Pretriangulated A_∞-categories

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Preface

Мудрець же Фізику провадив,
І толковав якихсь монадів,
І думав, відкіль взявся світ?

Іван Котляревський,
Вергілієва Енеїда, частина третя, 1798.[1]

Monoids associative up to homotopy appeared first in topology in works of Stasheff in early sixties. Simultaneously he started to study their differential graded algebraic analogues, called A_∞-algebras. In nineties mirror symmetry phenomenon discovered by physicists was spelled by Fukaya in a new language of A_∞-categories. They combine features of categories and of A_∞-algebras. The binary composition in A_∞-categories is associative up to a homotopy which satisfies an equation that holds up to another homotopy, etc. If these homotopies are trivial, we deal with a differential graded category. Bondal and Kapranov gave a notion of a pretriangulated envelope of a differential graded category. They indicated that pretriangulated differential graded categories are in the origin of the truncated notion – triangulated categories of Grothendieck and Verdier. Inspired by all these developments Kontsevich formulated the homological mirror conjecture: equivalence of pretriangulated A_∞-categories associated with a complex manifold and with its symplectic partner. Jointly with Soibelman he proved recently this conjecture for certain type of manifolds.

Summing up, the framework of differential graded categories and functors is too narrow for many problems, and it is preferable to consider wider class of A_∞-functors even dealing with differential graded categories. We have noticed that many features of A_∞-categories and A_∞-functors come from the fact that they form a symmetric closed multicategory. This structure is revealed in the language of comonads.

In the first part of this book the theory of multicategories is presented including its new ingredients: closed multicategories and multicategories enriched in symmetric multicategories. In the second part we apply this theory to (differential) graded $\Bbbk$-linear quivers. We deduce from it various properties of A_∞-categories, A_∞-functors and A_∞-transformations such as unitality – existence of unit or identity morphisms. Then we construct two

[1] A wise man there Physics honored,
He was explaining certain monads,
And thinking: where the world came from?

Ivan Kotlyarevsky, *Eneida*, part III, 1798.

ingredients of pretriangulated categories: the monad of shifts and the Maurer–Cartan monad. First some predecessors are defined, which are monads in the category of graded $\Bbbk$-linear quivers. Then we deduce commutation rules between these monads and the tensor comonad that defines A_∞-categories. This leads to monads on the multicategory of A_∞-categories. Commutation between the monad of shifts and the Maurer–Cartan monad is also important to us, since it turns their product into another monad, the monad of pretriangulated A_∞-categories. The latter A_∞-categories constitute the main subject of the book.

We prove that zeroth homology of a pretriangulated A_∞-category is a strongly triangulated category. As the name suggests, the latter notion due to Maltsiniotis is a triangulated category with stronger axioms. In appendices we do some algebra in 2-categories necessary for taking tensor products of A_∞-categories with differential graded categories.

The results proven in this book open up new possibilities for researchers in algebraic and symplectic geometry, homological algebra, algebraic homotopy theory and category theory. Graduate courses on mirror symmetry or representation theory would benefit from the considered topics in A_∞-category theory as well. We hope that the reader will find the developed techniques useful for his own research.

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Chapter 1

Introduction

A_∞-algebras were introduced by Stasheff in 1963 as algebraic counterpart of his theory of H-spaces, topological monoids associative up to homotopies which are in turn coherent up to higher homotopies, etc. [Sta63]. Since then the theory of A_∞-algebras underwent quiet development motivated by topological applications in works of Smirnov [Smi80, Smi85, Smi93, Smi01], Kadeishvili [Kad80, Kad82, Kad85, Kad88] and some other mathematicians. The interest to the subject grew significantly, when Fukaya suggested a construction of an A_∞-category arising from symplectic geometry data [Fuk93]. This idea was supported and developed further by Kontsevich [Kon95], who studied the phenomenon of mirror symmetry, discovered by physicists. He came up with the homological version of the mirror symmetry conjecture. It stated equivalence of two A_∞-categories, one coming from the symplectic structure of a manifold, another from the complex structure of its mirror manifold. The attempts to face this conjecture required significant efforts on the symplectic and complex sides of the story [Fuk03, KS09, BS04]. It already has led Kontsevich and Soibelman to the proof of homological mirror symmetry conjecture for certain type of manifolds [KS01]. These findings are expected to be summarized in the books by Fukaya, Oh, Ohta and Ono [FOOO09], by Kontsevich and Soibelman [KS07] and by Seidel [Sei08]. On the other hand, the basic notions of A_∞-categories and of A_∞-functors have been studied in works of Fukaya [Fuk02], Keller [Kel01], the second author [Lyu03] and Soibelman [Soi04].

In the present book the A_∞-category theory is elaborated aiming at another application. In 1963 Verdier has materialized the ideas of Grothendieck about homological algebra in the notion of triangulated category [Ver77]. It was, however, clear that the new concept was a truncation of something underlying it. An insight came from Bondal and Kapranov [BK90], who suggested to look for pretriangulated differential graded (dg) categories, whose 0-th homology would give the triangulated category in question. Drinfeld has succeeded in doing this for derived categories [Dri04] by relating quotient constructions for pretriangulated dg-categories and quotients (localizations) of triangulated categories.

Clearly, the class of differential graded functors between pretriangulated dg-categories is too small to provide a sufficient supply of morphisms. We propose to extend this class to unital A_∞-functors. It is advantageous to extend simultaneously the class of objects to pretriangulated A_∞-categories. The aim of this book is to define such notions and to study their properties. Taking 0-th homology we get from these properties some known results in the theory of triangulated (derived) categories. However, work with pretriangulated

A_∞-categories instead of triangulated categories opens up new possibilities. For instance, the quotient pretriangulated dg-category of complexes of quasicoherent sheaves is expected to encode more information about the variety than the corresponding derived category, cf. [BO02, Pol03].

1.1 Conventions and basic notions. We work within set theory in which all sets are elements of some universes [GV73]. In particular, a universe is an element of another universe. One of them, $\mathscr{U}$ (containing an element which is an infinite set) is considered basic. It is an element of some universe $\mathscr{U}' \ni \mathscr{U}$.

A structure is called *$\mathscr{U}$-small* if it consists of sets which are in bijection with elements of the universe $\mathscr{U}$ [GV73, Exposé I.1]. For instance, sets, rings, modules, categories, etc. can be $\mathscr{U}$-small. A category $\mathcal{V}$ is a *$\mathscr{U}$-category* if the sets of morphisms $\mathcal{V}(X,Y)$ are $\mathscr{U}$-small for all objects X, Y of $\mathcal{V}$. A $\mathscr{U}$-category $\mathcal{V}$ is $\mathscr{U}$-small if and only if $\operatorname{Ob}\mathcal{V}$ is a $\mathscr{U}$-small set.

Let $\Bbbk$ be a $\mathscr{U}$-small commutative associative ring with unity. Let **gr** denote the category of graded $\Bbbk$-modules. Let $\mathsf{C}_\Bbbk = \mathbf{dg}$ denote the abelian category of complexes of $\Bbbk$-modules with chain maps as morphisms. It is a closed monoidal category. A *graded quiver* $\mathcal{C}$ always means for us a $\mathscr{U}$-small set of objects $\operatorname{Ob}\mathcal{C}$ together with $\mathscr{U}$-small $\mathbb{Z}$-graded $\Bbbk$-modules of morphisms $\mathcal{C}(X,Y)$, given for each pair $X, Y \in \operatorname{Ob}\mathcal{C}$. A *differential graded category* is a graded quiver $\mathcal{C}$ equipped with differentials $m_1 : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$ of degree 1 and a category structure such that the binary compositions $m_2 : \mathcal{C}(X,Y) \otimes \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ and units $1_X : \Bbbk \to \mathcal{C}(X,X)$ are chain maps. In other terms, a differential graded category is a category enriched in **dg**, or a **dg**-category. The first example of a differential graded category is the **dg**-category $\underline{\mathsf{C}}_\Bbbk = \underline{\mathbf{dg}}$ of complexes of $\Bbbk$-modules.

For any graded $\Bbbk$-module M there is another graded $\Bbbk$-module $sM = M[1]$, its *suspension*, with the shifted grading $(sM)^k = M[1]^k = M^{k+1}$. The mapping $s : M \to sM$ given by the identity maps $M^k \Longrightarrow M[1]^{k-1}$ has degree -1.

Tensor product of homogeneous mappings $f : X \to Y$, $g : U \to V$ between graded $\Bbbk$-modules means the following mapping $X \otimes U \to Y \otimes V$ of the degree $\deg f + \deg g$:

$$(x \otimes u).(f \otimes g) = (-1)^{\deg u \cdot \deg f} x.f \otimes u.g = (-1)^{uf} x.f \otimes u.g.$$

As a rule, we shorten up the usual notation $(-1)^{\deg x}$ to $(-)^x$. In the same spirit, $(-)^{x+y}$ might mean $(-1)^{\deg x + \deg y}$, and $(-)^{xy}$ might mean $(-1)^{\deg x \cdot \deg y}$, etc.

An *A_∞-category* means for us a graded quiver $\mathcal{C}$ with n-ary compositions

$$b_n : s\mathcal{C}(X_0, X_1) \otimes \cdots \otimes s\mathcal{C}(X_{n-1}, X_n) \to s\mathcal{C}(X_0, X_n),$$

of degree 1 given for all $n \geqslant 1$ (we assume for simplicity that $b_0 = 0$) such that $b^2 = 0$ for

the $\Bbbk$-linear map $b : Ts\mathcal{C} \to Ts\mathcal{C}$, whose matrix components are maps

$$b = b_{kl} = \sum_{\substack{r+n+t=k\\ r+1+t=l}} 1^{\otimes r} \otimes b_n \otimes 1^{\otimes t} : T^k s\mathcal{C} \to T^l s\mathcal{C} \tag{1.1.1}$$

of degree 1, where the tensor quiver $Ts\mathcal{C} = \oplus_{n\geqslant 0} T^n s\mathcal{C} = \oplus_{n\geqslant 0}(s\mathcal{C})^{\otimes n}$ is defined as

$$Ts\mathcal{C}(X,Y) = \oplus_{n\geqslant 0} T^n s\mathcal{C}(X,Y) = \oplus^{n\geqslant 0}_{X_1,\dots,X_{n-1}\in \operatorname{Ob}\mathcal{C}} s\mathcal{C}(X,X_1)\otimes\cdots\otimes s\mathcal{C}(X_{n-1},Y).$$

In particular, $T^1 s\mathcal{C} = s\mathcal{C}$, $T^0 s\mathcal{C}(X,X) = \Bbbk$ and $T^0 s\mathcal{C}(X,Y) = 0$ if $X \neq Y$.

The equation $b^2 = 0$ is equivalent to the system of equations for $n > 0$:

$$\sum_{r+k+t=n} (1^{\otimes r} \otimes b_k \otimes 1^{\otimes t}) b_{r+1+t} = 0 : T^n s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y).$$

In particular, $b_1 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ satisfying the equation $b_1^2 = 0$ is a differential. Notice that maps here are composed from the left to the right. In general, the composition of maps, morphisms, functors, etc. is denoted in this book by $fg = f\cdot g = \xrightarrow{f}\xrightarrow{g} = g\circ f$. A function (or a functor) $f : X \to Y$ applied to an element is denoted $f(x) = xf = x.f = x{\bullet}f$ and occasionally fx. Preimage is denoted $f^{-1}(y) = f^{-1}y$.

The compositions b_n determine also operations

$$m_n = \bigl(\mathcal{C}^{\otimes n} \xrightarrow{s^{\otimes n}} (s\mathcal{C})^{\otimes n} \xrightarrow{b_n} s\mathcal{C} \xrightarrow{s^{-1}} \mathcal{C}\bigr)$$

of degree $2-n$. A differential graded category is an example of an A_∞-category for which $b_n = 0$ and $m_n = 0$ if $n > 2$. It has only the differential m_1 and an associative composition m_2, which is a chain map with respect to differentials $1\otimes m_1 + m_1 \otimes 1$ and m_1. For a general A_∞-category m_2 remains a chain map, but fails to be associative. Instead it is associative up to boundary of the homotopy m_3. Jointly m_2 and m_3 satisfy an equation, which holds up to higher homotopy m_4 and so on.

The tensor quiver $T\mathcal{C}$ becomes a counital coalgebra when equipped with the cut comultiplication $\Delta_0 : T\mathcal{C}(X,Y) \to \oplus_{Z\in\operatorname{Ob}\mathcal{C}} T\mathcal{C}(X,Z) \bigotimes_{\Bbbk} T\mathcal{C}(Z,Y)$, $h_1 \otimes h_2 \otimes \cdots \otimes h_n \mapsto \sum_{k=0}^n h_1 \otimes \cdots \otimes h_k \bigotimes h_{k+1} \otimes \cdots \otimes h_n$. The map b given by (1.1.1) is a coderivation with respect to this comultiplication. Thus b is a *codifferential*. Any coderivation $b : Ts\mathcal{C}(X,Y) \to Ts\mathcal{C}(X,Y)$ (thus, $b\Delta = \Delta(1\otimes b + b\otimes 1)$) has form (1.1.1) for $b_n = b\cdot \mathrm{pr}_1 : T^n s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ (in general, b_0 might be non-zero, but we do not consider such structures). Thus one of the approaches to A_∞-categories is to view them as differential graded coalgebras. This suggests to use morphisms of **dg**-coalgebras as A_∞-functors.

An *A_∞-functor* $f : \mathcal{A} \to \mathcal{B}$ is a map of objects $f = \operatorname{Ob} f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, $X \mapsto Xf$ and $\Bbbk$-linear maps $f : Ts\mathcal{A}(X,Y) \to Ts\mathcal{B}(Xf,Yf)$ of degree 0 which agree with the cut comultiplication and commute with the codifferentials b. Such f is determined in a unique

way by its components $f_n = f\,\mathrm{pr}_1 : T^n s\mathcal{A}(X,Y) \to s\mathcal{B}(Xf,Yf)$, $n \geqslant 1$ (we require that $f_0 = 0$):

$$f = f_{nm} = \sum_{i_1+\dots+i_k=n} f_{i_1} \otimes f_{i_2} \otimes \dots \otimes f_{i_k} : T^n s\mathcal{A}(X,Y) \to T^m s\mathcal{B}(Xf,Yf).$$

As shown e.g. in [Lyu03] 0-th homology of an A_∞-category $\mathcal{C}$ is a non-unital $\Bbbk$-linear category. To ensure that $H^0(\mathcal{C})$ were unital we may ask $\mathcal{C}$ to be unital. An A_∞-category $\mathcal{C}$ is *unital* if for any object X of $\mathcal{C}$ there is a cycle ${}_X\mathbf{i}_0^{\mathcal{C}} \in \mathcal{C}(X,X)[1]^{-1}$ such that the chain maps $-({}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$, $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 : s\mathcal{C}(X,Y) \to s\mathcal{C}(X,Y)$ are homotopic to 1. An A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is *unital* if ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 - {}_{Xf}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1$ for all objects X of $\mathcal{C}$.

A_∞-transformations $r : f \to g : \mathcal{A} \to \mathcal{B}$ (not necessarily natural) can be also introduced using coalgebra structure as (f,g)-coderivations, that is, maps $r : Ts\mathcal{A}(X,Y) \to Ts\mathcal{B}(Xf,Yg)$ of certain degree $\deg r$ such that $r\Delta = \Delta(f \otimes r + r \otimes g)$. They are restored from its components $r_n = r \cdot \mathrm{pr}_1 : T^n s\mathcal{A}(X,Y) \to s\mathcal{B}(Xf,Yg)$ via its matrix entries:

$$r_{nm} = \sum_{\substack{k+a+l=n \\ p+1+q=m}} f_{kp} \otimes r_a \otimes g_{lq} : T^n s\mathcal{A}(X,Y) \to T^m s\mathcal{B}(Xf,Yg).$$

Given two A_∞-categories $\mathcal{A}$, $\mathcal{B}$ one can form a third one $A_\infty(\mathcal{A},\mathcal{B})$ ("the category of functors"), whose objects are A_∞-functors $f : \mathcal{A} \to \mathcal{B}$ and graded $\Bbbk$-modules of morphisms $A_\infty(\mathcal{A},\mathcal{B})(f,g)[1]$ are formed by A_∞-transformations $r : f \to g$. The n-ary compositions B_n in $A_\infty(\mathcal{A},\mathcal{B})$ are given by explicit formulae [Fuk02, Lyu03, LH03]. In this book we are going to explain this phenomenon. We show that A_∞-categories form a closed category (not a monoidal one!), and $A_\infty(\mathcal{A},\mathcal{B})$ are inner homomorphism objects in this category. Actually, we show more: A_∞-categories form a closed symmetric multicategory, and we explore consequences of this fact.

Before describing these new features we recall some already known results about A_∞-categories. Objects X, Y of a unital A_∞-category $\mathcal{C}$ are said *isomorphic* if they are isomorphic in the ordinary category $H^0(\mathcal{C})$. If $\mathcal{B}$ is a unital A_∞-category then $\mathcal{C} = A_\infty(\mathcal{A},\mathcal{B})$ is unital as well. According to general terminology, A_∞-functors $f, g : \mathcal{A} \to \mathcal{B}$ are called *isomorphic* if they are isomorphic in $H^0(A_\infty(\mathcal{A},\mathcal{B}))$. Thus, there are natural A_∞-isomorphisms $r : f \to g$, $p : g \to f$. Naturality of r, p means that they are cycles of degree -1 in $A_\infty(\mathcal{A},\mathcal{B})[1]$, which correspond to cycles of degree 0 in $A_\infty(\mathcal{A},\mathcal{B})$. These cycles are mutually inverse in the sense of $H^0(A_\infty(\mathcal{A},\mathcal{B}))$. This gives meaning to the notions of A_∞-equivalences $f : \mathcal{A} \to \mathcal{B}$ (if both $\mathcal{A}$ and $\mathcal{B}$ are unital). The full subcategory of $A_\infty(\mathcal{A},\mathcal{B})$ consisting of unital A_∞-functors is denoted $A^u_\infty(\mathcal{A},\mathcal{B})$. Unital A_∞-categories are objects of a $\mathcal{C}at$-category (a 2-category) $\overline{A^u_\infty}$ with the categories of morphisms $H^0(A^u_\infty(\mathcal{A},\mathcal{B}))$. Natural A_∞-isomorphisms (resp. A_∞-equivalences) are precisely invertible 2-morphisms (resp. quasi-invertible 1-morphisms) in this 2-category.

