Banach Spaces IV:
Banach Spaces of Measurable Functions

Notes from the Functional Analysis Course (Fall 07 - Spring 08)

In this section we discuss another important class of Banach spaces arising from Measure and Integration theory.

Convention. Throughout this note \( K \) will be one of the fields \( \mathbb{R} \) or \( \mathbb{C} \), and all vector spaces are over \( K \).

A. Review of Measure and Integration Theory

In this section we recollect some basic notions and results, which can be found in most Real Analysis textbooks. The exposition is in the form of a sequence of Definitions and Facts.

Definitions. A measurable space is a pair \((X, A)\) consisting of a non-empty set \( X \) and a \( \sigma \)-algebra on \( X \). Our basic example will be of the form \((\Omega, \text{Bor}(\Omega))\), where \( \Omega \) is a topological Hausdorff space (most often locally compact), and \( \text{Bor}(\Omega) \) is the Borel \( \sigma \)-algebra. Given another measurable space \((Y, B)\), a map \( f : X \to Y \) is said to be a measurable transformation from \((X, A)\) to \((Y, B)\), if \( f^{-1}(B) \in A, \forall B \in B \). In practice, if one chooses a collection \( M \subset B \), which generates \( B \) as a \( \sigma \)-algebra, then a sufficient condition for \( f : (X, A) \to (Y, B) \) to be a measurable transformation is: \( f^{-1}(M) \in A, \forall M \in M \).

If we specialize to the case \( Y = K \), then a function \( f : (X, A) \to K \) is declared measurable, if it defines a measurable transformation \( f : (X, A) \to (K, \text{Bor}(K)) \). In the complex case, this is equivalent to the condition that both \( \text{Re} f, \text{Im} f : (X, A) \to \mathbb{R} \) are measurable. Since the Borel \( \sigma \)-algebra \( \text{Bor}(\mathbb{R}) \) is generated by various collections of intervals, such as for instance \( J = \{(s, \infty) : s \in \mathbb{R}\} \), measurability of \( f : (X, A) \to \mathbb{R} \) is equivalent to \( f^{-1}(J) \in A \), \( \forall J \in J \). On occasion one also employs measurable functions which take values in one of the extended intervals \([-\infty, \infty], [0, \infty], \text{or} [-\infty, 0] \), with the understanding that if \( T \) denotes any one of these spaces, one again uses the Borel \( \sigma \)-algebra \( \text{Bor}(T) \). (Of course any such \( T \) is homeomorphic to \([0, 1]\).)

We denote by \( m_K(X, A) \) the space of all \( K \)-valued measurable functions on \((X, A)\). It is well known that \( m_K(X, A) \) is a \( K \)-algebra, when equipped with point-wise addition and multiplication. As usually, when \( K = \mathbb{C} \), the subscript will be omitted from the notations.

The space of \([0, \infty]\)-valued measurable functions on \((X, A)\) is denoted by \( m_+(X, A) \).

The simplest measurable functions on a measurable space \((X, A)\) are the indicator functions \( \chi_A \), of sets \( A \in A \). (Recall that \( \chi_A(x) = 1 \), if \( x \in A \), and \( \chi_A(x) = 0 \), if \( x \not\in A \).) The next class of measurable functions are those in \( \text{span}_K \{ \chi_A : A \in A \} \subset m_K(X, A) \). These

\[ \text{This means that } B \text{ is the intersection of all } \sigma \text{-algebra that contain } M. \]
functions are referred to as elementary measurable functions, and the above mentioned span is denoted by $m_{\mathbb{K}}^{\text{elem}}(X, \mathcal{A})$. An equivalent description of this space is:

$$m_{\mathbb{K}}^{\text{elem}}(X, \mathcal{A}) = \{ f : X \to \mathbb{K} : \text{Range } f \text{ is finite, and } f^{-1}(\{\alpha\}) \in \mathcal{A}, \ \forall \alpha \in \text{Range } f \}.$$  

This characterization allows one to extend the notion of elementary measurable functions for functions $f : X \to Y$, where $Y$ is an arbitrary set. In particular, if we let $Y = [0, \infty]$, the corresponding space of functions is denoted by $m_{+}^{\text{elem}}(X, \mathcal{A})$.

The following result is a very basic one in Measure Theory.

**Fact 1.** Let $(X, \mathcal{A})$ be a measurable space, and let $T$ one of the spaces $\mathbb{R}$, $[-\infty, \infty]$, or $\mathbb{C}$. Define $\mathbb{K}$ to be either $\mathbb{R}$, when $T$ is $\mathbb{R}$ or $[-\infty, \infty]$, or $\mathbb{C}$, when $T = \mathbb{C}$. For a function $f : X \to T$, the following are equivalent:

- $f : (X, \mathcal{A}) \to T$ is measurable;
- there exists a sequence $(f_{n})_{n=1}^{\infty} \subset m_{\mathbb{K}}^{\text{elem}}(X, \mathcal{A})$, such that
  $$\lim_{n \to \infty} f_{n}(x) = f(x), \ \forall x \in X.$$  

In the case when $T = [-\infty, \infty]$ or $T = \mathbb{R}$, the functions $f_{n}$ can be chosen such that

$$\inf_{y \in X} f(y) \leq f_{n}(x) \leq \sup_{y \in X} f(y), \ \forall x \in X.$$  

Furthermore, in this case one has the following implications:

1. if $f(x) > -\infty, \ \forall x$, the sequence $(f_{n})_{n=1}^{\infty}$ can be chosen to be non-decreasing;
2. if $f(x) < \infty, \ \forall x$, the sequence $(f_{n})_{n=1}^{\infty}$ can be chosen to be non-increasing.

**Definitions.** Given a measurable space $(X, \mathcal{A})$, an “honest” measure on $\mathcal{A}$ is a $\sigma$-additive map $\mu : \mathcal{A} \to [0, \infty]$, with $\mu(\emptyset) = 0$. The $\sigma$-additivity condition means that, whenever $(A_{n})_{n=1}^{\infty} \subset \mathcal{A}$ is a sequence of disjoint sets, it follows that $\mu(\bigcup_{n=1}^{\infty} A_{n}) = \sum_{n=1}^{\infty} \mu(A_{n})$. When such a $\mu$ is identified, the triple $(X, \mathcal{A}, \mu)$ is referred to as a measure space.

Given a measure space $(X, \mathcal{A}, \mu)$, we define two maps

$$I_{\mu}^{+, \text{elem}} : m_{+}^{\text{elem}}(X, \mathcal{A}) \to [0, \infty],$$  

$$I_{\mu}^{+} : m_{+}(X, \mathcal{A}) \to [0, \infty]$$  

by the formulas:

$$I_{\mu}^{+, \text{elem}}(f) = \sum_{\alpha \in \text{Range } f} \alpha \cdot \mu(f^{-1}(\{\alpha\})), \ \ f \in m_{+}^{\text{elem}}(X, \mathcal{A}),$$  

$$I_{\mu}^{+}(g) = \sup \left\{ I_{\mu}^{+, \text{elem}}(f) : f \in m_{+}^{\text{elem}}(X, \mathcal{A}), \ 0 \leq f \leq g \right\}.$$  

(In the definition of $I_{\mu}^{+, \text{elem}}$ one uses the convention $0 \cdot \infty = \infty \cdot 0 = 0$.) The map $I_{\mu}^{+}$ is called the positive $\mu$-integral.

**Fact 2.** Let $(X, \mathcal{A}, \mu)$ be a measure space. The positive integral $I_{\mu}^{+} : m_{+}(X, \mathcal{A}) \to [0, \infty]$ has the following properties:
(i) \( I^+_\mu \) is positive-linear, in the sense that

- \( I^+_\mu(f + g) = I^+_\mu(f) + I^+_\mu(g), \forall f, g \in m_+(X, \mathcal{A}); \)
- \( I^+_\mu(\alpha f) = \alpha I^+_\mu(f), \forall f \in m_+(X, \mathcal{A}), \alpha \in [0, \infty]; \)

in particular, \( I^+_\mu \) is monotone, in the sense that, for \( f, g \in m_+(X, \mathcal{A}) \) one has the implication \( f \geq g \Rightarrow I^+_\mu(f) \geq I^+_\mu(g) \).

(ii) If \( f \in \mathcal{M}^{\text{elem}}_+(X, \mathcal{A}) \), then \( I^+_\mu(f) = I^+_{\mu, \text{elem}}(f) \). In particular, one has:

\[
I^+_\mu(\chi_A) = \mu(A), \quad \forall A \in \mathcal{A}.
\]

Another key feature of the positive integral, which is derived from its definition, is:

**Fact 3** (Lebesgue’s Monotone Convergence Theorem). If \( (f_n)_{n=1}^{\infty} \subset m_+(X, \mathcal{A}) \) is a non-decreasing sequence, then the function \( f : X \to [0, \infty] \) defined by \( f(x) = \lim_{n \to \infty} f_n(x), \) \( x \in X, \) belongs to \( m_+(X, \mathcal{A}) \) and has its positive integral given by \( I^+_\mu(f) = \lim_{n \to \infty} I^+_\mu(f_n). \)

**Definitions.** Suppose \((X, \mathcal{A}, \mu)\) is a measure space. A function \( f \in m_+(X, \mathcal{A}) \) is said to be \( \mu \)-integrable, if the function \( |f| \in m_+(X, \mathcal{A}) \) has finite positive integral: \( I_+^\mu(|f|) < \infty. \)

The space of \( \mu \)-integrable \( \mathbb{K} \)-valued functions is denoted by \( L^1_\mathbb{K}(X, \mathcal{A}, \mu). \) (As before, if \( \mathbb{K} = \mathbb{C} \), the subscript will be omitted.) The space of \( \mu \)-integrable functions \( f : X \to [0, \infty) \) is denoted by \( L^1_+(X, \mathcal{A}, \mu). \)

**Comment.** As it turns out, \( \mu \)-integrability makes sense even for measurable functions \( f : (X, \mathcal{A}) \to [-\infty, \infty]. \) The space of such functions is denoted by \( L^1_\mathbb{R}(X, \mathcal{A}, \mu). \) It turns out, however, that if \( f \in L^1_\mathbb{R}(X, \mathcal{A}, \mu) \), then the set \( A = \{ x \in X : f(x) = \pm \infty \} \) has \( \mu(A) = 0. \)

The role of sets of measure zero will be clarified a little later.

**Fact 4.** Assume \((X, \mathcal{A}, \mu)\) is a measure space.

(i) The space \( L^1_\mathbb{K}(X, \mathcal{A}, \mu) \) is a \( \mathbb{K} \)-linear subspace in \( m_+(X, \mathcal{A}) \), which can also be described as: \( L^1_\mathbb{K}(X, \mathcal{A}, \mu) = \text{SPAN}_\mathbb{K} L^1_+(X, \mathcal{A}, \mu). \)

(ii) A measurable function \( f : (X, \mathcal{A}) \) belongs to \( L^1_\mathbb{K}(X, \mathcal{A}, \mu) \), if and only of both \( \text{Re} f \) and \( \text{Im} f \) belong to \( L^1_\mathbb{R}(X, \mathcal{A}, \mu). \)

(iii) There exists a unique \( \mathbb{K} \)-linear map \( I_\mu : L^1_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \), such that \( I_\mu(f) = I^+_\mu(f), \forall f \in L^1_+(X, \mathcal{A}, \mu). \)

(iv) For an elementary function \( f \in m^{\text{elem}}_+(X, \mathcal{A}) \), the following are equivalent:

- \( f \) is \( \mu \)-integrable;
- \( \nu(f^{-1}(\{\alpha\})) < \infty, \forall \alpha \in \text{Range } f; \)
- \( f \in \text{SPAN}_\mathbb{K} \{ \chi_A : A \in \mathcal{A}, \mu(A) < \infty \}. \)

The linear map \( I_\mu : L^1_\mathbb{K}(X, \mathcal{A}, \mu) \to \mathbb{K} \) is referred to as the \( \mu \)-integral. The traditional notation is to write \( \int_X f \, d\mu \) instead of \( I_\mu(f) \). The space of elementary integrable functions, described in Fact 4 (iii) as

\[
m^{\text{elem}}_\mathbb{K}(X, \mathcal{A}) \cap L^1_\mathbb{K}(X, \mathcal{A}, \mu) = \text{SPAN}_\mathbb{K} \{ \chi_A : A \in \mathcal{A}, \mu(A) < \infty \},
\]
will be denoted by \( \mathcal{L}^\text{elem}_{\mathbb{K}}(X, \mathcal{A}, \mu) \). Given \( A_1, \ldots, A_n \in \mathcal{A} \), with \( \mu(A_1), \ldots, \mu(A_n) < \infty \), and \( \alpha_1, \ldots, \alpha_n \in \mathbb{K} \), the elementary integrable function \( \alpha_1 \chi_{A_1} + \cdots + \alpha_n \chi_{A_n} \) has its \( \mu \)-integral computed as:

\[
\int_X (\alpha_1 \chi_{A_1} + \cdots + \alpha_n \chi_{A_n}) \, d\mu = \alpha_1 \mu(A_1) + \cdots + \alpha_n \mu(A_n). \tag{1}
\]

The formula (1) is referred to as the *Elementary Integral Formula*.

Besides linearity, the integration map also has the continuity property:

\[
\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu, \quad \forall f \in \mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu). \tag{2}
\]

The integral notation will also be used in the case of the positive integral, that is, if \( f \in \mathcal{M}_+(X, \mathcal{A}) \), the positive integral \( I^+_\mu(f) \) will still be denoted by \( \int_X f \, d\mu \), although this quantity may equal \( \infty \). Using this notation, the space of integrable functions can be presented as:

\[
\mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{ f \in \mathcal{M}_{\mathbb{K}}(X, \mathcal{A}) : \int_X |f| \, d\mu < \infty \}.
\]

Integrability is preserved under negligible perturbations. In order to clarify this statement, one employs the following terminology.

**Definitions.** Suppose \((X, \mathcal{A}, \mu)\) is a measure space, We say that a set \( N \in \mathcal{A} \) is \( \mu \)-negligible, if \( \mu(N) = 0 \).

Assume one considers some “measurable” property \((P)\), which refers to points in \( x \). Here the term “measurable” means that the set \( \{ x \in X : x \text{ has property } (P) \} \) (and of course its complement) belongs to \( \mathcal{A} \). We say that \((P)\) holds *almost \( \mu \)-everywhere*, if the set

\[
N = \{ x \in X : x \text{ does not have property } (P) \}
\]

is \( \mu \)-negligible. In practice, one considers some very specific properties, based on some relation \( r \) on one of the spaces \( T = \mathbb{R}, \mathbb{C}, \mathbb{R}, \infty \), \( \mathbb{C}, \mathbb{R}, \infty \), and two measurable functions \( f, g : (X, \mathcal{A}) \to T \), so we write

\[
f \mathrel{r} g, \quad \mu\text{-a.e.},
\]

to indicate that \( f(x) \mathrel{r} g(x) \), almost \( \mu \)-everywhere.

**Fact 5.** Let \((X, \mathcal{A}, \mu)\) be a measure space, let \( T \) be one of the spaces \( \mathbb{R}, \mathbb{C}, \mathbb{R}, \infty \), \( \mathbb{C}, \mathbb{R}, \infty \), and let \( f : (X, \mathcal{A}) \to T \) be a measurable function.

(i) If \( f = 0 \), \( \mu \)-a.e., then \( f \) is integrable, and \( \int_X f \, d\mu = 0 \).

(ii) Conversely, if \( f \) is integrable, and \( \int_X |f| \, d\mu = 0 \), then \( f = 0 \), \( \mu \)-a.e..

As a consequence of the above properties, it follows that the space

\[
\mathfrak{M}_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{ f \in \mathcal{M}_{\mathbb{K}}(X, \mathcal{A}) : f = 0, \ \mu\text{-a.e.} \}
\]

is a linear subspace in \( \mathcal{L}^1_{\mathbb{K}}(X, \mathcal{A}, \mu) \).

Integrable functions can be produced via *dominance* conditions, as indicated by the following simple observation.
Fact 6. Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(T\) be one of the spaces \(\mathbb{R}, \mathbb{C},\) or \([-\infty, \infty]\), and let \(f : (X, \mathcal{A}) \to T\) be a measurable function. Assume \(g : (X, \mathcal{A}) \to [0, \infty]\) is integrable, and \(|f| \leq g, \mu\text{-a.e.}\) Then \(f\) is integrable, and:

\[
|\int_X f \, d\mu| \leq \int_X g \, d\mu.
\]

The ultimate refinement of the above statement is:

Fact 7 (Lebesgue’s Dominated Convergence Theorem). Let \((X, \mathcal{A}, \mu)\) be a measure space, let \(T\) be one of the spaces \(\mathbb{R}, \mathbb{C},\) or \([-\infty, \infty]\), and let \(f, f_n : (X, \mathcal{A}) \to T, n \geq 1\), be measurable functions. Assume the following.

(i) There exists an integrable function \(g : (X, \mathcal{A}) \to [0, \infty]\), such that \(|f_n| \leq g, \mu\text{-a.e.}\), \(\forall n \geq 1\).

In particular, all \(f_n, n \geq 1\), are integrable.

(ii) The equality \(\lim_{n \to \infty} f_n(x) = f(x)\) holds almost \(\mu\)-everywhere, that is, there exists some \(\mu\)-negligible set \(B \in \mathcal{A}\), such that \(\lim_{n \to \infty} f_n(x) = f(x), \forall x \in X \setminus B\).

Then \(f\) is integrable, and \(\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu\).

We end our review by recalling some standard Measure Theory terminology that will be employed a little later.

Definitions. Given a measure space \((X, \mathcal{A}, \mu)\), and some \(A \in \mathcal{A}\),

- we say that \(A\) is \(\mu\)-finite, if \(\mu(A) < \infty\);
- we say that \(A\) is \(\mu\)-\(\sigma\)-finite, if there exists a sequence\(^2\) \((A_n)_{n=1}^{\infty} \subseteq \mathcal{A}\) of \(\mu\)-finite sets, such that \(A = \bigcup_{n=1}^{\infty} A_n, \mu(A) < \infty\).

The whole measure space \((X, \mathcal{A}, \mu)\) is declared finite (resp. \(\sigma\)-finite), if the total set \(X\) is \(\mu\)-finite (resp. \(\mu\)-\(\sigma\)-finite). In this case it follows that all \(A \in \mathcal{A}\) are \(\mu\)-finite (resp. \(\mu\)-\(\sigma\)-finite).

