Hilbert Spaces II:  
Applications to Measure and Integration Theory

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

This section contains several fundamental results, which are proved using Hilbert space techniques. Since it is very likely that the reader has seen these results in the Real Analysis course, our presentation will be sketchy (in the form of Exercises), for most results.

**Convention.** Throughout this section $\mathbb{K}$ will be one of the fields $\mathbb{R}$ or $\mathbb{C}$, and $(X, \mathcal{A}, \mu)$ will denote a fixed measure space. The material presented in this section relies on results from BS IV, so the reader is urged to review that section.

**A. The Radon-Nikodim Theorems**

The results from this sub-section deal with the following feature.

**Definition.** Given a second measure $\nu$ on $\mathcal{A}$, we say that $\nu$ has the Radon-Nikodim property with respect to $\mu$, if there exists an $\mathcal{A}$-measurable function $g$ on $X$, such that

$$\nu(A) = \int_X g\chi_A \, d\mu,$$

for all (or just some) $A \in \mathcal{A}$. Using this terminology, a “Radon-Nikodim-type” theorem is one that has the Radon-Nikodim property as its conclusion. In other words, one looks for some simple conditions on $\nu$, which force (1).

**Comment.** The above definition is a bit imprecise due to the possible technical difficulties concerning the right-hand side of (1) and the fact that ultimately one also wants to cover the case when $\nu$ is a $\mathbb{K}$-valued measure. There are three instances, when one wants to make the above terminology a bit more precise, as follows.

A. If $\nu$ is an “honest” (i.e. $[0, \infty]$-valued) measure, we want $g$ to be non-negative (allowing $g$ even to take value $\infty$), in which case the right-hand side of (1) is understood as the positive $\mu$-integral, so it can also take value $\infty$. In this case, we could require (1) to hold either for all $A \in \mathcal{A}$, or for all $A$ that belong to one of the following collections:

$$\mathcal{A}_\text{fin}^\mu = \{ A \in \mathcal{A} : \mu(A) < \infty \},$$

$$\mathcal{A}_\sigma\text{-fin}^\mu = \{ A \in \mathcal{A} : \text{there exists } (A_n)_{n=1}^\infty, A_n \subset \mathcal{A}_\text{fin}^\mu, \text{ such that } A = \bigcup_{n=1}^\infty A_n \}.$$

B. If $\nu$ is $\mathbb{K}$-valued, we require $g$ to belong to $L_1^\mathbb{K}(X, \mathcal{A}, \mu)$, so that the right-hand side of (1) is always finite. As above, we may want to limit (1) to either $\mathcal{A}_\text{fin}^\mu$ or $\mathcal{A}_\sigma\text{-fin}^\mu$. 


Theorem 1 (“Easy” Radon-Nikodim Theorem). Assume \( \mu \) is finite and \( \nu \) is a \( \mathbb{C} \)-valued measure on \( \mathcal{A} \), such that
\[
|\nu(A)| \leq C \mu(A), \forall A \in \mathcal{A},
\] for some constant \( C \geq 0 \). Then there exists \( g \in L^1(X, \mathcal{A}, \mu) \), such that
\[
\nu(A) = \int_X g \chi_A \, d\mu, \forall A \in \mathcal{A}.
\] Moreover:

(i) any \( g \in L^1_{\mathbb{C}}(X, \mathcal{A}, \mu) \), that satisfies (3), also satisfies the inequality
\[
|g| \leq C, \mu\text{-a.e.}
\] (4)

(ii) the function \( g \) is essentially unique, in the sense that if \( g' \in L^1(X, \mathcal{A}, \mu) \) is another function that satisfies (3), then \( g = g' \), \( \mu \)-a.e., so in effect \( g = g' \), in \( L^1_\mathbb{C}(X, \mathcal{A}, \mu) \);

(iii) if \( \nu \) is \( \mathbb{R} \)-valued, then \( g \) can be taken in \( L^1_\mathbb{R}(X, \mathcal{A}, \mu) \);

(iv) if \( \nu \) is an “honest” measure, then \( g \) can be taken so that \( \text{Range } g \subset [0, \infty) \).

Sketch of Proof. For the existence part, it suffices to consider the real case \( K = \mathbb{R} \). Indeed, if we consider the \( \mathbb{R} \)-valued measures \( \nu_1 = \text{Re } \nu \) and \( \nu_2 = \text{Re } \nu \), then one clearly has the inequalities
\[
|\nu_k(A)| \leq |\nu(A)| \leq C \mu(A), \forall A \in \mathcal{A}.
\]
Assuming that \( \nu \) is \( \mathbb{R} \)-valued, we can write its Hahn-Jordan decomposition \( \nu = \nu^+ - \nu^- \). This means that \( \nu^\pm \) are two “honest” measures given by \( \nu^+(A) = \nu(A \cap B^+) \) and \( \nu^-(A) = -\nu(A \cap B^-) \), where \( (B^+, B^-) \) is a two-set \( \mathcal{A} \)-partition of \( X \). It is the trivial that
\[
0 \leq \nu^\pm(A) \leq C \mu(A), \forall A \in \mathcal{A},
\]
so for the existence part we can in fact assume that \( \nu \) is an “honest” finite measure.

Assuming that \( \nu \) is “honest” and satisfies (2), it is pretty obvious that one has the inclusions
\[
L^2(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \mu) \subset L^1(X, \mathcal{A}, \nu).
\] (5)
The first inclusion is in fact continuous, and this is due to the finiteness of \( \mu \), by Hölder’s inequality implies \( \|f\|_1 \leq \|f\|_2 \cdot \|\chi_X\|_2 = \|f\|_2 \sqrt{\mu(X)} \). The second inclusion in (5) follows from (2), which yields \( \int_X |f| \, d\nu \leq C \int_X |f| \, d\mu \), for all measurable functions \( f \). Using the inclusions (5), we can then define the map \( \phi : L^2(X, \mathcal{A}, \mu) \to \mathbb{C} \) by
\[
\phi(f) = \int_X f \, d\nu, \ f \in L^2(X, \mathcal{A}, \mu),
\]
which will be linear and continuous, since by the previous estimates, we have:
\[
|\phi(f)| \leq \int_X |f| \, d\nu \leq C \sqrt{\mu(X)} \cdot \|f\|_2, \ \forall f \in L^2(X, \mathcal{A}, \mu).
\]
Using Riesz' Theorem, there exists\(^1\) \(g \in \mathcal{L}^2(X, \mathcal{A}, \mu) \subset \mathcal{L}^1(X, \mathcal{A}, \mu)\), such that
\[
\int_X fg \, d\mu = \phi(f) = \int_X f \, d\nu, \forall f \in L^2(X, \mathcal{A}, \mu).
\]
Of course, if we specialize to \(f = \chi_A\), we now get (3), and we are done.

Having proved the existence part, we now focus on the other statement(s). To prove (i), we fix \(g \in \mathcal{L}^1(X, \mathcal{A}, \mu)\) satisfying (3), and prove that the set \(A = \{x \in X : |g(x)| > C\}\) has \(\mu(A) = 0\). If we take a countable set \(S \subset T\), which is dense in \(T\), then one can write
\[
A = \bigcup_{s \in S} \bigcup_{n \in \mathbb{N}} A_{ns},
\]
where
\[
A_{ns} = \{s \in X : \text{Re}[sg(x)] \geq C + \frac{1}{n}\},
\]
so in order to prove that \(\mu(A_{ns}) = 0\), it suffices to show that \(\mu(A_{ns}) = 0\), \(\forall s \in S, n \in \mathbb{N}\). Fix \(s \in S\) and \(n \in \mathbb{N}\), so that by (3) we have
\[
|\nu(A_{ns})| = |s\nu(A_{ns})| = \left| \int_X sg \chi_{A_{ns}} \, d\mu \right| \geq \int_X \text{Re}[sg] \chi_{A_{ns}} \, d\mu \geq \int_X \left[ C + \frac{1}{n}\right] \chi_{A_{ns}} \, d\mu = \left[ C + \frac{1}{n}\right] \mu(A_{ns}).
\]
Of course, by (2) this forces the inequality
\[
C \mu(A_{ns}) \geq \left[ C + \frac{1}{n}\right] \mu(A_{ns}),
\]
which by the finiteness of \(\mu\) yields \(\mu(A_{ns}) = 0\).

Statements (ii)-(iv) are left to the reader. \(\blacksquare\)

**Exercise 1**: Complete the proof of Theorem 1. Under the hypotheses of Theorem 1, also prove that, if \(g\) satisfies (3), then its norm in \(L^\infty\) is given by
\[
\|g\|_\infty = \min\{C \geq 0 : C \text{ satisfies (2)}\}.
\]

**Exercise 2**: Assume \(\nu\) is a \(K\)-valued measure, and let \(|\nu|\) be the variation measure, defined by
\[
|\nu|(A) = \sup \left\{ \sum_{n=1}^\infty |\nu(A_n)| : (A_n)_{n=1}^\infty \subset \mathcal{A} \text{ disjoint, and } \bigcup_{n=1}^\infty A_n = A\right\}, \forall A \in \mathcal{A}.
\]
Assume \(\mu\) is finite.

(i) Prove that, if \(\nu\) satisfies (2), then \(|\nu|\) also satisfies (2), and furthermore, if \(g \in \mathcal{L}^1(X, \mathcal{A}, \mu)\) satisfies (3), then
\[
|\nu|(A) = \int_X |g| \, d\mu, \forall A \in \mathcal{A}.
\]

(ii) Prove that, if \(|\nu| = \mu\), so in particular (2) holds with \(C = 1\), then any \(g\) that satisfies (3) also satisfies the equality: \(|g| = 1\), \(\mu\)-a.e.

\(^1\) Strictly speaking, by Riesz’ Theorem there exists \(h \in L^2(X, \mathcal{A}, \mu)\), such that \(\phi(f) = (h | f) = \int_X \bar{h} f \, d\mu\), \(\forall f \in L^2(X, \mathcal{A}, \mu)\), so we can set \(g = \bar{h}\).
Remark 1. Given a \( \mathbb{K} \)-valued measure on \( \mathcal{A} \), we know that its variation measure \( |\nu| \) is finite, and furthermore one can write \( \nu \) as a linear combination
\[
\nu = \sum_{k=1}^{n} \alpha_k \nu_k
\] (6)
of (up to four) “honest” finite measures \( \nu_1, \ldots, \nu_n \), all dominated by \( |\nu| \) i.e. satisfying \( \nu_k \leq C_k |\nu| \). This allows one to define, for every \( f \in L^1_{\mathbb{K}}(X, \mathcal{A}, |\nu|) \), its \( \nu \)-integral by
\[
\int_X f \, d\nu = \sum_{k=1}^{n} \alpha_k \int_X f \, d\nu_k.
\] (7)
As it turns out, the right-hand side is independent of the presentation (6) of \( \nu \) as a linear combination of “honest” finite measures dominated by \( |\nu| \), and this can be shown indirectly as follows. By Exercise 2, there exists an (essentially unique) \( \mathcal{A} \)-measurable function \( u : X \to \mathbb{K} \), with \( |u| = 1 \), \( |\nu| \)-a.e. on \( X \), such that \( \nu(A) = \int_X u \chi_A \, d|\nu| \), so for a given presentation of \( \nu \) as (6), one has the equality
\[
\sum_{k=1}^{n} \alpha_k \int_X \chi_A \, d\nu_k = \nu(A) = \int_X \chi_A \, u \, d|\nu|,
\] (8)
so instead of defining the \( \nu \)-integral by (7), we can define it by
\[
\int_X f \, d\nu = \int_X f u \, d|\nu|, \quad \forall f \in L^1_{\mathbb{K}}(X, \mathcal{A}, |\nu|)
\] (9)
and show that (7) holds for all possible presentations of \( \nu \) as (6). But this is pretty obvious, since by (8) the equality (7) holds for all \( f \in L^1_{\mathbb{K}}(X, \mathcal{A}, |\nu|) \), and \( L^1_{\mathbb{K}}(X, \mathcal{A}, |\nu|) \) is dense in \( L^1_{\mathbb{K}}(X, \mathcal{A}, |\nu|) \).

