1 The Setup.

Definition 1.1. We will use the following notations:

1. Let \( n \) and \( m \) be coprime positive integers. Let \( 0 \in \Gamma_{n,m} \subset \mathbb{Z}_{\geq 0} \) be the semigroup generated by \( n \) and \( m \).

2. \( M \subset \mathbb{Z} \) is a semi-module over \( \Gamma_{n,m} \), if \( M + \Gamma_{n,m} \subset M \). We denote the set of semimodules over \( \Gamma_{n,m} \) by \( S\text{Mod}_{n,m} \).

3. A semi-module \( M \) is called 0-normalized if \( 0 \in M \subset \mathbb{Z} \). We denote the set of 0-normalized semimodules over \( \Gamma_{n,m} \) by \( S\text{Mod}^0_{n,m} \).

4. A semi-module \( M \) is called a semi-ideal if \( M \subset \Gamma_{n,m} \). We denote the set of semi-ideals in \( \Gamma_{n,m} \) by \( S\text{Ideal}_{n,m} \).

According to Piontkowski, the Jacobi factor of the singularity \( x^m = y^n \) can be decomposed into affine cells enumerated by \( S\text{Mod}^0_{n,m} \):

\[
\dim(C(M)) = \sum_{0 \leq i < n} \#\{a_i < k < a_i + m : k \notin M \}.
\]

Let \( C = \{x^m = y^n\} \). Let \( \text{Hilb}(C, 0) = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \text{Hilb}^k(C, 0) \) be the local Hilbert scheme of points on \( C \), i.e. the space of ideals in \( \mathbb{C}[x^n, x^m] \). \( \text{Hilb}^k(C, 0) \) is the local Hilbert scheme of \( k \) points, i.e. space of ideals in \( I \subset \mathbb{C}[x^n, x^m] \), such that \( \dim(\mathbb{C}[x^n, x^m]/I) = k \).

According to Oblomkov and Shende, \( \text{Hilb}(C, 0) \) can be decomposed into affine cells enumerated by \( S\text{Ideal}_{n,m} \). The Piontkowski’s arguments can be extended (although it wasn’t written down anywhere yet), so that one gets the following formula for the dimension of the cell \( C(I) \) for \( I \in S\text{Ideal}_{n,m} \):

As before, let \( \{a_0, \ldots, a_{n-1}\} \) be the \( n \)-generators of \( I \). Then

\[
\dim(C(I)) = \sum_{0 \leq i < n} \#\{a_i < k < a_i + m : k \notin M, k \in \Gamma_{n,m} \}.
\]

Also, one naturally gets that \( C(I) \subset \text{Hilb}^{\#(\Gamma_{n,m}\setminus I)}(C, 0) \).

Definition 1.2. Let \( M \in S\text{Mod}_{n,m} \). Let \( \text{dim}^j(M) \) denote the dimension of the cell in the Jacobi factor, which correspond to \( M - \min(M) \).
This definition makes geometric sense only for $M \in SMod_{n,m}^0$, but it will convenient for us to use it for arbitrary semi-modules. Note also that $\dim^i(M + k) = \dim^i(M)$ for any $k \in \mathbb{Z}$

**Definition 1.3.** Let $I \in SIdeal_{n,m}$. Let $\dim^i(I)$ denote the dimension of the corresponding cell in the Hilbert scheme.

We are interested in the bigraded Poincaré series of the local Hilbert scheme, defined as follows:

$$P_{n,m}(q, t) = \sum_{I \in SIdeal_{n,m}} q^\text{#} \{I\} t^{\dim^i(I)},$$

where $P_{Hilb^k(C,0)}(t)$ is the Poincaré polynomial of $Hilb^k(C,0)$.