The following theorem gives a criterion of quasi-invertibility for an A_∞-functor ϕ with values in a unital A_∞-category: ϕ is an A_∞-equivalence iff $H^0(\phi)$ is essentially surjective on objects, and ϕ_1 consists of homotopy isomorphisms. Notice that the source A_∞-category is not assumed unital, but is proven to be such.

1.2 Theorem ([Lyu03, Theorem 8.8])**.** *Let $\mathcal{C}$ be an A_∞-category and let $\mathcal{B}$ be a unital A_∞-category. Let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-functor such that for all objects X, Y of $\mathcal{C}$ the chain map $\phi_1 : (s\mathcal{C}(X,Y), b_1) \to (s\mathcal{B}(X\phi, Y\phi), b_1)$ is homotopy invertible. Let $h : \operatorname{Ob}\mathcal{B} \to \operatorname{Ob}\mathcal{C}$ be a mapping. Assume that each object U of $\mathcal{B}$ is isomorphic to $Uh\phi$ in $H^0(\mathcal{B})$. Then $\mathcal{C}$ is unital, ϕ is an A_∞-equivalence, and there is an A_∞-functor $\psi : \mathcal{B} \to \mathcal{C}$ quasi-inverse to ϕ such that $\operatorname{Ob}\psi = h$.*

The theorem was originally proven in a more refined form, which allows to draw inductive conclusions. If the A_∞-functor ψ and the natural A_∞-isomorphism $r : \mathrm{id}_{\mathcal{B}} \to \psi\phi : \mathcal{B} \to \mathcal{B}$ are constructed up to k-th component, then the inductive procedure implies existence of the remaining components with the required properties.

The following proposition was proven in various forms by Gugenheim and Stasheff [GS86], Merkulov [Mer99], Kontsevich and Soibelman [KS01, Section 6.4], see also the survey [LM08a, Proposition 2.5].

1.3 Proposition. *Let $(s\mathcal{C}, d)$ be a differential graded quiver, let $\mathcal{B}$ be an A_∞-category, and let $f_1 : (s\mathcal{C}, d) \to (s\mathcal{B}, b_1)$ be a chain quiver morphism such that chain maps $f_1 : s\mathcal{C}(X,Y) \to s\mathcal{B}(Xf_1, Yf_1)$ are homotopy invertible for all pairs X, Y of objects of $\mathcal{C}$. Then there is an A_∞-category structure on $\mathcal{C}$ such that $b_1^{\mathcal{C}} = d$ and an A_∞-functor $f : \mathcal{C} \to \mathcal{B}$, whose first component is the given morphism f_1.*

Note that the proof is constructive. Since one knows the contracting homotopy of the cone of a certain homotopy isomorphism, one can write down recursive formulas for the components b_n, f_n, and to express them in terms of trees.

If in the above proposition the A_∞-category $\mathcal{B}$ is unital, the statement combines with the previous theorem. We find out that $\mathcal{C}$ is unital and $f : \mathcal{C} \to \mathcal{B}$ gives an A_∞-equivalence of $\mathcal{C}$ with a full A_∞-subcategory of $\mathcal{B}$.

The above applies, in particular, if the graded $\Bbbk$-modules $\mathcal{C}(X,Y) = H(\mathcal{B}(X,Y), m_1)$ with zero differential are homotopy isomorphic to the complexes $(\mathcal{B}(X,Y), m_1)$ (e.g. if $\Bbbk$ is a field). In this case it is possible to transfer the A_∞-category structure from $\mathcal{B}$ to its homology $\mathcal{C} = H(\mathcal{B})$, cf. Kadeishvili [Kad82]. Notice that $b_1^{\mathcal{C}} = 0$. A_∞-categories with such property are called *minimal*. Composition m_2 in minimal A_∞-categories is strictly associative.

This construction has consequences for the derived category of a $\Bbbk$-linear abelian category, when $\Bbbk$ is a field. The derived category can be presented as zeroth homology of a certain differential graded category $\mathcal{B}$, the quotient of all complexes over acyclic complexes, constructed by Drinfeld [Dri04, Section 3]. Massey products in the derived

category are related to the operations m_n for $H^\bullet(\mathcal{B})$, cf. Lu, Palmieri, Wu, and Zhang [LPWZ06].

Another application of A_∞-equivalences is related to the Yoneda Lemma. Recall that $\underline{\mathsf{C}}_\Bbbk$ denotes the differential graded category of complexes of $\Bbbk$-modules. For any unital A_∞-category $\mathcal{A}$ there is the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to A^u_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)$, constructed by Fukaya [Fuk02, Section 9]. It takes an object X of $\mathcal{A}$ to the A_∞-functor $H^X = \mathcal{A}(X, \_) : \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$, $Z \mapsto \mathcal{A}(X, Z)$. Higher components of H^X and $\mathscr{Y}$ do not vanish, they are given by explicit formulae, which we do not reproduce here. The first result about the Yoneda A_∞-functor is that $\mathscr{Y}$ is an A_∞-equivalence with its image, the full subcategory of the **dg**-category $A^u_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)$ consisting of A_∞-functors H^X. This theorem of Fukaya [Fuk02, Theorem 9.1] was extended to arbitrary A_∞-categories, unital in the above sense, in [LM08c, Proposition A.9]. It has an important corollary:

1.4 Corollary. *Let $\mathcal{A}$ be a unital A_∞-category. Then there is a differential graded category $\mathcal{D}$ with $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathcal{A}$ and an A_∞-equivalence $f : \mathcal{A} \to \mathcal{D}$ such that $\operatorname{Ob} f = \mathrm{id}$.*

More general A_∞-form of the Yoneda Lemma is the following:

1.5 Proposition ([LM08b, Proposition A.3])**.** *Let $\mathcal{A}$ be a unital A_∞-category, let X be an object of $\mathcal{A}$, and let $f : \mathcal{A} \to \underline{\mathsf{C}}_\Bbbk$ be a unital A_∞-functor. Then the natural chain map $Xf \to A^u_\infty(\mathcal{A}, \underline{\mathsf{C}}_\Bbbk)(H^X, f)$, defined by explicit formulae, is homotopy invertible.*

For the case of a ground field $\Bbbk$, this was also proved by Seidel [Sei08, Lemma 2.12].

The following proposition is proven in [LO06]. Several variations of hypotheses and conclusions are listed there.

1.6 Proposition ([LO06, Proposition 8.6])**.** *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories, and let $f : \mathcal{A} \to \mathcal{B}$ be a unital A_∞-functor. Let $g : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ be a mapping. Let cycles ${}_X r_0 s^{-1} \in Z^0\mathcal{B}(Xf, Xg)$ be invertible in $H^0\mathcal{B}$ for all objects X of $\mathcal{A}$. Then the map g extends to a unital A_∞-functor $g : \mathcal{A} \to \mathcal{B}$ and the given ${}_X r_0$ extend to an invertible natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$. Furthermore, such extensions up to level n can be continued to the full A_∞-notion. That is, given A_n-functor $(\operatorname{Ob} g, g_1, \dots, g_n)$ and A_n-transformation $(r_0, r_1, \dots, r_n)$ with invertible r_0 (such that the m-th components of $gb - bg$ and $rb + br$ vanish for all $0 \leqslant m \leqslant n$) extend to a unital A_∞-functor g and to an invertible natural A_∞-transformation r.*

The latter statement is due to inductive nature of the proof of Proposition 8.1 of [LO06].

There are five notions of quotient relevant to this book:

– Verdier's localization of triangulated categories from [Ver77, Ver96];

– Keller–Drinfeld's quotient of dg-categories from [Dri04]. If $\mathcal{B}$ is a full dg-subcategory of $\mathcal{C}$, then by Drinfeld's result there is an equivalence

$$H^0(\text{“Keller–Drinfeld”}\ \mathcal{C}^{\mathsf{tr}}/\mathcal{B}^{\mathsf{tr}}) \simeq \text{“Verdier”}\ H^0(\mathcal{C}^{\mathsf{tr}})/H^0(\mathcal{B}^{\mathsf{tr}});$$

– Drinfeld's quotient of dg-categories from [Dri04]. It is equivalent to Keller–Drinfeld's quotient under homotopy flatness assumptions on $\mathcal{C}$;

– Quotient $\mathsf{D}(\mathcal{C}|\mathcal{B})$ of A_∞-categories from [LO06], where $\mathcal{B}$ is a full A_∞-subcategory of $\mathcal{C}$. It coincides with Drinfeld's quotient if $\mathcal{C}$ is differential graded;

– Quotient $\mathsf{q}(\mathcal{C}|\mathcal{B})$ of unital A_∞-categories from [LM08c]. It is A_∞-equivalent to $\mathsf{D}(\mathcal{C}|\mathcal{B})$.

Let us discuss the last quotient in detail. Let $\mathcal{B}$ be a full A_∞-subcategory of a unital A_∞-category $\mathcal{C}$. For $\mathcal{A}$ a unital A_∞-category, denote by $A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}}$ the full A_∞-subcategory of $A^u_\infty(\mathcal{C},\mathcal{A})$, whose objects are unital A_∞-functors $\mathcal{C}\to\mathcal{A}$ which are contractible when restricted to $\mathcal{B}$. Contractibility means that the first components are null-homotopic.

1.7 Theorem ([LM08c, Theorem 1.3])**.** *In the above assumptions there exists a unital A_∞-category $\mathcal{D}=\mathsf{q}(\mathcal{C}|\mathcal{B})$ and a unital A_∞-functor $e:\mathcal{C}\to\mathcal{D}$ such that the composition $\mathcal{B}\hookrightarrow\mathcal{C}\xrightarrow{e}\mathcal{D}$ is contractible, and the strict A_∞-functor given by composition with e*

$$(e\boxtimes 1)M : A^u_\infty(\mathcal{D},\mathcal{A})\to A^u_\infty(\mathcal{C},\mathcal{A})_{\operatorname{mod}\mathcal{B}},\qquad f\mapsto ef,$$

is an A_∞-equivalence for an arbitrary unital A_∞-category $\mathcal{A}$.

Pretriangulated **dg**-categories appeared in the work of Bondal and Kapranov [BK90] under the name of enhanced differential graded categories. The reason to introduce them lied in deficiencies of triangulated categories. From the very beginning Grothendieck and Verdier [Ver77] understood that the notion of triangulated category is only a truncation of the real story. However they did their best to make triangulated categories into a powerful tool of homological algebra. The study of triangulated categories underwent a quiet development in the frame of the theory in works of Verdier [Ver96], Neeman [Nee01], Brown [Bro62]. On the other hand, there were attempts to build a system of axioms above the axioms of a triangulated category. Grothendieck initiated the theory of derivateurs which is developed further by Maltsiniotis [Mal07] and his collaborators. This modification intended to cure several drawbacks of the usual definition: non-functoriality of the cone, absence of any kind of product of triangulated categories etc. A less radical change originated in works of Neeman on K-theory of triangulated categories, see [Nee05] for a survey. These works combined with the work of Künzer [Kün07a] on Heller triangulated categories led Maltsiniotis to a new notion, that of strongly triangulated categories [Mal06]. It deals with higher dimensional analogs of triangles and octahedra, which brings the system of axioms closer to perfection, although the usual drawbacks are still present.

However, the approach of Bondal and Kapranov seems to be more fruitful. Instead of a mild modification of axioms they suggest to return to the origin of derived categories — to complexes, and to work with differential graded categories $\mathcal{D}$. The link with ordinary $\Bbbk$-linear categories is provided by taking the zeroth homology $H^0(\mathcal{D})$. Bondal and

Kapranov gave conditions for $\mathcal{D}$ which ensure that $H^0(\mathcal{D})$ be triangulated. Precisely, they constructed a monad Pre-Tr : $\mathbf{dg}$-$\mathcal{C}at \to \mathbf{dg}$-$\mathcal{C}at$, which we shall denote $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{tr}}$ and call the pretriangulated envelope in the sequel. The $\mathbf{dg}$-category $\mathcal{C}^{\mathsf{tr}}$ is obtained from $\mathcal{C}$ by adding formal shifts of objects and iterated cones of closed morphisms of degree 0, in other words, by considering twisted complexes, which are formed by solutions to Maurer–Cartan equations. Formally, an object X of $\mathcal{C}^{\mathsf{tr}}$ is a finite sequence of objects $X_1,\dots,X_p$ of $\mathcal{C}$ together with integers $n_1,\dots,n_p$ and elements $u_{ij} \in \mathcal{C}(X_i, X_j)^{n_j - n_i + 1}$, $1 \leqslant i < j \leqslant p$, satisfying the Maurer-Cartan equation

$$u_{ij}m_1^{\mathcal{C}} - \sum_{i<a<j} (u_{ia} \otimes u_{aj})m_2^{\mathcal{C}} = 0, \quad 1 \leqslant i < j \leqslant p, \tag{1.7.1}$$

where $m_1^{\mathcal{C}}$ and $m_2^{\mathcal{C}}$ are the differential and composition in $\mathcal{C}$ respectively. If X and Y are objects of $\mathcal{C}^{\mathsf{tr}}$ specified by data $(X_i, n_i, u_{ij})_{1\leqslant i<j\leqslant p}$ and $(Y_k, m_k, v_{kl})_{1\leqslant k<l\leqslant q}$, then the graded $\Bbbk$-module of morphisms between X and Y is given by

$$\mathcal{C}^{\mathsf{tr}}(X, Y) = \prod_{i=1}^{p} \prod_{k=1}^{q} \mathcal{C}(X_i, Y_k)[m_k - n_i].$$

Elements of $\mathcal{C}^{\mathsf{tr}}(X, Y)$ can be thought as $p \times q$-matrices; composition $m_2^{\mathcal{C}^{\mathsf{tr}}}$ in $\mathcal{C}^{\mathsf{tr}}$ is given by matrix multiplication. Putting $u_{ij} = 0$ if $i \geqslant j$, we may view elements u_{ij} as entries of a matrix $u \in \mathcal{C}^{\mathsf{tr}}(X, X)^1$. Denoting by d the naïve (componentwise) differential in $\mathcal{C}^{\mathsf{tr}}(X, Y)$, we can rewrite Maurer-Cartan equation (1.7.1) as $ud - u^2 = 0$. The differential in $\mathcal{C}^{\mathsf{tr}}$ is given by $fm_1^{\mathcal{C}^{\mathsf{tr}}} = fd - fv + (-)^{\deg f} uf$, for each $f \in \mathcal{C}^{\mathsf{tr}}(X, Y)$.

The multiplication $\mathrm{Tot}_{\mathcal{A}} : \mathcal{A}^{\mathsf{tr}\,\mathsf{tr}} \to \mathcal{A}^{\mathsf{tr}}$ in the monad tr is an equivalence of $\mathbf{dg}$-categories, which justifies the name of an envelope. The notation Tot is reminiscent of the total complex of a bicomplex. The unit of the monad tr is the natural embedding $u_{\mathsf{tr}} : \mathcal{C} \to \mathcal{C}^{\mathsf{tr}}$ of a $\mathbf{dg}$-category $\mathcal{C}$ into its pretriangulated envelope. If the corresponding fully faithful embedding $H^0(u_{\mathsf{tr}}) : H^0(\mathcal{C}) \to H^0(\mathcal{C}^{\mathsf{tr}})$ is essentially surjective on objects, $\mathcal{C}$ is called *pretriangulated* according to [Dri04, Section 2.4]. Equivalently, an object of $\mathcal{C}^{\mathsf{tr}}$ is isomorphic to an object of $\mathcal{C}$ in $H^0(\mathcal{C}^{\mathsf{tr}})$. Pretriangulated $\mathbf{dg}$-categories are called also *exact* by Keller [Kel06b, Section 4.5]. For a pretriangulated $\mathbf{dg}$-category $\mathcal{D}$ the category $H^0(\mathcal{D})$ is naturally triangulated.

The next question arose naturally: How can one obtain derived categories as zeroth homology of $\mathbf{dg}$-categories? The answer was given by Drinfeld [Dri04] who reproduced independently and pushed further an earlier construction of Keller [Kel99]. With an abelian category $\mathcal{A}$ the $\mathbf{dg}$-category $\mathcal{C} = \mathrm{Com}(\mathcal{A})$ of complexes in $\mathcal{A}$ and its $\mathbf{dg}$-subcategory $\mathcal{B} = \mathrm{AcycCom}(\mathcal{A})$ of acyclic complexes are associated. Then the zero homology of Keller-Drinfeld's quotient $\mathcal{D} = \mathcal{C}/\mathcal{B}$ is the unbounded derived category of $\mathcal{A}$.

The approach to differential graded categories developed by Toën is to consider a model structure on the category of small $\mathbf{dg}$-categories. His theory has some very practical consequences. For instance, Toën and Vaquié prove that a smooth compact complex

variety X is an algebraic space if and only if a certain **dg**-category $L_{parf}(X)$ is saturated [TV07]. This **dg**-category is formed by perfect complexes of sheaves of $\mathcal{O}_X$-modules.