Comment. The formula (9) can be generalized for signed measures, i.e. measures \( \nu \) that can take values in either \( (-\infty, \infty] \) or in \( [-\infty, \infty) \). (Of course, \( \mathbb{R} \)-valued measures are a special case of signed measures.) Using the Hahn-Jordan decomposition, we can write any signed measure \( \nu \) as a difference \( \nu^+ - \nu^- \) of two “honest” measures, one of them finite. More specifically, there exists an \( \mathcal{A} \)-partition of \( X \) into two sets \( (B^+, B^-) \), such that \( \nu^+(A) = \mu(A \cap B^+) \) and \( \nu^- = -\nu(A \cap B^-) \). The variation measure is then given by \( |\nu| = \nu^+ + \nu^- \), so if we define the function \( u = \chi_{B^+} - \chi_{B^-} \), the equality (9) still holds.

Remark 2. Using the above constructions, we now see that, if \( \mu \) is finite, and a \( \mathbb{C} \)-valued measure \( \nu \) satisfies (2), then the unique \( g \in L^1(X, \mathcal{A}, \mu) \) that satisfies (3) will automatically satisfy the following identities:
\[
\int_X f \, d\nu = \int_X fg \, d\mu,
\] (10)
\[
\int_X f \, d|\nu| = \int_X f |g| \, d\mu,
\] (11)
for all \( f \in L^1(X, \mathcal{A}, \mu) \). Such identities are often referred to as Change of Variable Formulas.
In what follows our goal will be to relax the hypotheses of Theorem 1 as much as possible.

**Definition.** An “honest” measure on $\mathcal{A}$ is said to be absolutely continuous with respect to $\mu$, if: every $\mu$-negligible set $A \in \mathcal{A}$ is also $\nu$-negligible, that is, one has the implication

$$\mu(A) = 0 \Rightarrow \nu(A) = 0.$$  (12)

In this case, one uses the notation $\nu \ll \mu$.

Of course, it makes sense to consider the implication (12) also in the case when $\nu$ is $\mathbb{K}$-valued or a signed measure. In that case condition (12) is equivalent to $|\nu| \ll \mu$.

**Comment.** It is pretty clear that, for an “honest” measure $\nu$, the existence of a constant $C \geq 0$ that satisfies

$$\nu(A) \leq C\mu(A), \ \forall A \in \mathcal{A},$$  (13)

forces $\nu \ll \mu$, so subsequent versions of Theorem 1, which have the hypothesis (2) replaced by $\nu \ll \mu$ (or $|\nu| \ll \mu$ in the $\mathbb{K}$-valued case), will be generalizations of Theorem 1. As it turns out (and this will be helpful for proving such generalizations), one has some sort of converse of the implication “(13) $\Rightarrow$ (12),” which goes as follows.

**Lemma 1.** If $\mu$ is finite, and $\nu$ is an “honest” measure on $\mathcal{A}$, such that $\nu \ll \mu$, then there exists a sequence $(\nu_n)_{n=1}^\infty$ of finite “honest” measures on $\mathcal{A}$, such that, for each $A \in \mathcal{A}$, the sequence $(\nu_n(A))_{n=1}^\infty \subset [0, \infty)$ has the following properties:

1. $(\nu_n(A))_{n=1}^\infty$ is non-decreasing;
2. $\lim_{n \to \infty} \nu_n(A) = \nu(A)$;
3. $\nu_n(A) \leq n\mu(A), \ \forall n \in \mathbb{N}$.

**Sketch of Proof.** The idea is to define $\nu_n = (n\mu) \wedge \nu$. In general, for any two measures $\eta$, $\nu$ with at least one of them finite, it is possible to define two measures $\eta \wedge \nu$ and $\eta \vee \nu$, by mimicking $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\max\{a, b\} = \frac{1}{2}(a + b - |a - b|)$, but using the signed measure $\eta - \nu$ and its variation measure $|\eta - \nu|$ instead. We will not discuss this construction in general, but instead we are going to describe it explicitly in the case when $\eta = n\mu$. For every $n \in \mathbb{N}$ we consider the signed measure $\lambda_n = n\mu - \nu$, and its Hahn-Jordan decomposition $\lambda_n = \lambda_n^+ - \lambda_n^-$, where $\lambda_n^+$ are two “honest” measures (with $\lambda_n^+$ in fact finite) given by $\lambda_n^+(A) = \lambda_n(A \cap B_n^+)$ and $\lambda_n^-(A) = -\lambda_n(A \cap B_n^-)$, where $(B_n^+, B_n^-)$ is a certain $\mathcal{A}$-partition of $X$. Based on the equality $\lambda_{n+1} = \lambda_n + \mu$ and the properties of the Hahn-Jordan decomposition, one can prove that, for every $A \in \mathcal{A}$ one has the inequalities

$$\lambda_n^+(A) \leq \lambda_{n+1}^+(A) \leq \lambda_n^+(A) + \mu(A),$$  (14)

$$\lambda_n^-(A) \geq \lambda_{n+1}^-(A) \geq \lambda_n^-(A) - \mu(A).$$  (15)

Since by construction we have

$$n\mu(A \cap B_n^+) = \nu(A \cap B_n^+) + \lambda_n^+(A) \geq \nu(A \cap B_n^+),$$

$$\nu(A \cap B_n^-) = n\mu(A \cap B_n^-) + \lambda_n^-(A) \geq n\mu(A \cap B_n^-),$$

\footnote{Since $\mu$ is assumed to be finite, $\lambda_n$ is $[-\infty, \infty)$-valued.}
we can think $B^+_n$ as “the region where $n\mu \geq \nu$” and $B^-_n$ as “the region where $n\mu \leq \nu$” (although the use of “the” is not justified, since the partition $(B^+_n, B^-_n)$ is not unique!), so we will define $\nu_n$ by

$$\nu_n(A) = \nu(A \cap B^+_n) + n\mu(A \cap B^-_n), \quad A \in \mathcal{A},$$

and it is pretty obvious that condition (iii) is satisfied.

Concerning the non-uniqueness of the partition $(B^+_n, B^-_n)$, as well as property (i), it turns out that, using (14) and (15), one can prove the $\mu$-inclusions

$$B^+_n \subset B^+_{n+1} \quad \text{and} \quad B^-_{n+1} \subset B^-_n, \quad \forall \ n \in \mathbb{N},$$

where the notation “$D \subset E$” means $\mu(D \setminus E) = 0$. Of course, by the absolute continuity assumption, we also have

$$B^+_n \subset \nu B^+_{n+1} \quad \text{and} \quad B^-_{n+1} \subset \nu B^-_n, \quad \forall \ n \in \mathbb{N}.$$  

Using (16) and (17), one can easily prove statement (i).

To prove statement (ii), we first notice that, by construction, we clearly have $\nu(A) \geq \nu_n(A), \ \forall \ n \in \mathbb{N}$, which yields

$$\nu(A) \geq \lim_{n \to \infty} \nu_n(A), \quad \forall \ A \in \mathcal{A}. \quad (18)$$

To prove that the above inequality is in fact an equality, we first consider the sets $B^+ = \bigcup_{n=1}^{\infty} B^+_n$ and $B^- = \bigcap_{n=1}^{\infty} B^-_n = X \setminus B^+$, and we notice that, using\(^3\) (17) we have the equality

$$\nu(A \cap B^+) = \lim_{n \to \infty} \nu(A \cap B^+_n). \quad (19)$$

Secondly, since by construction we have the inclusions $B^+_n \subset B^+$ and $B^- \supset B^-$, we also have the inequalities

$$\nu_n(A) = \nu(A \cap B^+_n) + n\mu(A \cap B^-_n) \geq \nu(A \cap B^+_n) + n\mu(A \cap B^-_n),$$

which by (19) yield

$$\lim_{n \to \infty} \nu_n(A) \geq \nu(A \cap B^+) + \lim_{n \to \infty} n\mu(A \cap B^-). \quad (20)$$

On the one hand, if $\mu(A \cap B^-) > 0$, this forces $\lim_{n \to \infty} \nu_n(A) = \infty$, so by (18) we must have $\nu(A) = \infty$. On the other hand, if $\mu(A \cap B^-) = 0$, then by absolute continuity we also have $\nu(A \cap B^-) = 0$, which forces $\nu(A) = \nu(A \cap B^+)$, and then (20) yields $\lim_{n \to \infty} \nu(A) \geq \nu(A)$.  

**Exercise 3.**\(^2\) Fill in the details of the proof of Lemma 1.

\(^3\) It is well known that if a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{A}$ is $\nu$-increasing, i.e. $A_n \subset A_{n+1}, \ \forall \ n$, then $\nu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n \to \infty} \nu(A_n)$.  

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Theorem 2 (Radon-Nikodim Theorem: The “Honest Vs. Finite” Case). Assume $\mu$ is finite and $\nu$ is an “honest” measure on $\mathcal{A}$, such that $\nu \ll \mu$. Then there exists an $\mathcal{A}$-measurable function $g : X \to [0, \infty]$, such that

$$\nu(A) = \int_X g \chi_A d\mu, \ \forall A \in \mathcal{A}. \tag{21}$$

Moreover:

(i) if $g' : X \to [0, \infty]$ is another $\mathcal{A}$-measurable function that satisfies (21), then $g = g'$, $\mu$-a.e.;

(ii) any $\mathcal{A}$-measurable function $g : X \to [0, \infty]$ that satisfies (21) also satisfies:

$$\int_X f d\nu = \int_X fg d\mu, \ \forall f \in M_+(X, \mathcal{A}). \tag{22}$$

Proof. We begin with the proof of the existence part. By Lemma 1, there exists a sequence $(\nu_n)_{n=1}^\infty$ of finite measures on $\mathcal{A}$, such that

(a) $(\nu_n(A))_{n=1}^\infty \subset [0, \infty)$ is non-decreasing,

(b) $\lim_{n \to \infty} \nu_n(A) = \nu(A),$

(c) $\nu_n(A) \leq n\mu(A), \ \forall n \in \mathbb{N},$

for all $A \in \mathcal{A}$. Since the “honest” finite measures $\nu - \nu_{n-1}$, also satisfy (c), we can then apply Theorem 1, to obtain, for every $n \in \mathbb{N}$ an $\mathcal{A}$-measurable function $h_n : X \to [0, n]$, such that

$$\nu_n(A) - \nu_{n-1}(A) = \int_X h_n \chi_A d\mu, \ \forall A \in \mathcal{A}, \tag{23}$$

for all $n \in \mathbb{N}$. (Here we use the convention $\nu_0 = 0$.) We can then define the non-decreasing sequence $(g_n)_{n=1}^\infty$ of measurable $\mathcal{A}$-functions by $g_n = \sum_{k=1}^n h_k$, which will now satisfy

$$\nu_n(A) = \int_X g_n \chi_A d\mu, \ \forall A \in \mathcal{A}, \tag{24}$$

for all $n \in \mathbb{N}$. The function $g : X \to [0, \infty]$, defined by $g(x) = \lim_{n \to \infty} g_n(x)$ is obviously $\mathcal{A}$-measurable, and by Lebesgue’s Monotone Convergence Theorem, combined with property (b) above, the identities (24) will clearly yield the identity (21).

