**Locally Convex Vector Spaces I:**

**Basic Local Theory**

*Notes from the Functional Analysis Course (Fall 07 - Spring 08)*

**Convention.** Throughout this note $\mathbb{K}$ will be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and all vector spaces are over $\mathbb{K}$.

**Definition.** A *locally convex vector space* is a pair $(X, \mathcal{T})$ consisting of a vector space $X$ and linear topology $\mathcal{T}$ on $X$, which is *locally convex*, in the sense that:

\[ \text{(lc) every } x \in X \text{ possesses a fundamental system of convex neighborhoods.} \]

A *locally convex topological vector space* is a locally convex vector space, whose topology is Hausdorff. Since convexity is translation invariant, for a linear topology $\mathcal{T}$, the local convexity condition (lc) needs only to be verified at $x = 0$.

The following result is a locally convex analogue of Proposition 2.B from TVS I.

**Proposition 1.** In a locally convex space $X$ there exists a basic neighborhood system for $0$, which consists of balanced open convex sets.

*Proof.* What we need to prove is that:

\[ (*) \text{ for every neighborhood } N \text{ of } 0, \text{ there exists a balanced open convex set } A \subset N. \]

First of all, by definition, there exists a convex neighborhood $V$ of $0$, such that $V \subset N$. Secondly, by Proposition 2.B from TVS I, there exists some open balanced set $B \subset V$. Now we are done, by taking $A = \text{conv}(B)$. (By CW I, $A$ is balanced open convex. Since $V$ is convex and contains $B$, it will also contain $A$.)

The next result is a locally convex analogue of the Corollary to Theorem 1 from TVS I. It should be noted, however, that the hypotheses are somehow “more economical” (the collection $\mathcal{C}$ could be, for instance, *finite*), and the conclusion is slightly weaker.

**Theorem 1.** Suppose $X$ is a vector space and $\mathcal{C}$ is a collection of balanced\(^1\) absorbing convex sets. Then there exists a unique locally convex linear topology $\mathcal{T}$ on $X$, such that the collection

\[ \mathcal{U} = \{ \varepsilon C : C \in \mathcal{C}, \varepsilon > 0 \} \]

constitutes a fundamental system of $\mathcal{T}$-neighborhoods of $0$.

*Proof.* Define the collection $\mathcal{V}$ consisting of *finite intersections of sets* in $\mathcal{U}$. We wish to apply Corollary for Theorem 1 from TVS I, so we need to check the following:

\[ \text{The fact that } C \text{ is balanced forces } C \ni 0. \]

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\(^1\) The fact that $C$ is balanced forces $C \ni 0$. 
(i) \( \mathfrak{V} \) is a filter, and all sets in \( \mathfrak{V} \) contain 0;

(ii) every \( \mathcal{V} \in \mathfrak{V} \) is absorbing;

(iii) for every \( \mathcal{V} \in \mathfrak{V} \), there exists \( \mathcal{W} \in \mathfrak{V} \), such that \( \mathcal{W} + \mathcal{W} \subset \mathcal{V} \);

(iv) every \( \mathcal{V} \in \mathfrak{V} \) is balanced.

Condition (i) is trivial, since all sets in \( \mathfrak{U} \) contain 0.

Condition (iv) is also clear, since every set in \( \mathfrak{U} \) is balanced, and (arbitrary) intersections of balanced sets are balanced.

Condition (ii) follows from Exercise 2 in CW II, since all sets in \( \mathfrak{U} \) are convex and absorbing.

To check condition (iii), start with some \( \mathcal{V} \in \mathfrak{V} \), so there exists \( C_1, \ldots, C_n \in \mathfrak{C} \) and \( \varepsilon_1, \ldots, \varepsilon_n > 0 \), such that

\[
V = (\varepsilon_1 C_1) \cap \ldots \cap (\varepsilon_n C_n).
\]

Consider then the set

\[
W = (\frac{\varepsilon_1}{2} C_1) \cap \ldots \cap (\frac{\varepsilon_n}{2} C_n),
\]

which again belongs to \( \mathfrak{V} \). Obviously \( 2W \subset V \), and since \( W \) is convex, we also have \( 2W = W + W \), so we are done.