1.8 The new features introduced in this book. Differential graded categories arise naturally in the algebraic geometry as the categories of complexes of sheaves. In the light of Drinfeld's results [Dri04], it is tempting to develop the entire homological algebra in the language of **dg**-categories. However, this seems hardly accessible if one confines to **dg**-functors only, simply because the supply of **dg**-functors between **dg**-categories is too poor. We share the confidence that the more flexible notion of unital A_∞-functor may help to get around the problem. For example, a **dg**-functor may not be invertible as a **dg**-functor, but it may have a quasi-inverse which is an A_∞-functor. Another example occurred in this book: there are two naturally arising **dg**-functors quasi-inverse to each other in A_∞-sense, but not in **dg**-sense, see Remark 13.34. We suggest therefore to enlarge the category of **dg**-categories, using unital A_∞-functors, and to consider A_∞-transformations between them. Then there is no reason not to extend the context to unital A_∞-categories!

Corollary 1.4 might create an impression that unital A_∞-categories being A_∞-equivalent to **dg**-categories are not even needed. This is false by several arguments. The first is natural appearance of Fukaya's A_∞-categories in symplectic geometry. Their objects are Lagrangian submanifolds with additional data. In order to replace an A_∞-category $\mathcal{A}$ with a **dg**-category via the Yoneda construction, one has to deal with A_∞-functors $\mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{C}}_{\Bbbk}$, which is suitable for theoretical reasoning, but not for practical computations. The study of A_∞-functors is inevitable anyway.

The second reason is that some constructions beginning with **dg**-categories end up in unital A_∞-categories. Such is the quotient $\mathsf{q}(\mathcal{C}|\mathcal{B})$ from [LM08c] which yields a unital A_∞-category even if $\mathcal{C}$ were a **dg**-category. Universality is proven precisely for this quotient. For an A_∞-equivalent quotient $\mathsf{D}(\mathcal{C}|\mathcal{B})$, which is a **dg**-category if $\mathcal{C}$ is, a direct proof of universality is not known. Notice also that $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is unital, but is not strictly unital, in general. Thus, strictly unital A_∞-categories do not cover all needs, and one should work with unital A_∞-categories.

The fundamental notions of category and functor were introduced by Eilenberg and Mac Lane in order to be able to consider natural transformations. In fact, once we have the concepts of category and functor, the notion of natural transformation just pops out. Indeed, the category $\mathcal{C}at$ of (small) categories is monoidal; it comes with the natural cartesian product of categories. In particular, we may consider functors of many arguments; these are functors from a cartesian product of categories. Now, for each pair of small categories $\mathcal{A}$ and $\mathcal{C}$, there is a natural category $\underline{\mathcal{C}at}(\mathcal{A}, \mathcal{C})$ with the following universal property: the functors from $\mathcal{A} \times \mathcal{B}$ to $\mathcal{C}$ are in bijection with the functors from $\mathcal{B}$ to $\underline{\mathcal{C}at}(\mathcal{A}, \mathcal{C})$, for each small category $\mathcal{B}$. The objects of the category $\underline{\mathcal{C}at}(\mathcal{A}, \mathcal{C})$ are just functors from $\mathcal{A}$ to $\mathcal{C}$, and the morphisms are natural transformations of these functors.

The category of A_∞-categories and A_∞-functors is not monoidal. There is no straightforward notion of tensor product of A_∞-categories. Nonetheless, there is a simple and

natural notion of A_∞-functor of many arguments! The totality of A_∞-categories and A_∞-functors forms a structure called *multicategory*. As in ordinary category theory, the notion of A_∞-transformation is implied by the notions of A_∞-category and A_∞-functor. Specifically, for each pair of A_∞-categories $\mathcal{A}$ and $\mathcal{C}$, there is a natural A_∞-category $A_\infty(\mathcal{A}, \mathcal{C})$ such that the A_∞-bifunctors $\mathcal{A}, \mathcal{B} \to \mathcal{C}$ are in bijection with the A_∞-functors $\mathcal{B} \to A_\infty(\mathcal{A}, \mathcal{C})$, for each A_∞-category $\mathcal{B}$. The objects of $A_\infty(\mathcal{A}, \mathcal{C})$ are just A_∞-functors $\mathcal{A} \to \mathcal{C}$, and the graded $\Bbbk$-modules of morphisms consist of A_∞-transformations.

The described phenomenon is captured by the concept of *closed multicategory*. It is a fairly natural notion, nonetheless, in spite of its naturality it seems not to be covered in the literature. The only reference we are aware of is the paper of Hyland and Power [HP02], where the notion of closed $\mathcal{C}at$-multicategory (i.e., a multicategory enriched in the category $\mathcal{C}at$ of categories) is implicitly present, although not spelled out. The first part of the book aims at filling this gap in the literature. Here we set up the framework of closed multicategories, in which the theory of A_∞-categories is developed in the second part of the book.

The observation that for a pair of A_∞-categories $\mathcal{A}$ and $\mathcal{C}$ there is an A_∞-category of A_∞-functors $A_\infty(\mathcal{A}, \mathcal{C})$ is not new. It is usually attributed to Kontsevich. The advantage of our approach is that it uncovers the nature of these A_∞-categories and places them into a wider context. Another way to understand the origin of the A_∞-category $A_\infty(\mathcal{A}, \mathcal{C})$ from the perspective of non-commutative geometry has been suggested in the recent paper of Kontsevich and Soibelman [KS09].

We do not try to define in this book the external tensor product of arbitrary A_∞-categories such that sets of objects are multiplied directly. Saneblidze and Umble succeeded to do it for A_∞-algebras [SU04]. For unital A_∞-categories there is a roundabout recipe: first replace them with A_∞-equivalent **dg**-categories, then take the external tensor product of **dg**-categories. Here a drawback is that the size of the categories increases. In this book we restrict to the following particular case: a (unital) A_∞-category is externally tensored with one or several **dg**-categories. The output is again a (unital) A_∞-category. We call this multiplication an action of (the Monoidal category of) differential graded categories on (the category of) A_∞-categories. It is given by explicit formulae not using the Yoneda imbedding. We apply this action in order to construct the multifunctor of shifts. Namely, this multifunctor is obtained as the external tensor product with a certain **dg**-category $\mathcal{Z}$ such that $\operatorname{Ob} \mathcal{Z} = \mathbb{Z}$. Monoidality of $\mathcal{Z}$ translates through the action into the multifunctor of shifts being a monad.

Treatment of A_∞-functors and A_∞-transformations of several variables given in this book had already allowed to study Serre A_∞-functors, whose mere definition requires a natural A_∞-isomorphism of two variables [LM08b]. We review this subject as well as A_∞-bimodules in Chapter 14. In order to relate ordinary Serre functors in (triangulated) categories with their A_∞-counterparts, it was crucial to have at hands the multifunctor $\mathsf{k} : \mathsf{A}_\infty^{\mathrm{u}} \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$ and the theory of $\mathcal{C}at$-enriched multicategories from this book. Further properties of k were discovered in [loc.cit.], which allowed to make it into a symmetric

$\mathcal{C}at$-multifunctor $\mathsf{k} : A_\infty^u \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$.

We have observed also that the 0-th homology of a pretriangulated A_∞-category is strongly triangulated in the sense of Maltsiniotis [Mal06]. In particular, it is triangulated in the sense of Grothendieck and Verdier [Ver77].

1.9 Synopsis of the book. In the first part of the book we establish categorical tools and gadgets necessary to approach our goal, A_∞-categories.

A lax (symmetric) Monoidal category is a category equipped with n-ary tensor product functors and functorial coherence morphisms. We start to study lax Monoidal categories in Chapter 2 in the enriched case. The reason to consider categories enriched in a Monoidal category $\mathcal{V}$ is motivated by an application in which $\mathcal{V} = \mathcal{C}at$, worked out in Appendices A–C. Besides, the non-enriched case (that is $\mathcal{V} = \mathcal{S}\mathrm{et}$) is not much easier than the enriched one, if one recalls that the Monoidal category $(\mathcal{S}\mathrm{et}, \times)$ is not strict. People are very much used to ignore that the direct product of sets is not strictly associative. This simplifies the computations indeed, however, most statements can be proven in enriched setting without significantly increasing the length of proofs. A few lengthy proofs are written down in the simplified non-enriched setting.

A (symmetric) *multicategory* [Lam69, Lam89] is a many object version of a (symmetric) operad [May72], or a *colored* (symmetric) operad. An operad is a one-object multicategory. A multicategory C has a class of objects $\operatorname{Ob}\mathsf{C}$ and sets of morphisms $\mathsf{C}(X_1, \dots, X_n; Y) \ni f : (X_i)_{i=1}^n \to Y$, associative compositions and units like in operads. Multicategories are considered in Chapter 3. A lax (symmetric) Monoidal category provides a good example of a (symmetric) multicategory.

In order to prepare the ground for closed multicategories we study in Chapter 4 multicategories enriched in multicategories. Then we pass to the main subject of the chapter. A multicategory C is called *closed* (cf. [Lam69, p. 106]) if for every sequence $X_1, \dots, X_n; Y$ there is an object $\underline{\mathsf{C}}(X_1, \dots, X_n; Y)$ of C and the evaluation morphism $\mathrm{ev} : X_1, \dots, X_n, \underline{\mathsf{C}}(X_1, \dots, X_n; Y) \to Y$, composition with which gives an isomorphism

$$\mathsf{C}\big((Y_j)_{j=1}^m; \underline{\mathsf{C}}((X_i)_{i=1}^n; Z)\big) \to \mathsf{C}\big((X_i)_{i=1}^n, (Y_j)_{j=1}^m; Z\big).$$

When C is a closed symmetric multicategory, the collection $\underline{\mathsf{C}}(X_1, \dots, X_n; Y)$ defines a symmetric multicategory $\underline{\mathsf{C}}$ enriched in C. Details are given in Chapter 4. We also study multifunctors between closed multifunctors and concentrate on their important consequences, closing transformations. These are tools that allow to obtain many devices for free. For instance, our system of notation for mappings between tensor products of graded $\Bbbk$-modules is nothing but the closing transformation for the tensor product functor in the symmetric Monoidal category **gr** of graded $\Bbbk$-modules.

An important working tool for us is the Kleisli construction for multicategories developed in Chapter 5. It uses as an input a *multicomonad* $(T, \Delta, \varepsilon) : \mathsf{C} \to \mathsf{C}$ defined as a multifunctor $T : \mathsf{C} \to \mathsf{C}$ and multinatural transformations $\Delta : T \to TT : \mathsf{C} \to \mathsf{C}$ and

$\varepsilon : T \to \mathrm{Id} : \mathsf{C} \to \mathsf{C}$ such that (T, Δ, ε) is a coalgebra in the strict monoidal category of multifunctors $\mathsf{C} \to \mathsf{C}$. *$T$-coalgebras* in general are objects X of C equipped with the coaction $\delta : X \to TX$ such that

$$\big(X \xrightarrow{\delta} TX \xrightarrow{T\delta} TTX\big) = \big(X \xrightarrow{\delta} TX \xrightarrow{\Delta} TTX\big),$$
$$\big(X \xrightarrow{\delta} TX \xrightarrow{\varepsilon} TTX\big) = \mathrm{id}\,.$$

However, we restrict our attention to free T-coalgebras, those which have the form $(TX, \Delta : TX \to TTX)$. The multicategory they form is what we mean by the Kleisli multicategory of T.

In the second part of the book we make our considerations concrete and step on the road leading to various A_∞-categories. First we study $\Bbbk$-linear graded quivers. The category of graded quivers $\mathscr{Q}$ is equipped in Chapter 6 with two symmetric Monoidal structures: $\boxtimes$ and $\boxtimes_u$. The respective symmetric Monoidal categories are denoted $\mathscr{Q}_p$ and $\mathscr{Q}_u$. We prove that both Monoidal categories are closed in Chapter 7. The symmetric multicategory $\widehat{\mathscr{Q}_u}$ associated with $\mathscr{Q}_u$ has the same objects as $\mathscr{Q}_u$ and the morphisms $\widehat{\mathscr{Q}_u}((\mathcal{A}_i)_{i=1}^n; \mathcal{B}) = \mathscr{Q}_u(\boxtimes_u^{i\in\mathbf{n}}\mathcal{A}_i, \mathcal{B})$. We also need the same constructions for the category ${}^d\mathscr{Q}$ of differential graded quivers, in particular, we use symmetric Monoidal categories ${}^d\mathscr{Q}_p$ and ${}^d\mathscr{Q}_u$. In order to unify the treatment we consider the category ${}^{\mathcal{V}}\mathscr{Q}$ of $\mathcal{V}$-quivers, $\mathcal{V}$ can be the category $\mathbf{gr}$ of graded $\Bbbk$-modules or the category $\mathsf{C}_\Bbbk = \mathbf{dg}$ of differential graded $\Bbbk$-modules (cochain complexes).

There is a lax symmetric Monoidal comonad $T^{\geqslant 1} : \mathscr{Q}_u \to \mathscr{Q}_u$, $\mathcal{C} \mapsto T^{\geqslant 1}\mathcal{C} = \oplus_{n>0} T^n\mathcal{C}$ considered in Chapter 6. The symmetric Monoidal category of $T^{\geqslant 1}$-coalgebras is also closed, as follows from results of Chapter 5 proved in an abstract context. We are mostly interested in the category $\mathscr{Q}_u{}^{T^{\geqslant 1}}$ of free $T^{\geqslant 1}$-coalgebras, that is, quivers of the form $T^{\geqslant 1}\mathcal{C}$. It is a closed category, and it can be obtained via Kleisli construction from $\mathscr{Q}_u$ and $T^{\geqslant 1}$, but this is not the point. What we are really interested in is its closed symmetric multicategory structure.

The symmetric multicategory $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ can be obtained via Kleisli construction for the associated multicomonad $T^{\geqslant 1} : \widehat{\mathscr{Q}_u} \to \widehat{\mathscr{Q}_u}$. It has the same objects as $\mathscr{Q}_u$ and the morphisms

$$\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}((\mathcal{A}_i)_{i=1}^n; \mathcal{B}) = \widehat{\mathscr{Q}_u}((T^{\geqslant 1}\mathcal{A}_i)_{i=1}^n; \mathcal{B}) = \mathscr{Q}_u(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}).$$

The main subject of this book, the symmetric multicategory A_∞ of A_∞-categories, is obtained from the above one via shift by $s = [1]$ and by imposing differentials b on objects in Chapter 8. Objects of A_∞ are A_∞-categories and morphisms $f : \mathcal{A}_1, \dots, \mathcal{A}_n \to \mathcal{B}$ are A_∞-functors with n entries, which can be identified with differential graded coalgebra maps $f : \boxtimes_{i=1}^n Ts\mathcal{A}_i \to Ts\mathcal{B}$, whose restriction to $\boxtimes_{i=1}^n T^0\mathcal{A}_i$ vanishes. The multicategory A_∞ is closed. Objects of the A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i=1}^n; \mathcal{B})$ are A_∞-functors with n entries. Its morphisms are A_∞-transformations between such A_∞-functors f and g, that is, (f, g)-coderivations.

The category of augmented counital coassociative coalgebras in $\mathscr{Q}$ is denoted $\mathrm{ac}\mathscr{Q}$. Similar category in ${}^d\mathscr{Q}$ (the category of augmented differential counital coassociative coalgebras in $\mathscr{Q}$) is denoted $\mathrm{ac}{}^d\mathscr{Q}$. One can define the symmetric multicategory A_∞ via pull-back squares

$$\begin{tikzcd}
\mathsf{A}_\infty \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{{}^d\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathrm{ac}{}^d\mathscr{Q}_p} \arrow[d] \\
\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \arrow[r, hook] & \widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, hook] & \widehat{\mathrm{ac}\mathscr{Q}_p}
\end{tikzcd}$$

where the horizontal arrows are full embeddings of symmetric multicategories and vertical arrows are faithful multifunctors.

A differential Hopf algebra $(Ts[A_\infty(\mathcal{A},\mathcal{A})(\mathrm{id},\mathrm{id})],\Delta_0,\mathrm{pr}_0,M,\mathrm{in}_0,B)$ is associated with an A_∞-category $\mathcal{A}$ due to closedness of A_∞. This structure turns the generalization $A_\infty(\mathcal{A},\mathcal{A})(\mathrm{id},\mathrm{id})$ of cohomological Hochschild complex into a homotopy Gerstenhaber algebra which is a particular case of a B_∞-algebra [GJ94, VG95, Vor00]. In particular, it has a Lie bracket, which is a generalization of the original Gerstenhaber bracket in Hochschild cochain complex [Ger63]. The cohomology $H^\bullet(sA_\infty(\mathcal{A},\mathcal{A})(\mathrm{id},\mathrm{id}),B_1)$ (generalization of Hochschild cohomology) is a non-unital Gerstenhaber algebra. All this we obtain as a consequence of A_∞-enriched multicategory structure of $\underline{\mathsf{A}_\infty}$.

The symmetric submulticategory $\mathsf{A}^{\mathrm{u}}_\infty$ consists of unital A_∞-categories and unital A_∞-functors $f:\mathcal{A}_1,\dots,\mathcal{A}_n\to\mathcal{B}$, most important for applications. A unital A_∞-functor of several arguments can be defined as such that its restriction to each entry is unital. This multicategory is also closed as shown in Chapter 9. Objects of the unital A_∞-category $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i=1}^n;\mathcal{B})$ are unital A_∞-functors. Its morphisms are all A_∞-transformations between such A_∞-functors.