To prove (ii) we first notice that the equality (21) immediately yields

$$\int_X f d\nu = \int_X fg d\mu, \ \forall f \in M_+^{\text{elem}}(X, \mathcal{A}). \tag{25}$$

For an arbitrary $\mathcal{A}$-measurable function $f : X \to [0, \infty]$, we know that there exists a non-decreasing sequence $(f_n)_{n=1}^\infty$ of elementary non-negative $\mathcal{A}$-measurable functions, with

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4 In both (21) and (22) we use the positive integral, so we do not exclude the possibility of infinite values.
\[ \lim_{n \to \infty} f_n(x) = f(x), \forall x \in X. \]

Since by (25), we know that \( \int_X f_n \, d\nu = \int_X f_n \, g \, d\mu, \forall n, \) the desired identity (22) follows from Lebesgue’s Monotone Convergence Theorem.

To prove the uniqueness property (i), we fix another \( \mathcal{A} \)-measurable function \( g' : X \to [0, \infty], \) satisfying (21), and we show that \( g' = g, \, \mu \text{-a.e.} \) By symmetry, it suffices to prove that, for any two rational numbers \( r > s \geq 0, \) the set \( A_{rs} = \{ x \in X \mid g(x) \geq r > s \geq g'(x) \} \) has \( \mu(A_{rs}) = 0. \) (If all the \( A_{rs} \)'s have \( \mu(A_{rs}) = 0, \) their union, which is the set \( A = \{ x \in X : g(x) > g'(x) \} \), will also have \( \mu(A) = 0, \) which means that \( g \leq g', \, \mu \text{-a.e.} \), so by reversing the roles one would also get \( g' \leq g, \, \mu \text{-a.e.} \)) Fix \( r > s \geq 0 \) and notice that, since \( g \) and \( g' \) both satisfy (21) we have

\[
\nu(A_{rs}) = \int_X g \chi_{A_{rs}} \, d\mu \geq \int_X r \chi_{A_{rs}} \, d\mu = r \mu(A_{rs}),
\]

\[
\nu(A_{rs}) = \int_X g' \chi_{A_{rs}} \, d\mu \leq \int_X s \chi_{A_{rs}} \, d\mu = s \mu(A_{rs}),
\]

thus forcing \( r \mu(A_{rs}) \leq s \mu(A_{rs}), \) which by the finiteness of \( \mu, \) in turn yields \( \mu(A_{rs}) = 0. \)

**Remark 3.** Given \( \nu, \mu, \) and \( g \) as in Theorem 3, the space of \( \nu \)-integrable functions can be characterized as:

\[
\mathcal{L}^1(X, \mathcal{A}, \nu) = \{ f \in m(X, \mathcal{A}) : |f|g \in \mathcal{L}^1(X, \mathcal{A}, \mu) \}.
\]

The Change of Variable Formula, which follows immediately from (22), has the form:

\[
\int_X f \, d\nu = \int_X f g \, d\mu, \, \forall f \in \mathcal{L}^1(X, \mathcal{A}, \nu).
\]  \( \tag{26} \)

The \( \mathbb{K} \)-valued version of Theorem 2 is as follows.

**Theorem 3** (Radon-Nikodim Theorem: The “\( \mathbb{K} \)-Valued Vs. Finite” Case). Assume \( \mu \) is finite and \( \nu \) is a \( \mathbb{C} \)-valued measure measure on \( \mathcal{A}, \) such that \( |\nu| \ll \mu. \) Then there exists \( g \in \mathcal{L}^1(X, \mathcal{A}, \mu), \) such that

\[
\nu(A) = \int_X g \chi_A \, d\mu, \, \forall A \in \mathcal{A}, \tag{27}
\]

Moreover:

(i) the function \( g \) is essentially unique, in the sense that if \( g' \in \mathcal{L}^1(X, \mathcal{A}, \mu) \) is another function that satisfies (27), then \( g = g', \, \mu \text{-a.e.} \), so in effect \( g = g', \) in \( L^1(X, \mathcal{A}, \mu). \)

(ii) if \( \nu \) is \( \mathbb{R} \)-valued, then \( g \) can be taken in \( \mathcal{L}^1_{\mathbb{R}}(X, \mathcal{A}, \mu); \)

(iii) if \( \nu \) is an “honest” finite measure, i.e. \( [0, \infty) \)-valued, then \( g \) can be taken so that \( \text{Range} \, g \subset [0, \infty); \)

(iv) if \( g \in \mathcal{L}^1(X, \mathcal{A}, \mu) \) satisfies (27), then

(a) \( |\nu|(A) = \int_X \chi_A |g| \, d\mu, \, \forall A \in \mathcal{A}; \)
(b) for every \( f \in \mathcal{L}^1(X, \mathcal{A}, |\nu|) \), one has the Change of Variable Formulas:

\[
\int_X f \, d\nu = \int_X f g \, d\mu, \tag{28}
\]
\[
\int_X f \, d|\nu| = \int_X f |g| \, d\mu. \tag{29}
\]

**Sketch of Proof.** We are only going to treat the existence part, leaving the rest for the reader to prove. By Theorem 2 and Remark 3, there exists an \( \mathcal{A} \)-measurable function \( h : X \to [0, \infty] \), such that

\[
|\nu|(A) = \int_X \chi_A h \, d\mu, \quad \forall A \in \mathcal{A};
\]
\[
\int_X f \, d|\nu| = \int_X f h \, d\mu, \quad \forall f \in \mathcal{L}^1(X, \mathcal{A}, |\nu|). \tag{31}
\]

Using the fact that all \( \mathbb{C} \)-valued measures \( \nu \) have finite variation measures, by (30) we get

\[
\int_X h \, d\mu = |\nu|(X) < \infty,
\]
so \( h \) is integrable. By Remark 1, one can also choose \( u \in \mathcal{M}(X, \mathcal{A}) \), with \( |u| = 1 \), such that

\[
\int_X f \, d\nu = \int_X f u \, d|\nu|, \quad \forall f \in \mathcal{L}^1(X, \mathcal{A}, |\nu|),
\]
so one can define the desired function as \( g = uh \).

**Exercise 4.** Complete the proof of Theorem 3.

Up to this point, we have only treated the case when \( \mu \) is finite. When trying to generalize Theorems 2 and 3 beyond this case, we will have to pay some price, namely we will have to limit the Radon-Nikodim property (1) to sets \( A \in \mathcal{A}_\text{fin}^\mu \). Concerning the Change of Variable Formulas (10) and (11), they will also have to be limited to some special type of measurable functions \( f \).

**Remark 4.** If \( \nu \) is “honest” and \( g : X \to [0, \infty] \) is an \( \mathcal{A} \)-measurable function, satisfying (1), for all \( A \in \mathcal{A}_\text{fin}^\mu \), then

(i) the identity (1) also holds for all \( A \in \mathcal{A}_\sigma^\mu \);  

(ii) the Change of Variable Formula (10) holds for all \( \mathcal{A} \)-measurable functions \( f : X \to [0, \infty] \), which have their support\(^5\) in \( \mathcal{A}_\sigma^\mu \).

Property (i) is due to the \( \sigma \)-additivity of both \( \nu \) and \( \mu \), combined with the fact that all sets in \( A \in \mathcal{A}_\sigma^\mu \) are countable disjoint unions of sets in \( A \in \mathcal{A}_\text{fin}^\mu \).

Property (ii) follows from the observation that if \( f : X \to [0, \infty] \) is an \( \mathcal{A} \)-measurable function such that \( \text{supp} \, f \) belongs to \( \mathcal{A}_\sigma^\mu \), then there exists a non-decreasing sequence

\(^5\) The support of \( f \) is defined as \( \text{supp} \, f = \{ x \in X : f(x) \neq 0 \} \).
\( f_n \in L_{\text{elem}}^e(X, \mathcal{A}, \mu) \), such that \( \lim_{n \to \infty} f_n(x) = f(x), \forall x \), and then by Lebesgue’s Monotone Convergence Theorem one has the equalities
\[
\int_X f \, d\nu = \lim_{n \to \infty} \int_X f_n \, d\nu \quad \text{and} \quad \int_X f g \, d\mu = \lim_{n \to \infty} \int_X f_n g \, d\mu.
\]
By this argument, it suffices to prove (10) only for \( f \in L_{\text{elem}}^e(X, \mathcal{A}, \mu) = \text{span} \{ \chi_A : \mathcal{A}_{\text{fin}}^\mu \} \), which is trivial from the assumption, which yields
\[
\int_X \chi_A \, d\nu = \nu(A) = \int_X \chi_A g \, d\mu, \quad \forall A \in \mathcal{A}_{\text{fin}}^\mu.
\]

**Comment.** To obtain the generalizations of Theorems 2 and 3, one replaces the finiteness assumption on \((X, \mathcal{A}, \mu)\) with decomposability. Recall (see BS IV) that the measure space \((X, \mathcal{A}, \mu)\) is said to be **decomposable**, if there exists an \(\mathcal{A}\)-partition \(\pi = (X_i)_{i \in I}\) of \(X\), such that

(A) \( \mathcal{A} = \{ A \subseteq \mathcal{A} : A \cap X_i \in \mathcal{A}, \forall i \in I \} \);

(B) \( \mu(X_i) < \infty, \forall i \in I \);

(c) for every \( A \in \mathcal{A} \) with \( \mu(A) < \infty \), one has:
\[
\mu(A) = \sum_{i \in I} \mu(A \cap X_i). \quad (32)
\]
In this case, we say that the partition \( \pi \) constitutes a **decomposition of** \((X, \mathcal{A}, \mu)\).

**Remarks 5-6.** Assume \((X, \mathcal{A}, \mu)\) is decomposable, and \(\pi = (X_i)_{i \in I}\) is one of its decompositions.

5. As pointed out in BS IV, using condition (A) we know that, if \( T \) is either one of the spaces \( \mathbb{R}, \mathbb{C}, [-\infty, \infty], \) of \([0, \infty]\), then for a function \( f : X \to T \), the following are equivalent
   (i) \( f \) is \( \mathcal{A} \)-measurable;
   (ii) \( f|_{X_i} \) is \( \mathcal{A}|_{X_i} \)-measurable, for each \( i \in I \).