Exercises 1-2. Let \((X, \mathcal{A}, \mu)\) be a measure space. For a measurable function \(f : (X, \mathcal{A})\), let \(S_f\) denote the set \(\{x \in X : f(x) \neq 0\}\). (Since \(f\) is measurable, \(S_f\) belongs to \(\mathcal{A}\)).

1. Prove that if \(f \in L^1_{\mathbb{R}}(X, \mathcal{A}, \mu)\), then the set \(S_f\) is \(\mu\)-\(\sigma\)-finite.

2. Prove that, if \(f\) is bounded, and \(S_f\) is \(\mu\)-finite, then \(f \in L^1_{\mathbb{R}}(X, \mathcal{A}, \mu)\).

B. The \(L^p\)-spaces

Throughout this sub-section we fix a measure space \((X, \mathcal{A}, \mu)\).

Notations. For every \(p \in [1, \infty)\) we define the space

\[
L^p_{\mathbb{R}}(X, \mathcal{A}, \mu) = \{f \in M_{\mathbb{R}}(X, \mathcal{A}) : \int_X |f|^p \, d\mu < \infty\}.
\]

\(^2\) In fact the sequence can be chosen to consist of disjoint sets.
(The integral |∫_X |f|^p dμ stands for the positive integral \( I^+_\mu \).) We also define the map \( Q_p : \mathfrak{L}_K(X, A, \mu) \to [0, \infty) \) by

\[
Q_p(f) = \left[ \int_X |f|^p d\mu \right]^{1/p}, \quad \forall f \in \mathfrak{L}_K^p(X, A, \mu).
\]

It is pretty clear that, when \( p = 1 \), we are dealing with the same space as the one introduced in sub-section A, which is already known to be a \( K \)-vector space. Furthermore, in this case the map \( Q_1 \) is obviously a seminorm on \( \mathfrak{L}_K^1(X, A, \mu) \). For \( 1 < p < \infty \), the same will be true, as a result of the following result.

**Theorem 1.** Assume \( 1 < p, q < \infty \) are Hölder conjugate, i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \).

(i) \( \mathfrak{L}_K^p(X, A, \mu) \) is a \( K \)-linear subspace in \( \mathfrak{m}_K(X, A) \).

(ii) If \( f \in \mathfrak{L}_K^p(X, A, \mu) \) and \( g \in \mathfrak{L}_K^q(X, A, \mu) \), then \( fg \in \mathfrak{L}_K^1(X, A, \mu) \), and furthermore one has the inequality

\[
Q_1(fg) \leq Q_p(f) \cdot Q_q(g).
\]

(iii) If \( f \in \mathfrak{L}_K^p(X, A, \mu) \), then

\[
Q_p(f) = \max \{ |\int_X fg d\mu| : g \in \mathfrak{L}_K^q(X, A, \mu), \ Q_q(g) \leq 1 \}.
\]

(iv) \( Q_p : \mathfrak{L}_K^p(X, A, \mu) \to [0, \infty) \) is a seminorm on \( \mathfrak{L}_K^p(X, A, \mu) \).

**Proof.** (i). It is pretty obvious that, if \( f \in \mathfrak{L}_K^p(X, A, \mu) \), then \( \alpha f \in \mathfrak{L}_K^p(X, A, \mu), \forall \alpha \in K \).

Assume now \( f_1, f_2 \in \mathfrak{L}_K^p(X, A, \mu) \), and let us show that \( f_1 + f_2 \in \mathfrak{L}_K^p(X, A, \mu) \). This is, however, pretty obvious, since the inequality\(^3\) \( |f_1 + f_2|^p \leq 2^p|f_1|^p + 2^p|f_2|^p \) implies (by dominance) the integrability of \(|f_1 + f_2|^p|\).

(ii). Fix \( f \in \mathfrak{L}_K^p(X, A, \mu) \) and \( g \in \mathfrak{L}_K^q(X, A, \mu) \). Use Fact 1, to choose two non-decreasing sequences \( (F_n)_{n=1}^\infty \) and \( (G_n)_{n=1}^\infty \) in \( \mathfrak{m}_K^\text{elem}(X, A) \), such that lim \( n \to \infty \) \( F_n(x) = |f(x)|^p \) and lim \( n \to \infty \) \( G_n(x) = |g(x)|^q, \forall x \in X \). Of course, since the two sequences are non-decreasing, we have the inequalities \( 0 \leq F_1 \leq F_2 \leq \cdots \leq |f|^p \) and \( 0 \leq G_1 \leq G_2 \leq \cdots \leq |g|^q \), so all the \( F_n \)'s and the \( G_n \)'s a integrable. If we define \( f_n = G_n^{1/p} \) and \( g_n = G_n^{1/q} \), then these will also be elementary integrable, and will satisfy the inequalities \( 0 \leq f_1 \leq f_2 \leq \cdots \leq |f|, \) \( 0 \leq g_1 \leq g_2 \leq \cdots \leq |g|, \) as well as \( \lim_{n \to \infty} f_n(x) = |f(x)| \) and \( \lim_{n \to \infty} g_n(x) = |g(x)|, \) \( \forall x \in X \). Since we clearly have \( 0 \leq f_1 g_1 \leq f_2 g_2 \leq \cdots \leq |f|g|, \) as well as \( \lim_{n \to \infty} f_n(x)g_n(x) = |f(x)g(x)|, \) \( \forall x \in X, \) by Lebesgue’s Monotone Convergence Theorem it follows that the (positive) integral \( \int_X |fg| d\mu \) can be computed as

\[
\int_X |fg| d\mu = \lim_{n \to \infty} \int_X f_n g_n d\mu.
\]

Fix for the moment \( n \), and let us write\(^4\) \( f_n = \sum_{j=1}^k \alpha_j \chi_{A_j} \) and \( g_n = \sum_{j=1}^k \beta_j \chi_{A_j} \) with all \( A \)'s \( \mu \)-finite, and all \( \alpha \)'s and \( \beta \)'s non-negative, so that

\[
\int_X f_n g_n d\mu = \sum_{j=1}^k \alpha_j \beta_j \mu(A_j) = \sum_{j=1}^k x_j y_j,
\]

\(^3\) Use the inequality \( a + q \leq 2 \max\{a, b\} \), which implies \( (a + b)^p \leq 2^p(a^p + b^p) \), \( \forall a, b \geq 0 \).

\(^4\) One can write separately \( f_n = \sum_{i=1}^r \alpha_i \chi_{D_i} \) with \( D_1, \ldots, D_r \) disjoint \( \mu \)-finite, and \( g_n = \sum_{m=1}^s \beta_m \chi_{E_m} \) with \( E_1, \ldots, E_s \) disjoint \( \mu \)-finite. The \( A \)'s will the intersections \( D_i \cap E_m \), which will also be \( \mu \)-finite.
where \( x_j = \alpha_j[\mu(A_j)]^{1/p} \) and \( y_j = \beta_j[\mu(A_j)]^{1/q} \), \( j = 1, \ldots, k \). Using the (classical) Hölder Inequality, we have

\[
\sum_{j=1}^{k} x_j y_j \leq \left( \sum_{j=1}^{k} x_j^p \right)^{1/p} \cdot \left( \sum_{j=1}^{k} y_j^q \right)^{1/q}.
\]  

(7)

Notice now that

\[
\sum_{j=1}^{k} x_j^p = \sum_{j=1}^{k} \alpha_j^p[\mu(A_j)] = \int_X F_n \, d\mu \leq \int_X |f|^p \, d\mu = Q_p(f)^p,
\]

\[
\sum_{j=1}^{k} y_j^q = \sum_{j=1}^{k} \beta_j^q[\mu(A_j)] = \int_X G_n \, d\mu \leq \int_X |g|^q \, d\mu = Q_q(g)^q,
\]

so now if we combine (7) and (6), we obtain

\[
\int_X f_n g_n \, d\mu \leq Q_p(f) \cdot Q_q(g).
\]  

(8)

Since the inequality (8) holds for all \( n \), using (5) it follows that

\[
\int_X |fg| \, d\mu \leq Q_p(f) \cdot Q_q(g).
\]

Of course, this proves that \( fg \) is indeed integrable, and also satisfies (3).

(iii). Fix \( f \in L^p_K(X, \mathcal{A}, \mu) \). Since, by (i), for every \( g \in L^q_K(X, \mathcal{A}, \mu) \) we clearly have the inequality \( |\int_X fg \, d\mu| \leq \int_X |fg| \, d\mu \leq Q_p(f) \cdot Q_q(g) \), it follows that

\[
\sup \{ |\int_X fg \, d\mu| : g \in L^q_K(X, \mathcal{A}, \mu), \ Q_q(g) \leq 1 \} \leq Q_p(f),
\]

so in order to prove (4), all we need to do is to produce some \( g \in L^q_K(X, \mathcal{A}, \mu) \), with \( Q_q(g) \leq 1 \), such that

\[
|\int_X fg \, d\mu| = Q_p(f).
\]  

(9)

The case when \( Q_p(f) = 0 \) is trivial (take \( g = 0 \)), so we are going to assume that \( Q_p(f) > 0 \). Define the function \( g_0 : X \to \mathbb{K} \) by:

\[
g_0(x) = \begin{cases} 
\frac{|f(x)|^p}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases}
\]

Obviously, \( g_0 \) is measurable, and satisfies the identity

\[ |g_0|^q = |f|^{pq-q} = |f|^p, \]

so \( g_0 \) belongs to \( L^q_K(X, \mathcal{A}, \mu) \), and

\[
Q_q(g_0) = \left[ \int_X |f|^p \, d\mu \right]^{1/q} = Q_p(f)^{p/q}.
\]  

(10)
Moreover, since \( fg_0 = |f|^p \), we also have
\[
\left| \int_X fg_0 \, d\mu \right| = \int_X |f|^p \, d\mu = Q_p(f)^p, \tag{11}
\]
so if we define \( g = Q_q(g_0)^{-1}g_0 \), then \( g \) also belongs to \( \mathcal{L}_q^p(X, A, \mu) \), but now has \( Q_q(g) = 1 \), and by (10) and (11) it satisfies
\[
\left| \int_X fg \, d\mu \right| = Q_q(g_0)^{-1}Q_p(f)^p = Q_p(f)^{-\frac{p}{q}}Q_p(f)^p = Q_p(f).
\]

(iv). The identity
\[ Q_p(\alpha f) = |\alpha| \cdot Q_p(f), \quad \forall f \in \mathcal{L}_q^p(X, A, \mu), \quad \alpha \in \mathbb{K} \]
is trivial from the definition of \( Q_p \), so all that is left to prove is the triangle inequality
\[
Q_p(f_1 + f_2) \leq Q_p(f_1) + Q_p(f_2), \quad \forall f_1, f_2 \in \mathcal{L}_q^p(X, A, \mu).
\]
This inequality can be derived from (ii) and (iii) as follows. Fix some \( g \in \mathcal{L}_q^p(X, A, \mu) \), such that \( Q_q(g) \leq 1 \) and \( Q_p(f_1 + f_2) = \int_X (f_1 + f_2)g \, d\mu \), and use linearity and continuity of the integral to obtain
\[
Q_p(f_1 + f_2) \leq \int_X |(f_1 + f_2)g| \, d\mu \leq \left[ \int_X |f_1g| \, d\mu \right] + \left[ \int_X |f_2g| \, d\mu \right] = Q_1(f_1g) + Q_1(f_2g) \leq Q_p(f_1) \cdot Q_q(g) + Q_p(f_2) \cdot Q_q(g) \leq Q_p(f_1) + Q_p(f_2). \]

**Definitions.** Fix \( p \in [1, \infty) \). By Fact 6, for \( f \in \mathcal{L}_q^p(X, A, \mu) \), one has the equivalence \( \int_X |f|^p \, d\mu = 0 \Longleftrightarrow f = 0 \), \( \mu \)-a.e., so one has the the equality
\[
\{ f \in \mathcal{L}_q^p(X, A, \mu) : Q_p(f) = 0 \} = \mathcal{N}_q(X, A, \mu). \tag{12}
\]
Define the quotient space
\[
L^p_q(X, A, \mu) = \mathcal{L}_q^p(X, A, \mu) / \mathcal{N}_q(X, A, \mu).
\]
In other words, \( L^p_q(X, A, \mu) \) is the collection of equivalence classes associated with the relation “\( = \), \( \mu \)-a.e.” For a function \( f \in \mathcal{L}_q^p(X, A, \mu) \) we denote by \([f]\) its equivalence class in \( L^p_q(X, A, \mu) \). So the equality \([f] = [g]\) is equivalent to \( f = g \), \( \mu \)-a.e. By (12) the seminorm \( Q_p \) induces a norm on \( L^p_q(X, A, \mu) \), which we denote by \( \| \cdot \|_p \), so by definition we have \( \|[f]\|_p = Q_p(f) \).

**Conventions.** We are going to abuse a bit the notation, by writing “\( f \in L^p_q(X, A, \mu) \)” if \( f \) belongs to \( \mathcal{L}_q^p(X, A, \mu) \). (We will always have in mind the fact that this notation signifies that \( f \) is almost uniquely determined.) Likewise, instead of writing \( \|[f]\|_p \), we are going to write \( \|f\|_p \), so that from now on \( Q_p(f) \) will be replaced by \( \|f\|_p \). With this notations, we now have \( f \in L^p_q(X, A, \mu) \Longleftrightarrow |f|^p \in L^1_q(X, A, \mu) \), in which case \( \|f\|_p = \| |f|^p \|_1 \).

The integration map \( \int_X : \mathcal{L}_q^1(X, A, \mu) \rightarrow \mathbb{K} \) gives rise to a correctly defined map, which we denote by the same symbol, so that by the continuity condition we can write:
\[
\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu = \|f\|_1, \quad \forall f \in L^1_q(X, A, \mu). \tag{13}
\]
With these notations, statements (ii) and (iii) in Theorem 1 can be restated as follows.
• If \( f \in L^p_K(X, \mathcal{A}, \mu) \) and \( g \in L^q_K(X, \mathcal{A}, \mu) \), then \( fg \in L^1_K(X, \mathcal{A}, \mu) \) and
\[
\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q. \tag{14}
\]

In particular, using (13), one also has
\[
\left| \int_X fg \, d\mu \right| \leq \|f\|_p \cdot \|g\|_q. \tag{15}
\]

• If \( f \in L^p_K(X, \mathcal{A}, \mu) \), then
\[
\|f\|_p = \max \left\{ \left| \int_X fg \, d\mu \right| : g \in L^q_K(X, \mathcal{A}, \mu), \|g\|_q \leq 1 \right\}. \tag{16}
\]

The inequality (15) is referred to as the Integral Hölder Inequality.

**Remarks 1-2.** Let \( p, q \in (1, \infty) \) be two Hölder conjugate numbers.

1. One has a bilinear map
\[
\Phi : L^p_K(X, \mathcal{A}, \mu) \times L^q_K(X, \mathcal{A}, \mu) \ni (f, g) \mapsto \int_X fg \, d\mu \in K,
\]
which satisfies the inequality
\[
|\Phi(f, g)| \leq \|f\|_p \cdot \|g\|_q. \tag{17}
\]
Using (16) it follows that \( \Phi \) establishes a dual pairing (see DT I), which is referred to as the Hölder dual pairing.

2. Using the Hölder dual pairing, one defines, for every \( f \in L^p_K(X, \mathcal{A}, \mu) \) the linear functional \( f^\# : L^q_K(X, \mathcal{A}, \mu) \ni g \mapsto \int_X fg \, d\mu \in K \). By (17), the functional \( f^\# \) is norm-continuous, and by (16) it follows that \( \|f^\#\| = \|f\|_p \). Therefore, the Hölder dual pairing produces an isometric linear map
\[
# : L^p_K(X, \mathcal{A}, \mu) \ni f \mapsto f^\# \in L^q_K(X, \mathcal{A}, \mu)^* \tag{18}.
\]

**Comment.** It will be shown in HS II that the map (18) is in fact a linear isomorphism. In particular, from this identification Theorem 2 below would follow. One can, however, obtain the same result directly, as shown in the direct proof presented here.

**Theorem 2.** All spaces \( L^p_K(X, \mathcal{A}, \mu) \), \( p \in [1, \infty) \), are Banach spaces.

**Proof.** Fix \( p \in [1, \infty) \). In order to prove that \( L^p_K(X, \mathcal{A}, \mu) \) we are going to verify the Summability Test, so we start with some sequence \( (f_n)_{n=1}^{\infty} \) in \( L^p_K(X, \mathcal{A}, \mu) \), such that the quantity \( S = \sum_{n=1}^{\infty} \|f_n\|_p \) is finite, and we show that the series \( \sum_{n=1}^{\infty} f_n \) is convergent in \( L^p_K(X, \mathcal{A}, \mu) \).

Define the function \( h : X \to [0, \infty) \) by \( h(x) = \sum_{n=1}^{\infty} |f_n(x)| \). Of course, we can write \( h(x) = \lim_{n \to \infty} h_n(x) \), where \( h_n = |f_1| + \cdots + |f_n| \). (Of course all \( h \)'s are measurable.)

