EXAM 2: SOLUTIONS  
MATHEMATICS 320  
SPRING 2004

(1) Show how you would compute \((2348 \times 3, 723, 234) + (7652 \times 3, 723, 234)\) mentally.  
(10 points)  
**Solution.** \((2348 \times 3, 723, 234) + (7652 \times 3, 723, 234) = (2348 + 7652) \times 3, 723, 234 = 10,000 \times 3, 723, 234 = 37,232,340,000\)

(2) Explain why a sequence that begins 2, 6, . . . could be either an arithmetic or a geometric sequence.  
(10 points)  
**Solution.** We could have an arithmetic sequence with initial term 2 and common difference of 4, that is, the sequence 2, 6, 10, 14, . . .  
We could also have a geometric sequence with initial term 2 and common ratio 3, that is, the sequence 2, 6, 12, 18, . . .

(3) Recall if a \(A\) is a set, then \(n(A)\) represents the number of elements in set \(A\). If \(n(A) = 71\), \(n(B) = 53\), \(n(A \cap B) = 27\), what is \(n(A \cup B)\)?  
(10 points)  
**Solution.** The answer is not \(n(A) + n(B) = 71 + 53 = 124\), because you would be counting the elements in \(A \cap B\) twice. Therefore,  
\[
n(A \cup B) = n(A) + n(B) - n(A \cap B) = 124 - 27 = 97.
\]

(4) The following is an example of the German low-stress algorithm. Find 5764 \(\times 43\) using this method and explain how it works.  
(12 points)  

\[
\begin{array}{c}
4967 \\
\times \\
35
\end{array}
\]

\[
\begin{array}{cccc}
2 & 1 \\
1 & 8 & 3 & 5 \\
2 & 7 & 3 & 0 \\
1 & 2 & 4 & 5 \\
2 & 0 \\
\hline
1 & 7 & 3 & 8 & 4 & 5
\end{array}
\]

**Hint:** You might compute 4967 \(\times 35\) using the intermediate algorithm and compare with this algorithm.  
**Solution.** To see what’s going on we first have to figure out the pattern.

\[
\begin{array}{c}
4967 \\
\times \\
35
\end{array}
\]

\[
\begin{array}{cccc}
2 & 1 \\
1 & 8 & 3 & 5 \\
2 & 7 & 3 & 0 \\
1 & 2 & 4 & 5 \\
2 & 0 \\
\hline
1 & 7 & 3 & 8 & 4 & 5
\end{array}
\]

Date: April 14, 2004.
The method works by breaking up the number being multiplied by place value, then multiplying and adding. This can be justified by the associative, commutative, and distributive properties.

Another way to justify this method works is to compute with the intermediate algorithm and rearrange the sums of the resulting products of the multiplication by place value, that is,

\[
\begin{array}{c}
4967 \\
35 \\
35 \\
300 \\
4500 \\
20000 \\
210 \\
1800 \\
27000 \\
120000 \\
173845 \\
\end{array}
\]

\[
\begin{array}{c}
4967 \\
35 \\
210 \\
1800 \\
27000 \\
120000 \\
20000 \\
173845 \\
\end{array}
\]

Using this method to compute \(5764 \times 43\) we have

\[
\begin{array}{c}
5764 \\
\times \\
43 \\
\hline \\
16 \\
2412 \\
2818 \\
2021 \\
15 \\
\hline \\
247852 \\
\end{array}
\]

(5) Show in detail why \(3^4 \cdot 5^4 = (15)^4\). (10 points)

**Solution.** It is not enough just to say that since the exponents are the same we have \(3^4 \cdot 5^4 = (15)^4\). This is true but doesn’t show what’s going on. Using the the definition of an exponents we write

\[
3^4 \cdot 5^4 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = (3 \cdot 5) \cdot (3 \cdot 5) \cdot (3 \cdot 5) \cdot (3 \cdot 5) = (3 \cdot 5)^4 = (15)^4.
\]

(6) (a) Complete the following multiplication table in base five. (6 points)

**Solution.**

\[
\begin{array}{c|cccc}
\times & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 4 & 11 & 13 \\
3 & 3 & 11 & 14 & 22 \\
4 & 4 & 13 & 22 & 31 \\
\end{array}
\]

(b) Compute each of the following division problems in base five. (2 points each)

(i) \(22_{\text{five}} \div 3_{\text{five}}\)

**Solution.** We can rewrite this problem as \(22_{\text{five}} = 3_{\text{five}} \times \_\_\_\_\_\_\_\_\_\_\text{five}. Then using the table in part (a) we see that the answer must be \(4_{\text{five}}\).

(ii) \(13_{\text{five}} \div 4_{\text{five}}\)

**Solution.** We can rewrite this problem as \(13_{\text{five}} = 4_{\text{five}} \times \_\_\_\_\_\_\_\_\_\_\text{five}. Then using the table in part (a) we see that the answer must be \(2_{\text{five}}\).
(7) A student in Junction City has done the following subtraction problem. Determine if the solution is correct. If it is correct, explain why it works. If not, explain what the student did wrong. (10 points)

\[
\begin{array}{ccc}
6 & 12 & 15 \\
- & A^2 & B \\
\hline
4 & 8 & 7
\end{array}
\]

Solution. This method is correct and is called the *equal-additions algorithm*, which was described in homework problem B12 in section 4.2. Some American children who attended school in Germany while their parents were stationed there while serving in the military learned to subtract this way.

(8) For each of the following draw the image of the quadrilateral under the specified transformation. (3 points each)

(a) Reflect about line \( m \).

(b) Rotate 90° counterclockwise about the point \( P \).

(c) Translate parallel to the directed line segment \( \overrightarrow{MN} \).
(9) Find the base that makes the following subtraction problem correct. (8 points)

\[
230 \quad - \quad 93 = 159
\]

**Solution.** The base has to be greater than 9, since the numeral 9 shows up in this problem. One way to figure this out is to set up the problem using the standard algorithm, that is,

\[
\begin{array}{c c c c c c c c c}
& & & & 1 & 12 \\
2 & 3 & 0 & \quad \quad \quad - & 9 & 3 \\
\hline
& & & & 1 & 5 & 9
\end{array}
\]

Now, in any base, 10 represents the base number. So you have to think

\[? - 9 = 5.\]

The number must be 12. Since \(10_{\text{twelve}} = 12\), the problem works out. So, this problem is done in base-12.

(10) Let \(A\) be the set of even whole numbers and let \(B\) be the set of whole numbers that are multiples of 3. (10 points)

(a) Describe the set \(A \cap B\).

**Solution.** \(A = \{0, 2, 4, 6, \ldots\}\) and \(B = \{0, 3, 6, 9, \ldots\}\). This means that \(A \cap B\) is the set of whole numbers that are even and multiples of 3, which is the set of whole numbers that are multiples of 6. Therefore, \(A \cap B = \{0, 6, 12, 18 \ldots\}\)

(b) Is the set \(A \cap B\) closed under multiplication? Is it closed under addition? You must justify your answers.

**Solution.** Since the product of any two multiples of 6 is also a multiple of 6, the set \(A \cap B\) is closed under multiplication. Similarly, since the sum of any two multiples of 6 is also a multiple of 6, the set \(A \cap B\) is closed under addition.