6. The equality (32) also holds for \( A \in \mathcal{A}_{\sigma, \text{fin}}^\mu \).

**Theorem 4** (Radon-Nikodim Theorem: The “Honest Vs. Decomposable” Case). Assume \((X, \mathcal{A}, \mu)\) is decomposable and \( \nu \) is an “honest” measure measure on \( \mathcal{A} \), such that \( \nu \ll \mu \). Then there exists an \( \mathcal{A} \)-measurable function \( g : X \to [0, \infty] \), such that\(^6\)
\[
\nu(A) = \int_X g \chi_A \, d\mu, \quad \forall A \in \mathcal{A}_{\text{fin}}^\mu. \quad (33)
\]
Moreover:

\(^6\) As in Theorem 2, in (33), (34) and (35), we use the positive integral, so we do not exclude the possibility of infinite values.
if \( g' : X \to [0, \infty] \) is another \( \mathcal{A} \)-measurable function that satisfies (33), then \( g = g' \), \( \mu \)-a.e.;

(ii) any \( \mathcal{A} \)-measurable function \( g : X \to [0, \infty] \) that satisfies (33) also satisfies:

\[
\int_X f \, d\nu = \int_X fg \, d\mu, \tag{34}
\]

for all \( f \in \mathcal{M}_+(X, \mathcal{A}) \), with \( \text{supp } f \in \mathcal{A}_\sigma^{\mathcal{A}-\text{fin}} \), so in particular it also satisfies

\[
\nu(A) = \int_X g \chi_A \, d\mu, \quad \forall A \in \mathcal{A}_\sigma^{\mathcal{A}-\text{fin}}, \tag{35}
\]

**Sketch of Proof.** To prove the existence part, we fix a decomposition \( \pi = (X_i)_{i \in I} \) for \((X, \mathcal{A}, \mu)\). For each \( i \in I \) consider the finite measure space \((X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})\) and the “honest” measure \( \nu|_{X_i} \), where \( \mathcal{A}|_{X_i} = \{ A \cap X_i : A \in \mathcal{A} \} \) and \( \mu|_{X_i} \) and \( \nu|_{X_i} \) denote the restrictions of \( \mu \) and \( \nu \) to \( \mathcal{A}|_{X_i} \). Since we clearly have \( \nu|_{X_i} \ll \mu|_{X_i} \), by Theorem 2, for each \( i \in I \), there exists an \( \mathcal{A}|_{X_i} \)-measurable function \( g_i : X_i \to [0, \infty] \), such that

\[
\nu(A \cap X_i) = \int_{X_i} g_i \chi_A \, d(\mu|_{X_i}), \quad \forall A \in \mathcal{A}. \tag{36}
\]

Using Remark 5, there exists a (unique) \( \mathcal{A} \)-measurable function \( g : X \to [0, \infty] \), such that \( g_i|_{X_i} = g_i, \forall i \in I \). To check that \( g \) satisfies (33), we start with some \( A \in \mathcal{A}_\sigma^{\mathcal{A}-\text{fin}} \) and we use the fact (see BS IV) that the set \( I(A) = \{ i \in I : \mu(A \cap X_i) \neq 0 \} \) is countable, and furthermore, the set \( A' = \bigcup_{i \in I(A)} A \cap X_i \subset A \) belongs to \( \mathcal{A} \), and satisfies \( \mu(A \setminus A') = 0 \). By absolute continuity, we also have the equality \( \nu(A \setminus A') = 0 \), so we get

\[
\nu(A) = \nu(A') = \sum_{i \in I(A)} \nu(A \cap X_i).
\]

Using (36) this clearly yields (33).

The rest of the proof is left to the reader. \( \square \)

**Exercise 5.** Complete the proof of Theorem 4.

**Theorem 5** (Radon-Nikodim Theorem: The “\( \mathbb{K} \)-Valued Vs. Decomposable” Case). Assume \((X, \mathcal{A}, \mu)\) is decomposable and \( \nu \) is a \( \mathbb{C} \)-valued measure measure on \( \mathcal{A} \), such that \( |\nu| \ll \mu \). Then there exists \( g \in \mathcal{L}^1(X, \mathcal{A}, \mu) \), such that

\[
\nu(A) = \int_X g \chi_A \, d\mu, \quad \forall A \in \mathcal{A}_\sigma^{\mathcal{A}-\text{fin}}, \tag{37}
\]

Moreover:

(i) the function \( g \) is essentially unique, in the sense that if \( g' \in \mathcal{L}^1(X, \mathcal{A}, \mu) \) is another function that satisfies (37), then \( g = g' \), \( \mu \)-a.e., so in effect \( g = g' \), in \( L^1(X, \mathcal{A}, \mu) \);

(ii) if \( \nu \) is \( \mathbb{R} \)-valued, then \( g \) can be taken in \( \mathcal{L}^1_{\mathbb{R}}(X, \mathcal{A}, \mu) \).
(iii) if $\nu$ is an "honest" finite measure, i.e. $[0,\infty)$-valued, then $g$ can be taken so that $\text{Range } g \subset [0,\infty)$;

(iv) if $g \in L^1(X, A, \mu)$ satisfies (37), then for every function $f \in L^1(X, A, |\nu|)$ with $\text{supp } f \in A^\mu_\text{fin}$, one has the Change of Variable Formulas:

$$\int_X f \, d\nu = \int_X fg \, d\mu, \tag{38}$$
$$\int_X f \, d|\nu| = \int_X f|g| \, d\mu, \tag{39}$$

so in particular, one also has the equalities

$$\nu(A) = \int_X g\chi_A \, d\mu, \tag{40}$$
$$|\nu|(A) = \int_X |g|\chi_A \, d\mu, \tag{41}$$

for all $A \in A^\mu_\text{fin}$.

**Sketch of Proof.** We are only going to treat the existence part, and the uniqueness statement (i), leaving the rest for the reader to prove. Exactly as in Theorem 3, it suffices to consider the case when $\nu$ is an "honest" finite measure. By Theorem 4, there exists an $A$-measurable function $g_0 : X \rightarrow [0,\infty]$, such that

$$\nu(A) = \int_X g_0\chi_A \, d\mu, \quad \forall A \in A^\mu_\text{fin}, \tag{42}$$

Fix a decomposition $\pi = (X_i)_{i \in I}$ for $(X, A, \mu)$, and consider the index set

$$J = \{i \in I : \nu(X_i) \neq 0\}.$$

Since $\nu$ is finite, the set $J$ is countable. For any set $A \in A$, let us denote the union $\bigcup_{i \in I} A \cap X_i$ by $A_J$.

**CLAIM:** For every $A \in A^\mu_\text{fin}$, the set $A_J$ belongs to $A^\mu_\text{fin}$, and satisfies $\nu(A \setminus A_J) = 0$.

First of all, since $A_J \subset A$ and $\mu(A) < \infty$, it is clear that $\mu(A_J) < \infty$. Secondly, if we use the notations from the proof of Theorem 4, we know that the index set

$$I(A) = \{i \in I : \mu(A \cap X_i) \neq 0\}$$

is countable, and the set $A' = \bigcup_{i \in I(A)} A \cap X_i \subset A$ satisfies $\mu(A \setminus A') = 0$, which by absolute continuity also gives $\nu(A \setminus A') = 0$. Of course, if we consider the countable index set $I(A) \cup J$, the union $B = \bigcup_{j \in I(A) \cup J} A \cap X_i$, will satisfy the inclusions $A', A_J \subset B \subset A$, as well as the equality $\mu(A \setminus B) = 0$, which by absolute continuity also gives $\nu(A \setminus B) = 0$, so in order to prove the Claim we only need to show that $\nu(B \setminus A_J) = 0$. But this is trivial, since by construction we have

$$\nu(B \setminus A_J) = \sum_{i \in I(A) \setminus J} \nu(A \cap X_i) \leq \sum_{i \in I(A) \setminus J} \nu(X_i) = 0.$$
Using the above Claim, we see that if we consider the set \( X_j = \bigcup_{i \in J} X_i \in \mathcal{A} \), and the function \( g = g_0 \chi_{X_j} \), then, using Lebesgue’s Monotone Convergence Theorem and \( \sigma \)-additivity, it follows that, for every \( A \in \mathcal{A}_{\mu}^n \), we have
\[
\int_X g \chi_A \, d\mu = \sum_{i \in J} \int_X g \chi_{A \cap X_i} \, d\mu = \sum_{i \in J} \nu(A \cap X_i) = \nu(A_j) = \nu(A),
\]
thus proving (37). Again by Lebesgue’s Monotone Convergence Theorem, using (37) with \( h \) and \( \nu \) is \( \nu \)

If one considers the sets \( A \) which both belong to \( \mathcal{A} \), prove that \( g \) is \( \nu \)-finite, it follows that, for every \( A \in \mathcal{A}_{\mu}^n \), we have
\[
\int_X g \, d\mu = \sum_{i \in J} \int_X g \chi_{X_i} \, d\mu = \sum_{i \in J} \nu(X_i) = \nu(X_j) < \infty,
\]
thus proving that \( g \) indeed belongs to \( \mathcal{L}^1(X, \mathcal{A}, \mu) \).

To prove the uniqueness statement (i), we assume (working now in the general case, when \( \nu \) is \( \mathbb{C} \)-valued) we have two functions \( g, g' \in \mathcal{L}^1(X, \mathcal{A}, \mu) \), both satisfying (37), and let us prove that \( g = g' \), \( \mu \)-a.e. Using real and imaginary parts, we have
\[
\begin{align*}
\int_X \Re g \chi_A \, d\mu &= \int_X \Re \nu\chi_A \, d\mu, \\
\int_X \Im g \chi_A \, d\mu &= \int_X \Im \nu\chi_A \, d\mu,
\end{align*}
\]
for all \( A \in \mathcal{A}_{\mu}^n \), which means that the functions \( h_1 = \Re g - \Re g' \) and \( h_2 = \Im g - \Im g' \), which both belong to \( \mathcal{L}^1_\mu(X, \mathcal{A}, \mu) \), satisfy
\[
\int_X h_1 \chi_A \, d\mu = \int_X h_2 \chi_A \, d\mu = 0, \quad \forall A \in \mathcal{A}_{\mu}^n.
\]
If one considers the sets \( A_{kn}^+ = \{ x \in X : h_k(x) \geq \frac{1}{n} \} \) and \( A_{kn}^- = \{ x \in X : h_k(x) \leq -\frac{1}{n} \} \), \( k = 1, 2, n \in \mathbb{N} \), then the inequalities \( |h_k| \leq \frac{1}{n} \chi_{A_{kn}^-} \), combined with the \( \mu \)-integrability of \( h_k \), force \( A_{kn}^\pm \) to belong to \( \mathcal{A}_{\mu}^n \), so using (43) with \( A_{kn}^\pm \), and the inequalities \( h_k \chi_{A_{kn}^-} \leq -\frac{1}{n} \chi_{A_{kn}^-} \), and \( h_k \chi_{A_{kn}^+} \geq \frac{1}{n} \chi_{A_{kn}^+} \), we obtain \( \frac{1}{n} \mu(A_{kn}^\pm) \leq 0 \), thus forcing \( \mu(A_{kn}^\pm) = 0, \forall n \in \mathbb{N}, k = 1, 2 \). This forces \( h_1 = h_2 = 0, \mu \)-a.e., which yields \( g = g' \), \( \mu \)-a.e.

Exercise 6: Complete the proof of Theorem 5.

Corollary 1 (The Radon-Nikodim Theorems: the “Any Vs. \( \sigma \)-finite” Case). If \( (X, \mathcal{A}, \mu) \) is \( \sigma \)-finite, then Theorems 2 and 3 hold word-for-word.

Proof. If \( (X, \mathcal{A}, \mu) \) is \( \sigma \)-finite, then it is decomposable, so we can use Theorems 4 and 5. In this case, however, we have the equality \( \mathcal{A}_{\mu}^n = \mathcal{A} \), and the notions of “\( \mu \)-l.a.e.” and “\( \mu \)-a.e.” are equivalent.

B. The duals of \( L^p \)-spaces

In this sub-section we apply the Radon-Nikodim Theorems to the calculation of the topological duals of the \( L^p \)-spaces, for \( 1 \leq p < \infty \). As customary, we declare two “numbers”
Let $p, q \in [1, \infty]$ H"{o}lder conjugate, if either both $p$ and $q$ are finite and $\frac{1}{p} + \frac{1}{q} = 1$, or $p = 1$ and $q = \infty$, or vice versa.