For any A_∞-category $\mathcal{A}$ there is a strictly unital A_∞-category $\mathcal{A}^{\mathsf{su}}$, containing $\mathcal{A}$ and formally added strict unit elements. Denote by $u_{\mathsf{su}}:\mathcal{A}\hookrightarrow\mathcal{A}^{\mathsf{su}}$ the natural embedding. We prove in Chapter 9 that the A_∞-functor $A_\infty(u_{\mathsf{su}},\mathcal{C}):A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C})\to A_\infty(\mathcal{A},\mathcal{C})$ is an A_∞-equivalence for an arbitrary unital A_∞-category $\mathcal{C}$. Moreover, this functor admits a one-sided inverse $F_{\mathsf{su}}:A_\infty(\mathcal{A},\mathcal{C})\to A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C})$, that is, $F_{\mathsf{su}}\cdot A_\infty(u_{\mathsf{su}},\mathcal{C})=\mathrm{id}_{A_\infty(\mathcal{A},\mathcal{C})}$. Consequently, the pair $(\mathcal{A}^{\mathsf{su}},u_{\mathsf{su}}:\mathcal{A}\hookrightarrow\mathcal{A}^{\mathsf{su}})$ unitally represents the A^u_∞-2-functor $A^u_\infty\to A^u_\infty$, $\mathcal{C}\mapsto A_\infty(\mathcal{A},\mathcal{C})$. We may interpret this as follows: the strictly unital envelope of an A_∞-category is simultaneously its unital envelope in the weak sense.

In order to explain what are pretriangulated A_∞-categories we describe the relevant monad $\mathcal{C}\mapsto\mathcal{C}^{\mathsf{tr}}$. It is a product (composition) of two monads: the monad of shifts and the Maurer–Cartan monad. The functor of shifts $-^{[]}$ from (unital) A_∞-categories to (unital) A_∞-categories assigns to an A_∞-category $\mathcal{C}$ the A_∞-category $\mathcal{C}^{[]}$ obtained by adding formal shifts of objects. Thus, $\mathrm{Ob}\,\mathcal{C}^{[]}=(\mathrm{Ob}\,\mathcal{C})\times\mathbb{Z}$. An object of $\mathcal{C}^{[]}$ is denoted $X[n]=(X,n)$. The functor of shifts is a monad. The A_∞-category $\mathcal{C}$ is embedded into $\mathcal{C}^{[]}$ via $X\mapsto X[0]$. This is the unit of the monad $-^{[]}$. The details are given in Chapter 10. We say that a

unital A_∞-category $\mathcal{C}$ is *closed under shifts* if any object of $\mathcal{C}^{[]}$ is isomorphic to an object of the type $X[0]$ which came from some object X of $\mathcal{C}$.

The Maurer–Cartan monad $-^{\mathsf{mc}}$ from (unital) A_∞-categories to (unital) A_∞-categories assigns to an A_∞-category $\mathcal{C}$ the A_∞-category $\mathcal{C}^{\mathsf{mc}}$. An object of $\mathcal{C}^{\mathsf{mc}}$ is an ordered sequence $(X_i)_{i=1}^n$ of objects of $\mathcal{C}$, together with elements $x_{ij} \in \mathcal{C}(X_i, X_j)[1]^0$ such that $x_{ij} = 0$ for $i \geqslant j$ (that is, the matrix $x = (x_{ij})$ is upper triangular) and the Maurer–Cartan equation holds:

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (x_{ik_1} \otimes x_{k_1k_2} \otimes \dots \otimes x_{k_{m-1}j}) b_m^{\mathcal{C}} = 0.$$

This agrees with the case of **dg**-categories. Using the shorthand $\otimes$ for the matrix–tensor product we may write this equation as $\sum_{m>0}(x^{\otimes m})b_m = 0$. The graded $\Bbbk$-modules of morphisms in $\mathcal{C}^{\mathsf{mc}}$ are

$$\mathcal{C}^{\mathsf{mc}}\big(((X_i)_{i=1}^n, x), ((Y_j)_{j=1}^m, y)\big) = \prod_{i=1}^n \prod_{j=1}^m \mathcal{C}(X_i, Y_j).$$

Its elements r can be viewed as $n \times m$-matrices. The compositions in $\mathcal{C}^{\mathsf{mc}}$ are given in matrix notation by

$$b_n^{\mathsf{mc}} : s\mathcal{C}^{\mathsf{mc}}(X^0, X^1) \otimes \dots \otimes s\mathcal{C}^{\mathsf{mc}}(X^{n-1}, X^n) \to s\mathcal{C}^{\mathsf{mc}}(X^0, X^n),$$
$$(r^1 \otimes \dots \otimes r^n) b_n^{\mathsf{mc}} = \sum_{t_0,\dots,t_n \geqslant 0} \big[(x^0)^{\otimes t_0} \otimes r^1 \otimes (x^1)^{\otimes t_1} \otimes \dots \otimes r^n \otimes (x^n)^{\otimes t_n}\big] b_{t_0+\dots+t_n+n}^{\mathcal{C}}.$$

Informally we may say that $\mathcal{C}^{\mathsf{mc}}$ is obtained from $\mathcal{C}$ by adding cones of morphisms of $\mathcal{C}$, cones of morphisms between cones, etc. The details are given in Chapter 11.

The embedding $\mathcal{C} \hookrightarrow \mathcal{C}^{\mathsf{mc}}$, $X \mapsto ((X), 0)$ is the unit of the monad $-^{\mathsf{mc}}$. We say that a unital A_∞-category $\mathcal{C}$ is mc-*closed* if any object of $\mathcal{C}^{\mathsf{mc}}$ is isomorphic to an object which came from $\mathcal{C}$. A *pretriangulated* A_∞-category is defined as a unital A_∞-category, closed under shifts and mc-closed.

The above description of $\mathcal{C}^{\mathsf{mc}}$ and $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ is rather close to the case of differential graded categories considered by Bondal and Kapranov [BK90]. However, there is another approach to $\mathcal{C}^{\mathsf{mc}}$ which reveals its A_∞-character even if $\mathcal{C}$ is differential graded. Define the A_∞-category $\mathbf{n}$ with objects $1, 2, \dots, n$ and the $\Bbbk$-modules of morphisms

$$\mathbf{n}(i,j)[1] = \begin{cases} 0, & i \geqslant j \\ \Bbbk, & i < j \end{cases}.$$

Necessarily, all compositions b_k for $\mathbf{n}$ vanish. Then $\mathcal{C}^{\mathsf{mc}}$ is constructed from A_∞-categories $A_\infty(\mathbf{n}, \mathcal{C})$, which are particular cases of the following construction.

The multifunctors of shifts for quivers induce the multifunctors

$$-^{[]} : \mathsf{A}_\infty \to \mathsf{A}_\infty, \qquad -^{[]} : \mathsf{A}_\infty^{\mathrm{u}} \to \mathsf{A}_\infty^{\mathrm{u}}$$

which are also monads. Notice that the unit of this monad $u_{[]} : \mathrm{Id} \to -^{[]}$ is a multinatural transformation (of multifunctors), while the multiplication $m_{[]} : -^{[][]} \to -^{[]}$ is only a natural transformation (of functors). We also view $-^{[]}$ as an A_∞-2-monad – a sort of 2-functor with a monad structure in Chapter 10.

The Maurer–Cartan functors

$$-^{\mathsf{mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty, \qquad -^{\mathsf{mc}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty$$

are monads as well. The above functors are not multifunctors. The composition $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ is also a monad:

$$-^{\mathsf{tr}} : \mathsf{A}_\infty \to \mathsf{A}_\infty, \qquad -^{\mathsf{tr}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty,$$

The latter is the monad of pretriangulated A_∞-categories.

For a unital A_∞-category $\mathcal{C}$ multiplications in all three monads

$$m_{[]} : \mathcal{C}^{[][]} \to \mathcal{C}^{[]}, \qquad m_{\mathsf{mc}} : \mathcal{C}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{C}^{\mathsf{mc}}, \qquad m_{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}}$$

are A_∞-equivalences, as shown in Chapters 10, 11. Hence, these functors are sort of completion. The A_∞-category $\mathcal{C}^{[]}$ is closed under shifts, $\mathcal{C}^{\mathsf{mc}}$ is mc-closed and $\mathcal{C}^{\mathsf{tr}}$ is pretriangulated.

The multifunctor version of the Maurer–Cartan functor is also defined:

$$-^{\mathsf{Mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty, \qquad -^{\mathsf{Mc}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty .$$

It is a multi-dimensional generalization of $-^{\mathsf{mc}}$. In Chapter 11 we study first the case of quivers, then consider A_∞-categories. Objects of $\mathcal{C}^{\mathsf{Mc}}$ are A_∞-functors $X : \mathbf{n_1}, \dots, \mathbf{n_k} \to \mathcal{C}$, where A_∞-categories $\mathbf{n_i} = \{1, 2, \dots, n_i\}$ were described above. The A_∞-category $\mathcal{C}^{\mathsf{Mc}}$ can be constructed from the A_∞-categories $\underline{\mathsf{A}_\infty}(\mathbf{n_1}, \dots, \mathbf{n_k}; \mathcal{C})$. It is a full A_∞-subcategory of the iterated mc-construction:

$$\mathcal{C}^{\mathsf{mc}^\infty} \overset{\mathrm{def}}{=} \varinjlim\bigl(\mathcal{C} \xhookrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}} \xhookrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}^2} \xhookrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}^3} \dots\bigr).$$

Here $u_{\mathsf{mc}} : \mathcal{C} \hookrightarrow \mathcal{C}^{\mathsf{mc}}$ is the unit of the comonad $-^{\mathsf{mc}}$. For unital A_∞-categories $\mathcal{C}$ all these embeddings but the first are A_∞-equivalences, therefore, $\mathcal{C}^{\mathsf{mc}} \hookrightarrow \mathcal{C}^{\mathsf{Mc}}$ is an A_∞-equivalence. As a consequence the functor $-^{\mathsf{mc}} : A^u_\infty \to A^u_\infty$ inherits some properties of the multifunctor $-^{\mathsf{Mc}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty$. The functor $-^{\mathsf{mc}}$ becomes an A^u_∞-2-monad – a sort of 2-functor with a monad structure. The same applies to $-^{[]} : A^u_\infty \to A^u_\infty$ and to $-^{\mathsf{tr}} : A^u_\infty \to A^u_\infty$.

Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories. If $\mathcal{C}$ is closed under shifts, then by Chapter 10

1) the A_∞-category $A_\infty(\mathcal{A}, \mathcal{C})$ is closed under shifts,

2) the A_∞-category $A^u_\infty(\mathcal{A}, \mathcal{C})$ is closed under shifts,

3) the A_∞-functor $A^u_\infty(u_{[]}, \mathcal{C}) : A^u_\infty(\mathcal{A}^{[]}, \mathcal{C}) \to A^u_\infty(\mathcal{A}, \mathcal{C})$ is an A_∞-equivalence which admits a one-sided inverse $F_{[]} : A^u_\infty(\mathcal{A}, \mathcal{C}) \to A^u_\infty(\mathcal{A}^{[]}, \mathcal{C})$, so that $F_{[]} \cdot A^u_\infty(u_{[]}, \mathcal{C}) = \mathrm{id}_{A^u_\infty(\mathcal{A},\mathcal{C})}$.

4) the A_∞-functor $-^{[]} : A^u_\infty(\mathcal{A}, \mathcal{B}) \to A^u_\infty(\mathcal{A}^{[]}, \mathcal{B}^{[]})$ is homotopy full and faithful, that is, its first component is homotopy invertible.

Similar properties hold for mc-closed $\mathcal{C}$ by Chapter 11 and for pretriangulated $\mathcal{C}$ by Chapter 12. Such properties for pretriangulated A_∞-categories were mentioned first by Kontsevich in his letter to Beilinson [Kon99].

In Chapter 13 we consider categories with translation structure $\mathcal{C}$ for which the translation endofunctor $\Sigma : \mathcal{C} \to \mathcal{C}$ is not an automorphism. Such generalization of Verdier's axioms [Ver77, Ver96] is dictated by our approach to A_∞-categories closed under shits. We recall the definition of strongly triangulated categories due to Maltsiniotis [Mal06]. We prove finally that the 0-th homology of a pretriangulated A_∞-category is strongly triangulated.

Let $\mathcal{B}$ be a full A_∞-subcategory of a unital A_∞-category $\mathcal{C}$. Then there are A_∞-equivalent quotients $\mathsf{D}(\mathcal{C}|\mathcal{B})$ from [LO06] and $\mathsf{q}(\mathcal{C}|\mathcal{B})$ from [LM08c]. We prove in Chapters 10, 11 that there are A_∞-equivalences

$$\mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}, \quad \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}, \quad \mathsf{q}(\mathcal{C}^{\mathsf{tr}}|\mathcal{B}^{\mathsf{tr}}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{tr}}.$$

Since $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is A_∞-equivalent to $\mathsf{D}(\mathcal{C}|\mathcal{B})$, the same holds for the latter quotient.

Chapter 14 is a survey of applications of results and methods of this book obtained in [LM08b]. The topics include A_∞-bimodules over A_∞-categories, Serre A_∞-functors and the generalized Yoneda Lemma for A_∞-categories.

1.10 Off the principal road.

The dependence of a chapter on other chapters is shown on the following scheme:

$$\begin{tikzcd}
\boxed{4} \arrow[r] & \boxed{5} \arrow[r] & \boxed{6} \arrow[r] \arrow[d] & \boxed{7} \arrow[r] & \boxed{8} \arrow[r] & \boxed{9} \arrow[r] \arrow[d] & \boxed{11} \arrow[d] & \\
\boxed{3} \arrow[u] \arrow[r] & \boxed{A} \arrow[r] & \boxed{B} \arrow[r] & \boxed{C} \arrow[rr] & & \boxed{10} \arrow[r] \arrow[d] & \boxed{12} \arrow[r] & \boxed{13} \arrow[d] \\
\boxed{2} \arrow[u] & & & & & \boxed{14} \arrow[rr] & & \boxed{1}
\end{tikzcd}$$

The way of actions in Kleisli multicategories described in Appendices A–C is parallel to the shorter way of monads in Kleisli multicategories presented in Chapter 5. However the longer way is more transparent and allows to obtain more results. That is why we have chosen to present both approaches.

In Chapter 5 we consider an abstract setting of a multicomonad $T : \mathsf{C} \to \mathsf{C}$ in a multicategory C and a monad $M : \mathsf{C} \to \mathsf{C}$. The functor M has to be a multifunctor, but

the multiplication natural transformation $m : MM \to M$ is not necessarily multinatural. We assume given also a multinatural transformation $\xi : MT \to TM : \mathsf{C} \to \mathsf{C}$, the commutation law. We propose sufficient conditions which make sure that the monad M lifts to a monad M^T in the Kleisli multicategory C^T. These conditions are similar to Beck's distributivity laws between two monads. This construction is applied to the tensor multicomonad $T^{\geqslant 1} : \widehat{\mathscr{Q}_u} \to \widehat{\mathscr{Q}_u}$ and the monad of shifts $-^{[]} : \widehat{\mathscr{Q}_u} \to \widehat{\mathscr{Q}_u}$. We find the corresponding commutation law ξ and prove that the monad of shifts $-^{[]}$ lifts to the Kleisli multicategory $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$.

The same multifunctor $-^{[]}$ is obtained in Appendix C via tensoring with a monoidal graded category $\mathcal{Z}$. The graded quiver $\mathcal{Z}$ has $\operatorname{Ob} \mathcal{Z} = \mathbb{Z}$ and $\mathcal{Z}(m, n) = \Bbbk[n-m]$. This $\mathcal{Z}$ is an algebra in the symmetric Monoidal category of graded categories $\mathscr{D} = \mathbf{gr}\text{-}\mathcal{C}at$ equipped with the tensor product $\boxtimes$ (Chapter 10). This category $\mathscr{D}$ acts via $\boxtimes$ on symmetric Monoidal categories of graded quivers $\mathscr{Q}_p$ and $\mathscr{Q}_u$ (Appendix B). As a corollary, the symmetric multicategory $\widehat{\mathscr{D}}$ acts on the symmetric multicategory $\widehat{\mathscr{Q}_u}$.

In order to formulate such statements rigorously and without drowning into lots of irrelevant isomorphisms and equations between them, we develop in Appendix A the language of internal lax symmetric Monoidal categories which live inside a symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{C}$. Mostly we are interested in the paradigmatic example of the category $\mathfrak{C} = $ sym-Mono-$\mathcal{C}at$ of symmetric Monoidal categories. It can be also viewed as a (weak) symmetric monoidal 2-category, but the point of view of weak 6-categories is hardly helpful.

By Proposition 2.36 commutative algebras in a symmetric Monoidal category $\mathcal{C}$ form a symmetric Monoidal category $\operatorname{ComAlg}(\mathcal{C})$ themselves. Moreover, each commutative algebra in $\mathcal{C}$ determines a commutative algebra in $\operatorname{ComAlg}(\mathcal{C})$. In complete analogy we show in Appendix A that internal symmetric Monoidal categories in a symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{C}$ form a symmetric Monoidal $\mathcal{C}at$-category sym-Mono-cat-$\mathfrak{C}$ themselves. Moreover, each symmetric Monoidal category in $\mathfrak{C}$ determines a symmetric Monoidal category in sym-Mono-cat-$\mathfrak{C}$. This implies that the symmetric multicategory $\widehat{\mathscr{D}} = \widehat{\mathbf{gr}\text{-}\mathcal{C}at}$ is an internal symmetric Monoidal category in the symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{SMCatm}$ of symmetric multicategories. In particular, it makes sense to discuss actions of $\widehat{\mathscr{D}}$ on other symmetric multicategories.