According to the above considerations, one has the following formula:

$$P_{n,m}(q, t) = \sum_{I \in SIdeal_{n,m}} q^\text{#} \{I\} t^{\dim^i(I)}.$$

### 2 Duality

There is a natural map $\tau : SMod_{n,m} \rightarrow SMod_{n,m}^0$ given by $I \mapsto I - \min(I)$. Let $\pi : SIdeal_{n,m} \rightarrow SMod_{n,m}^0$ be its restriction to $SIdeal_{n,m}$. The fiber $\pi^{-1}(M)$ consist of the semi-ideals of the form $M + k$ for some $k \in \mathbb{Z}_{\geq 0}$. This motivates the following definitions:

**Definition 2.1.** Let $M \in SMod_{n,m}$. We define the dual semi-module $\widehat{M}$ as follows:

$$\widehat{M} = \{k \in \mathbb{Z} : M + k \subset \Gamma_{n,m}\}$$

We also define the renormalized dual semimodule $M^*$ by shifting $\widehat{M}$ to the left, so that $\min(M^*) = 0$.

**Remark:** Easy to see that $M^* \in SMod_{n,m}^0$. For $M \in SMod_{n,m}^0$, one also gets $\widehat{M} \in SIdeal_{n,m}$.

One can use these notations to rewrite the formula for the bigraded Poincaré series:

$$P_{n,m}(q, t) = \sum_{I \in SIdeal_{n,m}} q^\text{#} \{I\} t^{\dim^i(I)} = \sum_{M \in SMod_{n,m}^0} \sum_{k \in \widehat{M}} q^\text{#} \{\Gamma_{n,m}\} t^{\dim^i(M + k)}.$$
In fact, one can replace $S\text{Mod}^{0}_{n,m}$ in the above formula by any subset $S \subset S\text{Mod}_{n,m}$, such that $\tau|_{S} : S \to S\text{Mod}^{0}_{n,m}$ is a bijection:

$$P_{n,m}(q,t) = \sum_{M \in S} \sum_{k \in M} q^{\#(\Gamma_{n,m}\setminus(M+k))} t^{\dim(M+k)}.$$ 

We will need the following Theorem:

**Theorem 2.1.** One has the following formulas:

$$M^* = -(\mathbb{Z}\setminus M) + \max(\mathbb{Z}\setminus M),$$

and

$$\widehat{M} = -(\mathbb{Z}\setminus M) + 2\delta - 1,$$

where $\delta = \frac{(n-1)(m-1)}{2}$.

**Proof.** To check that $M + k \subset \Gamma_{n,m}$, it is enough to check that $2\delta - 1 \notin M + k$. Indeed, $\mathbb{Z}\setminus(M + k)$ is invariant under subtracting $n$ and $m$, and $\mathbb{Z}\setminus\Gamma_{n,m}$ is generated by $2\delta - 1$ over $-n$ and $-m$. So, $2\delta - 1 \notin M + k$ implies $\mathbb{Z}\setminus\Gamma_{n,m} \subset \mathbb{Z}\setminus(M + k)$.

In turn, $2\delta - 1 \notin M + k$ is equivalent to $k \notin -M + 2\delta - 1$, or $k \in -(\mathbb{Z}\setminus M) + 2\delta - 1$. The first formula is then obtained by shifting $-(\mathbb{Z}\setminus M)$ so that it starts from 0. \hfill \Box

**Corollary:** The map $M \mapsto M^*$ is an involution on $S\text{Mod}^{0}_{n,m}$. The map $M \mapsto \widehat{M}$ is an involution on $S\text{Mod}_{n,m}$.

Using the above Theorem, one can further simplify the formula for the bigraded Poincaré polynomial. Take $S = \{M \in S\text{Mod}_{n,m} : \widehat{M} \in S\text{Mod}^{0}_{n,m}\}$. Clearly, $\tau|_{S} : S \to S\text{Mod}^{0}_{n,m}$ is a bijection. Therefore,

$$P_{n,m}(q,t) = \sum_{M \in S} \sum_{k \in M} q^{\#(\Gamma_{n,m}\setminus(M+k))} t^{\dim(M+k)} =$$

$$= \sum_{M \in S\text{Mod}^{0}_{n,m}} \sum_{k \in M} q^{\#(\Gamma_{n,m}\setminus(M+k))} t^{\dim(M+k)} =$$

$$= \sum_{M \in S\text{Mod}^{0}_{n,m}} \sum_{k \in M} q^{\#(\Gamma_{n,m}\setminus(-(\mathbb{Z}\setminus M)+2\delta-1+k))} t^{\dim(-(\mathbb{Z}\setminus M)+2\delta-1+k)}.$$
3 The $\delta$ grading.