Recall now that in BS IV, given H"{o}lder conjugate numbers $p$ and $q$, we defined the H"{o}lder dual paring

$$\Phi : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \times L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni (f, g) \longmapsto \int_X fg \, d\mu.$$ 

Moreover, we also showed that, for any $g \in L^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$, the linear map

$$g^# : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni f \longmapsto \int_X fg \, d\mu \in \mathbb{K}$$

is norm-continuous on $L^p$, and furthermore, the map

$$\# : L^q_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto g^# \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)^* \quad (44)$$

is linear and \textit{isometric}, if $1 < p \leq \infty$. Eventually we are going to show that this map is in fact an isometric \textit{isomorphism}, if $1 < p < \infty$.

\textbf{Comment.} The case $p = 1$ (when we use $q = \infty$) is a bit more complicated, and will require some additional restrictions on the ambient measure space $(X, \mathcal{A}, \mu)$. As pointed out in BS IV, in this case the map (44) may fail to be isometric, in general. Its correct version (which is always isometric) has $L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu)$ replaced with $L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu)$, so instead of (44), the appropriate map to consider is

$$\#_{\text{loc}} : L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni [g]_{\text{loc}} \longmapsto g^# \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)^* \quad (45)$$

where $g^#(f) = \int_X fg \, d\mu$, $g \in L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu)$, $f \in L^1_{\mathbb{K}}(X, \mathcal{A}, \mu)$. In the case when $(X, \mathcal{A}, \mu)$ is \textit{nowhere degenerate} (which happens, for instance, in the $\sigma$-finite case), the surjective contraction

$$L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni g \longmapsto [g]_{\text{loc}} \in L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu)$$

is in fact an isometric isomorphism, so in this case the map (45) is essentially the same as (44) with $p = 1$ and $q = \infty$.

\textbf{Notation.} Given $p \in [1, \infty]$ and $h \in L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu)$, we denote the multiplication map

$L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \ni f \longmapsto hf \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$ by $M_h$. As it turns out, since for every $\mathcal{A}$-measurable function $f : X \to \mathbb{K}$, one has the inequality $|hf| \leq \|h\|_{\infty} \cdot |f|$, $\mu$-a.e., the (obviously) linear operator $M_h$ is norm-continuous, and one has the inequality $\|M_h\| \leq \|h\|_{\infty}$.

\textbf{Comment.} In preparation for the proof or Theorem 6 below, it will be helpful to reduce everything to the \textit{real} case $\mathbb{K} = \mathbb{R}$. This is done in a manner similar to what we did in BS III, as follows.

\textbf{Definition.} Let $\mathcal{X}$ be either one of the complex Banach spaces $L^p(X, \mathcal{A}, \mu)$, $p \in [1, \infty]$, or $L^{\text{loc},\infty}(X, \mathcal{A}, \mu)$, and let $\mathcal{X}_{\mathbb{R}}$ denote its real counterpart: $L^p_{\mathbb{R}}(X, \mathcal{A}, \mu)$, $p \in [1, \infty]$, or $L^{\text{loc},\infty}_{\mathbb{R}}(X, \mathcal{A}, \mu)$. A $\mathbb{C}$-linear functional $\phi : \mathcal{X} \to \mathbb{C}$ is said to be \textit{hermitian}, if $\phi(\mathcal{X}_{\mathbb{R}}) \subset \mathbb{R}$. The features of hermitian functionals are similar to those discussed in BS III, as shown below.

\textbf{Exercise} 7\textsuperscript{a}. Use the notations as above.

(i) Prove that, for every $\mathbb{R}$-linear functional $\psi : \mathcal{X}_{\mathbb{R}} \to \mathbb{R}$, there exists a unique hermitian $\mathbb{C}$-linear functional $\phi : \mathcal{X} \to \mathbb{C}$, such that $\phi|_{\mathcal{X}_{\mathbb{R}}} = \psi$. 

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(ii) Prove that, if $\phi$ and $\psi$ are as above, the following are equivalent:

- $\phi$ is continuous;
- $\psi$ is continuous.

Moreover, in this case, prove the equality $\|\phi\| = \|\psi\|$. 

(iii) Prove that for any $\mathbb{C}$-linear continuous functional $\phi : \mathcal{X} \to \mathbb{C}$, there exist two hermitian $\mathbb{C}$-linear continuous functionals $\phi^\text{Re}, \phi^\text{Im} : \mathcal{X} \to \mathbb{C}$, such that $\phi = \phi^\text{Re} + i\phi^\text{Im}$.

**Theorem 6** (Hölder duality: the finite case). Assume $p \in [1, \infty)$ and let $q \in (1, \infty]$ be the Hölder conjugate of $p$. If $(X, \mathcal{A}, \mu)$ is a finite measure space, then the map (44) is an isometric linear isomorphism.

**Proof.** As previously discussed, we already know that the linear map (44) is isometric, so the only property we need to prove is surjectivity. Start with a linear continuous functional $\phi : L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathbb{K}$, and let us indicate how a function $g \in L^q_{\mathbb{K}}(X, \mathcal{A}, \mu)$ can be constructed, such that $\phi(f) = \int_X fg \, d\mu$, $\forall f \in L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$.

Using Exercise 7, we can assume that $\phi$ is hermitian, so in fact we can limit ourselves to the real case $\mathbb{K} = \mathbb{R}$.

**Claim 1:** The map $\nu : \mathcal{A} \ni A \mapsto \phi(\chi_A) \in \mathbb{R}$ defines an $\mathbb{R}$-valued measure on $\mathcal{A}$, which satisfies $|\nu(A)| \leq \|\phi\| \cdot \mu(A)^{1/p}$, $\forall A \in \mathcal{A}$. 

(47)

To check $\sigma$-additivity we start with a sequence $(A_n)_{n=1}^\infty \subset \mathcal{A}$ of disjoint sets, we let $A = \bigcup_{n=1}^\infty A_n$, and we must show that $\sum_{n=1}^\infty \nu(A_n) = \nu(A)$. Since $\phi$ is linear and continuous, it suffices to show that, when we work in $L^p_{\mathbb{K}}(X, \mathcal{A}, \mu)$, the series $\sum_{n=1}^\infty \chi_{A_n}$ converges to $\chi_A$, in the norm topology, which means that, when we consider the partial sums $f_N = \sum_{n=1}^N \chi_{A_n}$, we have $\lim_{N \to \infty} \|\chi_A - f_N\|_p = 0$. But this is pretty clear, since, on the one hand, we have $\chi_A - f_N = \chi_{B_N}$, where $B_N = \bigcup_{n=N+1}^\infty A_n$, so

$$\|\chi_A - f_N\|_p^p = \int_X |\chi_{B_N}|^p \, d\mu = \int_X \chi_{B_N} \, d\mu = \mu(B_N) = \sum_{n=N+1}^\infty \mu(A_n),$$

and on the other hand, by $\sigma$-additivity and finiteness of $\mu$ we also have $\sum_{n=1}^\infty \mu(A_n) = \mu(A) < \infty$, which means that

$$\sum_{n=N+1}^\infty \mu(A_n) = \mu(A) - \sum_{n=1}^N \mu(A_n) \xrightarrow{N \to \infty} 0.$$ 

The inequality (47) is obvious from the norm inequality $|\phi(\chi_A)| \leq \|\phi\| \cdot \|\chi_A\|_p$, and the equality $\|\chi_A\|_p^p = \int_X \chi_A \, d\mu = \mu(A)$. 

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Using Claim 1 and the Radon-Nikodim Theorem 4 (the "K-valued Vs. finite" case), there exists \( g \in \mathcal{L}_R^1(X, \mathcal{A}, \mu) \), such that \( \nu(A) = \int_X \chi_A g \, d\mu, \forall \, A \in \mathcal{A} \). By linearity, we get in fact the equality

\[
\phi(f) = \int_X fg \, d\mu, \quad \forall \, f \in L^\text{elem}_R(X, \mathcal{A}, \mu). \tag{48}
\]

To finish the proof of the Theorem, we must show that

(a) \( g \) in fact belongs to \( \mathcal{L}_R^q(X, \mathcal{A}, \mu) \), and

(b) the equality (48) holds in fact for all \( f \in L^p_R(X, \mathcal{A}, \mu) \).

As it turns out, if we prove property (a), then (b) will follow automatically, since condition (a) forces the continuity of the linear functional \( g^\# : L^p_R(X, \mathcal{A}, \mu) \ni f \mapsto \int_X fg \, d\mu \in \mathbb{R} \), so the desired equality follows from (48) and the fact that \( L^\text{elem}_R(X, \mathcal{A}, \mu) \) is dense in \( L^1_R(X, \mathcal{A}, \mu) \).

Property (a) is pretty clear if \( p = 1 \), when we must show that \( g \) belongs to \( \mathcal{L}_R^\infty(X, \mathcal{A}, \mu) \). Indeed, this follows immediately from the "easy" Radon-Nikodim Theorem 2, since in the case \( p = 1 \), by Claim 1 we have \( |\nu(A)| \leq \|\phi\| : \mu(A), \forall \, A \in \mathcal{A} \).

For the remainder of the proof we are going to assume that \( p \) belongs to \((1, \infty)\), so the Hölder conjugate \( q \) of \( p \) will also belong to the same interval, which means that the desired property (a) now reads: \( \int_X |g|^q \, d\mu < \infty \). Consider the sets \( X^+ = \{ x \in X : g(x) \geq 0 \} \) and \( X^- = X \setminus X^+ \), which both belong to \( \mathcal{A} \), and the functions \( g^+ = g \chi_{X^+} \) and \( g^- = -g \chi_{X^-} \), which both belong to \( \mathcal{L}_R^1(X, \mathcal{A}, \mu) \), and satisfy the equality

\[
g = g^+ - g^- . \tag{49}
\]

Remark now that, since the multiplication operators \( M_{g_{X^+}} \) are continuous, the functionals \( \phi^+ = \phi \circ M_{g_{X^+}} \) and \( \phi^- = -\phi \circ M_{g_{X^-}} \) are continuous as well, and they satisfy the equality

\[
\phi = \phi^+ - \phi^-.
\]

Moreover, since both multiplication operators \( M_{g_{X^+}} \) and \( M_{g_{X^-}} \) map the space \( L^\text{elem}_R(X, \mathcal{A}, \mu) \) into itself, using (48) we also have

\[
\phi^+(f) = \int_X fg^+ \, d\mu \quad \text{and} \quad \phi^-(f) = \int_X fg^- \, d\mu , \quad \forall \, f \in L^\text{elem}_R(X, \mathcal{A}, \mu). \tag{50}
\]

By (49) we see that it suffices to show that both \( g^+ \) and \( g^- \) belong to \( \mathcal{L}_R^q(X, \mathcal{A}, \mu) \). By (50), we see that everything reduces now to proving the Claim below. (Both pairs \((\phi^+, g^+)\) and \((\phi^-, g^-)\) satisfy (48). The added advantage here is the fact that both \( g^+ \) and \( g^- \) are non-negative.)

**Claim 2:** If \( p \in (1, \infty) \), and \( g \in \mathcal{L}^1_+(X, \mathcal{A}, \mu) \) satisfies (48), then \( \int_X g^q \, d\mu \leq \|g\|^q (< \infty) \).