In Appendix B we discuss actions of a symmetric multicategory D on a symmetric multicategory C equipped with a multicomonad $T : \mathsf{C} \to \mathsf{C}$. An intertwiner of the action and the multicomonad insures that the action lifts to an action of D on the Kleisli multicategory C^T. Such an intertwiner for the action $\boxdot$ of $\mathsf{D} = \widehat{\mathscr{D}}$ on $\widehat{\mathscr{Q}_u}$ and for the multicomonad $T^{\geqslant 1} : \widehat{\mathscr{Q}_u} \to \widehat{\mathscr{Q}_u}$ is constructed in Appendix C. Hence, $\widehat{\mathscr{D}}$ acts on the Kleisli symmetric multicategory $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ of graded quivers. This action extends to the action of symmetric multicategory $\widehat{\mathbf{dg}\text{-}\mathcal{C}at}$ of differential graded categories on (unital) A_∞-categories. The action $\boxdot : \mathsf{A}^{\mathrm{u}}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}^{\mathrm{u}}_\infty$ is a generalization of the tensor

product $\mathbf{dg}\text{-}\mathcal{C}at \boxtimes \mathbf{dg}\text{-}\mathcal{C}at \to \mathbf{dg}\text{-}\mathcal{C}at$ of differential graded categories. Any algebra A in $\mathscr{D}$ induces a multifunctor $1 \boxdot \widehat{A} : \mathsf{A}_\infty \to \mathsf{A}_\infty$. In particular, this holds for the algebra $A = \mathcal{Z} \in \operatorname{Ob}\mathscr{D}$. The multifunctor $1 \boxdot \widehat{\mathcal{Z}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$ is precisely the shift multifunctor $-^{[]}$. The tensor product $\otimes : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z}$ induces the natural transformation $m : -^{[][]} \to -^{[]}$, the multiplication. It is not multinatural, because $\mathcal{Z}$ is not a commutative algebra.

1.11 Unital structures for A_∞-categories. Definition of unital A_∞-categories used in this book is not the only approach to unitality. Another definition of unitality was given by Kontsevich and Soibelman [KS09, Definition 4.2.3.]. They define a *unital structure* of an A_∞-category $\mathcal{A}$ as an A_∞-functor $U_{\mathsf{su}}^{\mathcal{A}} : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ such that $\bigl(\mathcal{A} \xhookrightarrow{u_{\mathsf{su}}} \mathcal{A}^{\mathsf{su}} \xrightarrow{U_{\mathsf{su}}^{\mathcal{A}}} \mathcal{A}\bigr) = \mathrm{id}_{\mathcal{A}}$. It is proven in [LM06b] that an A_∞-category is unital if and only if it admits a unital structure. One more notion of unitality is proposed by Fukaya [Fuk02, Definition 5.11], *homotopy unitality*, whose essence is extension of A_∞-structure from $\mathcal{A}$ to a certain larger quiver. It is proven in [LM06b] that an A_∞-category is unital if and only if it homotopy unital. Thus, all known so far approaches to unitality agree. Let us provide the details now.

Given an A_∞-category $\mathcal{A}$, we associate a strictly unital A_∞-category $\mathcal{A}^{\mathsf{su}}$ with it. It has the same set of objects and for any pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$ the graded $\Bbbk$-module $s\mathcal{A}^{\mathsf{su}}(X,Y)$ is given by

$$s\mathcal{A}^{\mathsf{su}}(X,Y) = \begin{cases} s\mathcal{A}(X,Y), & \text{if } X \neq Y, \\ s\mathcal{A}(X,X) \oplus \Bbbk\,{}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}, & \text{if } X = Y, \end{cases}$$

where ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a new generator of degree -1. The element ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a strict unit by definition, and the canonical embedding $e_{\mathcal{A}} = u_{\mathsf{su}} : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathsf{su}}$ is a strict A_∞-functor.

1.12 Definition. A *unital structure* of an A_∞-category $\mathcal{A}$ is a choice of an A_∞-functor $U_{\mathsf{su}}^{\mathcal{A}} : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}$ such that

$$\bigl(\mathcal{A} \xhookrightarrow{e_{\mathcal{A}}} \mathcal{A}^{\mathsf{su}} \xrightarrow{U_{\mathsf{su}}^{\mathcal{A}}} \mathcal{A}\bigr) = \mathrm{id}_{\mathcal{A}}\,.$$

1.13 Proposition ([LM06b]). *Suppose that an A_∞-category $\mathcal{A}$ admits a unital structure. Then the A_∞-category $\mathcal{A}$ and the A_∞-functor $U_{\mathsf{su}}^{\mathcal{A}}$ are unital in the sense of Definitions 9.10 and 9.11.*

1.14 Theorem ([LM06b]). *Every unital A_∞-category admits a unital structure.*

1.15 Homotopy unital A_∞-categories. One more notion of unitality is dictated by combinatorial properties of associahedra, also known as the Deligne–Mumford–Stasheff compactifications of the moduli space of configurations of points on the circle.

1.16 Definition. An A_∞-category $\mathcal{C}$ is called *homotopy unital* in the sense of Fukaya [Fuk02, Definition 5.11] (reproduced in [FOOO09] and in [Sei08, Definition 2a]) if the graded $\Bbbk$-quiver
$$\mathcal{C}^+ = \mathcal{C} \oplus \Bbbk\mathcal{C} \oplus s\Bbbk\mathcal{C}$$
(with $\operatorname{Ob}\mathcal{C}^+ = \operatorname{Ob}\mathcal{C}$) has an A_∞-structure b^+ of the following kind. Denote the generators of the second and the third summands of $s\mathcal{C}^+ = s\mathcal{C} \oplus s\Bbbk\mathcal{C} \oplus s^2\Bbbk\mathcal{C}$ by ${}_X\mathbf{i}_0^{\mathcal{C}\mathsf{su}} = 1s$ and $\mathbf{j}_X^{\mathcal{C}} = 1s^2$ of degree respectively -1 and -2 for $X \in \operatorname{Ob}\mathcal{C}$. The conditions on b^+ are:

1) the elements ${}_X\mathbf{i}_0^{\mathcal{C}} \overset{\text{def}}{=} {}_X\mathbf{i}_0^{\mathcal{C}\mathsf{su}} - \mathbf{j}_X^{\mathcal{C}} b_1^+$ belong to $s\mathcal{C}(X, X)$ for all $X \in \operatorname{Ob}\mathcal{C}$;

2) the A_∞-category $\mathcal{C}^+$ is strictly unital with the strict units $\mathbf{i}_0^{\mathcal{C}\mathsf{su}}$;

3) the embedding $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is a strict A_∞-functor;

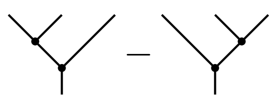
4) $(s\mathcal{C} \oplus s^2\Bbbk\mathcal{C})^{\otimes n} b_n^+ \subset s\mathcal{C}$ for any $n > 1$.

The distinguished cycles ${}_X\mathbf{i}_0^{\mathcal{C}} \in \mathcal{C}(X, X)[1]^{-1}$ turn any homotopy unital A_∞-category $\mathcal{C} \subset \mathcal{C}^+$ into a unital A_∞-category $\mathcal{C}$. Indeed, the identity $(1 \otimes b_1^+ + b_1^+ \otimes 1)b_2^+ + b_2^+ b_1^+ = 0$ applied to $s\mathcal{C} \otimes \mathbf{j}^{\mathcal{C}}$ or to $\mathbf{j}^{\mathcal{C}} \otimes s\mathcal{C}$ implies
$$\begin{aligned}
(1 \otimes \mathbf{i}_0^{\mathcal{C}})b_2^{\mathcal{C}} &= 1 + (1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ \quad : s\mathcal{C} \to s\mathcal{C}, \\
(\mathbf{i}_0^{\mathcal{C}} \otimes 1)b_2^{\mathcal{C}} &= -1 + (\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ b_1^{\mathcal{C}} + b_1^{\mathcal{C}}(\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ : s\mathcal{C} \to s\mathcal{C}.
\end{aligned}$$
Thus, $(1 \otimes \mathbf{j}^{\mathcal{C}})b_2^+ : s\mathcal{C} \to s\mathcal{C}$ and $(\mathbf{j}^{\mathcal{C}} \otimes 1)b_2^+ : s\mathcal{C} \to s\mathcal{C}$ are unit homotopies. Therefore, any homotopy unital A_∞-category is unital. Vice versa:

1.17 Theorem ([LM06b])**.** *Any unital A_∞-category $\mathcal{C}$ with unit elements $\mathbf{i}_0^{\mathcal{C}}$ admits a homotopy unital structure $(\mathcal{C}^+, b^+)$ with $\mathbf{j}^{\mathcal{C}} b_1^+ = \mathbf{i}_0^{\mathcal{C}\mathsf{su}} - \mathbf{i}_0^{\mathcal{C}}$.*

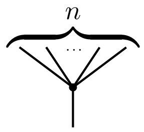
1.18 The operad of A_∞-algebras. By an *operad* we mean a multicategory (possibly enriched in a monoidal category) with one object. It is often called a 'non-Σ-operad' or 'non-symmetric operad'.

Denote by *Ass* the operad of associative algebras. It is defined as the quotient of the free operad generated by the operation  by the ideal generated by the element

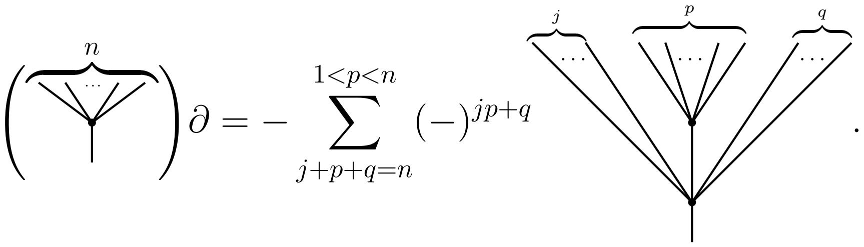


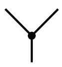
corresponding to the associativity axiom. It is easy to see that for each $n \geqslant 1$ the $\Bbbk$-module $Ass(n)$ is free of rank 1. It is generated by an 'n-fold product'. We may view *Ass* as a differential graded operad concentrated in degree 0.

The operad A_∞ is defined as follows: as a graded operad, it is freely generated by operations



of degree $2 - n$, $n \geqslant 1$. Thus the component of degree g of the graded $\Bbbk$-module $A_\infty(n)$ is a free $\Bbbk$-module generated by the set of plane rooted trees with $n + 1$ external and $n + g - 1$ internal vertices. The differential is given on the generators by the formula



There is a natural morphism of differential graded operads $\varepsilon : (A_\infty, \partial) \to (Ass, 0)$ specified by mapping the generator  to the '2-fold product' in $Ass(2)$ and by mapping the other generators to 0. More explicitly, the component of degree 0 of the graded $\Bbbk$-module $A_\infty(n)$ is the free $\Bbbk$-module generated by plane binary rooted trees with $n + 1$ external vertices. Each of these is mapped to the 'n-fold product' in $Ass(n)$. The other components of $A_\infty(n)$ are mapped to 0.

The following result should not be considered new. It is rather an application of Markl's theory of homotopy algebras and resolutions of operads [Mar96, Mar00]. We supply the details of the proof since they introduce notions related to A_∞-structures.

1.19 Proposition. *The chain map $A_\infty(n) \to Ass(n)$ is homotopy invertible, for each $n \geqslant 1$.*

Proof. It suffices to deal with the ground commutative ring $\Bbbk = \mathbb{Z}$. Markl proves in [Mar96] that $A_\infty(n) \to Ass(n)$ is a quasi-isomorphism if $\Bbbk$ is a field of characteristic zero. In Example 4.8 of the same article he remarks that the map remains a quasi-isomorphism over $\Bbbk = \mathbb{Z}$ as well. Since in this case $A_\infty(n)$ and $Ass(n)$ are complexes of free abelian groups of finite rank, being a quasi-isomorphism is equivalent to being homotopy invertible.

The above sketch of proof hides the operad of Stasheff associahedra [Sta63] inside. Let us reveal it now.

Denote by $D = \{z \in \mathbb{C} \mid |z| \leqslant 1\}$ the unit disc. The configuration space $Conf_{n+1}(\partial D) \subset (\partial D)^{n+1}$ consists of $(n + 1)$-tuples of points on the boundary circle whose numbering is compatible with their cyclic order. For $n \geqslant 2$ the complex automorphisms group $\operatorname{Aut}(D) \simeq PSL_2(R)$ acts freely and properly on this configuration space, and one sets

$$\mathcal{R}^{n+1} = Conf_{n+1}(\partial D)/\operatorname{Aut}(D).$$

Let T be a stable *n-leafed* tree (a tree with $n+1$ external legs, one of which is the root, other n are leaves). *Stability* means that the valency $|v|$ is at least 3 for all internal vertices v. The set of internal vertices is denoted $\mathrm{v}(T)$. One sets $\mathcal{R}^T = \prod_{v\in\mathrm{v}(T)} \mathcal{R}^{|v|}$. The Deligne–Mumford–Stasheff compactification of $\mathcal{R}^{n+1}$ (the *associahedron*) is

$$\bar{\mathcal{R}}^{n+1} = \coprod_T \mathcal{R}^T,$$

the union is over all stable n-leafed trees T. The associahedron is a part of the real locus $\bar{\mathcal{M}}_{0,n+1}(\mathbb{R})$ of the complex Deligne–Mumford space $\bar{\mathcal{M}}_{0,n+1}$. So $\bar{\mathcal{R}}^{n+1}$ comes with a canonical structure of smooth compact manifold with corners.

The A_∞-associativity relations are in great part due to the recursive structure of the Deligne–Mumford–Stasheff compactifications $\bar{\mathcal{R}}^{n+1}$. More precisely, each stable n-leafed tree with two vertices, $T(p,j) = (|^{\sqcup j} \sqcup \mathfrak{t}_p \sqcup |^{\sqcup q})\cdot \mathfrak{t}_{j+1+q}$ (where $q = n-j-p$ and $\mathfrak{t}_p$ is a corolla with p leaves), labels a codimension one boundary face

$$\mathcal{R}^{T(p,j)} \cong \mathcal{R}^{p+1} \times \mathcal{R}^{n+2-p} \subset \partial\bar{\mathcal{R}}^{n+1}; \tag{1.19.1}$$

those faces correspond to the summands $(1^{\otimes j}\otimes m_p \otimes 1^{\otimes q})m_{n-p+1}$ with $1<p<n$.

Seidel proves in his book [Sei08, (12.25)] that for certain chosen orientations of $\mathcal{R}^\bullet$ the orientations of the left and right sides of embedding (1.19.1) differ by $(-)^{jp+q}$. He explains also that smooth compact manifolds with corners $\bar{\mathcal{R}}^{n+1}$ are isomorphic to Stasheff's convex polytopes K_n, the associahedra [Sta63], see also [MSS02, Section II.1.6]. The same result is proven by Costello [Cos04, Proposition 3.2.2]. The operad $K_\bullet$ is known to be a cellular operad (i.e., an operad in the monoidal category of cell complexes, where the tensor product is just direct product). The cells (faces of the polytope) are identified with $\mathcal{R}^T$. This explicit description implies that the differential graded operad A_∞ is isomorphic to the differential graded operad $\{CC_\bullet(K_n)\}_{n\geqslant 1}$, where $CC_\bullet$ denotes the functor of cellular chains. In fact, the n-ary generators of A_∞ are m_n, $n\geqslant 2$, $\deg m_n = 2-n$, and the differential is

$$\partial m_n = -\sum_{j+p+q=n}^{1<p<n} (-)^{jp+q}(1^{\otimes j}\otimes m_p\otimes 1^{\otimes q})\cdot m_{j+1+q}.$$

Let T denote the (terminal) cellular operad with $T(n) = \{\cdot\}$ for each $n\geqslant 1$. Then it is easy to see that the operad $CC_\bullet(T)$ is precisely the operad *Ass*. Consider the natural map $K\to T$. Since each of the polyhedra K_n is contractible, the map $K_n \to T(n)$ is a homotopy equivalence, for each $n\geqslant 1$. It follows that the induced map $CC_\bullet(K_n)\to CC_\bullet(T(n))$ is a quasi-isomorphism. One may argue as follows: for cell complexes the cellular homology is naturally isomorphic to the singular homology, therefore the map in homology $H_m(CC_\bullet(K_n)) \to H_m(CC_\bullet(T(n)))$ identifies with the map $H_m(K_n)\to H_m(T(n))$ between singular homology groups, which is an isomorphism by elementary algebraic topology. Thus, the map in question

$$A_\infty(n) = CC_\bullet(K_n) \to CC_\bullet(T(n)) = Ass(n)$$

is a quasi-isomorphism of complexes of free abelian groups of finite rank. Therefore it is homotopy invertible. □

Thus, A_∞ is a free resolution of $\mathcal{A}ss$ over an arbitrary commutative ring $\Bbbk$ by Markl [Mar96, Example 4.8]. Differential graded algebras (A, m_1) over this operad are precisely A_∞-algebras. They satisfy the equation

$$\sum_{j+p+q=n}^{1\leqslant p\leqslant n} (-)^{jp+q}(1^{\otimes j} \otimes m_p \otimes 1^{\otimes q}) \cdot m_{j+1+q} = 0. \tag{1.19.2}$$

1.20 Operadic approach to A_∞-categories. The notion of A_∞-category also admits an 'operadic' description. Namely, an A_∞-category $\mathcal{A}$ consists of the following data:

- a set $\operatorname{Ob}\mathcal{A}$ of objects;
- for each pair of objects X and Y, a complex $(\mathcal{A}(X,Y), d)$ of morphisms;
- for each $n \geqslant 1$ and sequence of objects $X_0, \dots, X_n$, a morphism of complexes

  $$\alpha_n : \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n) \otimes A_\infty(n) \to \mathcal{A}(X_0, X_n),$$

  called *action*.