Let us have a closer look at the power of $q$. Since $M \mapsto \hat{M}$ is an involution, $\hat{M} + k \subset \Gamma_{n,m}$ for any $k \in M$. In particular, since $M \in SMod^{0}_{n,m}$, we get $\hat{M} \in SIdeal_{n,m}$. It follows immediately, that

$$\# \{ \Gamma_{n,m} \setminus (\hat{M} + k) \} = \# \{ \Gamma_{n,m} \setminus \hat{M} \} + k.$$  

So, it is enough to calculate $\# \{ \Gamma_{n,m} \setminus \hat{M} \}$. Note that $\Gamma_{n,m} \supset \hat{M} \supset \mathbb{Z}_{>2\delta}$. Therefore,

$$\# \{ \Gamma_{n,m} \setminus \hat{M} \} = \# \{ 0 \leq k < 2\delta : k \in \Gamma_{n,m}, k \notin \hat{M} \} =$$

$$= \# \{ 0 \leq k < 2\delta : k \in \Gamma_{n,m} \} - \{ 0 \leq k < 2\delta : k \in \hat{M} \} = \delta - \{ 0 \leq k < 2\delta : k \in \hat{M} \}.$$  

Using the Theorem 2.1, one gets

$$\{ 0 \leq k < 2\delta : k \in \hat{M} \} = \{ 0 \leq k < 2\delta : k \in - \mathbb{Z} \setminus \hat{M} + 2\delta - 1 \} =$$

$$= \{ 0 \leq k < 2\delta : 2\delta - 1 - k \in \mathbb{Z} \setminus \hat{M} \} = \{ 0 \leq k < 2\delta : 2\delta - 1 - k \notin \hat{M} \} =$$

$$= \{ -1 < k \leq 2\delta - 1 : k \notin \hat{M} \} = \{ 0 \leq k < 2\delta : k \notin \hat{M} \} =: \delta(M).$$  

Therefore,

$$\# \{ \Gamma_{n,m} \setminus (\hat{M} + k) \} = \delta - \delta(M) + k.$$

4 The homological grading.

Let us have a closer look at the power of $t$ in the formula for the bigraded Poincarè series. One does it by comparing $\dim^{h}(I)$ with $\dim^{j}(I)$ for $I \in SIdeal_{n,m}$.
Theorem 4.1. Let $I \in S\text{Ideal}_{n,m}$, and $k = \min(I) \iff \pi(I) + k = I$. Then

$$\dim^b(I) = \dim^i(I) - \#\{k \in I : k < 2\delta\}.$$ 

Proof. There is an equivalent way to compute $\dim^i(M)$ for a semi-module $M \in S\text{Mod}_{n,m}$. As before, let $\{a_0, \ldots, a_{n-1}\}$ be the $n$-generators of $M$. Let $\{b_0, \ldots, b_{m-1}\}$ be the $q$-cogenerators of $M$ (i.e. the $-q$-generators of $\mathbb{Z}\setminus M$). Then

$$\dim^i(M) = \#\{(a_i, b_j) : a_i < b_j\}.$$ 

Indeed, for any fixed $n$-generator $a_i$ there are exactly $\#\{a_i < k < a_i + m : k \notin M\}$ $m$-cogenerators $b_j$ greater than $a_i$.

Let now $M = I \in S\text{Ideal}_{n,m}$. Not all of the couples $(a_i, b_j)$, $a_i < b_j$ should be counted when computing $\dim^b(I)$. Indeed, it might happen that the corresponding number $k \in (a_i, a_i + m) \setminus M$ is not in $\Gamma_{n,m}$.

