not metrizable. (Hint: Without any loss of generality, one can assume that $I = \mathbb{N}$. This follows from LCVS II. Show that every Cauchy sequence in $(\mathcal{X}, \mathfrak{T}_\text{sum})$ is bounded, thus by Exercise 5* convergent. Now if $(\mathcal{X}, \mathfrak{T}_\text{sum})$ were metrizable, it would be a Frechet space, and at the same time a countable union of finite dimensional subspaces. Use Baire’s Theorem to reach a contradiction.)