On the one hand, using the triangle inequality in \( L^p_K(X, \mathcal{A}, \mu) \) we have
\[
\int_X h_n^p = \|f_1\|_p + \cdots + \|f_n\|_p \leq [\|f_1\|_p + \cdots + \|f_n\|_p]^p \leq S^p, \ \forall n. \tag{19}
\]
On the other hand, since we clearly have \( 0 \leq h_1(x)^p \leq h_2(x)^p \leq \cdots \leq h(x)^p \), as well as \( \lim_{n \to \infty} h_n(x)^p = h(x)^p \), by Lebesgue’s Monotone Convergence Theorem it follows that
\[
\int_X h^p \, d\mu = \lim_{n \to \infty} \int_X h_n^p \, d\mu,
\]
so going back to (19) it follows that \( f \) is integrable, it follows that set \( A = \{ x \in X : h(x) = \infty \} \in A \) has \( \mu(A) = 0 \). (The obvious inequality \( tx \leq h^p \) will force \( \mu(A) \leq S^p \), \( \forall t > 0 \), and this can only happen when \( \mu(A) = 0 \).) For \( x \in X \setminus A \), the series \( \sum_{n=1}^{\infty} |f_n(x)| \) is convergent (to \( h(x) < \infty \)), so the series \( \sum_{n=1}^{\infty} f_n(x) \) is also convergent (in \( \mathbb{K} \)). Define then the measurable function \( g : X \to \mathbb{K} \) by
\[
g(x) = \begin{cases} 
\sum_{n=1}^{\infty} f_n(x) & \text{if } x \in X \setminus A \\
0 & \text{if } x \in A
\end{cases}
\]
so that if we define the partial sums \( g_n = f_1 + \cdots + f_n \), we have \( \lim_{n \to \infty} g_n(x) = g(x), \mu\text{-a.e.} \)

We now wish to prove that \( g \in L^p \) and that \( \lim_{n \to \infty} \|g - g_n\|_p = 0 \). To prove that \( g \) is in \( L^p \) we notice that, since we obviously have \( |g_n| \leq h_n \leq h \), we also get \( |g_n|^p \leq h^p, \mu\text{-a.e.} \) and since \( \lim_{n \to \infty} |g_n(x)|^p = |g(x)|^p, \mu\text{-a.e.} \), we can apply Lebesgue’s Dominated Convergence Theorem, to conclude that \( |g|^p \) is integrable. Likewise, for any \( n, k \geq 1 \), we have \( |g_{n+k} - g_n| \leq h_{n+k} - h_n \leq h - h_n \leq h \), taking \( \lim_{n \to \infty} \) and the \( p \)th power yields
\[
|g - g_n|^p \leq h^p, \mu\text{-a.e.}
\]
Since \( \lim_{n \to \infty} |g(x) - g_n(x)|^p = 0, \mu\text{-a.e.} \), again by Lebesgue’s Dominated Convergence Theorem we conclude that \( \lim_{n \to \infty} \int_X |g - g_n|^p \, d\mu = 0 \), which means precisely that \( \|g - g_n\|_p \to 0 \), thus proving the desired conclusion.

**Example 1.** Start with some non-empty set \( X \), let \( \mathcal{A} = \mathcal{P}(X) \), and let \( \mu \) be the *counting measure*, defined as
\[
\mu(A) = \begin{cases}
\text{Card } A & \text{if } A \text{ is finite} \\
\infty & \text{if } A \text{ is infinite}
\end{cases}
\]
Then the Banach space \( L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \) is precisely the space \( L^p_{\mathbb{K}}(X) \) discussed in BS I.

**Example 2.** Start with some non-empty set \( X \), let \( \mathcal{A} = \mathcal{P}(X) \), and but let \( \mu \) be the *degenerate counting measure*, defined as
\[
\mu(A) = \begin{cases}
0 & \text{if } A = \emptyset \\
\infty & \text{if } A \neq \emptyset
\end{cases}
\]
Then \( L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{0\} \). As this example suggests, unless some local finiteness condition is assumed, integration theory can be extremely trivial.

**Notation.** Let \((X, \mathcal{A}, \mu)\) be a measure space. It is not hard to see that \( \mathcal{L}^\text{elem}_{\mathbb{K}}(X, \mathcal{A}, \mu) \) – the space of elementary integrable functions – is contained (as a linear subspace) in \( \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \), for every \( p \in [1, \infty) \). Therefore one has a correctly defined linear map \( T_p : \mathcal{L}^\text{elem}_{\mathbb{K}}(X, \mathcal{A}, \mu) \to \mathcal{L}^p_{\mathbb{K}}(X, \mathcal{A}, \mu) \). As it turns out, \( \ker T_p \) is independent of \( p \), since it can be described as \( \mathcal{N}_{\mathbb{K}}(X, \mathcal{A}, \mu) \cap \mathcal{L}^\text{elem}_{\mathbb{K}}(X, \mathcal{A}, \mu) \). This means that the spaces \( \text{Range} T_p \) are
all isomorphic, and for this reason all of them will be denoted by \( L^\text{elem}_K(X,A,\mu) \), and we will understand that this space is contained in all \( L^p \)-spaces. Using the (abusive) notation convention introduced earlier, we can think

\[
L^\text{elem}_K(X,A,\mu) = \text{span}_K\{\chi_A : A \in A, \mu(A) < \infty\}, \tag{20}
\]

with the understanding that

- two functions \( f \) and \( g \) in the linear span on the right-hand of (20) are the same, if \( f = g, \mu\text{-a.e.} \)
- the linear span on the right-hand side of (20) can be computed in \( L^p_K(X,A,\mu) \), for any \( p \in [1,\infty) \).

**Exercise 3.** With these conventions, prove that \( L^\text{elem}_K(X,A,\mu) \) is dense in \( L^p_K(X,A,\mu) \), in the norm topology, for every \( p \in [1,\infty) \).

**Exercise 4.** Prove that, if \( p, q, r \in [1,\infty) \) are such that \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \), and if \( f \in L^p_K(X,A,\mu) \) and \( g \in L^q_K(X,A,\mu) \), then \( fg \in L^r_K(X,A,\mu) \), and furthermore, one has the inequality \( \|fg\|_r \leq \|f\|_p \cdot \|g\|_q \).

**Exercise 5.** Assume \((X,A,\mu)\) is a finite measure space. Prove that, if \( p, q \in [1,\infty) \) are such that \( p \geq q \), then \( L^p_K(X,A,\mu) \subset L^q_K(X,A,\mu) \). In particular, this gives rise to a linear injective map \( J : L^p_K(X,A,\mu) \hookrightarrow L^q_K(X,A,\mu) \). Prove that \( J \) is continuous.

C. The spaces \( m^b, L^\infty \) and \( L^{\text{loc},\infty} \)

The Hölder dual pairing between \( L^p \) and \( L^q \) can be extended to include the case \( p = 1 \). There are several “candidate” spaces that can be paired with \( L^1 \). What we seek is a Banach space \( \mathcal{X} \) of measurable functions on \((X,A)\), such that

(i) for any \( f \in L^1_K(X,A,\mu) \) and any \( g \in \mathcal{X} \), the function \( fg \) is in \( L^1_K(X,A,\mu) \), and furthermore one has the inequality \( \|fg\|_1 \leq \|f\|_1 \cdot \|g\|_1 \);

(ii) the correspondence

\[
\Phi : L^1_K(X,A,\mu) \times \mathcal{X} \ni (f,g) \longmapsto \int_X fg \, d\mu \in K
\]

establishes a dual pairing. (By (i) the map \( \Phi \) is automatically bilinear.)

In this sub-section we will investigate three natural choices for \( \mathcal{X} \), which satisfy (i). Concerning condition (ii), in the order we introduce these spaces, they will behave better and better.

**Definitions.** The most obvious one is the space \( m^b_K(X,A) \) of all bounded measurable functions \( g : (X,A) \to \mathbb{K} \), which is a Banach space, when equipped with the norm \( \| \cdot \|_{\text{sup}} \), defined by

\[
\|g\|_{\text{sup}} = \sup_{x \in X} |g(x)|, \quad g \in m^b_K(X,A).
\]
(Of course, \( m^b_b(X, \mathcal{A}) \) is contained in the Banach space \( \ell^\infty_b(X) \) of all bounded functions \( g : X \to \mathbb{K} \), and the norm \( \| . \|_{\sup} \) is precisely the induced norm. Since the uniform limit of a sequence of bounded measurable functions is again bounded and measurable, it follows that \( \ell^\infty_b(X) \) is a Banach space.)

Of course, if \( f \in L^1_b(X, \mathcal{A}, \mu) \) and \( g \in m^b_b(X, \mathcal{A}) \), then there exists a constant, for instance \( C = \| g \|_{\sup} \), such that \( |fg| \leq C|f| \). This will force \( fg \in L^1_b(X, \mathcal{A}) \), as well as the inequality \( \| fg \|_1 \leq C \| f \|_1 \), so one indeed has a bilinear map

\[
\Phi : L^1_b(X, \mathcal{A}, \mu) \times m^b_b(X, \mathcal{A}) \ni (f, g) \mapsto \int_X fg \, d\mu \in \mathbb{K},
\]

which satisfies the inequality

\[
|\Phi(f, g)| \leq \| f \|_1 \cdot \| g \|_{\sup}, \quad \forall f \in L^1_b(X, \mathcal{A}, \mu), \ g \in m^b_b(X, \mathcal{A}).
\]

**Exercise 6**. Prove that, for any \( f \in L^1_b(X, \mathcal{A}, \mu) \), one has the equality

\[
\| f \|_1 = \max \{ |\Phi(f, g)| : g \in m^b_b(X, \mathcal{A}), \| g \|_{\sup} \leq 1 \}.
\]

**Comment.** In general, the map (21) does not give rise to a dual pairing. Since this map is bilinear, it still gives rise to two linear continuous maps

\[
# : L^1_b(X, \mathcal{A}, \mu) \ni f \mapsto f^\# \in m^b_b(X, \mathcal{A})^*,
\]

\[
# : m^b_b(X, \mathcal{A}) \ni g \mapsto g^\# \in L^1_b(X, \mathcal{A}, \mu)^*
\]

given by \( f^\#(g) = g^\#(f) = \int_X fg \, d\mu \).

By (22) both maps (23) and (24) are contractive. By Exercise 6, the map (23) is in fact isometric. The only drawback of the choice of \( m^b_b \) as our “candidate” is the fact that the map (24) might fail to be injective. (Ultimately, with an adequate space \( X \) in place of \( \ell^\infty_b(X, \mathcal{A}) \) we are going to make both (23) and (24) isometric.) The non-injectivity of (24) can be explained by the possible existence of non-zero functions \( g \in m^b_b(X, \mathcal{A}) \), with the property that \( g = 0, \mu\text{-a.e.} \). Of course, for such a \( g \) we also have \( fg = 0, \mu\text{-a.e.} \), \( \forall f \in L^1(X, \mathcal{A}), \) thus \( g^\# = 0 \). Going back to argument that justified the inequality (22), we clearly see that if we take \( C \) to be any constant, such that \( |g| \leq C, \mu\text{-a.e.} \), we will have \( \| fg \| \leq C |f|, \mu\text{-a.e.} \), so the inequality (22) will hold with \( \| g \|_{\sup} \) replaced by \( C \). This justifies the following terminology.

**Definition.** Let \( (X, \mathcal{A}, \mu) \) be a measure space. A function \( g \in m_b(X, \mathcal{A}) \) is said to be essentially \( \mu \)-bounded, if there exists some \( C \geq 0 \), such that \( |g| \leq C, \mu\text{-a.e.} \). For such a function, one can show that the set \( \{ C \geq 0 : |g| \leq C, \mu\text{-a.e.} \} \) has in fact a minimum point, which we denote by \( Q_\infty(g) \). One defines the space

\[
\mathcal{L}^\infty_b(X, \mathcal{A}, \mu) = \{ g \in m_b(X, \mathcal{A}) : g \text{ essentially } \mu\text{-bounded} \},
\]

which is clearly a vector space. The map \( Q_\infty : \mathcal{L}^\infty_b(X, \mathcal{A}, \mu) \to [0, \infty) \) is clearly a seminorm. As was the case with the \( \mathcal{L}^p \)-spaces for \( p \in [1, \infty) \), one has the equality

\[
\{ g \in \mathcal{L}^\infty_b(X, \mathcal{A}, \mu) : Q_\infty(g) = 0 \} = \mathfrak{N}_X(X, \mathcal{A}, \mu).
\]
Consider the space \( L_\infty^b(X, \mathcal{A}, \mu) \) defined as:

\[
L_\infty^b(X, \mathcal{A}, \mu) = L_\infty^b(X, \mathcal{A}, \mu) / \mathfrak{N}_\infty(X, \mathcal{A}, \mu),
\]

where the seminorm \( Q_\infty \) induces a norm, which we denote by \( \| \cdot \|_\infty \). As was the case with the \( L^p \)-spaces introduced in sub-section B, we are going to use the same notation conventions, so when we write “\( g \in L_\infty^b(X, \mathcal{A}, \mu) \)” we mean that \( g \) belongs to \( L_\infty^b(X, \mathcal{A}, \mu) \).

With these notations, we have a better “candidate” for \( X \), specifically we can consider the bilinear map

\[
\Phi : L_\infty^b(X, \mathcal{A}, \mu) \times L_\infty^b(X, \mathcal{A}) \ni (f, g) \mapsto \int_X fg\, d\mu \in \mathbb{K},
\]

which satisfies the inequality

\[
|\Phi(f, g)| \leq \|f\|_1 \cdot \|g\|_\infty, \quad \forall f \in L_\infty^b(X, \mathcal{A}, \mu), \ g \in L_\infty^b(X, \mathcal{A}, \mu).
\]

**Remark 3.** For every \( g \in L_\infty^b(X, \mathcal{A}, \mu) \), there exists \( h \in M_\infty^b(X, \mathcal{A}) \), such that \( h = g \), \( \mu \)-a.e. and \( \|h\|_{\text{sup}} = Q_\infty(g) \). Indeed, by the definition of \( Q_\infty \), we know that the set

\[
A = \{ x \in X : |g(x)| > Q_\infty(g) \} \in \mathcal{A}
\]

is \( \mu \)-negligible, so if we define the function

\[
h(x) = \begin{cases} 
  g(x) & \text{if } x \in X \setminus A \\
  0 & \text{if } x \in A
\end{cases}
\]

then clearly \( h \in M_\infty^b(X, \mathcal{A}) \), and satisfies \( h = g \), \( \mu \)-a.e., as well as \( \|h\|_{\text{sup}} \leq Q_\infty(g) \). Of course, we also have the inequality \( |g| \leq \|h\|_{\text{sup}} \), \( \mu \)-a.e., so in fact we have the equality \( \|h\|_\infty = Q_\infty(g) \).

From this observation, it follows that the inclusion map \( M_\infty^b(X, \mathcal{A}) \hookrightarrow L_\infty^b(X, \mathcal{A}, \mu) \) gives rise to a surjective linear contraction

\[
\Pi_\infty : M_\infty^b(X, \mathcal{A}) \to L_\infty^b(X, \mathcal{A}, \mu),
\]

which satisfies the condition

\[
\|g\|_\infty = \min \{ \|h\|_{\text{sup}} : h \in M_\infty^b(X, \mathcal{A}), \ \Pi h = g \}, \quad \forall g \in L_\infty^b(X, \mathcal{A}, \mu).
\]

**Proposition 1.** If we consider the space

\[
\mathfrak{N}_\infty^b(X, \mathcal{A}, \mu) = M_\infty^b(X, \mathcal{A}) \cap \mathfrak{N}_\infty(X, \mathcal{A}, \mu)
\]

and the map \( \Pi_\infty \) defined above, then:

(i) \( \text{Ker} \Pi_\infty = \mathfrak{N}_\infty^b(X, \mathcal{A}, \mu) \), hence \( \mathfrak{N}_\infty^b(X, \mathcal{A}, \mu) \) is norm-closed in \( (M_\infty^b(X, \mathcal{A}), \| \cdot \|_{\text{sup}}) \);

(ii) when we equip the quotient space \( M_\infty^b(X, \mathcal{A}) / \mathfrak{N}_\infty^b(X, \mathcal{A}, \mu) \) with the quotient norm, the induced map

\[
\hat{\Pi}_\infty : M_\infty^b(X, \mathcal{A}) / \mathfrak{N}_\infty^b(X, \mathcal{A}, \mu) \to L_\infty^b(X, \mathcal{A}, \mu)
\]

is an isometric linear isomorphism.
In particular, $L^\infty_X(X, \mathcal{A}, \mu)$ is a Banach space.

Proof. Statement (i) is immediate. The fact that $\hat{\Pi}_\infty$ is isometric follows from (27). \hfill \Box

Comments. As Remark 3 and Proposition 1 show, in the definition of $L^\infty_X(X, \mathcal{A}, \mu)$, instead of starting with the space of essentially bounded functions in $m^b_X(X, \mathcal{A})$, we could have started already with the space $m^b_X(X, \mathcal{A})$ equipped with the seminorm $Q_\infty$, so that

$$\{ f \in m^b_X(X, \mathcal{A}) : Q_\infty(f) = 0 \} = \mathfrak{m}^b_X(X, \mathcal{A}, \mu),$$

and then we can define $L^\infty_X(X, \mathcal{A}, \mu)$ as the quotient $m^b_X(X, \mathcal{A})/\mathfrak{m}^b_X(X, \mathcal{A}, \mu)$.

By (26), the bilinear map (25) induces, of course, two linear contractions

$$\begin{align*}
\#: & \ L^1_X(X, \mathcal{A}, \mu) \ni f \longmapsto f^\# \in L^\infty_X(X, \mathcal{A})^*, \\
\#: & \ L^\infty_X(X, \mathcal{A}) \ni g \longmapsto g^\# \in L^1_X(X, \mathcal{A}, \mu)^*
\end{align*}$$

given by $f^\#(g) = g^\#(f) = \int_X fg \, d\mu$. As before (see Exercise 7 below), the map (28) is still isometric. The map (29), however, may still fail to be injective (see Exercise 8 below), so we will need one more “adjustment.”

Exercise 7. Prove that the map (28) is isometric.

Exercise 8. Consider the measure space $(X, \mathcal{A}, \mu)$, where $X = \{1, 2\}$, $\mathcal{A} = \mathcal{P}(X)$, and $\mu(\emptyset) = 0$, $\mu(\{1\}) = 1$, $\mu(\{2\}) = \mu(\{1, 2\}) = \infty$. It is pretty clear that $m^b_X(X, \mathcal{A})$ is the 2-dimensional space of all functions $h : \{1, 2\} \to \mathbb{K}$, and every such function is bounded. Let $f = \chi_{\{1\}}$ and $g = \chi_{\{2\}}$, and prove that:

(i) $\mathfrak{m}^b_X(X, \mathcal{A}, \mu) = \{0\}$, so $L^p_X(X, \mathcal{A}, \mu)$ is identified with $L^p_X(X, \mathcal{A}, \mu)$, for every $p \in [1, \infty]$;

(ii) $L^1_X(X, \mathcal{A}, \mu) = \mathbb{K}f$;

(iii) $L^\infty_X(X, \mathcal{A}, \mu) = m^b_X(X, \mathcal{A})$;

(iv) $\|f\|_1 = \|g\|_\infty = 1$, but $\int_X hg \, d\mu = 0$, $\forall h \in L^1_X(X, \mathcal{A}, \mu)$.