Since \( g \geq 0 \), by the definition of the (positive) integral, we know that

\[
\int_X g^q \, d\mu = \sup \left\{ \int_X h \, d\mu : h \in \mathcal{L}^\text{elem}_+(X, \mathcal{A}, \mu), \, h \leq g^q \right\},
\]
so in order to prove the Claim it suffices to prove that, for any $h \in \mathcal{L}^{\text{elem}}(X, \mathcal{A}, \mu)$, such that $h \leq g^q$, one has the inequality $\int_X h \, d\mu \leq \|\phi\|^q$. Fix such an $h$, and observe that, since $\frac{1}{p} + \frac{1}{q} = 1$, we can write $h = h^{1/p}h^{1/q}$. Furthermore, since $h^{1/p}$ is elementary, and $h^{1/q} \leq g$, we have

$$\int_X h \, d\mu = \int_X h^{1/p}h^{1/q} \, d\mu \leq \int_X h^{1/p}g \, d\mu = \phi(h^{1/p}).$$

By the norm inequality, we also have

$$\phi(h^{1/p}) \leq \|\phi\| \cdot \|h^{1/p}\|_p = \left( \int_X |h^{1/p}|^p \, d\mu \right)^{1/p} = \left( \int_X h \, d\mu \right)^{1/p},$$

so if we denote $\int_X h \, d\mu$ by $J$ and we combine (51) with (52) we now have $J \leq \|\phi\| \cdot J^{1/p}$, which yields $J^{1/q} = J^{1-1/p} \leq \|\phi\|$, thus proving the desired inequality $J \leq \|\phi\|^q$. □

In our approach to the generalization of Theorem 6 will use the following constructions.

**Notations.** Assume $p \in [1, \infty)$. Given a set $Y \in \mathcal{A}$, we can regard the Banach space $L^p(Y, \mathcal{A}|_Y, \mu|_Y)$ both as a quotient and as a closed linear sub-space of $L^p(X, \mathcal{A}, \mu)$. When viewing $L^p(Y, \mathcal{A}|_Y, \mu|_Y)$ as a quotient, we employ the restriction map $Q_Y^p : f \mapsto f|_Y$. When viewing $L^p(Y, \mathcal{A}|_Y, \mu|_Y)$ as a closed linear subspace, we employ the isometric linear map $T_Y : L^p(Y, \mathcal{A}|_Y, \mu|_Y) \hookrightarrow L^p(X, \mathcal{A}, \mu)$, defined as follows. For every $f \in L^p(Y, \mathcal{A}|_Y, \mu|_Y)$, we define the function $T_Y f : X \to \mathbb{K}$ by

$$(T_Y f)(x) = \begin{cases} f(x) & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases}$$

The composition $T_Y \circ Q_Y^p : L^p(X, \mathcal{A}, \mu) \to L^p(X, \mathcal{A}, \mu)$ coincides with the multiplication operator $M_{\chi_Y}$. Of course, the composition $Q_Y^p \circ T_Y : L^p(Y, \mathcal{A}, \mu) \to L^p(Y, \mathcal{A}, \mu)$ is the identity map.

Given a linear functional $\phi : L^p(X, \mathcal{A}, \mu) \to \mathbb{K}$, we denote the composition $\phi \circ M_{\chi_Y}$ by $\phi_Y$. We can also define the restricted map $\phi|_Y = \phi \circ T_Y : L^p(Y, \mathcal{A}|_Y, \mu|_Y) \to \mathbb{K}$, which will satisfy the equality $\phi|_Y \circ Q_Y^p = \phi_Y$.

**Remarks 7-10.** Suppose $p \in [1, \infty)$, and $\pi = (X_i)_{i \in I}$ is an $\mathcal{A}$-partition of $X$, which is $\mu$-integrable, i.e. such that $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$, for all $A \in \mathcal{A}_{\text{fin}}$.

7. By Proposition-Definition 5 from BS IV, the correspondence $D_p : f \mapsto (Q_{X_i}^p f)_{i \in I}$ establishes an isometric linear isomorphism

$$D_p : (L^p(X, \mathcal{A}, \mu), \| \cdot \|_p) \to \ell^p(L^p(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \| \cdot \|_p)_{i \in I}. \tag{53}$$

8. If we take $q \in (1, \infty]$ to be the Hölder conjugate of $p$, then using the isometric isomorphism (53) and Theorem 8 from BS II, one obtains an isometric linear isomorphism

$$\Sigma : L^p(X, \mathcal{A}, \mu)^* \to \ell^q(L^p(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})^*, \ell^q)_{i \in I}, \tag{54}$$

defined as $\Sigma = (D_p^{-1})^* \circ \Psi$, where $(D_p^{-1})^*$ is the transpose of the inverse of (53), and

$$\Psi : (\ell^p(L^p(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \| \cdot \|_p)_{i \in I})^* \to \ell^q(L^p(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})^*, \| \cdot \|_p)_{i \in I}$$

is the isometric isomorphism from Theorem 8 in BS II.
9. Upon writing down all the identifications from the preceding remarks, the isometric isomorphism (54) is simply given as \( \Sigma(\phi) = (\phi|_{X_i})_{i \in I} \). Therefore, for a linear functional \( \phi : L^p_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \), the following are equivalent:

(i) \( \phi \) is continuous;

(ii) all restricted functionals \( \phi|_{X_i} = \phi \circ T_{X_i} : L^p_\mathbb{K}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \to \mathbb{K}, i \in I, \) are continuous, and the \( I \)-tuple \((\|\phi|_{X_i}\|)_{i \in I}\) belongs to \( \ell^q(I) \), i.e.

\[ \bullet \sum_{i \in I} \|\phi|_{X_i}\|^q < \infty, \text{ if } p, q \in (1, \infty), \]

\[ \bullet \sup_{i \in I} \|\phi|_{X_i}\| < \infty, \text{ if } p = 1 \text{ and } q = \infty. \]

Furthermore, if this is the case, then

\[ \|\phi\| = \begin{cases} \left[ \sum_{i \in I} \|\phi|_{X_i}\|^q \right]^{1/q} & \text{if } p, q \in (1, \infty) \\
\sup_{i \in I} \|\phi|_{X_i}\| < \infty & \text{if } p = 1 \text{ and } q = \infty \end{cases} \]  

(55)

10. In the preceding remark, the equivalence \((i) \Leftrightarrow (ii)\), as well as the formulas (55) hold, if instead of the restricted functionals \( \phi|_{X_i} \), we use the functionals \( \phi_{X_i} = \phi \circ M_{X_i} : L^p_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \). This follows from the identities \( \phi_{X_i} = (\phi|_{X_i}) \circ Q_{X_i} \) and \( \phi|_{X_i} = \phi \circ T_{X_i} = \phi_{X_i} \circ T_{X_i} \), which show that \( \phi_{X_i} \) is continuous, if and only if \( \phi|_{X_i} \) is continuous, and furthermore one has the equality \( \|\phi_{X_i}\| = \|\phi|_{X_i}\| \).

**Lemma 2.** Assume \( p \in (1, \infty) \), and \( \phi : L^p_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \) is a \( \mathbb{K} \)-linear continuous functional. If we regard the collection \( \mathcal{A}_{\text{fin}}^\mu \) as a directed set (with “\( \supset \)” as the order relation), then the nets \( (\phi_Y)_{Y \in \mathcal{A}_{\text{fin}}^\mu} \subset L^p_\mathbb{K}(X, \mathcal{A}, \mu)^* \) and \( (\|\phi_Y\|)_{Y \in \mathcal{A}_{\text{fin}}^\mu} \subset [0, \infty) \) have the following properties.

(i) The net \((\|\phi_Y\|)_{Y \in \mathcal{A}_{\text{fin}}^\mu}\) is increasing, i.e. for \( Y_1, Y_2 \in \mathcal{A}_{\text{fin}}^\mu \), one has the implication \( Y_1 \supset Y_2 \Rightarrow \|\phi_{Y_1}\| \geq \|\phi_{Y_2}\| \).

(ii) \( \lim_{Y \in \mathcal{A}_{\text{fin}}^\mu} \|\phi_Y\| = \|\phi\| \).

(iii) If \( 1 < p < \infty \), then \( (\phi_Y)_{Y \in \mathcal{A}_{\text{fin}}^\mu} \) converges to \( \phi \) in the norm topology, i.e.

\[ \lim_{Y \in \mathcal{A}_{\text{fin}}^\mu} \|\phi - \phi_Y\| = 0. \]

**Proof.** (i) This statement is pretty obvious, since the inclusion \( Y_1 \supset Y_2 \) implies the equality \( M_{X_{Y_2}} = M_{X_{Y_1}} \circ M_{X_{Y_2}} \), which yields \( \phi_{Y_2} = \phi_{Y_1} \circ M_{X_{Y_2}} \), so by the norm inequality we get \( \|\phi_{Y_2}\| \leq \|\phi_{Y_1}\| \cdot \|M_{X_{Y_2}}\| \leq \|\phi_{Y_1}\| \).

(ii) Since \((\|\phi_Y\|)_{Y \in \mathcal{A}_{\text{fin}}^\mu}\) is increasing, we have \( \lim_{Y \in \mathcal{A}_{\text{fin}}^\mu} \|\phi_Y\| = \sup_{Y \in \mathcal{A}_{\text{fin}}^\mu} \|\phi_Y\| \), so it suffices to prove the equality

\[ \sup_{Y \in \mathcal{A}_{\text{fin}}^\mu} \|\phi_Y\| = \|\phi\|. \]  

(56)

Denote the left-hand side of (56) by \( S \). Since \( \phi_Y = \phi \circ M_{X_Y} \), with \( M_{X_Y} \) a linear contraction, it follows that \( S \leq \|\phi\| \). To prove the other inequality, we fix for the moment some \( \varepsilon > 0 \),
and we use the definition of the norm of $\phi$, combined with the density of $L^p_{\mathcal{K}} \ell_1(X, \mathcal{A}, \mu)$ in $L^p_{\mathcal{K}}(X, \mathcal{A}, \mu)$, to find an elementary integrable function $f = \alpha_1\chi_{A_1} + \cdots + \alpha_n\chi_{A_n}$ (with $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ and $A_1, \ldots, A_n \in \mathcal{A}_{\text{fin}}$), such that $\|f\|_p \leq 1$, and

$$\|\phi(f)\| \geq \|\phi\| - \varepsilon. \tag{57}$$

Since the set $Y = A_1 \cup \cdots \cup A_n$ obviously belongs to $\mathcal{A}_{\text{fin}}$, and we clearly have $f = M_{XY}f$, it follows that $\phi(f) = \phi_Y(f)$, so we have $\|\phi(f)\| = \|\phi_Y(f)\| \leq \|\phi_Y\| \cdot \|f\|_p \leq \|\phi_Y\|$, so if we go back to (57) we now get $\|\phi_Y\| \geq \|\phi\| - \varepsilon$. This shows, of course, that

$$S \geq \sup_{\phi \in \mathcal{A}_{\text{fin}}^p} \|\phi_Y\| \geq \|\phi\| - \varepsilon,$$

which forces (by the fact that the above inequality holds for all $\varepsilon > 0$) the inequality $S \geq \|\phi\|$.

