Convexity “Warm-up” II: 
The Minkowski Functional

**Convention.** Throughout this note \( K \) will be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and all vector spaces are over \( K \). In this section, however, when \( K = \mathbb{C} \), it will be only the real linear structure that will play a significant role.

**Proposition-Definition 1.** Let \( X \) be a vector space, and let \( A \subset X \) be a convex absorbing set. Define, for every element \( x \in X \) the quantity

\[
q_A(x) = \inf \{ \tau > 0 : x \in \tau A \}. \tag{1}
\]

(i) The map \( q_A : X \to \mathbb{R} \) is a quasi-seminorm, i.e.

\[
q_A(x + y) \leq q_A(x) + q_A(y), \quad \forall x, y \in X; \tag{2}
\]

\[
q_A(tx) = tq_A(x), \quad \forall x \in X, \ t \geq 0. \tag{3}
\]

(ii) One has the inclusions:

\[
\{ x \in X : q_A(x) < 1 \} \subset A \subset \{ x \in X : q_A(x) \leq 1 \}. \tag{4}
\]

(iii) If, in addition to the above hypothesis, \( A \) is balanced, then \( q_A \) is a seminorm, i.e. besides (2) and (3) it also satisfies the condition:

\[
q_A(\alpha x) = |\alpha| \cdot q_A(x), \quad \forall x \in X, \ \alpha \in K. \tag{5}
\]

The map \( q_A : X \to \mathbb{R} \) is called the **Minkowski functional associated to** \( A \).

**Proof.** (i). To prove (2) start with two elements \( x, y \in X \) and some \( \varepsilon > 0 \). By the definition (1) there exist positive real numbers \( \alpha < q_A(x) + \varepsilon \) and \( \beta < q_A(y) + \varepsilon \), such that \( x \in \alpha A \) and \( y \in \beta A \). By Exercise ??? from LCTVS I, it follows that

\[
x + y \in \alpha A + \beta A = (\alpha + \beta)A,
\]

so by the definition (1) we get

\[
q_A(x + y) \leq \alpha + \beta < q_A(x) + q_A(y) + 2\varepsilon.
\]

Since the inequality \( q_A(x + y) < q_A(x) + q_A(y) + 2\varepsilon \) holds for all \( \varepsilon > 0 \), the desired inequality (2) follows.

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\(^1\) Since \( A \) is absorbing, the set on the right-hand side of (1) is non-empty.
To prove (3), let us first observe that, since \( A \) is absorbing, it contains the zero vector, so in fact \( 0 \in \tau A, \forall \tau > 0 \), which means that \( q_A(0) = 0 \). This shows that in order to prove (3) we may restrict to the case when \( t > 0 \). In that case we clearly have the equivalence:

\[
 tx \in \tau A \iff x \in (\tau/t)A,
\]

which shows that the map

\[
 \Theta : \{ \tau > 0 : tx \in \tau A \} \ni \tau \mapsto \tau/t \in \{ \gamma > 0 : x \in \gamma A \}
\]

is a bijection. Taking the infimum then yields the equality

\[
 q_A(tx)/t = q_A(x),
\]

which is exactly (3).

(ii). To prove the first inclusion in (4), suppose \( x \in X \) satisfies the inequality \( q_A(x) < 1 \), and let us show that \( x \in A \). By the definition (1), there exists some \( \tau \in (0,1) \), such that \( x \in \tau A \). In particular, the vector \( a = \frac{1}{\tau}x \) belongs to \( A \), so the vector \( \tau a + (1 - \tau)0 = x \) also belongs to \( A \).

The second inclusion in (4) is quite obvious, since for any \( a \in A \), the set \( \{ \tau > 0 : a \in \tau A \} \) clearly contains 1.

(iii). Assume \( A \) is balanced. In particular, if we consider the multiplicative group

\[
 G = \{ \gamma \in K : |\alpha| = 1 \},
\]

we know that

\[
 \gamma A = A, \ \forall \gamma \in G.
\]

Therefore, for any \( x \in X, \tau > 0, \) and \( \gamma \in G, \) one has the equivalences

\[
 x \in \tau A \iff \gamma x \in \tau \gamma A \iff \gamma x \in \tau A.
\]

By the definition (1) this yields

\[
 q_A(\gamma x) = q_A(x), \ \forall x \in X, \gamma \in G. \tag{6}
\]

To prove (5) we notice that for any \( \alpha \in K \) we can find \( \gamma \in G \) and \( t \geq 0 \), with

\[
 \alpha = \gamma t, \tag{7}
\]

so using (6) and (3) we get

\[
 q_A(\alpha x) = q_A(\gamma tx) = q_A(tx) = tq_A(x),
\]

and the desired equality follows from (7) which forces \( t = |\alpha| \).

**Remark.** Call a subset \( A \subset X \) *openly absorbing*, if for every \( x \in X \), the set

\[
 T(x) = \{ t > 0 : tx \in A \}
\]

is non-empty\(^2\) and *open* in \((0, \infty)\).

With this terminology, statement (ii) from Proposition-Definition 1 can be slightly improved, in the following sense.

**Proposition 2.** For a convex absorbing subset \( A \subset X \), the following are equivalent:

\^[2\] Of course, the condition \( T(x) \neq \emptyset, \forall x \in X \), means that \( A \) is absorbing.
(i) $\mathcal{A}$ is openly absorbing;

(ii) $\mathcal{A} = \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}$.