Conclude the, for this measure space, the map (29) fails to be injective, simply because $g^\# = 0$.

As Exercise 8 suggests, the Banach space $L^\infty_X(X, \mathcal{A}, \mu)$ is still an inadequate “candidate” to be paired with $L^1_X(X, \mathcal{A}, \mu)$. In preparation for our final adjustment we introduce the following terminology.

Definitions. Given a measure space $(X, \mathcal{A}, \mu)$, we say that a set $A \in \mathcal{A}$ is locally $\mu$-negligeable, if $\mu(A \cap F) = 0$, for all $\mu$-finite sets $F \in \mathcal{A}$.

For a “measurable” property (P), we say that (P) holds locally almost $\mu$-everywhere, if the set

$$N = \{ x \in X : x \text{ does not have property (P)} \}$$

is locally $\mu$-negligeable. For this purpose we use the notation “$\mu$-l.a.e.”

For a measurable function $g : (X, \mathcal{A}) \to \mathbb{K}$, we say that $g$ is locally essentially $\mu$-bounded, if there exists some $C \geq 0$, such that $|g| \leq C$, $\mu$-l.a.e.. Exactly as before, for such a function,
one can show that the set \( \{ C \geq 0 : |g| \leq C, \ \mu\text{-l.a.e.} \} \) has in fact a minimum point, which we denote by \( Q_{\text{loc}, \infty}(g) \).

Following the same process as the one for the construction of \( L^\infty \) we consider the vector spaces

\[
\mathcal{L}_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) = \{ g \in \mathcal{M}_{k}(X, \mathcal{A}) : g \text{ locally essentially } \mu\text{-bounded} \},
\]

\[
\mathcal{N}_{k}^{\text{loc}}(X, \mathcal{A}, \mu) = \{ g \in \mathcal{M}_{k}(X, \mathcal{A}) : g = 0, \ \mu\text{-l.a.e.} \},
\]

so that \( \mathcal{M}_{k}^{\text{loc}}(X, \mathcal{A}, \mu) = \{ g \in \mathcal{L}_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) : Q_{\text{loc}, \infty}(g) = 0 \} \), and we define the quotient space

\[
L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) = \mathcal{L}_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu)/\mathcal{N}_{k}^{\text{loc}}(X, \mathcal{A}, \mu),
\]

on which \( Q_{\text{loc}, \infty} \) induces a norm, which we denote by \( \| \cdot \|_{\text{loc}, \infty} \).

**Remark 4.** Exactly as in Remark 3, one can show that, for every \( g \in \mathcal{L}_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \), there exists \( h \in \mathcal{M}_{k}(X, \mathcal{A}) \), such that \( h = g, \ \mu\text{-l.a.e.} \) and \( \| h \|_{\text{sup}} = Q_{\text{loc}, \infty}(g) \). In particular, this gives rise to a surjective linear contraction

\[
\Pi_{\text{loc}, \infty} : \mathcal{M}_{k}^{b}(X, \mathcal{A}) \to L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu),
\]

whose kernel is the space

\[
\mathcal{N}_{k}^{\text{loc}, b}(X, \mathcal{A}, \mu) = \{ g \in \mathcal{M}_{k}^{b}(X, \mathcal{A}) : Q_{\text{loc}, \infty}(g) = 0 \} = \{ g \in \mathcal{M}_{k}^{b}(X, \mathcal{A}) : g = 0, \ \mu\text{-l.a.e} \},
\]

which is therefore closed in \( \mathcal{M}_{k}^{b}(X, \mathcal{A}) \).

Exactly as in Proposition 1, this yields an isometric linear isomorphism

\[
\hat{\Pi}_{\text{loc}, \infty} : \mathcal{M}_{k}^{b}(X, \mathcal{A})/\mathcal{N}_{k}^{\text{loc}, b}(X, \mathcal{A}, \mu) \to L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu), \tag{30}
\]

so in particular \( L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \) is a Banach space.

**Remark 5.** It is trivial that we have the inclusions

\[
\mathcal{N}_{k}(X, \mathcal{A}, \mu) \subset \mathcal{L}_{k}^{\infty}(X, \mathcal{A}, \mu)
\]

\[
\bigcap \mathcal{N}_{k}^{\text{loc}}(X, \mathcal{A}, \mu) \subset \mathcal{L}_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu)
\]

so taking quotients we obtain a linear map

\[
\Pi_{\text{loc}} : L_{k}^{\infty}(X, \mathcal{A}, \mu) \to L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu). \tag{32}
\]

Since both \( L_{k}^{\infty}(X, \mathcal{A}, \mu) \) and \( L_{k}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \) are is a quotients of \( \mathcal{M}_{k}^{b}(X, \mathcal{A}) \) – by Remarks 3 and 4 – it follows that the map (32) is surjective, and satisfies the equality \( \Pi_{\text{loc}} \circ \Pi_{\infty} = \Pi_{\text{loc}, \infty} \).

Going back to the inclusions (31), we remark also that, if \( f, g \in \mathcal{L}_{k}^{\infty}(X, \mathcal{A}, \mu) \) are such that \( f = g, \ \mu\text{-a.e.} \), then one clearly has the equality \( Q_{\text{loc}, \infty}(f) = Q_{\text{loc}, \infty}(g) \). This means that, passing to quotients one has a correctly defined seminorm \( Q_{\text{loc}} \) on \( L_{k}^{\infty}(X, \mathcal{A}, \mu) \). In fact, using the map (32), this seminorm is simply given by

\[
Q_{\text{loc}}(g) = \| \Pi_{\text{loc}}(g) \|_{\text{loc}, \infty}, \ g \in L_{k}^{\infty}(X, \mathcal{A}, \mu),
\]
so in particular \( \ker \Pi_{\text{loc}} = \{ g \in L^\infty_K(X,A,\mu) : Q_{\text{loc}}(g) = 0 \} \). In other words, if we denote this kernel by \( N_{K,\text{loc}}^{\infty}(X,A,\mu) \), the Banach space \( L_{K,\text{loc}}^{\infty}(X,A,\mu) \) is isometrically isomorphic to the quotient space \( L_{K,\text{loc}}^{\infty}(X,A,\mu)/N_{K,\text{loc}}^{\infty}(X,A,\mu) \), thus giving another proof of the fact that \( L_{K,\text{loc}}^{\infty}(X,A,\mu) \) is a Banach space.

**Notation.** For a measurable function \( f : (X,\mathcal{X}) \to \mathbb{K} \), the notation “\( f \in L_{K,\text{loc}}^{\infty}(X,A,\mu) \)” will be avoided. If \( f \) belongs to \( L_{K,\text{loc}}^{\infty}(X,A,\mu) \), its image \( \Pi_{\text{loc}}(f) \) in \( L_{K,\text{loc}}^{\infty}(X,A,\mu) \) will be denoted simply by \([f]_{\text{loc}}\). We use the same notation for the image \( \Pi_{\text{loc}}(f) \) of an element \( f \in L_{K}^{\infty}(X,A,\mu) \).

**Exercise 9** Let \((X,A,\mu)\) be a measure space. Prove that the space \( m_{K,\text{elem}}^{\text{elem}}(X,A) \) is a norm- dense linear subspace in \((m_{K}^{b}(X,A),\| \cdot \|_{\text{sup}})\). Conclude that

(i) the space \( E_{K,\text{elem}}^{\infty}(X,A,\mu) = \Pi_{\infty}(m_{K,\text{elem}}^{\text{elem}}(X,A)) \) is norm-dense in \( L_{K}^{\infty}(X,A,\mu) \);

(ii) the space \( E_{K,\text{elem}}^{\text{loc},\infty}(X,A,\mu) = \Pi_{\text{loc},\infty}(m_{K,\text{elem}}^{\text{elem}}(X,A)) \) is norm-dense in \( L_{K,\text{loc}}^{\infty}(X,A,\mu) \).

Using our notation conventions, these two sub-spaces are

\[
E_{K}(X,A,\mu) = \text{SPAN}_{K}\{\chi_{A} : A \in A\} \text{ (in } L^{\infty}), \\
E_{K,\text{loc}}^{\infty}(X,A,\mu) = \text{SPAN}_{K}\{[\chi_{A}]_{\text{loc}} : A \in A\} \text{ (in } L^{\text{loc},\infty}).
\]

**Proposition 2.** If \( g \in L_{K,\text{loc}}^{\infty}(X,A,\mu) \), if \( p \in [1, \infty) \), and \( f \in L_{K}^{p}(X,A,\mu) \), then

\[
|fg| \leq Q_{\text{loc},\infty}(g)|f|, \quad \mu\text{-a.e.}, \quad (33)
\]

so in particular \( fg \) also belongs to \( L_{K}^{p}(X,A,\mu) \), and satisfies the inequality

\[
Q_{p}(fg) \leq Q_{\text{loc},\infty}(g) \cdot Q_{p}(f). \quad (34)
\]

**Proof.** Fix \( p \in [1, \infty) \), as well as \( f \in L_{K}^{p}(X,A,\mu) \) and \( g \in L_{K,\text{loc}}^{\infty}(X,A,\mu) \). To prove (33), we must show that the set \( N = \{ x \in X : |f(x)g(x)| > Q_{\text{loc},\infty}(g)|f(x)| \} \in A \) is \( \mu \)-negligeable. It is pretty clear that one can write \( N = S_{f} \cap A \), where \( S_{f} = \{ x \in X : f(x) \neq 0 \} \in A \), and \( A = \{ x \in X : |g(x)| > Q_{\text{loc},\infty}(g) \} \in A \). On the one hand, by Exercise 1, we know that \( S_{f} \) is \( \mu \)-\( \sigma \)-finite, so there exists a sequence \( (F_{n})_{n=1}^{\infty} \subset A \) of disjoint \( \mu \)-finite sets, such that \( S_{f} = \bigcup_{n=1}^{\infty} \). On the other hand, we know that \( A \) is locally \( \mu \)-negligeable, so in particular \( \mu(F_{n} \cap A) = 0, \forall n \), so by \( \sigma \)-additivity we have

\[
\mu(N) = \mu\left( \bigcup_{n=1}^{\infty} F_{n} \cap A \right) = \sum_{n=1}^{\infty} \mu(F_{n} \cap A) = 0.
\]

The second statement and the inequality (34) now follow immediately from (33), which implies

\[
\int_{X} |fg|^{p} \, d\mu \leq Q_{\text{loc},\infty}(g)^{p} \int_{X} |f|^{p} \, d\mu. \quad \square
\]

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Remark 6. Using Proposition 2 with \( p = 1 \), we obtain the existence of a bilinear map \( \Psi_{\text{loc}} : L^1_K(X, \mathcal{A}, \mu) \times L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu) \to \mathbb{K} \), defined (implicitly) by

\[
\Phi_{\text{loc}}(f, [g]_{\text{loc}}) = \int_X fg \, d\mu, \quad \forall f \in L^1_K(X, \mathcal{A}, \mu), \ g \in L^{\infty}_K(X, \mathcal{A}, \mu),
\]

which satisfies the inequality

\[
|\Phi_{\text{loc}}(f, v)| \leq \|f\|_1 \cdot \|v\|_{\text{loc,} \infty}, \quad \forall f \in L^1_K(X, \mathcal{A}, \mu), \ v \in L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu).
\]  

Using (35), it follows that

(i) For every \( f \in L^1_K(X, \mathcal{A}, \mu) \), the map \( f^\#: L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu) \ni v \mapsto \Phi_{\text{loc}}(f, v) \in \mathbb{K} \) is a linear continuous functional, and the linear map

\[
#_{\text{loc}} : L^1_K(X, \mathcal{A}, \mu) \ni f \mapsto f^\# \in L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu)^*
\]

is contractive.

(ii) For every \( v \in L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu) \), the map \( v^\# : L^1_K(X, \mathcal{A}, \mu) \ni f \mapsto \Phi_{\text{loc}}(f, v) \in \mathbb{K} \) is a linear continuous functional, and the linear map

\[
#_{\text{loc}} : L^{\text{loc},\infty}_K(X, \mathcal{A}, \mu) \ni v \mapsto v^\# \in L^1_K(X, \mathcal{A}, \mu)^*
\]

is contractive.

Proposition 3. With the notations as above, both maps (36) and (37) are isometric.

Proof. Everything can be traced back in the spaces \( \mathfrak{L}^1 \) and \( \mathfrak{L}^{\text{loc},\infty} \), where it suffices to prove the inequalities

\[
Q_1(f) \leq \sup \left\{ \int_X f k \, d\mu : k \in \mathfrak{L}^{\text{loc,}\infty}_K(X, \mathcal{A}, \mu), \ Q_{\text{loc,}\infty}(k) \leq 1 \right\}, \quad \forall f \in \mathfrak{L}^1_K(X, \mathcal{A}, \mu); \tag{38}
\]

\[
Q_{\text{loc,}\infty}(g) \leq \sup \left\{ \int_X h g \, d\mu : k \in \mathfrak{L}^1_K(X, \mathcal{A}, \mu), \ Q_1(h) \leq 1 \right\}, \quad \forall g \in \mathfrak{L}^{\text{loc,}\infty}_K(X, \mathcal{A}, \mu). \tag{39}
\]

(The reverse inequalities “\( \geq \)” in both (38) and (39) hold trivially by (35).)

To prove (38) we fix \( f \in \mathfrak{L}^1_K(X, \mathcal{A}, \mu) \) and we define the function \( k : X \to \mathbb{K} \) by

\[
k(x) = \begin{cases} 
\frac{|f(x)|}{f(x)} & \text{if } f(x) \neq 0 \\
0 & \text{if } f(x) = 0
\end{cases}
\]

Since \( k \) is measurable, and \( |k(x)| \leq 1, \forall x \in X \), it is obvious that \( k \) belongs to \( \mathfrak{L}^{\text{loc,}\infty}_K(X, \mathcal{A}, \mu) \), and has \( Q_{\text{loc,}\infty}(k) \leq 1 \). Of course, by construction we have \( fk = |f| \), so \( Q_1(f) = \int_X |f| \, d\mu = \int_X fk \, d\mu \), and (38) follows immediately.

To prove (39), fix \( g \in \mathfrak{L}^{\text{loc,}\infty}_K(X, \mathcal{A}, \mu) \) and denote the supremum in the right-hand side of (39) by \( C \), so that in order to prove (39), it suffices to prove that for every \( \varepsilon > 0 \), one has

\[
|g| \leq C + \varepsilon, \quad \mu\text{-a.e.} \tag{40}
\]
Argue by contradiction, assuming the existence of some \( \varepsilon > 0 \), for which the inequality (40) fails, which means that the set \( A = \{ x \in X : |g(x)| > C + \varepsilon \} \in \mathcal{A} \) is not locally \( \mu \)-negligible, thus there exists some \( F \in \mathcal{A} \), such that \( F \subset A \) and \( 0 < \mu(F) < \infty \). Consider then the function \( h_0 : X \to \mathbb{K} \), defined by

\[
h_0(x) = \begin{cases} 
|g(x)| & \text{if } x \in F \\
 0 & \text{if } x \in X \setminus F 
\end{cases}
\]

(Of course, if \( x \in F \), then \( |g(x)| > C + \varepsilon > 0 \), so \( g(x) \neq 0 \).) Notice that \( h_0 \) is measurable, and \( |h_0| = \chi_F \), so in particular it follows that \( h_0 \) is integrable, and \( Q_1(h_0) = \int_X |h_0| \, d\mu = \int_X \chi_F \, d\mu = \mu(F) > 0 \). With this choice of \( h_0 \) we have

\[
h_0g = |g|\chi_F \geq (C + \varepsilon)\chi_F
\]

(the inequality follows from the inclusion \( F \subset A \)), so integrating over \( X \) yields

\[
\int_X h_0g \, d\mu \geq (C + \varepsilon)\mu(F). \tag{41}
\]

Of course, the function \( h = \mu(F)^{-1}h_0 \) is also integrable, and has \( Q_1(h) = 1 \). But now, if we divide the inequality (41) by \( \mu(F) \) we obtain

\[
\int_X hg \, d\mu \geq C + \varepsilon,
\]

which is impossible, since the left-hand side is \( \leq C \).

\[\square\]

**Remark 7.** The maps (37), (29) and (32) are linked by the identity \( \#_{\text{loc}} \circ \Pi_{\text{loc}} = \# \). This identity, combined with the next Exercise, explains how \( \Pi_{\text{loc}} \) measures the injectivity failure of \( \# \).

**Exercise 10.** Prove that, for a measure space \((X, \mathcal{A}, \mu)\), the following are equivalent:

(i) the map (32) is injective;

(ii) the map (32) is isometric;

(iii) every locally \( \mu \)-negligible set \( A \in \mathcal{A} \) is \( \mu \)-negligible;

(iv) for every \( A \in \mathcal{A} \), with \( \mu(A) > 0 \), there exists \( F \in \mathcal{A}, F \subset A \) with \( 0 < \mu(F) < \infty \).

**Definitions.** A measure space \((X, \mathcal{A}, \mu)\), satisfying the above equivalent conditions, is called **nowhere degenerate.**

At the other extreme, we say that a measure space \((X, \mathcal{A}, \mu)\) is **degenerate**, if \( \mu(A) \in \{0, \infty\} \), \( \forall A \in \mathcal{A} \). In this case, one has \( L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{0\} \) and \( L^p_{\mathbb{K}}(X, \mathcal{A}, \mu) = \{0\} \), \( \forall p \in [1, \infty) \), although the space \( L^\infty_{\mathbb{K}}(X, \mathcal{A}, \mu) \) might be non-trivial, as seen for instance in Example 2.
Remark 8. If \((X, \mathcal{A}, \mu)\) is \(\sigma\)-finite, then it is nowhere degenerate. Indeed, if we write \(X = \bigcup_{n=1}^{\infty} F_n\), with \((F_n)_{n=1}^{\infty} \subset \mathcal{A}\) disjoint and \(\mu\)-finite, then for every \(A \in \mathcal{A}\) we have \(\mu(A) = \sum_{n=1}^{\infty} \mu(A \cap F_n)\), with \(\mu(A \cap F_n) < \infty, \forall n\). Therefore, if \(\mu(A) > 0\), then at least one of the sets \(A \cap F_n\) will have \(\mu(A \cap F_n) > 0\).