(iii). Assume now $1 < p < \infty$. (The case $p = 1$ is discussed in Exercise 8 below.) Given any set $Y \in \mathcal{A}_{\text{fin}}^p$, it is pretty obvious that $(Y, X \setminus Y)$ constitutes a $\mu$-integrable $\mathcal{A}$-partition of $X$, so by Remarks 9-10 one has the equality

$$\|\phi\|^q = \|\phi_Y\|^q + \|\phi_{X \setminus Y}\|^q. \tag{58}$$

Since we also have $M_{XY} + M_{X \setminus Y} = I$, the identity operator on $L^p_{\mathcal{K}}(X, \mathcal{A}, \mu)$, we also have the equality $\phi_Y + \phi_{X \setminus Y} = \phi$, so if we replace $\phi_{X \setminus Y}$ with $\phi - \phi_Y$, the equality (58) now reads:

$$\|\phi - \phi_Y\|^q = \|\phi\|^q - \|\phi_Y\|^q, \tag{59}$$

so by part (ii) we immediately get $\lim_{Y \in \mathcal{A}_{\text{fin}}^p} \|\phi - \phi_Y\|^q = 0$, and we are done. $\square$

**Exercise 8.** Use the notations from Lemma 2, and assume $p = 1$.

(i) Prove that the net $(\phi_Y)_{Y \in \mathcal{A}_{\text{fin}}^p}$ converges to $\phi$ in the $w^*$-topology, i.e. $\lim_{Y \in \mathcal{A}_{\text{fin}}^p} \phi_Y(f) = \phi(f), \forall f \in L^1_{\mathcal{K}}(X, \mathcal{A}, \mu)$.

(ii) Let $X$ be an infinite set, and consider the measure space $(X, \mathcal{A}, \mu)$, with $\mathcal{A} = \mathcal{P}(X)$ and $\mu$ is the counting measure, so that $L^p_{\mathcal{K}}(X, \mathcal{A}, \mu) = \ell_1^p(X)$, and $\mathcal{A}_{\text{fin}}^\mu = \mathcal{P}_{\text{fin}}(X)$. Give an example of a linear continuous functional $\phi : \ell_1^p(X) \to \mathbb{K}$, such that $\|\phi\| = 1$, but $\|\phi - \phi_Y\| = 1, \forall Y \in \mathcal{P}_{\text{fin}}(X)$, thus showing that statement (iii) from Lemma 2 is not true, in general, for $p = 1$.

The next result show that, if $1 < p < \infty$, then Theorem 6 holds, without the finiteness condition. The case $p = 1$, in which we require additional restrictions, will be treated separately (see Theorem 8 and Corollary 2 below).

**Theorem 7** (Riesz-Fischer). *If $1 < p, q < \infty$ are Hölder conjugate of $p$, then map (44) is an isometric linear isomorphism.*

**Proof.** Since the map (44) is linear and isometric, its range, which we denote here by $\mathcal{Z}$, is norm closed in the dual Banach space $L^p_{\mathcal{K}}(X, \mathcal{A}, \mu)^*$. By Lemma 2, the set

$$\mathcal{W} = \{\phi_Y : \phi \in L^p_{\mathcal{K}}(X, \mathcal{A}, \mu)^*, Y \in \mathcal{A}_{\text{fin}}^\mu\}$$
is norm dense in \( L^p_\mathbb{K}(X, \mathcal{A}, \mu)^* \), so the proof will be finished once we show that \( \mathcal{W} \subset \mathcal{Z} \). Fix some \( \mathbb{K} \)-linear continuous functional \( \phi : L^p_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \), and some set \( Y \in \mathcal{A}_\text{fin}^\mu \), and let us indicate how a function \( g \in \mathcal{L}^q_\mathbb{K}(X, \mathcal{A}, \mu) \) can be constructed, so that

\[
\phi_Y(f) = \int_X fg \, d\mu, \quad \forall f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu).
\]

(60)

Consider the linear continuous functional \( \phi|_Y = \phi \circ T_Y \in L^p_\mathbb{K}(Y, \mathcal{A}|_Y, \mu|_Y) \), to which we can apply Theorem 6 (the fact that \( Y \in \mathcal{A}_\text{fin}^\mu \) is essential here), so there exists \( g_0 \in \mathcal{L}^q_\mathbb{K}(Y, \mathcal{A}|_Y, \mu|_Y) \), such that

\[
\phi_Y(f_0) = \int_Y f_0 g_0 \, d(\mu|_Y), \quad \forall f_0 \in L^p_\mathbb{K}(Y, \mathcal{A}|_Y, \mu|_Y).
\]

(61)

If we consider the extension of \( g = T_Y g_0 \) of \( g_0 \) to the whole \( X \), then using (61) we get

\[
\phi_Y(f) = \phi(f \chi_Y) = (\phi|_Y)(f|_Y) = \int_Y (f|_Y) g_0 \, d(\mu|_Y) = \int_X fg \, d\mu, \quad \forall f \in L^p_\mathbb{K}(X, \mathcal{A}, \mu).
\]

\[\Box\]

Our next goal is now to identify the dual Banach space \( L^1_\mathbb{K}(X, \mathcal{A}, \mu)^* \). As discussed at the beginning of this sub-section the natural candidate for this identification is the isometric linear map (45).

**Theorem 8.** If the measure space \((X, \mathcal{A}, \mu)\) is decomposable, then the map (45) is an isometric isomorphism.

**Proof.** We already know that (45) is isometric, so we only need to prove surjectivity.

Start with a linear continuous functional \( \phi : L^1_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \), and let us indicate how \( g \in \mathcal{L}^{\infty,\text{loc}}_\mathbb{K}(X, \mathcal{A}, \mu) \) can be constructed, so that

\[
\phi(f) = \int_X fg \, d\mu, \quad \forall f \in L^1_\mathbb{K}(X, \mathcal{A}, \mu).
\]

(62)

Fix a decomposition \( \pi = (X_i)_{i \in I} \) for \((X, \mathcal{A}, \mu)\). For each \( i \in I \), by applying Theorem 6 to the finite measure space \((X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})\), there exists \( g_i \in \mathcal{L}^\infty_\mathbb{K}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \subset \mathcal{L}^{\infty,\text{loc}}_\mathbb{K}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \), such that

\[
(\phi|_{X_i})(f) = \int_{X_i} f g_i \, d(\mu|_{X_i}), \quad \forall f \in L^1_\mathbb{K}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}).
\]

(63)

Of course, \( g_i : X_i \to \mathbb{K} \) can be chosen to be “honestly” bounded, i.e. such that \(|g_i(x)| \leq \|g_i\|_\infty\), so using Remark 9, we can assume that

\[
|g_i(x)| \leq \|\phi\|, \quad \forall i \in I, \, x \in X_i.
\]

(64)

Using the fact that \( \pi \) is a decomposition (see also sub-section D in BS IV), there exists a bounded \( \mathcal{A} \)-measurable function \( g : X \to \mathbb{K} \), such that \( g|_{X_i} = g_i \), \( \forall i \in I \). In order to check (62), we first observe that, since \( g \) is bounded, the linear functional \( g^* : L^1_\mathbb{K}(X, \mathcal{A}, \mu) \ni f \mapsto \int_X fg \, d\mu \in \mathbb{K} \) is continuous, so the desired equality (62) simply reads: \( \phi = g^* \).

Using Remark 8, we know that we have an isometric linear isomorphism

\[
\Sigma : L^1_\mathbb{K}(X, \mathcal{A}, \mu)^* \to \ell^\infty(L^1_\mathbb{K}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})^*)_{i \in I}.
\]

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The equalities (65) follow trivially from (63) and from the equalities \( g_i = g|_{X_i}. \)

**Corollary 2.** If \((X, \mathcal{A}, \mu)\) is \(\sigma\)-finite, then the map

\[
\# : L^\infty_K(X, \mathcal{A}, \mu) \ni g \longmapsto g^\# \in L^1_K(X, \mathcal{A}, \mu)^*
\]

is an isometric linear isomorphism.

**Proof.** If \((X, \mathcal{A}, \mu)\) is \(\sigma\)-finite, then it is also decomposable, and the quotient map \(L^\infty_K(X, \mathcal{A}, \mu) \ni g \longmapsto [g]_{\text{loc}} \in L^{\infty, \text{loc}}_K(X, \mathcal{A}, \mu)\) is an isometric linear isomorphism.

**Remark 11.** As pointed out in BS IV, for \(p \in [1, \infty)\), the \(L^p\)-spaces do not change, if we replace \((X, \mathcal{A}, \mu)\) with the Caratheodory completion \((X, \mathcal{A}^c, \mu^c)\), that is, the natural inclusion \(L^p_K(X, \mathcal{A}, \mu) \subset L^p_K(X, \mathcal{A}^c, \mu^c)\) yields an (isometric) equality

\[
L^p_K(X, \mathcal{A}, \mu) = L^p_K(X, \mathcal{A}^c, \mu^c), \ 1 \leq p < \infty.
\]

In particular, when \(p = 1\), a “better candidate” to represent the dual Banach space \(L^1_K(X, \mathcal{A}, \mu)^*\) is the Banach space \(L^{\infty, \text{loc}}_K(X, \mathcal{A}^c, \mu^c)\).

As it turns out (see BS IV), if \((X, \mathcal{A}, \mu)\) is only assumed to be quasi-decomposable, i.e. there exists a \(\mu\)-integrable partition \(\pi = (X_i)_{i \in I}\), with all the \(X_i\)’s in \(\mathcal{A}^c_{\text{fin}}\), then \(\pi\) is a decomposition for \((X, \mathcal{A}^c, \mu^c)\), so by Theorem 8 we obtain an isometric linear isomorphism

\[
L^{\infty, \text{loc}}_K(X, \mathcal{A}^c, \mu^c)^\# \overset{\#_{\text{loc}}}{\longrightarrow} L^1_K(X, \mathcal{A}^c, \mu^c)^* \overset{\sim}{\longrightarrow} L^1_K(X, \mathcal{A}, \mu)^*.
\]

Theorem 8 has a slight generalization, which goes as follows.

**Exercise 9.** Prove that if there exists an \(\mathcal{A}\)-partition \((X_0, X_1)\) of \(X\), such that

- \((X_1, \mathcal{A}|_{X_1}, \mu|_{X_1})\) is decomposable,

- \((X_0, \mathcal{A}|_{X_0}, \mu|_{X_0})\) is degenerate, i.e. \(\mu(A \cap X_0) \in \{0, \infty\}, \forall A \in \mathcal{A}\),

then (45) is again an isomorphism.

The next five Exercises explain how the dual Banach spaces \(M^b_K(X, \mathcal{A}), L^\infty_K(X, \mathcal{A}, \mu)^*\) and \(L^{\infty, \text{loc}}_K(X, \mathcal{A}, \mu)^*\) are computed. These characterizations are analogous to the one for \(\ell^\infty_K(S)^*\), given in BS II.

**Exercises 12-16.** Denote by \(F_K(\mathcal{A})\) the space of all bounded, finitely additive maps \(\tau : \mathcal{A} \rightarrow K\).

12. For any \(\tau \in F_K(\mathcal{A})\), define the map \(|\tau| : \mathcal{A} \rightarrow [0, \infty]\) by

\[
|\tau|(A) = \sup \left\{ \sum_{j=1}^n |\tau(A_j)| : n \in \mathbb{N}, (A_j)_{j=1}^n \subset \mathcal{A} \text{ disjoint, } \bigcup_{j=1}^n A_n = A \right\}.
\]

Prove that \(|\tau|\) is also finitely additive and bounded, thus \(|\tau| \in F_K(\mathcal{A})\).
13. Prove that $\mathfrak{F}_K(A)$ becomes a vector space, when we define
\[ (\tau_1 + \tau_2)(A) = \tau_1(A) + \tau_2(A), \quad \tau_1, \tau_2 \in \mathfrak{F}_K(A), \]
\[ (\alpha \tau)(A) = \alpha \tau(A), \quad \tau \in \mathfrak{F}_K(A), \quad \alpha \in K, \]
for every $A \in A$. Prove that the map
\[
\| \cdot \|_{\text{var}} : \mathfrak{F}_K(A) \ni \tau \mapsto |\tau|(X) \in [0, \infty)
\]
defines a norm on $\mathfrak{F}_K(A)$, referred to as the variation norm, and furthermore, $(\mathfrak{F}_K(A), \| \cdot \|_{\text{var}})$ is a Banach space.