The action is required to be compatible with the structure of the operad A_∞ in the obvious sense. In particular, the action is required to be unital, i.e., the tree $|$ must act as the identity. In other words, $(1 \otimes |)\alpha_1 = 1 : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y)$. Following the notation of [LM06a], denote by $\mathcal{T}^n_{\geqslant 2}$ the set of stable n-leafed plane rooted trees t (trees with $n+1$ external vertices such that each internal vertex is adjacent to at least 3 edges). The graded $\Bbbk$-module $A_\infty(n)$ is freely generated by the set $\mathcal{T}^n_{\geqslant 2}$, where a tree $t \in \mathcal{T}^n_{\geqslant 2}$ is assigned degree $|t| + 1 - n$. Therefore, the action morphism α_n amounts to a collection of $\Bbbk$-linear maps

$$m_t = (1^{\otimes n} \otimes t)\alpha_n : \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{A}(X_0, X_n)$$

of degree $|t| + 1 - n$, for each $t \in \mathcal{T}^n_{\geqslant 2}$, satisfying certain compatibility conditions. In particular, for each $n \geqslant 2$, there is a $\Bbbk$-linear map

$$m_n = m_{\mathsf{t}_n} : \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_2) \otimes \dots \otimes \mathcal{A}(X_{n-1}, X_n) \to \mathcal{A}(X_0, X_n)$$

of degree $2-n$. Together with the differential $m_1 \overset{\text{def}}{=} d : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y)$, these $\Bbbk$-linear maps satisfy the A_∞-identity. Conversely, the compatibility of α_n with composition in the operad A_∞ implies that, for each tree $t \in \mathcal{T}^n_{\geqslant 2}$ with the canonical decomposition

$$(t, \leqslant) = (1^{\sqcup\alpha_1} \sqcup \mathsf{t}_{k_1} \sqcup 1^{\sqcup\beta_1}) \cdot (1^{\sqcup\alpha_2} \sqcup \mathsf{t}_{k_2} \sqcup 1^{\sqcup\beta_2}) \cdot \dots \cdot \mathsf{t}_{k_N},$$

where $N = |t|$, the $\Bbbk$-linear map m_t is given by the composite

$$m_t = (1^{\otimes\alpha_1} \otimes m_{k_1} \otimes 1^{\otimes\beta_1}) \cdot (1^{\otimes\alpha_2} \otimes m_{k_2} \otimes 1^{\otimes\beta_2}) \cdot \dots \cdot m_{k_N} : \mathcal{A}^{\otimes n} \to \mathcal{A}.$$

Define also

$$b_t = (1^{\otimes\alpha_1} \otimes b_{k_1} \otimes 1^{\otimes\beta_1}) \cdot (1^{\otimes\alpha_2} \otimes b_{k_2} \otimes 1^{\otimes\beta_2}) \cdot \dots \cdot b_{k_N} : (s\mathcal{A})^{\otimes n} \to s\mathcal{A}.$$

It is easy to see that $s^{\otimes n} b_t = (-)^\sigma m_t s$, where $\sigma = \sum_{i=1}^n (\beta_i + \alpha_i k_i)$.

In [LM06a], the free A_∞-category $\mathcal{FQ}$ is associated with a differential graded quiver $\mathcal{Q}$. The freeness of $\mathcal{FQ}$ is expressed by an A_∞-equivalence

$$\operatorname{restr} : A_\infty(\mathcal{FQ}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{C}),$$

for each unital A_∞-category $\mathcal{C}$, where the target A_∞-category is formed by chain quiver maps $\mathcal{Q} \to \mathcal{C}$, see [LM06a] for details.

Let us show that the free A_∞-category $\mathcal{FQ}$ admits an equivalent description in terms of the operad A_∞. We associate with $\mathcal{Q}$ the differential graded quiver

$$\widetilde{\mathcal{F}}\mathcal{Q} \overset{\text{def}}{=} \bigoplus_{k\geqslant 1} T^k\mathcal{Q} \otimes A_\infty(k),$$

'a free algebra over the operad A_∞'. It admits the following structure of an A_∞-category. For $n > 1$, the n-fold tensor product $(\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n}$ decomposes into a direct sum

$$(\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} = \bigoplus_{k_1,\dots,k_n\geqslant 1} \otimes^{i\in\mathbf{n}}(T^{k_i}\mathcal{Q} \otimes A_\infty(k_i)).$$

For each $k, k_1, \dots, k_n \geqslant 1$ and $t \in \mathcal{T}^n_{\geqslant 2}$, the matrix coefficient

$$(m_t^{\widetilde{\mathcal{F}}\mathcal{Q}})_{(k_i);k} = \bigl[\otimes^{i\in\mathbf{n}}(T^{k_i}\mathcal{Q} \otimes A_\infty(k_i)) \xrightarrow{\operatorname{in}_{(k_i)}} (\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \xrightarrow{m_t^{\widetilde{\mathcal{F}}\mathcal{Q}}} \widetilde{\mathcal{F}}\mathcal{Q} \xrightarrow{\operatorname{pr}_k} T^k\mathcal{Q} \otimes A_\infty(k)\bigr]$$

of $m_t^{\widetilde{\mathcal{F}}\mathcal{Q}}$ vanishes unless $k = k_1 + \dots + k_n$, in which case it is given by

$$(m_t^{\widetilde{\mathcal{F}}\mathcal{Q}})_{(k_i);k} = \bigl[\otimes^{i\in\mathbf{n}}(T^{k_i}\mathcal{Q} \otimes A_\infty(k_i)) \xrightarrow{\sigma_{(12)}} T^{k_1+\dots+k_n}\mathcal{Q} \otimes (A_\infty(k_1) \otimes \dots \otimes A_\infty(k_n)) \\ \xrightarrow{1\otimes(-\cdot t)} T^{k_1+\dots+k_n}\mathcal{Q} \otimes A_\infty(k_1 + \dots + k_n)\bigr]. \quad (1.20.1)$$

The differential $m_1^{\widetilde{\mathcal{F}}\mathcal{Q}} : \widetilde{\mathcal{F}}\mathcal{Q} \to \widetilde{\mathcal{F}}\mathcal{Q}$ is given by the sum of morphisms

$$d_{T^k\mathcal{Q}} \otimes 1 + 1 \otimes \partial : T^k\mathcal{Q} \otimes A_\infty(k) \to T^k\mathcal{Q} \otimes A_\infty(k), \quad k \geqslant 1.$$

The natural embedding $\mathrm{in}_1 : \mathcal{Q} \cong T^1\mathcal{Q} \otimes A_\infty(1) \hookrightarrow \widetilde{\mathcal{F}}\mathcal{Q}$ is a chain quiver map, therefore it extends uniquely to a strict A_∞-functor $f = \widehat{\mathrm{in}_1} : \mathcal{F}\mathcal{Q} \to \widetilde{\mathcal{F}}\mathcal{Q}$. Its first component is described in [LM06a, Section 2.6]:

$$f_1|_{s\mathcal{F}_t\mathcal{Q}} = \bigl[s\mathcal{F}_t\mathcal{Q} = (s\mathcal{Q})^{\otimes n}[-|t|] \xrightarrow{s^{|t|}} (s\mathcal{Q})^{\otimes n} \xrightarrow{\mathrm{in}_1^{\otimes n}} (s\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \xrightarrow{b_t^{\widetilde{\mathcal{F}}\mathcal{Q}}} s\widetilde{\mathcal{F}}\mathcal{Q}\bigr].$$

The composite of the second and the third arrows in the right hand side of the above formula fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{Q}^{\otimes n} \arrow[r, "\mathrm{in}_1^{\otimes n}"] \arrow[d, "s^{\otimes n}"'] & (\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \arrow[r, "(-)^\sigma m_t^{\widetilde{\mathcal{F}}\mathcal{Q}}"] \arrow[d, "s^{\otimes n}"] & \widetilde{\mathcal{F}}\mathcal{Q} \arrow[d, "s"] \\
(s\mathcal{Q})^{\otimes n} \arrow[r, "\mathrm{in}_1^{\otimes n}"] & (s\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \arrow[r, "b_t^{\widetilde{\mathcal{F}}\mathcal{Q}}"] & s\widetilde{\mathcal{F}}\mathcal{Q}
\end{tikzcd}$$

The upper row of the diagram is equal, up to sign $(-)^\sigma$, to the composite

$$\mathcal{Q}^{\otimes n} \xrightarrow{s^{n-|t|-1}} \mathcal{Q}^{\otimes n}[n-|t|-1] \cong \mathcal{Q}^{\otimes n} \otimes \Bbbk[n-|t|-1] \hookrightarrow \widetilde{\mathcal{F}}\mathcal{Q}, \tag{1.20.2}$$

as formula (1.20.1) shows. Taking into account the decomposition

$$A_\infty(n) = \bigoplus_{t \in \mathcal{T}^n_{\geqslant 2}} \Bbbk[n-|t|-1],$$

we conclude that f_1 is the sum of morphisms

$$(-)^\sigma \cdot \bigl[(s\mathcal{Q})^{\otimes n}[-|t|] \xrightarrow{s^{|t|}} (s\mathcal{Q})^{\otimes n} \xrightarrow{(s^{\otimes n})^{-1}} \mathcal{Q}^{\otimes n} \xrightarrow{s^{n-|t|-1}}$$
$$\mathcal{Q}^{\otimes n}[n-|t|-1] \cong \mathcal{Q}^{\otimes n} \otimes \Bbbk[n-|t|-1] \hookrightarrow \widetilde{\mathcal{F}}\mathcal{Q} \xrightarrow{s} s\widetilde{\mathcal{F}}\mathcal{Q}\bigr],$$

over all $t \in \mathcal{T}^n_{\geqslant 2}$. In particular, f_1 is an isomorphism. Thus, A_∞-categories $\mathcal{F}\mathcal{Q}$ and $\widetilde{\mathcal{F}}\mathcal{Q}$ are isomorphic.

Part I

Closed multicategories

Chapter 2

Lax Monoidal categories

Usual notion of a monoidal category is convenient due to coherence theorem of Mac Lane [Mac63] which allows to work with a monoidal category as if it were strict. Often one pretends that a monoidal functor is strict and skips the corresponding isomorphisms. This can not be done with lax Monoidal functors which play a crucial rôle in this book. The term 'Monoidal' as opposed to 'monoidal' indicates that categories are equipped with n-ary tensor products, related by many associativity morphisms. The latter are invertible for Monoidal categories and not necessarily invertible for lax Monoidal categories. The same convention applies to functors. Technically (lax) Monoidal categories and functors fit better into the framework of multicategories and multifunctors which is widely used in the sequel. We also develop the corresponding notions for categories enriched in symmetric Monoidal categories $\mathcal{V}$. For instance, $\mathcal{V}$ can be the category of small categories.

2.1 Conventions for finite totally ordered sets. Whenever I and J_i, $i \in I$, are finite totally ordered sets, the Cartesian product $\prod_{i\in I} J_i$ is always equipped with the linear lexicographic ordering: $(j_i) < (j'_i)$ if and only if there exists $k \in I$ such that $j_i = j'_i$ for $i < k$ and $j_k < j'_k$. The disjoint union of J_i

$$\bigsqcup_{i\in I} J_i = \bigl\{(i,j) \in I \times \bigcup_{i\in I} J_i \mid j \in J_i\bigr\} \tag{2.1.1}$$

is also equipped with the standard linear ordering:

$$(i,j) < (i',j') \Longleftrightarrow i < i' \text{ or } (i = i' \text{ and } j < j').$$

In particular cases like $I = \mathbf{n} = \{1 < 2 < \cdots < n\}$ the same totally ordered set can be also denoted as $I_1 \sqcup I_2 \sqcup \cdots \sqcup I_n$. We always equip the disjoint union $A \sqcup B \sqcup \cdots \sqcup Z$ of finite totally ordered sets with the standard linear ordering, even if the indexing set I is not indicated explicitly. Here it means that $a < b$ for all $a \in A$, $b \in B$ etc.

When $f : I \to J$ is a map between totally ordered sets, the subsets $f^{-1}(j)$ get a linear ordering, induced by the embedding $f^{-1}(j) \hookrightarrow I$.

For any finite set I denote by $|I|$ the number of elements of I.

Let $\mathcal{O}$ be the category of finite ordinals, whose objects are totally ordered finite sets including the empty set (elements of a universe $\mathscr{U}$) and morphisms are *isotonic maps*, that is, non-decreasing maps, i.e. mappings which preserve the non-strict order. Let $\mathcal{O}_s$ be its skeleton subcategory with objects $\mathbf{n} = \{1 < 2 < \cdots < n\}$ for $n \geqslant 0$, where $\mathbf{0} = \varnothing$.

Let $\mathcal{S}$ denote the category, whose objects are the same as in $\mathcal{O}$, but the morphisms are all mappings of finite sets. Denote by $\mathcal{S}_s$ its skeleton subcategory with objects $\mathbf{n}$, $n \in \mathbb{Z}_{\geqslant 0}$.

The definitions of various notions below that rely on $\mathcal{O}$ do not depend on the concrete choice of the universe $\mathscr{U}$. Indeed, we may replace everywhere $\mathcal{O}$ with its skeleton subcategory $\mathcal{O}_s$. Each functor or natural transformation can be restricted from $\mathcal{O}$ to $\mathcal{O}_s$, or can be canonically extended from $\mathcal{O}_s$ to $\mathcal{O}$ using the only isotonic isomorphism

$$\mathbf{n} \ni k \mapsto i_k \in I \tag{2.1.2}$$

for each $I \in \operatorname{Ob} \mathcal{O}$ with $|I| = n$.

Let $[n]$ denote the category constructed from the totally ordered set $[n] = \{0 < 1 < 2 < \dots < n-1 < n\}$. Its objects are integers m, $0 \leqslant m \leqslant n$. The set of morphisms $[n](k,m)$ is empty if $k > m$, and has unique element $(k \to m)$ if $k \leqslant m$. The conventional $\boldsymbol{\Delta}$ denotes the full subcategory in $\mathcal{O}$ with objects $[n]$ for $n \geqslant 0$. It can be convenient to imagine an element $i \in \mathbf{n}$ as an arrow $(i-1) \to i$ in the category $[n]$. This suggests a faithful functor $[\,] : \mathcal{O}_s^{\mathrm{op}} \to \boldsymbol{\Delta}$ which maps $(\pi : \mathbf{n} \to \mathbf{m}) \in \mathcal{O}_s$ to $([\pi] : [m] \to [n]) \in \boldsymbol{\Delta}$, where $[\pi](j) = \sum_{i \leqslant j} |\pi^{-1}(i)| = |\pi^{-1}(]0,j])|$. Note that $[\pi](0) = 0$, $[\pi](m) = n$, and this property characterizes the image of the above functor. Moreover, this functor restricts to an anti-isomorphism between the subcategory of isotonic surjections $\pi : \mathbf{n} \to \mathbf{m}$ in $\mathcal{O}_s$ and the subcategory of isotonic injections $[\pi] : [m] \to [n]$ in $\boldsymbol{\Delta}$ such that $[\pi](0) = 0$, $[\pi](m) = n$.

We associate a totally ordered set $[I] = \{0\} \sqcup I$ with an arbitrary $I \in \operatorname{Ob} \mathcal{O}$. We view elements of I as generating arrows of the category $[I]$. Let $f : I \to J$ be an isotonic map. For an arbitrary subset $K \subset J$ the linear order on $f^{-1}(K)$ is induced by the embedding $f^{-1}(K) \hookrightarrow I$. With the map f the following map is associated: $[f]^* = \mathrm{id}_{\{0\}} \sqcup f : [I] \to [J]$. We view it as a functor. It has a right adjoint functor

$$[f] : [J] \to [I], \qquad y \mapsto [f](y) \overset{\mathrm{def}}{=} \max([f]^*)^{-1}([0,y]).$$

Here $[0,y] = \{z \in [J] \mid z \leqslant y\} \subset [J]$. Indeed, for any $x \in [I]$, $y \in [J]$ the following inequalities are equivalent:

$$x \leqslant [f](y) \Longleftrightarrow [f]^*(x) \leqslant y.$$

Therefore, there is natural bijection $[I](x,[f]y) \simeq [J]([f]^*x,y)$.

Given a non-empty set S, for each category $\mathcal{I} = \mathcal{O}, \mathcal{O}_s, \mathcal{S}, \mathcal{S}_s$ we define the category $\mathcal{I}(S)$, where an object is an indexed family $(X_i \in S)_{i \in I}$ for some $I \in \operatorname{Ob} \mathcal{I}$, and a morphism $(X_i)_{i \in I} \to (Y_j)_{j \in J}$ is just a morphism $f : I \to J$ in $\mathcal{I}$. The category $\mathcal{I}(S)$ is equivalent to $\mathcal{I}$. The canonical equivalence takes $(X_i \in S)_{i \in I}$ to I and it is identity on morphisms.

Recall that the *nerve* $\mathsf{N}\mathcal{I}$ of a category $\mathcal{I}$ is the simplicial set (a functor) $\boldsymbol{\Delta}^{\mathrm{op}} \to \mathcal{S}et$, $[n] \mapsto \mathsf{N}_n\mathcal{I} = \mathcal{C}at([n],\mathcal{I})$. Elements of $\mathsf{N}_n\mathcal{I}$ are functors $f : [n] \to \mathcal{I}$, $(i \to j) \mapsto (f_{i \to j} : f(i) \to f(j))$. Clearly, such a morphism f is unambiguously specified by a composable

sequence

$$(f_i = f_{(i-1)\to i})_{i\in\mathbf{n}} = \big(f(0) \xrightarrow{f_1} f(1) \xrightarrow{f_2} \cdots f(n-1) \xrightarrow{f_n} f(n)\big). \tag{2.1.3}$$

We turn the nerve of $\mathcal{I}$ into a 2-category as follows.

2.2 Definition. Objects of a 2-category $\mathsf{N}\mathcal{I}$ are objects of $\mathcal{I}$, 1-morphisms are elements of $\mathsf{N}\mathcal{I}(I,J) = \coprod_{n\geqslant 0} \mathsf{N}_n\mathcal{I}(I,J)$, where $\mathsf{N}_n\mathcal{I}(I,J) = \{f \in \mathcal{C}at([n],\mathcal{I}) \mid f(0) = I,\ f(n) = J\}$, a 2-morphism $(f : [n] \to \mathcal{I}) \to (g : [m] \to \mathcal{I})$ is a morphism $(\pi : \mathbf{n} \to \mathbf{m}) \in \mathcal{O}$ such that $g = [\pi]\cdot f$. (Horizontal) composition of 1-morphisms is the concatenation of sequences of morphisms in $\mathcal{I}$. The unit 1-morphism 1_I is the only functor $[0] \to \mathcal{I}$ which maps object 0 to I. (Vertical) composition of 2-morphisms is that of $\mathcal{O}$. Multiplying a 2-morphism $\pi : \mathbf{n} \to \mathbf{m}$ on the left (resp. on the right) with a 1-morphism of height k gives the 2-morphism $\mathrm{id}_{\mathbf{k}} \sqcup \pi : \mathbf{k}+\mathbf{n} \to \mathbf{k}+\mathbf{m}$ (resp. $\pi \sqcup \mathrm{id}_{\mathbf{k}} : \mathbf{n}+\mathbf{k} \to \mathbf{m}+\mathbf{k}$).