Proof. $(i) \Rightarrow (ii)$. Assume $\mathcal{A}$ is openly absorbing. By (4) we only need to prove the inclusion $\mathcal{A} \subset \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}$.

Start with some $x \in \mathcal{A}$, so that, using the notation from the above Definition, the set $T(x)$ contains 1. Since $T(x)$ is open, it will contain a small open interval around 1. In particular, $T(x)$ contains some $t > 1$. For such a $t$, it follows that $x \in t^{-1} \mathcal{A}$, so $q_{\mathcal{A}}(x) \leq t - 1 < 1$.

$(ii) \Rightarrow (i)$. Assume now condition (ii), and let us prove that $\mathcal{A}$ is openly absorbing.

Fix some $x \in \mathcal{X}$, and some $t \in T(x)$. We wish to produce an open interval $J$, such that $t \in J \subset T(x)$. On the one hand, since $tx \in \mathcal{A}$, by (ii) it follows that $q_{\mathcal{A}}(tx) < 1$, or equivalently $t < q_{\mathcal{A}}(x) - 1$ (with the convention that $q_{\mathcal{A}}(x) - 1 = \infty$, if $q_{\mathcal{A}}(x) = 0$). This suggests that we could define $J$ to be the interval $(0, q_{\mathcal{A}}(x) - 1)$. To check that this works, start with some $s \in J$, i.e. $0 < s < q_{\mathcal{A}}(x)^{-1}$. This forces $q_{\mathcal{A}}(sx) = sq_{\mathcal{A}}(x) < 1$, so by (ii) $sx \in \mathcal{A}$, which means that $s$ indeed belongs to $T(x)$.

Exercises 1-9. Let $\mathcal{X}$ be a vector space.

1. Suppose $\mathcal{A}, \mathcal{B} \subset \mathcal{X}$ are convex absorbing sets, such that $\mathcal{A} \subset \mathcal{B}$. Show that $q_{\mathcal{A}}(x) \geq q_{\mathcal{B}}(x)$, $\forall x \in \mathcal{X}$.

2. Prove that if $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{X}$ are convex and absorbing, then $\bigcap_{i=1}^{n} \mathcal{A}_i$ is also absorbing. Give an example of two (non-convex) absorbing sets $\mathcal{A}_1, \mathcal{A}_2 \subset \mathbb{R}$, such that $\mathcal{A}_1 \cap \mathcal{A}_2$ is not absorbing.

3. Prove that, if $\mathcal{A}_1, \ldots, \mathcal{A}_n \subset \mathcal{X}$ are openly absorbing, then so is $\bigcap_{i=1}^{n} \mathcal{A}_i$.

4. Prove that given an arbitrary collection $(\mathcal{A}_i)_{i \in I}$ of openly absorbing subsets, the union $\bigcup_{i \in I} \mathcal{A}_i$ is again openly absorbing.

5. Show that, if $\mathcal{A}$ is openly absorbing, then so is its convex hull $\text{conv}(\mathcal{A})$.

6. Suppose $\mathcal{X}$ is equipped with a linear topology, and $\mathcal{A} \subset \mathcal{X}$ is an open set containing 0. Prove that $\mathcal{A}$ is openly absorbing.

7. Suppose $\mathcal{A} \subset \mathcal{X}$ is convex and absorbing. Define the pseudo-interior of $\mathcal{A}$ to be the set $\mathcal{A}^\diamond = \{x \in \mathcal{X} : q_{\mathcal{A}}(x) < 1\}$. Prove the following statements.

   (i) $\mathcal{A}^\diamond$ is the largest openly absorbing set contained in $\mathcal{A}$.

   (ii) $\mathcal{A}^\diamond$ is convex.

   (iii) $q_{\mathcal{A}^\diamond} = q_{\mathcal{A}}$. 

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8. Let \( p : \mathcal{X} \to \mathbb{R} \) be a quasi-seminorm on a real vector space \( \mathcal{X} \). Consider the set \( \mathcal{A} = \{ x \in \mathcal{X} : p(x) < 1 \} \). Show that:

(i) \( \mathcal{A} \) is convex and openly absorbing.
(ii) If \( p \) is a seminorm, then \( \mathcal{A} \) is also balanced.
(iii) The Minkowski functional \( q_{\mathcal{A}} \) is given by

\[
q_{\mathcal{A}}(x) = \max\{p(x), 0\}, \quad \forall x \in \mathcal{X}.
\]

9\( \diamond \). Assume \( \mathcal{A} \subset \mathcal{X} \) is non-empty, convex and balanced. Prove that the following conditions are equivalent:

(i) the Minkowski functional \( q_{\mathcal{A}} \) is a norm, i.e. \( q_{\mathcal{A}}(x) = 0 \Rightarrow x = 0 \);
(ii) \( \bigcap_{t>0} t\mathcal{A} = \{0\} \).

Exercise 10. Let \( n \geq 1 \) be an integer. Consider the following sets in \( \mathbb{K}^n \):

\[
\mathcal{A} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 < 1 \};
\]

\[
\mathcal{B} = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1 \}.
\]

Show that for any \( x = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \), one has the equalities

\[
q_{\mathcal{A}}(x) = q_{\mathcal{B}}(x) = \sqrt{\alpha_1^2 + \cdots + \alpha_n^2}.
\]