D*. Measure completions, partitions, and decomposability

Although this sub-section is optional (with most of the material presented in the form of Exercises), it contains some important results that are often overlooked, or entirely omitted from most Real Analysis textbooks. Our whole discussion is motivated by the following observation. Suppose one has two measure space structures \((X, \mathcal{A}, \mu)\) and \((X, \mathcal{B}, \nu)\) on the same set \(X\), such that \(\mathcal{A} \subset \mathcal{B}\) and \(\mu(A) = \nu(A), \forall A \in \mathcal{A}\), in which case we use the notation \((X, \mathcal{A}, \mu) \subset (X, \mathcal{B}, \nu)\). Then it is pretty obvious that we have the inclusions

\[
\begin{align*}
    m_{\mathcal{K}}(X, \mathcal{A}) &\subset m_{\mathcal{K}}(X, \mathcal{B}), \\
    m_{\mathcal{K}}^{b}(X, \mathcal{A}) &\subset m_{\mathcal{K}}^{b}(X, \mathcal{B}), \\
    \mathfrak{m}_{\mathcal{K}}(X, \mathcal{A}, \mu) &\subset \mathfrak{m}_{\mathcal{K}}(X, \mathcal{B}, \nu), \\
    \mathcal{L}_{\mathcal{K}}^{p}(X, \mathcal{A}, \mu) &\subset \mathcal{L}_{\mathcal{K}}^{p}(X, \mathcal{B}, \nu), \quad p \in [1, \infty],
\end{align*}
\]

which give rise to linear maps

\[
    T_p : L_{\mathcal{K}}^{p}(X, \mathcal{A}, \mu) \rightarrow L_{\mathcal{K}}^{\infty}(X, \mathcal{B}, \nu). \tag{46}
\]

As it turns out, the maps (46) are isometric, for all \(p \in [1, \infty]\), so using the conventions from the previous sub-sections, we are going to write them as inclusions:

\[
    L_{\mathcal{K}}^{p}(X, \mathcal{A}, \mu) \subset L_{\mathcal{K}}^{p}(X, \mathcal{B}, \nu). \tag{47}
\]

Although (42)-(45) may all be strict inclusions (if \(\mathcal{A} \subset \mathcal{B}\)), we shall see that it is not impossible for the inclusion (47) to be an equality.

Since it is not the goal here to include a comprehensive treatment of Measure Theory, most of the material is presented in the form of Exercises.

Comment. In general, given \((X, \mathcal{A}, \mu) \subset (X, \mathcal{B}, \nu)\), it may be impossible to construct a “natural” map

\[
    L_{\mathcal{K}}^{\text{loc,}\infty}(X, \mathcal{A}, \mu) \rightarrow L_{\mathcal{K}}^{\text{loc,}\infty}(X, \mathcal{B}, \nu),
\]

because the implication “\(A \text{ locally } \mu\)-negligable (in } \mathcal{A}\)” \(\Rightarrow \text{ “}A \text{ locally } \nu\)-negligable (in } \mathcal{B}\)” may fail. ( The measure space \((X, \mathcal{A}, \mu)\) may be degenerate, but very small, for instance \(\mathcal{A} = \{\emptyset, X\}\), with \(\mu(X) = \infty\), but the measure space \((X, \mathcal{B}, \nu)\) can be very large, with plenty of \(\nu\)-finite sets, for instance \(\mathcal{B} = \mathcal{P}(X)\), with \(\nu\) the counting measure, with \(X\) infinite.)

Notation. For two subsets \(M, N \subset X\), we define the set \(M \triangle N = (M \setminus N) \cup (N \setminus M)\), which is referred to as the symmetric difference of \(M \text{ and } N\). It is pretty clear that the indicator function of \(M \triangle N\) is given by \(\chi_{M \triangle N} = |\chi_M - \chi_N|\).

Exercises 11-13. Assume \((X, \mathcal{A}, \mu) \subset (X, \mathcal{B}, \nu)\).

11. Prove that the maps (46) are isometric, for all \(p \in [1, \infty]\).
12. Prove that the following are equivalent:

(i) there exists $p \in [1, \infty)$, such that $L^p_{\underline{X}}(X, \mathcal{A}, \mu) = L^p_{\underline{X}}(X, \mathcal{B}, \nu)$;
(ii) for every $p \in [1, \infty)$, one has $L^p_{\underline{X}}(X, \mathcal{A}, \mu) = L^p_{\underline{X}}(X, \mathcal{B}, \nu)$;
(iii) for every $\nu$-finite set $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$, such that $\nu(B \triangle A) = 0$.

(HINT: By Exercise 11, $L^p_{\underline{X}}(X, \mathcal{A}, \mu)$ is closed in $L^p_{\underline{X}}(X, \mathcal{B}, \nu)$. Prove that condition (iii) is equivalent to the equality $L^p_{\underline{X}}(X, \mathcal{A}, \mu) = L^p_{\underline{X}}(X, \mathcal{B}, \nu)$, and then use Exercise 3.)

13. Prove that the following are equivalent:

(i) $L^\infty_{\underline{K}}(X, \mathcal{A}, \mu) = L^\infty_{\underline{K}}(X, \mathcal{B}, \nu)$;
(ii) for every $B \in \mathcal{B}$, there exists $A \in \mathcal{A}$, such that $\nu(B \triangle A) = 0$.

(HINT: Same as above, but show that condition (ii) is equivalent to the equality $E^\infty_{\underline{K}}(X, \mathcal{A}, \mu) = E^\infty_{\underline{K}}(X, \mathcal{B}, \nu)$, and then use Exercise 9.)

**Example 3.** Given a measure space $(X, \mathcal{A}, \mu)$, we define the collection

$$\mathcal{N}(\mathcal{A}, \mu) = \{N \subset X : \text{there exists a } \mu\text{-negligible set } D \in \mathcal{A}, \text{ with } D \supseteq N \} \subset \mathcal{P}(X).$$

One can show that, for a set $B \in \mathcal{P}(X)$, the following conditions are equivalent:

(i) there exists $A \in \mathcal{A}$, such that $B \triangle A \in \mathcal{N}(\mathcal{A}, \mu)$;
(ii) there exist $A \in \mathcal{A}$ and $N \in \mathcal{N}(\mathcal{A}, \mu)$, such that $B = A \cup N$;
(iii) there exist $A \in \mathcal{A}$ and $N \in \mathcal{N}(\mathcal{A}, \mu)$, such that $B = A \setminus N$.

Moreover, in this case the sets $A$ are $\mu$-essentially unique, in the sense that if $A_1, A_2 \in \mathcal{A}$ each satisfy any one of the conditions (i)-(iii), then $\mu(A_1 \triangle A_2) = 0$, so in particular one has the equality $\mu(A_1) = \mu(A_2)$. In particular, this means that, if we denote by $\tilde{\mathcal{A}}$ the collection of all $B \subset X$ that satisfy any one the above conditions, we can define a map $\tilde{\mu} : \tilde{\mathcal{A}} \rightarrow [0, \infty]$ by $\tilde{\mu}(B) = \mu(A)$, where $A \in \mathcal{A}$ is any set that satisfies one of the conditions (i)-(iii). One can show that this way we obtain a measure space $(X, \tilde{\mathcal{A}}, \tilde{\mu})$, called the completion of $(X, \mathcal{A}, \mu)$, which will clearly (by Exercises 12 and 13) satisfy the equalities

$$L^p_{\underline{X}}(X, \mathcal{A}, \mu) = L^p_{\underline{X}}(X, \tilde{\mathcal{A}}, \tilde{\mu}), \quad \forall p \in [1, \infty].$$

The terminology one uses in connection with this construction is as follows. We say that a measure space $(X, \mathcal{A}, \mu)$ is complete, if $\mathcal{N}(\mathcal{A}, \mu) \subset \mathcal{A}$, or equivalently $\tilde{\mathcal{A}} = \mathcal{A}$. As it turns out, if $(X, \mathcal{A}, \mu)$ is not complete, then $(X, \tilde{\mathcal{A}}, \tilde{\mu})$ is complete (see Exercise 16 below).

**Exercise 14.** Prove all above statements. (This material can be found in most Real Analysis textbooks.)

Regarding the above construction, it is natural to ask, given a measure space $(X, \mathcal{A}, \mu)$, if there exists other measure space structures $(X, \mathcal{B}, \nu) \supseteq (X, \tilde{\mathcal{A}}, \tilde{\mu})$, for which the inclusions (47) are also equalities. One natural candidate is described as follows.
Definitions. Fix a measure space \((X, \mathcal{A}, \mu)\), and define the map \(\mu^* : \mathcal{P}(X) \to [0, \infty] \) by
\[
\mu^*(S) = \inf \{\mu(A) : A \in \mathcal{A}, A \supset S\}, \quad S \in \mathcal{P}(X).
\]
One can show that \(\mu^*\) is an outer measure\(^5\) on \(X\), which will be called the Caratheodory outer measure associated with \((X, \mathcal{A}, \mu)\). It is pretty obvious that \(\mu^*(A) = \mu(A), \forall A \in \mathcal{A}\). If we apply then the Caratheodory construction, we obtain a \(\sigma\)-algebra \(\mathcal{M}(\mu^*)\) of all \(\mu^*\)-measurable sets, on which the restriction of \(\mu^*\) is a measure. From now on we are going to denote the \(\sigma\)-algebra \(\mathcal{M}(\mu^*)\) by \(\mathcal{A}^c\), and the measure \(\mu^*|_{\mathcal{A}^c}\) by \(\mu^c\). The resulting measure space \((X, \mathcal{A}^c, \mu^c)\) is called the Caratheodory completion of \((X, \mathcal{A}, \mu)\). One can show (see Exercise 17) that \((X, \mathcal{A}, \mu)\) is complete, but the Caratheodory completion \((X, \mathcal{A}^c, \mu^c)\) is also complete, thus justifying the above terminology. (The inclusion \(\mathcal{A} \subset \mathcal{A}^c\) may be strict, however. See Exercise 15.)

In order to clarify the distinction between \(\mathcal{A}\) and \(\mathcal{A}^c\) it is useful to introduce the following notation. For a measure space \((X, \mathcal{B}, \nu)\) we define the collection
\[
\mathcal{N}^{\text{loc}}(\mathcal{B}, \nu) = \{N \subset X : \text{there exists a locally } \nu\text{-negligible set } D \subset \mathcal{B}, \text{with } D \supset N\},
\]
and we say that \((X, \mathcal{B}, \nu)\) is locally complete, if \(\mathcal{N}^{\text{loc}}(\mathcal{B}, \nu) \subset \mathcal{B}\). It is pretty obvious that every “locally complete” \(\Rightarrow\) “complete.”

With this terminology, one can show (see Exercise 17) that, for any measure space \((X, \mathcal{A}, \mu)\), the Caratheodory completion \((X, \mathcal{A}^c, \mu^c)\) is locally complete (but \(\mathcal{A}\) may fail to be locally complete).

Exercise 15. Let \(X\) be an infinite set, and let \(\mathcal{A} = \{\emptyset, X\}\) be equipped with the measure \(\mu\) defined by \(\mu(X) = \infty\). Prove that \((X, \mathcal{A}, \mu)\) is complete, but the Caratheodory completion \((X, \mathcal{A}^c, \mu^c)\) is given by \(\mathcal{A}^c = \mathcal{P}(X)\), and \(\mu^c\) the degenerate counting measure \(\mu^c(A) = \infty, \forall A \neq \emptyset\).

Exercise 16. Let \(\eta\) on \(X\) be an outer measure on \(S\), and let \(\mathcal{F}(\eta)\) denote the collection \(\{F \subset X : \eta(F) < \infty\}\).

(i) Prove that, for a set \(A \subset X\), the following are equivalent:
- \(A\) is \(\eta\)-measurable, i.e. \(\eta(S) = \eta(S \cap A) + \eta(S \setminus A), \forall S \in \mathcal{P}(X)\);
- \(\eta(F) = \eta(F \cap A) + \eta(F \setminus A), \forall F \in \mathcal{F}(\eta)\).

(ii) Prove that, if we consider the collections
\[
\mathcal{M}(\eta) = \{A \subset X : A \text{ } \eta\text{-measurable}\},
\]
\[
\mathcal{N}(\eta) = \{N \subset X : \eta(N) = 0\},
\]
\[
\mathcal{N}^{\text{loc}}(\eta) = \{N \subset X : \eta(N \cap F) = 0, \forall F \in \mathcal{F}(\eta)\},
\]
then one has the inclusions \(\mathcal{N}(\eta) \subset \mathcal{N}^{\text{loc}}(\eta) \subset \mathcal{M}(\eta)\).

(iii) Assume \((X, \mathcal{A}, \mu)\) is a measure space, and let \(\mu^*\) be the associated Caratheodory outer measure.

\(^5\) At this point, the reader is urged to go back to BS III, sub-section ??, where all the meaningful features of outer measures are explained.
• Prove that, if $A \subset X$ is such that $\mu(F) = \mu^*(F \cap A) + \mu^*(F \setminus A)$, for all $F \in A$ with $\mu(F) < \infty$, then $A \in \mathcal{M}(\mu^*)$.

• If $N \subset X$ is such that, $\mu^*(N \cap F) = 0$, for all $F \in A$ with $\mu(F) < \infty$, then $N \in \mathcal{N}_{loc}(\mu^*)$.

**Exercises 17-20.** Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $\mu^*$ be the associated Caratheodory outer measure.

17. Using the notations from the previous Exercise, prove that:

(i) $\mathcal{N}(\mathcal{A}, \mu) = \mathcal{N}(\tilde{\mathcal{A}}, \tilde{\mu}) = \mathcal{N}(\mathcal{A}^c, \mu^c) = \mathcal{N}(\mu^*)$;

(ii) $\mathcal{N}_{loc}(\mathcal{A}, \mu) = \mathcal{N}_{loc}(\tilde{\mathcal{A}}, \tilde{\mu}) = \mathcal{N}_{loc}(\mathcal{A}^c, \mu^c) = \mathcal{N}_{loc}(\mu^*)$.

Conclude, using Exercise 16, that

(a) both $(X, \tilde{\mathcal{A}}, \tilde{\mu})$ and $(X, \mathcal{A}^c, \mu^c)$ are complete;

(b) $(X, \mathcal{A}^c, \mu^c)$ is locally complete.

18. Prove that, if a set $S \in \mathcal{P}(X)$ has $\mu^*(S) < \infty$, then the following are equivalent: (i) $S \in \mathcal{A}^c$; (ii) $S \in \tilde{\mathcal{A}}$. Conclude that $L^p_{K}(X, \mathcal{A}, \mu) = L^p_{K}(X, \mathcal{A}^c, \mu^c), \forall \, p \in [1, \infty)$.

19. Prove that, for a set $A \in \mathcal{A}$, the following are equivalent:

• $A$ is locally $\mu$-negligible (in $\mathcal{A}$);

• $A$ is locally $\tilde{\mu}$-negligible (in $\tilde{\mathcal{A}}$);

• $A$ is locally $\mu^c$-negligible (in $\mathcal{A}^c$);

• $A \in \mathcal{N}_{loc}(\mu^*)$.

Conclude that one has the inclusions

$$
\mathcal{N}_{loc}(X, \mathcal{A}, \mu) \subset \mathcal{N}_{loc}(X, \tilde{\mathcal{A}}, \tilde{\mu}) \subset \mathcal{N}_{loc}(X, \mathcal{A}^c, \mu^c),
$$

therefore, by passing to quotient spaces, there exists natural linear maps

$$
L^\infty_{loc}(X, \mathcal{A}, \mu) \xrightarrow{T'} L^\infty_{loc}(X, \tilde{\mathcal{A}}, \tilde{\mu}) \xrightarrow{T''} L^\infty_{loc}(X, \mathcal{A}^c, \mu^c).
$$

20. Concerning the maps in (51), prove that:

(i) both $T'$ and $T''$ are isometric, so from now on we can write them as inclusions

$$
L^\infty_{loc}(X, \mathcal{A}, \mu) \subset L^\infty_{loc}(X, \tilde{\mathcal{A}}, \tilde{\mu}) \subset L^\infty_{loc}(X, \mathcal{A}^c, \mu^c);
$$

(ii) the first inclusion in (i) is an equality, i.e. $L^\infty_{loc}(X, \mathcal{A}, \mu) = L^\infty_{loc}(X, \tilde{\mathcal{A}}, \tilde{\mu})$. 

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Remark 9. If \((X, \mathcal{A}, \mu)\) is \(\sigma\)-finite, then \(\mathcal{A}^c = \bar{\mathcal{A}}\). Indeed, if we write \(X = \bigcup_{n=1}^{\infty} F_n\), where \((F_n)_{n=1}^{\infty} \subset \mathcal{A}\) have \(\mu(F_n) < \infty\), \(\forall n\), and we start with some set \(S \in \mathcal{A}^c\), then \(S \cap F_n \in \mathcal{A}\) has \(\mu^*(S \cap F_n) \leq \mu^*(F_n) = \mu(F_n) < \infty\), so by Exercise 17 it follows that \(S \cap F_n \in \bar{\mathcal{A}}\), \(\forall n\), and then \(S = \bigcup_{n=1}^{\infty} S \cap F_n\) will also belong to \(\bar{\mathcal{A}}\).

Comment. To summarize what we have obtained so far, the inclusions \((X, \mathcal{A}, \mu) \subset (X, \bar{\mathcal{A}}, \bar{\mu}) \subset (X, \mathcal{A}^c, \mu^c)\) give rise to:

\[
L_p^p(X, \mathcal{A}, \mu) = L_p^p(X, \bar{\mathcal{A}}, \bar{\mu}) = L_p^p(X, \mathcal{A}^c, \mu^c), \quad \forall p \in [1, \infty); \quad (52)
\]

\[
L_\infty^\infty(X, \mathcal{A}, \mu) = L_\infty^\infty(X, \bar{\mathcal{A}}, \bar{\mu}) \subset L_\infty^\infty(X, \mathcal{A}^c, \mu^c); \quad (53)
\]

\[
L_{loc}^\infty(X, \mathcal{A}, \mu) = L_{loc}^\infty(X, \bar{\mathcal{A}}, \bar{\mu}) \subset L_{loc}^\infty(X, \mathcal{A}^c, \mu^c). \quad (54)
\]

Without any additional (significant) restrictions on \((X, \mathcal{A}, \mu)\), one expects to have strict inclusion in both (53) and (54), as seen for instance in Exercise 15.