2.3 Definition. Objects of a full subcategory $\mathsf{B}\mathcal{S}$ of the category $\mathsf{N}\mathcal{S}_{12}$ formed by 1- and 2-morphisms of $\mathsf{N}\mathcal{S}$ are elements $f \in \mathcal{C}at([n],\mathcal{S})$ of $\mathsf{N}\mathcal{S}$ such that

$$\text{for all } 0 \leqslant p < q < r \leqslant n,\ a,b \in f(p) \text{ inequalities } a < b \text{ and } f_{p\to q}(a) > f_{p\to q}(b) \text{ imply } f_{p\to r}(a) \geqslant f_{p\to r}(b). \tag{2.3.1}$$

A morphism $(f : [n] \to \mathcal{S}) \to (g : [m] \to \mathcal{S})$ of $\mathsf{B}\mathcal{S}$ is a morphism $(\pi : \mathbf{n} \to \mathbf{m}) \in \mathcal{O}$ such that $g = [\pi]\cdot f$.

The category $\mathsf{B}\mathcal{S}$ is not a 2-subcategory, because it is not closed with respect to composition of 1-morphisms in $\mathsf{N}\mathcal{S}$. Similarly a full subcategory $\mathsf{B}(\mathcal{S}(S))$ of the category $\mathsf{N}(\mathcal{S}(S))_{12}$ is defined for an arbitrary set S.

2.4 Lax Monoidal categories and functors. Let $\mathcal{V}$ be a category, and let I be a finite totally ordered set. View I as a discrete category, so $I(i,j) = \varnothing$ if $i \neq j$ and $I(i,i) = \{1\}$. Denote by $\mathcal{V}^I$ the category $\mathcal{C}at(I,\mathcal{V})$ of functors from I to $\mathcal{V}$:

$$\operatorname{Ob}\mathcal{V}^I = \{\text{ maps } I \to \operatorname{Ob}\mathcal{V} : i \mapsto C_i\}, \qquad \mathcal{V}^I((B_i)_{i\in I},(C_i)_{i\in I}) = \prod_{i\in I}\mathcal{V}(B_i,C_i).$$

The category $\mathcal{S}et$ of sets, equipped with functors $\otimes^I_{\mathcal{S}et} = \prod_I : \mathcal{S}et^I \to \mathcal{S}et$, $(C_i)_{i\in I} \mapsto \prod_{i\in I} C_i$, and natural transformations $\lambda^f_{\mathcal{S}et} : \prod_{i\in I} C_i \xrightarrow{\sim} \prod_{j\in J}\prod_{i\in f^{-1}j} C_i$ for any map $f : I \to J$ in $\operatorname{Mor}\mathcal{S}$, is an example of a symmetric Monoidal category $\mathcal{V} = (\mathcal{V},\otimes^I_{\mathcal{V}},\lambda^f_{\mathcal{V}})$. We give its definition following Day and Street [DS03], Leinster [Lei03, Definition 3.1.1], and Definitions 1.2.2, 1.2.14 and 1.2.16 of [Lyu99]. We apologize for calling oplax monoidal categories of Day–Street and Leinster by the name of lax Monoidal categories. Lax and oplax notions differ by passing to the opposite category. Since we need in this book only one choice of direction for structure morphisms, we have decided to use the term 'lax'. We introduce lax symmetric and braided Monoidal categories simultaneously with plain ones.

2.5 Definition. A *lax (symmetric, braided) Monoidal category* $(\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}})$ consists of the following data:

1. A category $\mathcal{V}$.
2. A functor $\otimes^I = \otimes^I_{\mathcal{V}} : \mathcal{V}^I \to \mathcal{V}$, for every set $I \in \operatorname{Ob} \mathcal{S}$. In particular, a map $\otimes^I_{\mathcal{V}} : \prod_{i\in I} \mathcal{V}(X_i, Y_i) \to \mathcal{V}(\otimes^{i\in I} X_i, \otimes^{i\in I} Y_i)$ is given.

   For a map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$) introduce a functor $\otimes^f = \otimes^f_{\mathcal{V}} : \mathcal{V}^I \to \mathcal{V}^J$ which to a function $X : I \to \operatorname{Ob} \mathcal{V}$, $i \mapsto X_i$ assigns the function $J \to \operatorname{Ob} \mathcal{V}$, $j \mapsto \otimes^{i\in f^{-1}(j)} X_i$. The linear order on $f^{-1}(j)$ is induced by the embedding $f^{-1}(j) \hookrightarrow I$. The functor $\otimes^f_{\mathcal{V}} : \mathcal{V}^I \to \mathcal{V}^J$ acts on morphisms via the map

$$\prod_{i\in I} \mathcal{V}(X_i, Y_i) \xrightarrow{\sim} \prod_{j\in J} \prod_{i\in f^{-1}j} \mathcal{V}(X_i, Y_i) \xrightarrow{\prod_{j\in J} \otimes^{f^{-1}j}} \prod_{j\in J} \mathcal{V}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i).$$

3. A morphism of functors

$$\lambda^f : \otimes^I \to \otimes^J \circ \otimes^f : \mathcal{V}^I \to \mathcal{V}, \qquad \lambda^f : \otimes^{i\in I} X_i \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i,$$

   for every map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$).

4. A morphism of functors $\rho^L : \otimes^L \to \operatorname{Iso} : \mathcal{V}^L \to \mathcal{V}$, for each 1-element set L, where functor Iso is the obvious isomorphism of categories.

These data are subject to the following axioms:

(i) for all sets $I \in \operatorname{Ob} \mathcal{O}$, for all 1-element sets J

$$\left[\otimes^{i\in I} X_i \xrightarrow{\lambda^{\mathrm{id}_I}} \otimes^{i\in I} (\otimes^{\{i\}} X_i) \xrightarrow{\otimes^{i\in I} \rho^{\{i\}}} \otimes^{i\in I} X_i\right] = \mathrm{id}, \tag{2.5.1}$$

$$\left[\otimes^{i\in I} X_i \xrightarrow{\lambda^{I\to J}} \otimes^J (\otimes^{i\in I} X_i) \xrightarrow{\rho^J} \otimes^{i\in I} X_i\right] = \mathrm{id}; \tag{2.5.2}$$

(ii) for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\mathcal{O}$ (resp. from $\mathcal{S}$, resp. a pair from $\mathcal{S}$ that satisfies the following condition)

> For any pair of elements $a, b \in I$ inequalities $a < b$ and $f(a) > f(b)$ imply $gf(a) \geqslant gf(b)$ (2.5.3)

the following equation holds:

$$\begin{tikzcd}
\otimes^{i\in I} X_i \arrow[rr, "\lambda^f"] \arrow[d, "\lambda^{fg}"'] & & \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i \arrow[d, "\lambda^g"] \\
\otimes^{k\in K} \otimes^{i\in f^{-1}g^{-1}k} X_i \arrow[rr, "\otimes^{k\in K} \lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k}"] \arrow[urr, phantom, "="] & & \otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i
\end{tikzcd} \tag{2.5.4}$$

A *Monoidal* (resp. *symmetric Monoidal*, resp. *braided Monoidal*) *category* is a lax one for which all λ^f and ρ^L are isomorphisms.

If $f : I \to J$ is a bijection in $\mathcal{S}$, then the canonical morphism $\otimes^f_{\mathcal{V}} : \mathcal{V}^I \to \mathcal{V}^J$, $\prod_{i\in I} \mathcal{V}(X_i, Y_i) \longrightarrow \prod_{j\in J} \mathcal{V}(X_{f^{-1}j}, Y_{f^{-1}j})$ is the canonical isomorphism.

We shall mostly need the category $\mathcal{V} = \mathcal{S}et$ of sets, the category of graded $\Bbbk$-modules $\mathcal{V} = \mathbf{gr} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$, the category of differential graded $\Bbbk$-modules $\mathcal{V} = \mathbf{dg} = \mathbf{dg}(\Bbbk\text{-}\mathbf{Mod})$ and the category of small categories $\mathcal{V} = \mathcal{C}at$.

A Monoidal category $(\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}})$ is called *strict* if $\lambda^f_{\mathcal{V}} : \otimes^I_{\mathcal{V}} \to \otimes^f_{\mathcal{V}} \cdot \otimes^J_{\mathcal{V}}$ are identity morphisms for all isotonic maps $f : I \to J$, $\otimes^L = \mathrm{Id}$, and $\rho^L : \otimes^L \to \mathrm{Id}$ is the identity morphisms, for each 1-element set L. Any plain (resp. symmetric, resp. braided) Monoidal category (so all structure maps $\lambda^f_{\mathcal{V}}$ and $\rho^L_{\mathcal{V}}$ are isomorphisms) is equivalent to a plain (resp. symmetric, resp. braided) strict Monoidal category due to Leinster [Lei03, Theorem 3.1.6]. Plain (resp. symmetric, resp. braided) strict Monoidal categories are in bijection with plain (resp. symmetric, resp. braided) strict monoidal categories [Lyu99, Propositions 1.2.15, 1.2.17]. To explain the notion of equivalence, we recall the definition of Monoidal functors and transformations given in the plain case by Leinster [Lei03, Definitions 3.1.3 and 3.1.4].

2.6 Definition. A *lax (symmetric, braided) Monoidal functor* between lax (symmetric, braided) Monoidal categories

$$(F, \phi^I) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$$

consists of

i) a functor $F : \mathcal{C} \to \mathcal{D}$,

ii) a functorial morphism for each set $I \in \operatorname{Ob} \mathcal{S}$

$$\phi^I : \otimes^I_{\mathcal{D}} \circ F^I \to F \circ \otimes^I_{\mathcal{C}} : \mathcal{C}^I \to \mathcal{D}, \qquad \phi^I : \otimes^{i\in I}_{\mathcal{D}} FX_i \to F \otimes^{i\in I}_{\mathcal{C}} X_i,$$

such that

$$\rho^L = \bigl[\otimes^L FX \xrightarrow{\phi^L} F \otimes^L X \xrightarrow{F\rho^L} FX\bigr],$$

for each 1-element set L, and for every map $f : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) and all families $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ the following equation holds:

$$\begin{tikzcd}
\otimes^{i\in I}_{\mathcal{D}} FX_i \arrow[rrr, "\phi^I"] \arrow[d, "\lambda^f_{\mathcal{D}}"'] & & & F \otimes^{i\in I}_{\mathcal{C}} X_i \arrow[d, "F\lambda^f_{\mathcal{C}}"] \\
& & = & \\
\otimes^{j\in J}_{\mathcal{D}} \otimes^{i\in f^{-1}j}_{\mathcal{D}} FX_i \arrow[r, "\otimes^{j\in J}_{\mathcal{D}} \phi^{f^{-1}j}"] & \otimes^{j\in J}_{\mathcal{D}} F \otimes^{i\in f^{-1}j}_{\mathcal{C}} X_i \arrow[rr, "\phi^J"] & & F \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} X_i
\end{tikzcd} \tag{2.6.1}$$

2.7 Definition. A *Monoidal transformation* (morphism of lax (symmetric, braided) Monoidal functors)

$$t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$$

is a natural transformation $t : F \to G$ such that for every $I \in \operatorname{Ob} \mathcal{S}$

$$\begin{tikzcd} \otimes^{i\in I}_{\mathcal{D}} FX_i \arrow[r, "\phi^I"] \arrow[d, "\otimes^I t"'] \arrow[dr, phantom, "="] & F \otimes^{i\in I}_{\mathcal{C}} X_i \arrow[d, "t"] \\ \otimes^{i\in I}_{\mathcal{D}} GX_i \arrow[r, "\psi^I"] & G \otimes^{i\in I}_{\mathcal{C}} X_i \end{tikzcd} \tag{2.7.1}$$

Assume from now on that $\mathcal{V}$ is a symmetric Monoidal category such that all ρ^L are identity morphisms. Let C be an object of $\mathcal{V}$. An *element* of C is a morphism $\mathbb{1}_{\mathcal{V}} \to C$ in $\mathcal{V}$. If $\mathcal{C}$ is a $\mathcal{V}$-category, then a morphism $f : X \to Y$ in $\mathcal{C}$ is an element f of $\mathcal{C}(X, Y)$, that is, a morphism $f : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(X, Y)$. An ordinary category $\overline{\mathcal{C}}$ is associated with $\mathcal{C}$, namely, $\operatorname{Ob} \overline{\mathcal{C}} = \operatorname{Ob} \mathcal{C}$ and $\overline{\mathcal{C}}(X, Y) = \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{C}(X, Y))$.

There is a lax symmetric Monoidal functor $(F, \phi^I) : \mathcal{V} \to \mathcal{S}et$, $F : C \mapsto \mathcal{V}(\mathbb{1}_{\mathcal{V}}, C)$,

$$\phi^I : \prod_{i\in I} \mathcal{V}(\mathbb{1}_{\mathcal{V}}, C_i) \xrightarrow{\otimes^I_{\mathcal{V}}} \mathcal{V}(\otimes^I \mathbb{1}_{\mathcal{V}}, \otimes^{i\in I} C_i) \xrightarrow{\mathcal{V}(\lambda^{\varnothing\to I}_{\mathcal{V}}, 1)} \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \otimes^{i\in I} C_i). \tag{2.7.2}$$

It is used to define a lax symmetric Monoidal $\mathcal{C}at$-functor $\mathcal{V}$-$\mathcal{C}at \to \mathcal{C}at$, $\mathcal{C} \to \overline{\mathcal{C}}$. The latter, in particular, can be viewed as a 2-functor. Moreover, any lax symmetric Monoidal functor $(B, \beta^I) : (\mathcal{V}, \otimes^I_{\mathcal{V}}, \lambda^f_{\mathcal{V}}) \to (\mathcal{W}, \otimes^I_{\mathcal{W}}, \lambda^f_{\mathcal{W}})$ between symmetric Monoidal categories gives rise to a lax symmetric Monoidal $\mathcal{C}at$-functor $(B_*, \beta^I_*) : \mathcal{V}$-$\mathcal{C}at \to \mathcal{W}$-$\mathcal{C}at$ [Man07, Section 1.1.14].

2.8 Lax Monoidal $\mathcal{V}$-categories. For a $\mathcal{V}$-category $\mathcal{C}$ and a finite totally ordered set I define $\mathcal{C}^I$ to be the $\mathcal{V}$-category of functions on I with values in $\mathcal{C}$:

$$\operatorname{Ob} \mathcal{C}^I = \{ \text{ maps } I \to \operatorname{Ob} \mathcal{C} : i \mapsto X_i \},$$
$$\mathcal{C}^I((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes^I_{\mathcal{V}} \mathcal{C}(X_i, Y_i) = \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i).$$

In particular, $\mathcal{C}^\varnothing$ is the $\mathcal{V}$-category $\mathbb{1}$ with one object $*$ (the only map $\varnothing \to \operatorname{Ob} \mathcal{C}$), whose endomorphism object is the unit object $\mathbb{1}_{\mathcal{V}}$ of $\mathcal{V}$. For each 1-element set I we identify $\mathcal{C}^I$ with $\mathcal{C}$ via the obvious isomorphism.

A *natural transformation* ν between $\mathcal{V}$-functors $F, G : \mathcal{C} \to \mathcal{D}$ is a collection of elements $\nu_X \in \mathcal{D}(FX, GX)$, $X \in \operatorname{Ob} \mathcal{C}$ (that is, elements $\nu_X \in \overline{\mathcal{D}}(FX, GX)$) which satisfies an equation

$$\begin{tikzcd} \mathcal{C}(X, Y) \arrow[r, "\lambda^{!\cdot}_{\mathcal{V}}"] \arrow[d, "\lambda^{\cdot!}_{\mathcal{V}}"'] & \mathcal{C}(X, Y) \otimes \mathbb{1}_{\mathcal{V}} \arrow[r, "F_{X,Y} \otimes \nu_Y"] \arrow[d, phantom, "="] & \mathcal{D}(FX, FY) \otimes \mathcal{D}(FY, GY) \arrow[d, "\mu^{\mathcal{D}}"] \\ \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(X, Y) \arrow[r, "\nu_X \otimes G_{X,Y}"] & \mathcal{D}(FX, GX) \otimes \mathcal{D}(GX, GY) \arrow[r, "\mu^{\mathcal{D}}"] & \mathcal{D}(FX, GY) \end{tikzcd} \tag{2.8.1}$$

in $\mathcal{V}$, where $\mu^{\mathcal{D}}$ is the composition in $\mathcal{D}$. One can show that it induces a natural transformation between ordinary functors $\nu : \overline{F} \to \overline{G} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$.

2.9 Remark. Let $\nu, \varkappa : F \to G : \mathcal{C} \to \mathcal{D}$ be natural $\mathcal{V}$-transformations. They coincide if and only if the induced ordinary natural transformations $\nu, \varkappa : \overline{F} \to \overline{G} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ coincide. Indeed, enriched and ordinary transformations are specified by the same data. Thus, while considering equations between natural $\mathcal{V}$-transformations, we may assume that $\mathcal{V} = \mathcal{S}\mathrm{et}$. On the other hand, to prove that given collection $\nu_X : \mathbb{1}_{\mathcal{V}} \to \mathcal{D}(FX, GX) \in \mathcal{V}$ is a natural $\mathcal{V}$-transformation, one has to prove an equation in $\mathcal{V}$.