14. Prove that the map
\[
\Phi : m^\text{elem}_K(X, A) \times \mathfrak{F}_K(A) \ni (f, \tau) \mapsto \sum_{\alpha \in \text{Range } f} \alpha \tau(f^{-1}(\{\alpha\})) \in K
\]
establishes a dual pairing between $m^\text{elem}_K(X, A)$ and $\mathfrak{F}_K(A)$. Prove that, for every $\tau \in \mathfrak{F}_K(A)$, the linear functional
\[
\tau^\# : m^\text{elem}_K(X, A) \ni f \mapsto \Phi(f, \tau) \in K
\]
satisfies the inequality
\[
|\tau^\#(f)| \leq \|\tau\|_{\text{var}} \cdot \|f\|_{\text{sup}}.
\]
Conclude that, for every $\tau \in \mathfrak{F}_K(A)$, the functional $\tau^\#$ can be uniquely extended to a linear norm-continuous functional, still denoted by
\[
\tau^\# : m^b_K(X, A) \to K.
\]
(It is understood here that on $m^b_K(X, A)$ one uses the Banach space norm $\| \cdot \|_{\text{sup}}$.)

15. Prove that the correspondence
\[
\mathfrak{F}_K(A) \ni \tau \mapsto \tau^\# \in m^b_K(X, A)^*, \quad (66)
\]
is an isometric linear isomorphism. Prove that the inverse of (66) is the map
\[
T : m^b_K(X, A)^* \ni \phi \mapsto \tau_{\phi} \in \mathfrak{F}_K(A), \quad (67)
\]
defined by: $\tau_{\phi}(A) = \phi(\chi_A)$, $A \in A$.

16. Consider the surjective contractions
\[
\Pi_{\infty} : m^b_K(X, A) \to L^\infty_K(X, A, \mu),
\]
\[
\Pi_{\infty, \text{loc}} : m^b_K(X, A) \to L^\infty_{K, \text{loc}}(X, A, \mu),
\]
and the compositions of their transposes with the map (67):
\[
\Sigma_{\infty} : L^\infty_K(X, A, \mu)^* \xrightarrow{\Pi_{\infty}} m^b_K(X, A)^* \xrightarrow{T} \mathfrak{F}_K(A),
\]
\[
\Sigma_{\infty, \text{loc}} : L^\infty_{K, \text{loc}}(X, A, \mu)^* \xrightarrow{\Pi_{\infty, \text{loc}}} m^b_K(X, A)^* \xrightarrow{T} \mathfrak{F}_K(A).
Prove that both $\Sigma_\infty$ and $\Sigma_{\infty,\text{loc}}$ are isometric linear maps, and their ranges are:

Range $\Sigma_\infty = \{ \tau \in \mathcal{F}_K(A) : \tau(A) = 0, \text{for all } \mu\text{-negligeable sets } A \in \mathcal{A} \}$,

Range $\Sigma_{\infty,\text{loc}} = \{ \tau \in \mathcal{F}_K(A) : \tau(A) = 0, \text{for all locally } \mu\text{-negligeable sets } A \in \mathcal{A} \}$.

**Conclusion.** If we denote the Ranges of $\Sigma_\infty$ and $\Sigma_{\infty,\text{loc}}$ by $\mathcal{F}_K^\infty(A, \mu)$ and $\mathcal{F}_K^{\infty,\text{loc}}(A, \mu)$, respectively, then we have two isometric linear isomorphisms $\Sigma_\infty : L^\infty_K(X, \mathcal{A}, \mu)^* \rightarrow \mathcal{F}_K^\infty(A, \mu)$ and $\Sigma_{\infty,\text{loc}} : L^\infty_K(X, \mathcal{A}, \mu)^* \rightarrow \mathcal{F}_K^{\infty,\text{loc}}(A, \mu)$.

**Comment.** In connection with Exercise 14, given $\tau \in \mathcal{F}_K(A)$ and $f \in m^b_K(X, \mathcal{A})$, the quantity $\tau^*(f)$ is sometimes denoted by $S_X f \, d\tau$. This notation was first introduced in sub-section D from BS II. Everything discussed there is a special case of the results from Exercises 12-16. In particular, the space $\mathcal{F}_K(S)$ introduced in BS II coincides with $\mathcal{F}_K(P(S))$.

We conclude with a series of Exercises that contain an important construction in Probability Theory.

**Exercises 17-20.** Assume a second $\sigma$-algebra $\mathcal{B} \subset \mathcal{A}$ is given, which is equipped with the measure $\mu|_B$.

17. Assume $1 < q < \infty$, and let $p = q/(q - 1)$ be the Hölder conjugate of $q$. Prove that for every $f \in L^p_K(X, \mathcal{A}, \mu)$, there exists a (unique $\mu$-a.e.) function $\tilde{f} \in L^q_K(X, \mathcal{B}, \mu|_B)$ such that

$$\int_X fg \, d\mu = \int_X \tilde{f}g \, d\mu, \quad \forall g \in L^p_K(X, \mathcal{B}, \mu|_B).$$

Prove that the map

$$\mathbb{E}^q_{A|B} : L^p_K(X, \mathcal{A}, \mu) \ni f \longmapsto \tilde{f} \in L^q_K(X, \mathcal{B}, \mu|_B)$$

has the following properties:

1. $\mathbb{E}^q_{A|B}$ is linear continuous, and has $\|\mathbb{E}^q_{A|B}\| \leq 1$;
2. $\mathbb{E}^q_{A|B}(f) = f, \forall f \in L^p_K(X, \mathcal{B}, \mu|_B)$;
3. $\mathbb{E}^q_{A|B}(h \cdot f) = h \cdot \mathbb{E}^q_{A|B}(f), \forall h \in L^{\infty,\text{loc}}_K(X, \mathcal{B}, \mu|_B), f \in L^q_K(X, \mathcal{A}, \mu)$.

(HINT: Consider the inclusion $J : L^p_K(X, \mathcal{B}, \mu|_B) \hookrightarrow L^p_K(X, \mathcal{A}, \mu)$, and its transpose $J^* : L^p_K(X, \mathcal{A}, \mu)^* \rightarrow L^p(X, \mathcal{B}, \mu|_B)^*$, and apply Theorem 7, by which the two dual Banach spaces are identified with the corresponding $L^q$-spaces.)

18. Prove that, in the case $q = 2 (= p)$, the map $\mathbb{E}^2_{A|B}$ coincides with the orthogonal projection of $L^2_K(X, \mathcal{A}, \mu)$ onto the closed linear subspace $L^2_K(X, \mathcal{B}, \mu|_B)$.

19. Assume $(X, \mathcal{B}, \mu|_B)$ is decomposable. Prove that for every $f \in L^p_K(X, \mathcal{A}, \mu)$, there exists a (unique $\mu$-a.e.) function $\tilde{f} \in L^p_K(X, \mathcal{B}, \mu|_B)$ such that, whenever $g \in m^b_K(X, \mathcal{B})$ has $\text{supp} \, g \in B_{\sigma,\text{fin}}$ and $fg \in L^p_{\sigma}(X, \mathcal{A}, \mu)$, it follows that $\tilde{f}g \in L^p_{\sigma}(X, \mathcal{B}, \mu|_B)$, and one has the equality

$$\int_X fg \, d\mu = \int_X \tilde{f}g \, d\mu.$$
Prove that the map
\[ \mathbb{E}^1_{A \upharpoonright B} : L^1_K(X, \mathcal{A}, \mu) \ni f \mapsto \tilde{f} \in L^1_K(X, \mathcal{B}, \mu|_B) \] (69)
has the following properties:

(i) \( \mathbb{E}^1_{A \upharpoonright B} \) is linear continuous, and has \( \| \mathbb{E}^1_{A \upharpoonright B} \| \leq 1 \);

(ii) \( \mathbb{E}^1_{A \upharpoonright B}(f) = f, \forall f \in L^1_K(X, \mathcal{B}, \mu|_B) \);

(iii) \( \mathbb{E}^1_{A \upharpoonright B}(h \cdot f) = h \cdot \mathbb{E}^1_{A \upharpoonright B}(f), \forall h \in L^\infty_{K, loc}(X, \mathcal{B}, \mu|_B), f \in L^1_K(X, \mathcal{A}, \mu) \).

(Hint: For every \( f \in L^1_K(X, \mathcal{A}, \mu) \), define the map \( \nu_f : \mathcal{B} \ni B \mapsto \int_X f \chi_B d\mu \in K \), and show that \( \nu_f \) is a \( K \)-valued measure on \( \mathcal{B} \), with \( \nu_f \ll \mu|_B \). Use Radon-Nikodim Theorem 5.)

20. Assume either

(a) \( q = 1 \) and \( (X, \mathcal{B}, \mu|_B) \) is decomposable, or

(b) \( 1 < q < \infty \) (without any additional restrictions).

Prove that the map (68) has the following properties:

(i) \( \mathbb{E}^q_{A \upharpoonright B} \) is positive, i.e. if \( f \in L^q_K(X, \mathcal{A}, \mu) \) is such that \( f \geq 0 \), then \( \mathbb{E}^q_{A \upharpoonright B}(f) \geq 0 \);

(ii) \( \mathbb{E}^q_{A \upharpoonright B} \) is faithful, i.e. if \( f \in L^q_K(X, \mathcal{A}, \mu) \) is such that \( f \geq 0 \) and \( \mathbb{E}^q_{A \upharpoonright B}(f) = 0 \), then \( f = 0 \).

The maps \( \mathbb{E}^q_{A \upharpoonright B} : L^q_K(X, \mathcal{A}, \mu) \to L^q_K(X, \mathcal{B}, \mu|_B) \), discussed above, are called the conditional expectation maps.

Comment. It is possible to construct a conditional expectation map
\[ \mathbb{E}^{\infty, loc}_{A \upharpoonright B} : L^{\infty, loc}_K(X, \mathcal{A}, \mu) \to L^{\infty, loc}_K(X, \mathcal{B}, \mu|_B), \]
in a manner analogous to the construction from Exercise 17. For such a construction, however, one needs some additional assumptions on the measure spaces \( (X, \mathcal{A}, \mu) \) and \( (X, \mathcal{B}, \mu|_B) \), which ensure that

(i) \( L^{\infty, loc}_K(X, \mathcal{B}, \mu|_B) \) gets identified as a closed linear subspace in \( L^{\infty, loc}_K(X, \mathcal{A}, \mu) \);

(ii) \( L^{\infty, loc}_K(X, \mathcal{B}, \mu|_B) \simeq L^1_K(X, \mathcal{B}, \mu|_B)^* \).

One sufficient condition for (i) and (ii) is the existence of a quasi-decomposition \( \pi = (X_i)_{i \in I} \) for \( (X, \mathcal{A}, \mu) \), such that

- all \( X_i \) belong to \( \mathcal{B} \),
- \( \pi \) constitutes a decomposition of \( (X, \mathcal{B}, \mu|_B) \).

The details, as well as the correct formulation of Exercise 17, are left to the reader.