Having checked the above conditions, we invoke the above mentioned result, to conclude that there exists a unique linear topology \( \mathfrak{T} \) that has \( \mathfrak{V} \) as a basic system of neighborhoods for 0. By construction, \( \mathfrak{T} \) is also the unique linear topology which has \( \mathfrak{U} \) as a fundamental system of neighborhoods for 0. Finally, since all sets in \( \mathfrak{V} \) are convex, it follows that \( \mathfrak{T} \) is indeed locally convex.

Remark 1. With \( \mathfrak{C} \) and \( \mathfrak{T} \) as in Theorem 1, the following are equivalent:

(i) \( \mathfrak{T} \) is Hausdorff;

(ii) \( \bigcap_{\varepsilon \in \mathcal{C}} (\varepsilon C) = \{0\} \).

This is quite clear since, using the notations from the proof, condition (ii) is equivalent to the condition \( \bigcap_{\mathcal{V} \in \mathfrak{V}} \mathcal{V} = \{0\} \), which in turn is (see Theorem 1 in TVS I) equivalent to (i).

Exercise 1. Suppose that \( \mathcal{X} \) is locally convex. Prove that the collection of all closed, convex, balanced neighborhoods of 0, constitutes a basis system of neighborhoods of 0. That is, for every neighborhood \( \mathcal{V} \) of 0, there exists a closed, convex, balanced neighborhood \( \mathcal{W} \) of 0, such that \( \mathcal{W} \subset \mathcal{V} \).

Exercises 2-4. Suppose \( \mathcal{X} \) is an infinite dimensional vector space.

2. Let \( \mathfrak{C} \) be the collection of all absorbing balanced convex subsets of \( \mathcal{X} \), and let \( \mathfrak{T} \) be the corresponding locally convex linear topology described by Theorem 1. Prove that:

(i) \( \mathfrak{T} \) is Hausdorff;


(ii) if \( \mathcal{Y} \) is a locally convex vector space, then all linear maps \( T : \mathcal{X} \to \mathcal{Y} \) are \( \mathcal{T} \)-continuous;

(iii) all linear subspaces are \( \mathcal{T} \)-closed;

(iv) \( \mathcal{T} \) is the strongest locally convex linear topology on \( \mathcal{X} \).

3. Consider the algebraic dual of \( \mathcal{X} \), i.e. the space
\[
\mathcal{X}' = \{ \phi : \mathcal{X} \to \mathbb{K} : \phi \text{ linear} \},
\]
and define, for \( \phi \in \mathcal{C} \), the set \( \mathcal{C}_\phi = \{ x \in \mathcal{X} : |\phi(x)| < 1 \} \). Let \( \mathcal{C}' = \{ \mathcal{C}_\phi : \phi \in \mathcal{X}' \} \), and let \( \mathcal{T}' \) be the corresponding locally convex linear topology described by Theorem 1. Prove that:

(i) \( \mathcal{T}' \) is Hausdorff;

(ii) all maps \( \phi \in \mathcal{X}' \) are \( \mathcal{T}' \)-continuous;

(iii) all linear subspaces are \( \mathcal{T}' \)-closed.

4. Show that the topology \( \mathcal{T}' \) is strictly weaker than \( \mathcal{T} \), i.e. there exist \( \mathcal{T} \)-open sets which are not \( \mathcal{T}' \)-open. (Hint: Every \( \mathcal{T}' \)-neighborhood contains an infinite dimensional linear subspace. There are, however, \( \mathcal{T} \)-neighborhoods of 0 without this property.)

**Exercises 5-6** Let \( 0 < p < 1 \). We denote by \( D_p \) the metric on \( \ell^p \), defined (see TVS I) by
\[
D_p(x, y) = \sum_{n=1}^\infty |x_n - y_n|^p, \quad \forall x = (x_n), y = (y_n) \in \ell^p.
\]
For every \( \rho > 0 \) we define the ball
\[
\mathcal{B}_\rho = \{ x \in \ell^p : D_p(x, 0) < \rho \}.
\]

5. Show that: \( \sup \{ D_p(x, 0) : x \in \text{conv} (\mathcal{B}_\rho) \} = \infty, \forall \rho > 0 \).

6. Use the above Exercise to prove that the metric topology on \( \ell^p \) is not locally convex.