The reason we focus on the Caratheodory completion, rather than the (ordinary) completion, is two-fold. On the one hand, if one is interested in the \(L^p\)-spaces, for \(p \in [1, \infty)\), the equalities (52) show that one loses nothing by replacing \((X, \mathcal{A}, \mu)\) with \((X, \mathcal{A}^c, \mu^c)\). On the other hand, although the inclusion (54) is strict, it shows that a “better pair” (in the sense discussed in the previous section) for \(L_{1}^1(X, \mathcal{A}, \mu)\) is the space \(L_{loc}^{\infty}(X, \mathcal{A}^c, \mu^c)\). The advantage of using this space, instead of \(L_{loc}^{\infty}(X, \mathcal{A}, \mu)\), is the increased flexibility. Ultimately, under some additional (but not too restrictive) conditions, it will be shown that \(L_{loc}^{\infty}(X, \mathcal{A}^c, \mu^c)\) is isometrically isomorphic to the dual Banach space \(L_1^1(X, \mathcal{A}, \mu)^*\).

We now turn our attention to the disassembling/reassembling problem for \(L^p\)-spaces. In order to formulate it we need to introduce some terminology.

**Notation.** Given a \(\sigma\)-algebra \(\mathcal{A}\) on \(X\), and a subset \(S \subset X\), we denote by \(\mathcal{A}|_S\) the collection \(\{A \cap S : A \in \mathcal{A}\}\), which is a \(\sigma\)-algebra on \(S\). In the case when \(S \in \mathcal{A}\), we clearly have the inclusion \(\mathcal{A}|_S \subset \mathcal{A}\), so given a measure \(\mu\) on \(\mathcal{A}\), its restriction to \(\mathcal{A}|_S\) will be a measure on \(\mathcal{A}|_S\), which we will denote by \(\mu|_S\).

**Exercise 21.** Let \((X, \mathcal{A}, \mu)\) be a measure space, and let \(S \in \mathcal{A}\).

(i) Prove that the restriction operator \(R_S : f \longmapsto f|_S\) maps (linearly):

(a) \(m_{K}(X, \mathcal{A})\) onto \(m_{K}(S, \mathcal{A}|_S)\),

(b) \(m_{bK}(X, \mathcal{A})\) onto \(m_{bK}(S, \mathcal{A}|_S)\),

(c) \(\mathfrak{m}_{K}(X, \mathcal{A}, \mu)\) onto \(\mathfrak{m}_{K}(S, \mathcal{A}|_S, \mu|_S)\),

(d) \(\mathfrak{m}_{loc}^{\infty}(X, \mathcal{A}, \mu)\) onto \(\mathfrak{m}_{loc}^{\infty}(S, \mathcal{A}|_S, \mu|_S)\),

(e) \(L_{loc}^{\infty}(X, \mathcal{A}, \mu)\) onto \(L_{loc}^{\infty}(S, \mathcal{A}|_S, \mu|_S)\), and

(f) \(L_{p}^{p}(X, \mathcal{A}, \mu)\) onto \(L_{p}^{p}(S, \mathcal{A}|_S, \mu|_S)\), for all \(p \in [1, \infty]\).

In particular, one obtains the surjective linear maps

\[
Q_{S}^{p} : L_{p}^{p}(X, \mathcal{A}, \mu) \rightarrow L_{p}^{p}(S, \mathcal{A}|_S, \mu|_S), \quad p \in [1, \infty), \quad (55)
\]

\[
Q_{S}^{loc, \infty} : L_{loc}^{\infty}(X, \mathcal{A}, \mu) \rightarrow L_{loc}^{\infty}(S, \mathcal{A}|_S, \mu|_S). \quad (56)
\]
(ii) Prove that the maps (55) and (56) are contractions.

**Definition.** Given a non-empty set \( X \), a system \( (X_i)_{i \in I} \) of nonempty subsets of \( X \) is called a *partition* of \( X \), if all \( X_i \)'s are disjoint, and \( \bigcup_{i \in I} X_i \). In the case when \( X \) comes equipped with a \( \sigma \)-algebra \( \mathcal{A} \), and all \( X_i \)'s belong to \( \mathcal{A} \), we say that \( (X_i)_{i \in I} \) is an \( \mathcal{A} \)-partition of \( X \).

**Exercise 22** Let \( (X, \mathcal{A}, \mu) \) be a measure space, and let \( (X_i)_{i \in I} \) be an \( \mathcal{A} \)-partition of \( X \). Using the maps (55) and (56) defined in the previous Exercise, prove the following statements.

(i) If \( p \in [1, \infty] \), and \( u \in L^p_K(X, \mathcal{A}, \mu) \), then the I-tuple

\[
D_p u = (Q^p_{X_i} u)_{i \in I} \in \prod_{i \in I} L^p_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})
\]

belongs to the Banach space \( \ell^p(L^p_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \parallel \cdot \parallel_p)_{i \in I} \). Moreover, the map

\[
D_p : (L^p_K(X, \mathcal{A}, \mu), \parallel \cdot \parallel_p) \rightarrow \ell^p(L^p_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \parallel \cdot \parallel_p)_{i \in I}
\]  

(57)

is a linear contraction.

(ii) For every \( u \in L^{loc, \infty}_K(X, \mathcal{A}, \mu) \), the I-tuple

\[
D_{loc, \infty} u = (Q^{loc, \infty}_{X_i} u)_{i \in I} \in \prod_{i \in I} L^{loc, \infty}_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i})
\]

belongs to the Banach space \( \ell^{\infty}(L^{loc, \infty}_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \parallel \cdot \parallel_{loc, \infty})_{i \in I} \). Moreover, the map

\[
D_{loc, \infty} : (L^{loc, \infty}_K(X, \mathcal{A}, \mu), \parallel \cdot \parallel_{loc, \infty}) \rightarrow \ell^{\infty}(L^{loc, \infty}_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \parallel \cdot \parallel_{loc, \infty})_{i \in I}
\]  

(58)

is a linear contraction.

(iii) If \( I \) is countable, then the maps (57) and (58) are isometric isomorphisms. (Note: For the map (57) with \( p \in [1, \infty) \), as well as for the map (58), this will be generalized shortly, so the reader should only focus on the map (57) for \( p = \infty \).)

The maps (57) and (58) are called the *disassembly maps* associated with given partition. With this terminology, the disassembly/reassembly problem is to decide when the maps (57) and (58) are in fact isometric isomorphisms. In this case, their inverses (which will also be isometric) will be called the *reassembly maps*.

This problem is extremely complicated for (57) with \( p = \infty \), and we are not going to address it at all. Of course, by Exercise 22 (iii) the disassembly/reassembly problem has a positive answer, when \( I \) is countable.

The next two results give a complete solution for (57) with \( p \in [1, \infty) \).

**Proposition 4.** If \( (X, \mathcal{A}, \mu) \) is a measure space, and \( (X_i)_{i \in I} \) is an \( \mathcal{A} \)-partition of \( X \), then the map (57) is surjective, for every \( p \in [1, \infty) \).
Proof. Fix \( p \in [1, \infty) \), and \( v = (v_i)_{i \in I} \in \prod_{i \in I} L^p_{\mathbb{R}}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \), with \( \sum_{i \in I} \|v_i\|_p^p < \infty \), and let us prove the existence of some \( u \in L^p_{\mathbb{R}}(X, \mathcal{A}, \mu) \), such that \( D_p u = v \). By summability, the set \( I_v = \{ i \in I : v_i \neq 0 \} \) is countable, so the set \( A = \bigcup_{i \in I_v} X_i \) belongs to \( \mathcal{A} \). Choose now, for every \( i \in I_v \), a function \( f_i \in L^p_{\mathbb{R}}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \), such that \( [f_i] = v_i \), and define the function \( f : X \to \mathbb{K} \) by

\[
f(x) = \begin{cases} f_i(x) & \text{if } x \in X_i \text{ and } i \in I_v \\ 0 & \text{if } x \not\in A \end{cases}
\]

It is pretty clear that \( f \) is measurable, and furthermore

\[
f|_{X_i} = \begin{cases} f_i & \text{if } i \in I_v \\ 0 & \text{if } i \not\in I_v \end{cases}
\]

so if we prove that \( f \) belongs to \( L^p_{\mathbb{R}}(X, \mathcal{A}, \mu) \), then we will have the equality \( v = D_p[f] \).

To prove that \( f \) belongs to \( L^p_{\mathbb{R}}(X, \mathcal{A}, \mu) \), we list \( I_v = \{ i_n : n \in \mathbb{N} \} \), with the \( i_n \)’s all different, where either \( M = \mathbb{N} \), or \( M = \{1, 2, \ldots, N\} \), and we define the increasing sequence of sets \( (A_n)_{n=1}^{\infty} \subset \mathcal{A} \), by

\[
A_n = \begin{cases} \bigcup_{k=1}^{n} X_{i_k} & \text{if } n \in M \\ A & \text{if } n \not\in M \end{cases}
\]

which clearly satisfies the equality \( \bigcup_{n=1}^{\infty} A_n = A \), so the sequence \( (\chi_{A_n}|f|^p)_{n=1}^{\infty} \) of non-negative measurable functions on \( (X, \mathcal{A}) \) is non-decreasing, and satisfies

\[
\lim_{n \to \infty} \chi_{A_n}(x)|f(x)|^p = |f(x)|^p, \quad \forall x \in X.
\]

By Lebesgue’s Monotone Convergence Theorem, it follows that

\[
\int_X |f|^p \, d\mu = \lim_{n \to \infty} \int_X \chi_{A_n}|f|^p \, d\mu. \tag{59}
\]

Of course, by construction, for every \( n \in \mathbb{N} \) we have

\[
\int_X \chi_{A_n}|f|^p \, d\mu = \begin{cases} \sum_{k=1}^{n} \int_{X_{i_k}} \chi_{X_{i_k}} |f|^p \, d\mu = \sum_{k=1}^{n} \int_{X_{i_k}} |f_{i_k}|^p \, d(\mu|_{X_{i_k}}), & \text{if } n \in M \\ \sum_{i \in I_v} \int_{X_i} \chi_{X_i} |f|^p \, d\mu = \sum_{i \in I_v} \int_{X_i} |f_i|^p \, d(\mu|_{X_i}), & \text{if } n \not\in M \end{cases}
\]

so by (59) we get:

\[
\int_X |f|^p \, d\mu = \lim_{n \to \infty} \int_X \chi_{A_n}|f|^p \, d\mu = \sum_{i \in I_v} \int_{X_i} |f_i|^p \, d(\mu|_{X_i}) = \sum_{i \in I_v} \|v_i\|_p^p = \|v\|_p^p < \infty. \quad \square
\]

**Proposition-Definition 5.** Let \((X, \mathcal{A}, \mu)\) be a measure space. For an \( \mathcal{A} \)-partition \((X_i)_{i \in I}\), the following are equivalent:

(i) the disassembly map (57) is isometric, for every \( p \in [1, \infty) \);
(i') the disassembly map (57) is isometric, for at least one \( p \in [1, \infty) \);

(ii) for any \( A \in \mathcal{A} \), with \( \mu(A) < \infty \), one has the equality\(^6\)

\[
\mu(A) = \sum_{i \in I} \mu(A \cap X_i). \tag{60}
\]

An \( \mathcal{A} \)-partition \((X_i)_{i \in I}\), satisfying the above equivalent conditions, is called \( \mu \)-integrable.

**Proof.** The implication \((i) \Rightarrow (i')\) is trivial.

\((i') \Rightarrow (ii)\). Fix \( p \in [1, \infty) \), for which the map (57) is isometric. This means that,

\[
\int_X |f|^p \, d\mu = \sum_{i \in I} \int_X |f| \, d\mu, \quad \forall f \in \mathcal{L}^p(X, \mathcal{A}, \mu). \tag{61}
\]

Fix now some set \( A \in \mathcal{A} \) with \( \mu(A) < \infty \), and let us prove the equality (60). This follows immediately from (61), applied to the characteristic function \( f = \chi_A \).

\((ii) \Rightarrow (i)\). Assume condition (ii) holds, fix some arbitrary \( p \in [1, \infty) \), and let us prove that the map (57) is isometric. Since \( D_p \) is norm-continuous, it suffices to prove the equality \( \|D_p u\|_p = \|u\|_p \), for all \( u \) in the dense subspace \( \mathcal{L}^p_{\text{elem}}(X, \mathcal{A}, \mu) \subset \mathcal{L}^p(X, \mathcal{A}, \mu) \). In turn, for this condition it suffices to prove that:

\[
\int_X g \, d\mu = \sum_{i \in I} \int_X \chi_{X_i} g \, d\mu, \quad \forall g \in \mathcal{L}^p_{\text{elem}}(X, \mathcal{A}, \mu), \tag{62}
\]

so basically we can assume that \( p = 1 \). Finally, by linearity, in order to prove (62) it suffices to consider the case when \( g = \chi_A \), for \( A \in \mathcal{A} \) with \( \mu(A) < \infty \), in which case the equality (62) reduces to (60).

**Comment.** Concerning the above definition, the reader is warned that the equality (60) is limited to sets of finite measure. In particular, if \( A \in \mathcal{A} \) has the property that \( \mu(A \cap X_i) = 0 \), \( \forall i \in I \), it does not follow that \( \mu(A) = 0 \).

**Remark 10.** Of course, if the index set \( I \) is countable, then every \( \mathcal{A} \)-partition \((X_i)_{i \in I}\) is \( \mu \)-integrable, since in this case the equality (60) holds for all \( A \in \mathcal{A} \).

In the general case \((I \text{ arbitrary})\), if \((X_i)_{i \in I}\) is a \( \mu \)-integrable \( \mathcal{A} \)-partition, then for every \( A \in \mathcal{A} \) with \( \mu(A) < \infty \), the fact that the sum in the right-hand side of (60) is finite forces the set \( I(A) = \{ i \in I : \mu(A \cap X_i) \neq 0 \} \) to be countable, so in particular, if one considers the set \( A' = \bigcup_{i \in I(A)} A \cap X_i \subset A \), then \( A' \) belongs to \( \mathcal{A} \), and by construction and by (60) it follows that \( \mu(A \setminus A') = 0 \).

**Comment.** The disassembly/reassembly problem for (58) is a bit more complicated, and will require some extra conditions imposed on the partition. One of them – \( \mu \)-integrability – is a very natural one to impose, especially if keep in mind the dual pairing between \( L^1 \) and \( L^{\text{loc}, \infty} \), so it is not surprising that we have the following result.

**Proposition 6.** Let \((X, \mathcal{A}, \mu)\) be a measure space, and \((X_i)_{i \in I}\) is a \( \mu \)-integrable \( \mathcal{A} \)-partition of \( X \), then the disassembly map (58) is isometric.

\(^6\) The right-hand side of (60) is understood in the sense of summability.
Proof. Fix some element \( u \in L_{\text{loc}}^{\infty}(X, \mathcal{A}, \mu) \), and let us show that the element
\[
D_{\text{loc}}^{\infty}u \in \ell^\infty(L_{\text{loc}}^{\infty}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \| \cdot \|_{\text{loc}, \infty})_{i \in I}
\]
has \( \|D_{\text{loc}}^{\infty}u\|_\infty = \|u\|_{\text{loc}, \infty} \). By Exercise 22 we already know that \( \|D_{\text{loc}}^{\infty}u\|_\infty \leq \|u\|_{\text{loc}, \infty} \), so all we need to do is to prove the inequality
\[
\|D_{\text{loc}}^{\infty}u\|_\infty \geq \|u\|_{\text{loc}, \infty}.
\]
(63)

If we choose a representative \( f \in L_{\text{loc}}^{\infty}(X, \mathcal{A}, \mu) \) i.e. \( u = [f] \), then the element \( D_{\text{loc}}^{\infty}u \) is the \( I \)-tuple \( ([f_i])_{i \in I} \), where \( f_i = f|_{X_i} \), so the inequality (63) simply reads:
\[
\sup_{i \in I} Q_i^{\ell_{\text{loc}}}(f_i) \geq Q_{\text{loc}}^{\infty}(f),
\]
(64)
where, for every \( i \in I \), the superscript in \( Q_i^{\ell_{\text{loc}}}(f_i) \) indicates that the seminorm \( Q_{\text{loc}}^{\infty}(f_i) \) is computed in \( L_{\text{loc}}^{\infty}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \). Denote the supremum in the left-hand side of (63) by \( C \), so that, in order to prove the inequality (63) we must show that the set
\[
N = \{ x \in X : |f(x)| > C \} \in \mathcal{A}
\]
is locally \( \mu \)-negligible\(^7\), that is, \( \mu(F \cap N) = 0 \), for all \( F \in \mathcal{A} \) with \( \mu(F) < \infty \).

Of course, for each \( i \in I \), we have \( N \cap X_i = \{ x \in X_i : |f_i(x)| > C \} = Q_i^{\ell_{\text{loc}}}(f_i) \in \mathcal{A}|_{X_i} \), which forces \( N \cap X_i \) to be locally \( (\mu|_{X_i}) \)-negligible. In particular, if we start with some set \( F \in \mathcal{A} \) with \( \mu(F) < \infty \), then the set \( F \cap X_i \in \mathcal{A}|_{X_i} \) has \( \mu|_{X_i}(F \cap X_i) < \infty \), thus forcing
\[
\mu((F \cap N) \cap X_i) = \mu|_{X_i}(((F \cap N) \cap (N \cap X_i)) = 0, \forall i \in I.
\]
But now, using (60) with \( A = F \cap N \), we get \( \mu(F \cap N) = \sum_{i \in I} \mu((F \cap N) \cap X_i) = 0 \), and we are done. \( \square \)

Comment. In order to solve the disassembly/reassembly problem for (58), our approach will be to try to construct the reassembly map. As it will turn out, one can construct a reassembly map \( R_{\text{loc}}^{\infty} : \ell^\infty(L_{\text{loc}}^{\infty}(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \| \cdot \|_{\text{loc}, \infty})_{i \in I} \to L_{\text{loc}}^{\infty}(X, \mathcal{B}, \nu) \), where \( (X, \mathcal{B}, \nu) \supset (X, \mathcal{A}, \mu) \) is a slightly larger measure space structure. This reassembly map will be constructed in such a way that the composition \( R_{\text{loc}}^{\infty} \circ D_{\text{loc}}^{\infty} \) agrees with the inclusion \( L_{\text{loc}}^{\infty}(X, \mathcal{A}, \mu) \subset L_{\text{loc}}^{\infty}(X, \mathcal{B}, \nu) \), and then the solution to the problem will depend on whether this inclusion is an equality.