Let $\{-m, n - m, 2n - m, \ldots, (m - 1)n - m\}$ be the $m$-cogenerators of $\Gamma_{n,m}$. Note that $(m - 1)n - m = 2\delta - 1$. Let us reorder $\{b_0, \ldots, b_{m-1}\}$ in such a way that $jn - m \equiv b_j \pmod{m}$ for all $0 \leq j < m$. It is easy to see from the above consideration that

$$\dim^i(I) - \dim^b(I) = \#\{(a_i, j) : a_i < jn - m, 0 \leq j < m\}.$$ 

Indeed, $jn - m \leq b_j$, and $a_i < jn - m$ iff the corresponding number $k \in (a_i, a_i + m) \setminus M$ is not in $\Gamma_{n,m}$.

Therefore, to prove the Theorem it is suffices to show that

$$\#\{k \in I : k < 2\delta\} = \#\{(a_i, j) : a_i < jn - m, 0 \leq j < m\}.$$ 

Since all $n$-generators are positive, $(m - 1)n - m = 2\delta - 1$, and $2\delta - 1 \notin I$, one gets

$$\#\{(a_i, j) : a_i < jn - m, 0 \leq j < m\} = \sum_{a_i < 2\delta - 1} \left\lfloor \frac{2\delta - 1 - a_i}{n} \right\rfloor.$$ 

On the other hand

$$\#\{k \in I : k < 2\delta\} = \sum_{0 \leq i < n} \#\{k \in I : k < 2\delta, k \equiv a_i \pmod{n}\} =$$
\[
= \sum_{a_i < 2\delta - 1} \left\lceil \frac{2\delta - 1 - a_i}{n} \right\rceil.
\]

So, for \( M \in SMod_{n,m}^0 \) and \( k \in M \) we get

\[
\dim^h(\widehat{M} + k) = \dim^j(\widehat{M} + k) - \#\{l \in \widehat{M} + k : l < 2\delta\} =
\]

\[
= \dim^j(\widehat{M}) - \#\{l \in -(\Z \setminus M) + 2\delta - 1 + k : l < 2\delta\} = \dim^j(\widehat{M}) - \#\{l \notin M : l > k - 1\} =
\]

\[
= \dim^j(\widehat{M}) - \#\{l \notin M : l > k\}.
\]

(The last equality follows from \( k \in M \).)

For the final step we need the following Lemma

**Lemma 4.1.** For any \( M \in SMod_{n,m} \) one has

\[
\dim^j(M) = \dim^j(-(\Z \setminus M)).
\]

**Proof.** This follows immediately from the fact that the dimension of the Piontkowski’s cell of the Jacobi factor is independent on the choice of order on the semigroup generators \( n, m \). Indeed, if \( \{a_0, \ldots, a_{n-1}\} \) are the \( n \)-generators of \( M \), and \( \{b_0, \ldots, b_{m-1}\} \) are the \( m \)-cogenerators, then \( \{-a_0, \ldots, -a_{n-1}\} \) are the \( n \)-cogenerators of \( \Z \setminus M \), and \( \{-b_0, \ldots, -b_{m-1}\} \) are the \( m \)-generators of \( \Z \setminus M \).

However, there is a purely combinatorial proof as well. \(\square\)

**Corollary:** For any \( M \in SMod_{n,m} \) one has

\[
\dim^j(M) = \dim^j(\widehat{M}) = \dim^j(M^*).
\]
5 The Conclusion.

Combining the results of the two previous sections one gets the following Theorem:

**Theorem 5.1.** One has the following formula for the bigraded Poincaré polynomial of the local Hilbert scheme of the singularity $x^m = y^n$:

$$P_{n,m}(q, t) = \sum_{M \in \text{Mod}^0_{n,m}} \sum_{k \in M} q^{\delta(M) + k \cdot \dim^1(M) - \# \{l \in M : l > k \}}$$