A very natural framework for reassembling \( I \)-tuples of measurable functions is provided by the following construction.

Definition. Suppose \( (X_i)_{i \in I} \) is a partition of \( X \). Assume on each \( X_i, i \in I \), one is given a \( \sigma \)-algebra \( \mathcal{B}_i \). On can then form the direct sum \( \sigma \)-algebra
\[
\bigvee_{i \in I} \mathcal{B}_i = \{ B \subset X : B \cap X_i \in \mathcal{B}_i, \forall i \in I \}.
\]
\(^7\) This will force \( \|u\|_{\text{loc}, \infty} = Q_{\text{loc}, \infty}(f) \leq C \).
It is pretty clear that this is the largest among all \( \sigma \)-algebras \( \mathcal{B} \) that satisfy \( \mathcal{B}|_{X_i} = \mathcal{B}, \forall i \in I \).

One should be aware of the fact that \( \bigvee_{i \in I} \mathcal{B}_i \) is larger than \( \sigma\text{-alg}(\bigcup_{i \in I} \mathcal{B}_i) \) – the \( \sigma \)-algebra generated by \( \bigcup_{i \in I} \mathcal{B}_i \). If \( I \) is countable, however, these two \( \sigma \)-algebras coincide. The next Exercises clarify this matter.

**Exercises 23-24.** Let \((X_i)_{i \in I}\) be a partition of \( X \), and assume that on each \( X_i \) one is given a \( \sigma \)-algebra \( \mathcal{B}_i \).

**23.** Prove that:

1. the map \( \prod_{i \in I} \mathcal{B}_i \ni (B_i)_{i \in I} \mapsto \bigcup_{i \in I} B_i \in \bigvee_{i \in I} \mathcal{B}_i \) is a bijection;
2. given \((B_i)_{i \in I} \in \prod_{i \in I} \mathcal{B}_i\), the set \( \bigcup_{i \in I} B_i \) belongs to \( \sigma\text{-alg}(\bigcup_{i \in I} \mathcal{B}_i) \), if and only if there exists a *countable* set of indices \( I_0 \subset I \), such that \( B_i \in \{\emptyset, X_i\}, \forall i \in I \setminus I_0 \).

Conclude that, if \( I \) is *countable*, then \( \bigvee_{i \in I} \mathcal{B}_i = \sigma\text{-alg}(\bigcup_{i \in I} \mathcal{B}_i) \).

**24.** Prove that for a measurable space \((Y, \mathcal{D})\), and a map \( f : X \to Y \), the following are equivalent:

- \( f \) is measurable as a map \( f : (X, \bigvee_{i \in I} \mathcal{B}_i) \to (Y, \mathcal{D}) \);
- \( f|_{X_i} : (X_i, \mathcal{B}_i) \to (Y, \mathcal{D}) \) is measurable, for each \( i \in I \).

Conclude that

1. the map
   \[
   D : m_\mathcal{X}(X, \bigvee_{i \in I} \mathcal{B}_i) \ni f \mapsto (f|_{X_i})_{i \in I} \in \prod_{i \in I} m_\mathcal{X}(X_i, \mathcal{B}_i)
   \]  
   (65)

   is a linear isomorphism;
2. when restricted to the space of bounded measurable functions, the map \( D \) yields an *isometric* linear isomorphism
   \[
   D^b : m_\mathcal{X}^b(X, \bigvee_{i \in I} \mathcal{B}_i) \ni f \mapsto (f|_{X_i})_{i \in I} \in \ell^\infty(m_\mathcal{X}^b(X_i, \mathcal{B}_i), \|\cdot\|_{\sup})_{i \in I}
   \]  
   (66)

   The maps (65) and (66) will also be called *disassembly* maps. Their inverses will also be called *reassembly* maps.

**Definition.** Assume \( \mathcal{A} \) is a \( \sigma \)-algebra on \( X \), and \( \pi = (X_i)_{i \in I} \) is an \( \mathcal{A} \)-partition. The direct sum \( \sigma \)-algebra \( \bigvee_{i \in I} (\mathcal{A}|_{X_i}) \), which we denote by \( \mathcal{A}_\pi \), is called the \( \pi \)-*enhancement* of \( \mathcal{A} \).

It is obvious that \( \mathcal{A}_\pi|_{X_i} = \mathcal{A}|_{X_i}, \forall i \in I \), and one has the inclusion \( \mathcal{A}_\pi \supset \mathcal{A} \). In general, this inclusion is strict, but if \( I \) is *countable*, then \( \mathcal{A}_\pi = \mathcal{A} \).

**Comment.** The problem with direct sum \( \sigma \)-algebras is that, given a measure space \((X, \mathcal{A}, \mu)\) and an arbitrary \( \mathcal{A} \)-partition \( \pi = (X_i)_{i \in I} \) of \( X \), it might be impossible\(^8\) to construct a measure \( \mu \) on \( \mathcal{A}_\pi \), that agrees with \( \mu \) on \( \mathcal{A} \).

---

\(^8\) For instance, if one starts with the Lebesgue structure \((\mathbb{R}, \mathcal{A}, \lambda)\) – where \( \mathcal{A} \) is the \( \sigma \)-algebra of Lebesgue measurable sets, and \( \lambda \) is the Lebesgue measure – and the partition \( \pi \) consists of singletons, then \( \mathcal{A}_\pi = \mathcal{P}(\mathbb{R}) \), and there exists no measure \( \nu \) on \( \mathcal{P}(\mathbb{R}) \), that agrees with \( \lambda \) on \( \mathcal{A} \).
In the case when \( \pi \) is \( \mu \)-integrable, the situation is much nicer, as shown below.

**Theorem 3.** Let \((X,A,\mu)\) be a measure space, let \(\mu^*\) be the associated Caratheodory outer measure, and let \((X,A^c,\mu^c)\) denote the Caratheodory completion. Assume \(\pi = (X_i)_{i \in I}\) is a \(\mu\)-integrable \(A\)-partition.

(i) The \(\pi\)-enhancement \(A_\pi\) is contained in \(A^c\), so when one restricts \(\mu^c\) to \(A_\pi\), one obtains a measure \(\mu_\pi\) on \(A_\pi\), such that: \((X,A,\mu) \subset (X,A_\pi,\mu_\pi) \subset (X,A^c,\mu^c)\).

(ii) The measure spaces \((X,A,\mu)\) and \((X,A_\pi,\mu_\pi)\) have identical associated Caratheodory outer measures, therefore they have the same Caratheodory completion: \((X,A^c,\mu^c)\).

(iii) The partition \(\pi\) is both \(\mu^c\)-integrable (when viewed as an \(A^c\)-partition) and \(\mu_\pi\)-integrable (when viewed as an \(A_\pi\)-partition).

(iv) The disassembly map\(^9\)

\[
D^\pi_{loc,\infty} : L^\infty_{\mu,\infty}(X,A_\pi,\mu_\pi) \to \ell^\infty(L^\infty_{\mu,\infty}(X_i,A_i|_{X_i},\mu_i|_{X_i}), \|\|_{loc,\infty})_{i \in I}
\]

is an isometric isomorphism.

(v) Using the notations from Exercise 16, for a set \(N \subset X\), the following are equivalent:

(a) \(N \in N^loc(\mu^*)\);

(b) \(X_i \cap N \in N^loc(\mu^*), \forall i \in I\);

(c) \(N\) belongs to \(A^c\) and is locally \(\mu^c\)-negligeable.

**Sketch of proof.** Let us first prove statement (v). We already know that (a) \(\Leftrightarrow\) (c), by Exercise 17. The implication (a) \(\Rightarrow\) (b) is trivial. Assume \(N \subset X\) has property (b), and let us show that \(N\) belongs to \(N^loc(\mu^*)\). By Exercise 16 it suffices to show that \(\mu^*(N \cap F) = 0\), for all \(F \in A\), with \(\mu(F) < \infty\). Fix such an \(F\) and let us first notice that, since \(\mu^*(N \cap F) \leq \mu^*(F) = \mu(F) < \infty\), by the definition of the outer measure \(\mu^*\), there exists \(G \in A\), with \(\mu(G) < \infty\), such that \(G \supset N \cap F\), so if we consider the countable set \(I(G) = \{i \in I : \mu(X_i \cap G) > 0\}\), then the set \(G' = \bigcup_{i \in I(G)} X_i \cap G\) belongs to \(A\) and satisfies \(\mu(G' \cap G') = 0\), which implies that

\[
\mu^*(N \cap F) \leq \mu^*(G' \cap N \cap F) + \mu^*((G \setminus G') \cap N \cap F) \leq \mu^*(G' \cap N \cap F) + \mu^*(G \setminus G') = \mu^*(G' \cap N \cap F).
\]

Of course, since \(G \supset N \cap F\), we have

\[
G' \cap N \cap F = \bigcup_{i \in I(G)} X_i \cap G \cap N \cap F = \bigcup_{i \in I(G)} X_i \cap N \cap F,
\]

which, by the countability of \(I(G)\), the \(\sigma\)-subadditivity of \(\mu^*\), and (68), gives

\[
\mu^*(N \cap F) \leq \mu^*(G' \cap N \cap F) \leq \sum_{i \in I(G)} \mu^*(X_i \cap N \cap F).
\]

---

\(^9\) The map \(D^\pi_{loc,\infty}\) is the map (58), with \(A\) replaced by \(A_\pi\).
Of course, by condition (b) we know that \( \mu^*(X_i \cap N \cap F) = 0, \forall i \in I \), so the above inequalities now force \( \mu^*(N \cap F) = 0 \).

(i). Start with some set \( A \in \mathcal{A}_\pi \), which means that \( A = \bigcup_{i \in I} A_i \), with \( A_i \in \mathcal{A} \) and \( A_i \subset X_i, \forall i \in I \), and let us show that \( A \) is \( \mu^* \)-measurable. According to Exercise 16, it suffices\(^{10} \) to show that

\[
\mu^*(F \cap A) + \mu^*(F \setminus A) \leq \mu(F), \tag{69}
\]

for every \( F \in \mathcal{A} \), with \( \mu(F) < \infty \). Fix such an \( F \), and define, for each \( i \in I \), the sets \( F_i = F \cap X_i \), so that the set \( I(F) = \{ i \in I : \mu(F_i) > 0 \} \) is countable, and the set \( F' = \bigcup_{i \in I(F)} F_i \subset F \) belongs to \( \mathcal{A} \), and satisfies \( \mu(F \setminus F') = 0 \). Remark now that, on the one hand we have the equalities

\[
F' \cap A = \bigcup_{i \in I(F)} (F_i \cap A_i) \quad \text{and} \quad F' \setminus A = \bigcup_{i \in I(F)} (F_i \setminus A_i), \tag{70}
\]

with \( F_i \cap A_i \) and \( F_i \setminus A_i \) in \( \mathcal{A}_i \subset \mathcal{A} \), so using the fact that \( I(F) \) is countable, it follows that both \( F' \cap A \) and \( F' \setminus A \) belong to \( \mathcal{A} \), and therefore we have

\[
\mu^*(F' \cap A) + \mu^*(F' \setminus A) = \mu^*(F' \cap A) + \mu^*(F' \setminus A) = \mu(F') = \mu(F) \tag{71}
\]

On the other hand, since both \( (F \setminus F') \cap A \) and \( (F \setminus F') \setminus A \) are contained in \( F \setminus F' \), which is \( \mu \)-negligeable, it follows that \( \mu^*((F \setminus F') \cap A) = \mu^*((F \setminus F') \setminus A) = 0 \), so using the properties of outer measures we get the inequalities

\[
\mu^*(F \cap A) \leq \mu^*(F' \cap A) + \mu^*((F \setminus F') \cap A) = \mu^*(F' \cap A), \\
\mu^*(F \setminus A) \leq \mu^*(F' \setminus A) + \mu^*((F \setminus F') \setminus A) = \mu^*(F' \setminus A),
\]

The desired inequality (69) now follows immediately, by adding these two inequalities, and using (71).

Statements (ii) and (iii) are left to the reader.

To prove statement (iv), it suffices to prove surjectivity. of \( D_{\text{loc}, \infty} \). (By (iii) and Proposition 6, the map \( D_{\text{loc}, \infty} \) is isometric.) Start with some element

\[
v = (v_i)_{i \in I} \in \ell^\infty(L^\text{loc,\infty}_K(X_i, \mathcal{A}_i, \mu_i), \| \cdot \|_{\text{loc,\infty}})_{i \in I},
\]

and let us construct an element \( u \in L^\text{loc,\infty}_K(X, \mathcal{A}_\pi, \mu_\pi) \), such that \( D_{\text{loc,\infty}} u = v \). First of all, we know that any element in \( L^\text{loc,\infty}_K \) can be represented by a bounded measurable function with same norm (see Remark 4), so for every \( i \in I \), one can choose some function \( f_i \in M^b_K(X_i, \mathcal{A}|_{X_i}) \), such that \( v_i = [f_i] \), in \( L^\text{loc,\infty}_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}) \) and also \( \|f_i\|_{\sup} = \|v_i\|_{\text{loc,\infty}} \). Since \( \|f_i\|_{\sup} \leq \|v_i\|_{\text{loc,\infty}} \leq \|v\|_{\infty} \), it follows that the \( I \)-tuple \( (f_i)_{i \in I} \) belongs to \( \ell^\infty(M^b_K(X_i, \mathcal{A}|_{X_i}, \mu|_{X_i}), \| \cdot \|_{\sup})_{i \in I} \). Since by definition, \( \bigvee_{i \in I} (\mathcal{A}|_{X_i}) = \mathcal{A}_\pi \), by Exercise 24, it follows that there exists \( f \in M^b_K(X, \mathcal{A}_\pi) \), such that \( f_i = f|_{X_i}, \forall i \in I \), and then the element \( u = [f] \in L^\text{loc,\infty}_K(X, \mathcal{A}_\pi, \mu_\pi) \) will clearly satisfy the equality \( D_{\text{loc,\infty}} u = v \). \( \square \)

\(^{10}\) The reverse inequality is always true, by the properties of outer measures.
Exercise 25. Complete the proof of Theorem 3.

Comment. Theorem 3 clarifies the reassembly/disassembly problem for (58), as follows. If \( \pi = (X_i)_{i \in I} \) is \( \mu \)-integrable, then using the isometric identification (67), we can think the disassembly map (58) as an inclusion

\[
L^\text{loc,}\infty_{K}(X, \mathcal{A}, \mu) \subseteq L^\text{loc,}\infty_{K}(X, \mathcal{A}_\pi, \mu_\pi),
\]

so the condition that (58) is an isomorphism is equivalent to the equality

\[
L^\text{loc,}\infty_{K}(X, \mathcal{A}, \mu) = L^\text{loc,}\infty_{K}(X, \mathcal{A}_\pi, \mu_\pi).
\] (72)

Of course, again by Theorem 3, we also have the inclusion

\[
L^\text{loc,}\infty_{K}(X, \mathcal{A}_\pi, \mu_\pi) \subseteq L^\text{loc,}\infty_{K}(X, \mathcal{A}^c, \mu^c),
\]

so a sufficient condition for (72) is the equality

\[
L^\text{loc,}\infty_{K}(X, \mathcal{A}, \mu) = L^\text{loc,}\infty_{K}(X, \mathcal{A}^c, \mu^c).
\] (73)

Of course, one way to achieve (72) is to require the equality \( \mathcal{A}_\pi = \mathcal{A} \), which is automatic if \( I \) is countable. In general, one is also interested in having (73), and then the following terminology will be helpful (see Theorems 4 and 5 below).

**Definition.** Suppose \((X, \mathcal{A}, \mu)\) is a measure space. An \( \mathcal{A} \)-partition \( \pi = (X_i)_{i \in I} \) is called a quasi-decomposition of \((X, \mathcal{A}, \mu)\), if:

(A) \( \pi \) is \( \mu \)-integrable;

(b) \( \mu(X_i) < \infty, \forall i \in I \).

If in addition to these conditions, one also has

(c) \( \mathcal{A}_\pi = \mathcal{A} \),

then \( \pi \) is called a decomposition of \((X, \mathcal{A}, \mu)\).

**Theorem 4.** Let \( \pi = (X_i)_{i \in I} \) be a quasi-decomposition of the measure space \((X, \mathcal{A}, \mu)\).

(i) For a subset \( S \subseteq X \), the following are equivalent:

(a) \( S \) belongs to \( \mathcal{A}^c \);

(b) \( S \cap X_i \) belongs to \( \mathcal{A}^c \), for each \( i \in I \);

(c) there exist \( A \in \mathcal{A}_\pi \) and a locally \( \mu^c \)-negligible set \( N \in \mathcal{A}^c \), such that \( S = A \setminus N \).

(ii) The inclusion \( L^\text{loc,}\infty_{K}(X, \mathcal{A}_\pi, \mu_\pi) \subseteq L^\text{loc,}\infty_{K}(X, \mathcal{A}^c, \mu^c) \) is an equality.

(iii) When viewed as an \( \mathcal{A}_\pi \)-partition, \( \pi \) is a decomposition of \((X, \mathcal{A}_\pi, \mu_\pi)\).

(iii’) When viewed as an \( \mathcal{A}^c \)-partition, \( \pi \) is a decomposition of \((X, \mathcal{A}^c, \mu^c)\).
Proof. (i). Let \( \mu^* \) be the Caratheodory outer measure associated with \((X, \mathcal{A}, \mu)\), which by Theorem 3 also coincides with the Caratheodory outer measure associated with \((X, \mathcal{A}, \mu)\). The implications (c) \(\Rightarrow\) (a) \(\Rightarrow\) (b) are trivial. To prove the implication (b) \(\Rightarrow\) (c), fix \( S \subset X \), satisfying condition (b), and let us indicate how the sets \( A \) and \( N \) can be constructed, so that (c) is satisfied. First of all, for every \( i \in I \), the set \( S_i = S \cap X_i \) is \( \mu^* \)-measurable (i.e. \( S_i \) belongs to \( \mathcal{A}^c \)), and has \( \mu^*(S_i) \leq \mu^*(X_i) = \mu(X_i) < \infty \). So in particular, there exists \( A_i \in \mathcal{A} \), such that \( A_i \supseteq S_i \), and \( \mu^*(S_i) = \mu^*(S_i) = \mu(A_i) \). Replacing \( A_i \) with \( A_i \cap X_i \), we can also assume that \( A_i \subset X_i \). Of course, by finiteness, the set \( N_i = A_i \cap S_i \in \mathcal{A}^c \) has \( \mu^*(N_i) = \mu^*(N_i) = 0 \). Define then the sets \( A = \bigcup_{i \in I} A_i \) and \( N = \bigcup_{i \in I} N_i \). On the one hand, since \( A \cap X_i = A_i \in \mathcal{A} \), it is clear that \( A \) belongs to the direct sum \( \sigma \)-algebra \( \bigvee_{i \in I} (A|_{X_i}) = \mathcal{A}^\pi \). On the other hand, for every \( i \in I \), the set \( X_i \cap N = N_i \) has \( \mu^*(N_i) = \mu^*(N_i) = 0 \), so we can use Theorem 3 to conclude that \( N \) belongs to \( \mathcal{A}^c \) and is locally \( \mu^c \)-neglijeable. Of course, by construction we also have \( S = A \setminus N \).

(ii). Since we already have an isometric inclusion \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}_\pi, \mu_\pi) \subset L_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) \), all we need to do is to show that \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}_\pi, \mu_\pi) \) contains the dense subspace \( E_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) \), which means that, for every \( S \in \mathcal{A}^c \), there exists \( A \in \mathcal{A}^c \), such that \( |\chi_A| = |\chi_S| \), in \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) \). But this is immediate from part (i).

Statement (iii) is immediate from Theorem 3. To prove statement (iii') we again invoke Theorem 3, which gives \( \mu^c \)-integrability of \( \pi \), so the only feature we must show is the equality \( (\mathcal{A}^c)^\pi = \mathcal{A}^c \). But this equality is trivial from statement (i).

\[ \square \]

Definition. A measure space \((X, \mathcal{A}, \mu)\) is said to be (quasi-)decomposable, it it admits a (quasi-)decomposition. With this terminology, the reassembly/disassembly problem for (58) has the following solution.

**Theorem 5.** If \((X, \mathcal{A}, \mu)\) is a decomposable measure space, then the inclusion

\[ L_{K}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \subset L_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) \]

is an equality. In particular, for any \( \mu \)-integrable \( \mathcal{A} \)-partition \((X_i)_{i \in I}\) of \( X \), the disassembly map \( D_{\text{loc}, \infty} : L_{K}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \to \ell^\infty(L_{K}^{\text{loc}, \infty}(X_i, \mathcal{A}_i, \mu|_{X_i}), \| \cdot \|_{\text{loc}, \infty})_{i \in I} \) is an isometric isomorphism.

Proof. Fix a decomposition \( \kappa = (Y_i)_{i \in J} \) for \((X, \mathcal{A}, \mu)\). On the one hand, by Theorem 4 we know that \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) = L_{K}^{\text{loc}, \infty}(X, \mathcal{A}_\kappa, \mu_\kappa) \). On the other hand, since \( \mathcal{A}_\kappa = \mathcal{A} \), we obviously have \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}_\kappa, \mu_\kappa) = L_{K}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \), so the preceding equality also gives the equality: \( L_{K}^{\text{loc}, \infty}(X, \mathcal{A}^c, \mu^c) = L_{K}^{\text{loc}, \infty}(X, \mathcal{A}, \mu) \). The second statement follows from Theorem 3, and the Comment that followed it.

**Comments.** Decomposability is perhaps the nicest notion in Measure Theory. Among other things, it provides the correct framework for the celebrated Radon-Nikodim Theorem, which will be discussed in HS II, as well as the “correct” dual pairing between the \( L^1 \)-spaces and the \( L_{\text{loc}, \infty} \)-spaces. The two (easy) examples below will be complemented by another one, in sub-section E.

**Example 5.** Every \( \sigma \)-finite measure space \((X, \mathcal{A}, \mu)\) is decomposable. Indeed, if we choose a countable \( \mathcal{A} \)-partition \( \pi = (X_i)_{i \in I} \), with all \( X_i \) of finite measure, then obviously \( \pi \) is a decomposition.
Example 6. If \((X, \mathcal{A}, \mu)\) is a quasi-decomposable measure space, then by Theorem 4, the Caratheodory completion \((X, \mathcal{A}^c, \mu^c)\) is decomposable.

E. Decomposable of Radon measures.

In this sub-section we specialize to the measure spaces \((\Omega, \text{Bor}(\Omega), \mu)\), where \(\Omega\) is a locally compact Hausdorff space, \(\text{Bor}(\Omega)\) is the Borel \(\sigma\)-algebra, and \(\mu\) is a Radon measure. A measure space of this form will be called a locally compact measure space. As in the previous sub-section, to such a measure space one associates the Caratheodory outer measure \(\mu^c\), and the Caratheodory completion \((\Omega, \text{Bor}(\Omega)^c, \mu^c)\). In order not to overload the notation excessively, the \(\sigma\)-algebra \(\text{Bor}(\Omega)^c\) will be simply denoted by \(M(\mu)\). Likewise, the measure \(\mu^c\) will be denoted by \(\mu^c_M\).

In preparation for the main result (Theorem 6 below), it is useful to introduce some terminology, and we do so in the Exercises below.

Exercises 26-27. Suppose \(\Omega\) is a locally compact Hausdorff space, and \(\mu\) is a Radon measure on \(\Omega\). A non-empty compact set \(K \subset \Omega\) is said to be \(\mu\)-tight, if for every non-empty compact proper subset \(L \subset K\), one has the strict inequality \(\mu(L) < \mu(K)\).

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\text{(HINT: If } \mu(K) = 0, \text{ one can take } K_0 \text{ to be a singleton. If } \mu(K) > 0 \text{ and } K \text{ is not } \mu\text{-tight, then the collection } \Lambda(K) = \{L \subset K : L \text{ non-empty, compact, and } \mu(L) = \mu(K)\}\} \text{ is non-empty. Prove that if } L_1, \ldots, L_n \in \Lambda(K), \text{ then } L_1 \cap \ldots \cap L_n \in \Lambda(K), \text{ so using the Finite Intersection Property, the compact set } K_0 = \bigcap_{L \in \Lambda(K)} L \text{ is non-empty. Prove that } K_0 \text{ also belongs to } \Lambda(K), \text{ by showing that } \mu(T) = 0, \text{ for all compact subsets } T \subset K \setminus K_0. \text{ Indeed, if one starts with such a } T, \text{ then } T \cap \bigcap_{L \in \Lambda(K)} L = \emptyset, \text{ so by the Finite Intersection Property and by the above-mentioned property of } \Lambda(K), \text{ there exists } L \in \Lambda(K) \text{ with } T \cap L = \emptyset, \text{ i.e. } T \subset K \setminus L, \text{ thus forcing } \mu(T) \leq \mu(K \setminus L) = 0. \text{ Since } K_0 \text{ is the smallest element in } \Lambda(K), \text{ it is pretty obvious that } K_0 \text{ is } \mu\text{-tight.})}
\)

27\(
\text{Prove that, for every non-empty compact set } K \subset \Omega, \text{ there exists at least one compact } \mu\text{-tight subset } K_0 \subset K, \text{ with } \mu(K \setminus K_0) = 0.
\)

\text{(HINT: If } \mu(K) = 0, \text{ one can take } K_0 \text{ to be a singleton. If } \mu(K) > 0 \text{ and } K \text{ is not } \mu\text{-tight, then the collection } \Lambda(K) = \{L \subset K : L \text{ non-empty, compact, and } \mu(L) = \mu(K)\}\} \text{ is non-empty. Prove that if } L_1, \ldots, L_n \in \Lambda(K), \text{ then } L_1 \cap \ldots \cap L_n \in \Lambda(K), \text{ so using the Finite Intersection Property, the compact set } K_0 = \bigcap_{L \in \Lambda(K)} L \text{ is non-empty. Prove that } K_0 \text{ also belongs to } \Lambda(K), \text{ by showing that } \mu(T) = 0, \text{ for all compact subsets } T \subset K \setminus K_0. \text{ Indeed, if one starts with such a } T, \text{ then } T \cap \bigcap_{L \in \Lambda(K)} L = \emptyset, \text{ so by the Finite Intersection Property and by the above-mentioned property of } \Lambda(K), \text{ there exists } L \in \Lambda(K) \text{ with } T \cap L = \emptyset, \text{ i.e. } T \subset K \setminus L, \text{ thus forcing } \mu(T) \leq \mu(K \setminus L) = 0. \text{ Since } K_0 \text{ is the smallest element in } \Lambda(K), \text{ it is pretty obvious that } K_0 \text{ is } \mu\text{-tight.})
\)

Definition. Suppose \(\eta\) is an outer measure on some set \(X\), and \(\pi = (X_i)_{i \in I}\) is a partition of \(X\). Consider the index set \(I_0 = \{i \in I : \eta(X_i) > 0\}\). We say that a set \(S \subset \Omega\) is \(\eta\)-essentially countably covered by \(\pi\), if:

\(a\) the index set \(I_0(S) = \{i \in I : S \cap X_i \neq \emptyset\}\) is countable;

\(b\) the set \(S_0 = \bigcup_{i \in I_0(S)} S \cap X_i \subset S\) satisfies the equality \(\eta(S \setminus S_0) = 0\).

Note that, by construction (without any conditions imposed on the partition \(\pi\), or on the set \(S\)), one has the equality

\[ S_0 = \bigcup_{i \in I_0} S \cap X_i, \quad \forall S \subset X. \] (74)
Theorem 6 (Compact Decomposability of Radon Measures). Suppose $\Omega$ is a locally compact Hausdorff space, and $\mu$ is a Radon measure on $\Omega$. 

(i) There exists a partition $\pi = (K_i)_{i \in I}$ of $\Omega$, such that

- all $K_i$, $i \in I$, are compact;
- every open set $D \subset \Omega$ with $\mu(D) < \infty$ is $\mu^*$-essentially countably covered by $\pi$.

(ii) If $\pi$ is any partition with the two properties stated above, then:

(A) $\pi$ is a quasi-decomposition\footnote{Since compact sets are Borel, it is obvious that $\pi$ is a $\text{Bor}(\Omega)$-partition.} of $(\Omega, \text{Bor}(\Omega), \mu)$;

(B) when viewed as a $\text{M}(\mu)$-partition, $\pi$ constitutes a decomposition of the measure space $(\Omega, \text{M}(\mu), \mu_{\text{M}})$.

Proof. To prove statement (i), we consider the set $\Theta$ of all collections $U \subset \mathcal{P}(\Omega)$, with the following property:

(*) all sets in $K \in U$ are disjoint, compact, $\mu$-tight, and have $\mu(K) > 0$.

Clearly the empty collection belongs to $\Theta$. One can easily see that if $U \subset \Theta$ is totally ordered by inclusion, then the collection $\bigcup_{U \in \Theta} U$ again satisfies property (*), so using Zorn Lemma, there exists a maximal collection $U_0$ with property (*). In general this does not exclude $U_0$ from being the empty collection. Let us list $U_0 = (K_i)_{i \in I_0}$ (therefore $i \neq j \Rightarrow K_i \cap K_j = \emptyset$), and let us complete the collection $(K_i)_{i \in I_0}$ to a partition $\pi = (K_i)_{i \in I}$ by adding all singleton sets in $\Omega \setminus \left[ \bigcup_{i \in I_0} K_i \right]$, so that, for $i \in I \setminus I_0$, the set $K_i$ is a singleton, and has $\mu(K_i) = 0$. (Of course, for $i \in I \setminus I_0$, the singleton set $K_i$ is $\mu$-tight and is disjoint from all sets in $U_0$, so if $\mu(K_i) > 0$ this will contradict the maximality of $U_0$.) Trivially, $\pi$ satisfies the first condition in (i), and the index set $I_0$ (which could be empty!) is precisely the set of all $i$'s with $\mu(K_i) > 0$. To prove the second condition, we fix an open set $D \subset \Omega$, with $\mu(D) < \infty$, and we must show that:

(a) the index set $I_0(D) = \{ i \in I_0 : D \cap K_i \neq \emptyset \}$ is countable;

(b) the set $D_0 = \bigcup_{i \in I_0(D)} D \cap K_i \subset D$ satisfies the equality $\mu^*(D \setminus D_0) = 0$.

To prove (a) we notice that, since all $K_i$, $i \in I_0$, are $\mu$-tight and have $\mu(K_i) > 0$, by Exercise 26 it follows that, for every $i \in I_0(D)$, we have $\mu(D \cap K_i) > 0$. In particular, for every finite set $F \subset I_0(D)$, we have $\sum_{j \in F} \mu(D \cap K_i) = \mu(\bigcup_{i \in F} D \cap K_i) \leq \mu(D)$, so we get $\sum_{i \in I_0(D)} \mu(D \cap K_i) \leq \mu(D) < \infty$, which clearly forces $I_0(D)$ to be countable.

To prove (b) first notice that, by construction, the set $D_0$ is Borel, being a countable union of the Borel sets $D \cap K_i$, $i \in I_0(D)$, so $\mu^*(D \setminus D_0) = \mu(D \setminus D_0)$. To prove the $\mu(D \setminus D_0) = 0$, we argue by contradiction, assuming that $\mu(D \setminus D_0) > 0$. Since $\mu(D \setminus D_0) \leq \mu(D) < \infty$, by the properties of Radon measures, we have

$$\mu(D \setminus D_0) = \sup \{ \mu(L) : L \subset D \setminus D_0, \; L \text{ compact} \},$$

where $L$ is a compact subset of $D \setminus D_0$. Since $\mu(D \setminus D_0) > 0$, there is a compact subset $L$ of $D \setminus D_0$ such that $\mu(L) > 0$. But then $\mu(D \setminus (D_0 \cup L)) = \mu(D \setminus D_0) - \mu(L) < \mu(D \setminus D_0)$, which is a contradiction, since $\mu(D \setminus D_0) \leq \mu(D) < \infty$.

Therefore, $\mu(D \setminus D_0) = 0$, and we have proven that $\pi$ satisfies the second condition (b).

To prove (B), we first notice that $\pi$ is a $\text{M}(\mu)$-partition, since it satisfies the countable additivity property:

$$\mu \left( \bigcup_{i \in I} K_i \right) = \sum_{i \in I} \mu(K_i).$$

This follows from the fact that $\pi$ is a decomposition of the measure $\mu$.

Hence, $\pi$ is a decomposition of the measure space $(\Omega, \text{M}(\mu), \mu_{\text{M}})$, as claimed.
so in particular there exists some compact set \( L \subset D \setminus D_0 \), with \( \mu(L) > 0 \). By Exercise 27, such an \( L \) can be chosen to be \( \mu \)-tight. But now we reached a contradiction, since \( L \) is disjoint from all sets in \( U_0 \), so the collection \( U_0 \cup \{ L \} \) will contradict the maximality of \( U_0 \).

(ii) Fix \( \pi = (K_i)_{i \in I} \) as in (i). To prove property (A), we must show that

\[
\mu(A) = \sum_{i \in I} \mu(A \cap K_i),
\]

for all Borel sets \( A \), with \( \mu(A) < \infty \). Since, the \( K_i \)'s are disjoint, for every finite set \( F \subset I \) one has \( \sum_{i \in F} \mu(K_i \cap A) = \mu(\bigcup_{i \in F} K_i \cap A) \leq \mu(A) \), and this shows that \( \sum_{i \in I} \mu(K_i \cap A) \leq \mu(A) \).

To prove the reverse inequality, we recall that, by the definition of Radon measures, we know that \( \mu(A) = \inf \{ \mu(D) : D \text{ open, } D \supseteq A \} \), so in particular there exists an open set \( D \supseteq A \), with \( \mu(D) < \infty \). By condition (i) the open set \( D \) is \( \mu \)-essentially countably covered. As argued before, the set \( D_0 = \bigcup_{i \in I_0} D \cap K_i = \bigcup_{i \in I_0(D)} D \cap K_i \) is Borel, and satisfies \( \mu(D \setminus D_0) = 0 \). Of course, the set \( A_0 = A \cap D_0 = \bigcup_{i \in I_0} A \cap K_i \) is also Borel, and because of the inclusion \( A \setminus A_0 \subset D \setminus D_0 \), we also have \( \mu(A \setminus A_0) = 0 \), which yields

\[
\mu(A) = \mu(A_0).
\]

On the other hand, since we can also write \( A_0 = \bigcup_{i \in I(D)} A \cap K_i \) (countable disjoint union), by \( \sigma \)-additivity, combined with (76), we now have

\[
\mu(A) = \mu(A_0) = \sum_{i \in I_0(D)} \mu(A \cap K_i) \leq \sum_{i \in I} \mu(A \cap K_i).
\]

By Theorem 4, property (B) immediately follows from (A).  \( \square \)

**Remark 11.** If \( \Omega \) is a locally compact Hausdorff space, and \( \mu \) is a Radon measure, then for every \( p \in [1, \infty) \), the space \( C_{c,\mathbb{K}}(\Omega) \), of all continuous functions \( f : \Omega \to \mathbb{K} \), with compact support, is dense in \( L^p_\mathbb{K}(\Omega, \mathbf{M}(\mu), \mu_M) \). Indeed, for every \( A \in \mathbf{M}(\mu) \), with \( \mu(A) < \infty \), and every \( \varepsilon > 0 \), there exist a compact set \( K \subset A \), and an open set \( D \supseteq A \), such that \( \mu(D \setminus K) < \varepsilon \), so if we choose, by Urysohn’s Lemma, a continuous function \( f : \Omega \to [0, 1] \), with compact support, such that \( f|_K = 0 \) and \( f|_{\Omega \setminus D} = 0 \), then one has the inequalities \( \chi_K \leq f, \chi_A \leq \chi_D \), which imply that \( \|f - \chi_A\|_p \leq \|\chi_D - \chi_K\|_p \leq \varepsilon^{1/p} \), thus proving that the norm-closure of \( C_{c,\mathbb{K}}(\Omega) \) in \( L^p_\mathbb{K}(\Omega, \mathbf{M}(\mu), \mu_M) \) contains all elementary integrable functions. By Exercise 3 this norm-closure coincides with \( L^p_\mathbb{K}(\Omega, \mathbf{M}(\mu), \mu_M) \).