2.10 Definition. A *lax (symmetric, braided) Monoidal $\mathcal{V}$-category* $(\mathcal{C}, \otimes^I, \lambda^f)$ consists of

1. A $\mathcal{V}$-category $\mathcal{C}$.
2. A $\mathcal{V}$-functor $\otimes^I = \otimes^I_{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}$, for every set $I \in \operatorname{Ob} \mathcal{S}$. In particular, a morphism $\otimes^I : \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i) \to \mathcal{C}(\otimes^{i\in I} X_i, \otimes^{i\in I} Y_i)$ is given.

   For a map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$) introduce a functor $\otimes^f = \otimes^f_{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}^J$ which to a function $X : I \to \operatorname{Ob} \mathcal{C}$, $i \mapsto X_i$ assigns the function $J \to \operatorname{Ob} \mathcal{C}$, $j \mapsto \otimes^{i\in f^{-1}(j)} X_i$. The linear order on $f^{-1}(j)$ is induced by the embedding $f^{-1}(j) \hookrightarrow I$. The functor $\otimes^f_{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}^J$ acts on morphisms via the map

$$\otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i) \xrightarrow[\sim]{\lambda^f_{\mathcal{V}}} \otimes^{j\in J}_{\mathcal{V}} \otimes^{i\in f^{-1}j}_{\mathcal{V}} \mathcal{C}(X_i, Y_i) \xrightarrow{\otimes^{j\in J}_{\mathcal{V}} \otimes^{f^{-1}j}} \otimes^{j\in J}_{\mathcal{V}} \mathcal{C}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i).$$

3. A morphism of $\mathcal{V}$-functors $\lambda^f : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(\otimes^{i\in I} X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i)$ for every map $f : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. $\operatorname{Mor} \mathcal{S}$):

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\otimes^f_{\mathcal{C}}"] \arrow[dr, "\otimes^I"'] & \mathcal{C}^J \arrow[d, "\otimes^J"] \\
\arrow[ur, phantom, "\Uparrow \lambda^f"] & \mathcal{C}
\end{tikzcd}$$

4. A morphism of $\mathcal{V}$-functors $\rho^L : \otimes^L \to \mathrm{Id}$, $\rho^L : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(\otimes^L X, X)$, for each 1-element set L

such that

(i) for all sets $I \in \operatorname{Ob} \mathcal{O}$, for all 1-element sets J

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\otimes^I"] \arrow[dr, "\otimes^{\mathrm{id}_I}"] \arrow[dr, bend right=60, "\mathrm{Id}"'] \arrow[dr, phantom, bend right=30, "\swarrow \otimes^{i\in I}\rho^{\{i\}}"'] & \mathcal{C} \arrow[d, phantom, "\searrow \lambda^{\mathrm{id}_I}"] \\
& \mathcal{C}^I \arrow[u, "\otimes^I"']
\end{tikzcd} = \mathrm{id}, \qquad
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\otimes^I"] \arrow[d, "\otimes^{I\to J}"'] \arrow[d, phantom, shift left=4, "\swarrow \lambda^{I\to J}"] & \mathcal{C} \\
\mathcal{C}^J \arrow[ur, "\otimes^J"] \arrow[ur, bend right=60, "\mathrm{Id}"'] \arrow[ur, phantom, bend right=30, "\searrow \rho^J"'] &
\end{tikzcd} = \mathrm{id}\,. \tag{2.10.1}$$

Here the transformation $\lambda^{(I)} \overset{\text{def}}{=} \otimes^{i\in I}\rho^{\{i\}} : \otimes^{\mathrm{id}_I} \to \mathrm{Id}$ means the composition

$$\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to I}} \otimes_{\mathcal{V}}^{i\in I} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{V}}^{i\in I}\rho^{\{i\}}} \otimes_{\mathcal{V}}^{i\in I} \mathcal{C}(\otimes^{\{i\}}X_i, X_i). \tag{2.10.2}$$

(ii) for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\mathcal{O}$ (resp. from $\mathcal{S}$, resp. a pair from $\mathcal{S}$ that satisfies condition (2.5.3)) the following equation holds:

$$\begin{tikzcd} \mathcal{C}^J \arrow[r, "\otimes^g_{\mathcal{C}}"] \arrow[dr, "\otimes^J" description] & \mathcal{C}^K \arrow[d, "\otimes^K"] \arrow[dl, phantom, "\lambda^g"] \\ \mathcal{C}^I \arrow[u, "\otimes^f_{\mathcal{C}}"] \arrow[r, "\otimes^I"'] \arrow[ur, phantom, "\lambda^f"] & \mathcal{C} \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{C}^J \arrow[r, "\otimes^g_{\mathcal{C}}"] \arrow[dr, phantom, "\otimes^K_{\mathcal{V}}\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k}"] & \mathcal{C}^K \arrow[d, "\otimes^K"] \\ \mathcal{C}^I \arrow[u, "\otimes^f_{\mathcal{C}}"] \arrow[ur, "\otimes^{g\circ f}_{\mathcal{C}}" description] \arrow[r, "\otimes^I"'] \arrow[ur, phantom, bend right, "\lambda^{g\circ f}"] & \mathcal{C} \end{tikzcd} \tag{2.10.3}$$

Here the transformation $\lambda^{(f,g)} \overset{\text{def}}{=} \otimes_{\mathcal{V}}^{K}\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k} : \otimes_{\mathcal{C}}^{g\circ f} \to \otimes_{\mathcal{C}}^{g} \circ \otimes_{\mathcal{C}}^{f}$ means the composition

$$\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to K}} \otimes_{\mathcal{V}}^{K} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{V}}^{k\in K}\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k}} \otimes_{\mathcal{V}}^{k\in K} \mathcal{C}(\otimes^{i\in f^{-1}g^{-1}k}X_i, \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i). \tag{2.10.4}$$

A *Monoidal* (resp. *symmetric Monoidal*, resp. *braided Monoidal*) *$\mathcal{V}$-category* is a lax one for which all λ^f and all ρ^L are isomorphisms.

In particular, the $\mathcal{V}$-functor $u_{\mathcal{C}} = \otimes^{\varnothing} : \mathbb{1} = \mathcal{C}^{\varnothing} \to \mathcal{C}$, $* \mapsto \mathbb{1}_{\mathcal{C}}$ defines the unit object $\mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}$.

For a bijective map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$), introduce a functor $\mathcal{C}^f : \mathcal{C}^I \to \mathcal{C}^J$ which permutes the objects. Namely, $\mathcal{C}^f((X_i)_{i\in I}) = (X_{f^{-1}j})_{j\in J}$, and the action on morphisms is given by

$$\begin{aligned} \mathcal{C}^I((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i, Y_i) \xrightarrow{\lambda_{\mathcal{V}}^{f}} \otimes_{\mathcal{V}}^{j\in J} \mathcal{C}(X_{f^{-1}j}, Y_{f^{-1}j}) \\ = \mathcal{C}^J((X_{f^{-1}j})_{j\in J}, (Y_{f^{-1}j})_{j\in J}). \end{aligned}$$

For two bijections $I \xrightarrow{f} J \xrightarrow{g} K$, we have $\mathcal{C}^{f\cdot g} = \mathcal{C}^f \cdot \mathcal{C}^g$. Since $\mathcal{C}^{\mathrm{id}_I} = \mathrm{Id}_{\mathcal{C}^I}$, the functor $\mathcal{C}^f$ is invertible with the inverse $\mathcal{C}^{(f^{-1})} : \mathcal{C}^J \to \mathcal{C}^I$. Consider the functorial morphism

$$\ell^f = \big[\otimes^I \xrightarrow{\lambda^f} \otimes^J \circ \otimes^f \xrightarrow{\otimes^{j\in J}\rho^{\{f^{-1}j\}}} \otimes^J \circ \mathcal{C}^f\big],$$

$$\ell^f = \big[\otimes^{i\in I}X_i \xrightarrow{\lambda^f} \otimes^{j\in J} \otimes^{\{f^{-1}j\}}X_{f^{-1}j} \xrightarrow{\otimes^{j\in J}\rho^{\{f^{-1}j\}}} \otimes^{j\in J} X_{f^{-1}j}\big].$$

2.11 Remark. Suppose that $\otimes^L = \mathrm{Id}$ and $\rho^L : \otimes^L \to \mathrm{Id}$ is an identity morphism, for each 1-element set L. Then $\mathcal{C}^f = \otimes^f_{\mathcal{C}} : \mathcal{C}^I \to \mathcal{C}^J$ and $\ell^f = \lambda^f : \otimes^I \to \otimes^J \circ \mathcal{C}^f$.

2.12 Proposition. *For each pair of composable bijections $I \xrightarrow{f} J \xrightarrow{g} K$, the equation*

$$\bigl[\otimes^I \xrightarrow{\ell^f} \otimes^J \circ \mathcal{C}^f \xrightarrow{\ell^g \circ \mathcal{C}^f} \otimes^K \circ \mathcal{C}^g \circ \mathcal{C}^f = \otimes^K \circ \mathcal{C}^{fg}\bigr] = \ell^{fg}$$

holds true. Furthermore, $\ell^{\mathrm{id}_I} = \mathrm{id} : \otimes^I \to \otimes^I$.

Proof. The equation in question expands out to the exterior of the following commutative diagram:

$$\begin{tikzcd}
\otimes^{i\in I} X_i \arrow[r, "\lambda^{fg}"] \arrow[d, "\lambda^f"'] & \otimes^{k\in K} \otimes^{\{f^{-1}g^{-1}k\}} X_{f^{-1}g^{-1}k} \arrow[r, equal] \arrow[d, "\otimes^{k\in K}\lambda^{\{f^{-1}g^{-1}k\}\to\{g^{-1}k\}}"] & \otimes^{k\in K} \otimes^{\{f^{-1}g^{-1}k\}} X_{f^{-1}g^{-1}k} \arrow[d, "\otimes^{k\in K}\rho^{\{f^{-1}g^{-1}k\}}"] \\
\otimes^{j\in J} \otimes^{\{f^{-1}j\}} X_{f^{-1}j} \arrow[r, "\lambda^g"] \arrow[d, "\otimes^{j\in J}\rho^{\{f^{-1}j\}}"] & \otimes^{k\in K} \otimes^{\{g^{-1}k\}} \otimes^{\{f^{-1}g^{-1}k\}} X_{f^{-1}g^{-1}k} \arrow[ur, "\otimes^{k\in K}\rho^{\{g^{-1}k\}}"'] \arrow[d, "\otimes^{k\in K}\otimes^{\{g^{-1}k\}}\rho^{\{f^{-1}g^{-1}k\}}"] & \otimes^{k\in K} X_{f^{-1}g^{-1}k} \\
\otimes^{j\in J} X_{f^{-1}j} \arrow[r, "\lambda^g"] & \otimes^{k\in K} \otimes^{\{g^{-1}k\}} X_{f^{-1}g^{-1}k} \arrow[ur, "\otimes^{k\in K}\rho^{\{g^{-1}k\}}"'] &
\end{tikzcd}$$

Here the left upper square is a particular case of equation (2.10.3), the left lower square commutes by the naturality of λ^g, and the right lower square commutes by the naturality of ρ. The triangle is commutative by the second of equations (2.10.1).

The second assertion is precisely the first of equations (2.10.1). $\square$

2.13 Corollary. *For each bijection $f : I \to J$, the morphism ℓ^f is invertible with the inverse $\ell^{f^{-1}} \circ \mathcal{C}^f : \otimes^J \circ \mathcal{C}^f \to \otimes^I \circ \mathcal{C}^{f^{-1}} \circ \mathcal{C}^f = \otimes^I$.*

A lax (symmetric, braided) Monoidal $\mathcal{V}$-category $(\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f)$ induces a lax (symmetric, braided) Monoidal category $(\overline{\mathcal{C}}, \otimes^I_{\overline{\mathcal{C}}}, \lambda^f)$. For instance,

$$\otimes^I_{\overline{\mathcal{C}}} : \prod_{i\in I} \overline{\mathcal{C}}(X_i, Y_i) = \prod_{i\in I} \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{C}(X_i, Y_i)) \xrightarrow{\otimes^I_{\mathcal{V}}} \mathcal{V}(\otimes^I_{\mathcal{V}} \mathbb{1}_{\mathcal{V}}, \otimes^{i\in I}_{\mathcal{V}} \mathcal{C}(X_i, Y_i))$$
$$\xrightarrow{\mathcal{V}(\lambda^{\varnothing\to I}_{\mathcal{V}}, \otimes^I_{\mathcal{C}})} \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{C}(\otimes^{i\in I} X_i, \otimes^{i\in I} Y_i)) = \overline{\mathcal{C}}(\otimes^{i\in I} X_i, \otimes^{i\in I} Y_i).$$

Due to Remark 2.9 equations (2.10.3) for $\mathcal{C}$ and $\overline{\mathcal{C}}$ are equivalent. The latter, written in terms of morphisms $\lambda^f : \otimes^{i\in I} X_i \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i$ in $\overline{\mathcal{C}}$ takes the form of equation (2.5.4).

We put $\otimes^{(f_i)_{i\in\mathbf{n}}}$ to be the composition of $\mathcal{C}^{f(0)} \xrightarrow{\otimes^{f_1}} \mathcal{C}^{f(1)} \xrightarrow{\otimes^{f_2}} \dots \xrightarrow{\otimes^{f_n}} \mathcal{C}^{f(n)}$ for each $(f_i)_{i\in\mathbf{n}}$ in $\mathsf{N}_n\mathcal{O}$ (resp. in $\mathsf{N}_n\mathcal{S}$, resp. in $\mathsf{B}_n\mathcal{S}$). Thus we have a functor (of ordinary categories) $\otimes_{\mathcal{C}} : \mathsf{N}\mathcal{O}_{01} \to \mathcal{V}\text{-}\mathcal{C}at$ (resp. $\otimes_{\mathcal{C}} : \mathsf{N}\mathcal{S}_{01} \to \mathcal{V}\text{-}\mathcal{C}at$). Here the category $\mathsf{N}\mathcal{I}_{01}$ consists of

objects and 1-morphisms of the 2-category $\mathsf{N}\mathcal{I}$. For every element (I) of $\mathsf{N}_0\mathcal{O}$, we have a morphisms of $\mathcal{V}$-functors $\curlywedge^{(I)} : \otimes^{\mathrm{id}_I} \to \mathrm{Id}$ given by (2.10.2). Furthermore, for every element $(f,g) = \big(I \xrightarrow{f} J \xrightarrow{g} K\big)$ of $\mathsf{N}_2\mathcal{O}$ (resp. $\mathsf{N}_2\mathcal{S}$, resp. $\mathsf{B}_2\mathcal{S}$) we have a morphism of $\mathcal{V}$-functors $\curlywedge^{(f,g)} : \otimes^{fg}_{\mathcal{C}} \to \otimes^{(f,g)}_{\mathcal{C}}$ given by (2.10.4). In particular, if K is a 1-element set, then for the only map $\triangleright : J \to K$ and the element $(f,\triangleright) = \big(I \xrightarrow{f} J \xrightarrow{\triangleright} K\big)$ we have $\curlywedge^{(f,\triangleright)} = \lambda^f : \otimes^I_{\mathcal{C}} \to \otimes^{(f,\triangleright)}_{\mathcal{C}}$. We define $\curlywedge$ also for an element $I \xrightarrow{f} J$ of height 1 of the nerve as $\curlywedge^{(f)} = \mathrm{id} : \otimes^f_{\mathcal{C}} \to \otimes^f_{\mathcal{C}}$.

If $f : [n] \to \mathcal{I}$, $(i \to j) \mapsto (f_{i\to j} : f(i) \to f(j))$ is an element of the nerve $\mathsf{N}_n\mathcal{I}$, specified by $(f_i)_{i\in\mathbf{n}}$ as in (2.1.3), and $k \in f(n)$, we denote by $(f_1,\dots,f_n|k) \in \mathsf{N}_n\mathcal{I}$ the tree $f^{|k} : [n] \to \mathcal{I}$, $i \mapsto f^{-1}_{i\to n}k$, with $f^{|k}_{i\to j}$ given by restriction of $f_{i\to j}$. In particular, $f^{|k}(n) = \{k\}$. Explicit presentation of $f^{|k}$ in the form (2.1.3) is

$$
\begin{aligned}
(f_1\dots f_n)^{-1}(k) \xrightarrow{f_1|_{(f_1\dots f_n)^{-1}k}} (f_2\dots f_n)^{-1}(k) \xrightarrow{f_2|_{(f_2\dots f_n)^{-1}k}} \cdots \\
\cdots \xrightarrow{f_{n-1}|_{(f_{n-1}f_n)^{-1}k}} f_n^{-1}(k) \xrightarrow{f_n|_{f_n^{-1}k}} \{k\}.
\end{aligned}
$$

Correspondingly, $\otimes^{(f_1,\dots,f_n|k)} = \otimes^{f^{|k}} : \mathcal{C}^{f^{|k}(0)} \to \mathcal{C}$.

2.14 Proposition. *Let $\mathcal{C}$ be a lax (resp. lax symmetric, resp. lax braided) Monoidal $\mathcal{V}$-category. For any (f_1,f_2,f_3) in $\mathsf{N}_3\mathcal{O}$ (resp. in $\mathsf{N}_3\mathcal{S}$, resp. in $\mathsf{B}_3\mathcal{S}$) the following associativity equation holds:*

$$
\begin{tikzcd}
\mathcal{C}^{f(1)} \arrow[r, "\otimes^{f_2}"] \arrow[dr, "\otimes^{f_2f_3}" description] & \mathcal{C}^{f(2)} \arrow[d, "\otimes^{f_3}"] \\
\mathcal{C}^{f(0)} \arrow[u, "\otimes^{f_1}"] \arrow[r, "\otimes^{f_1f_2f_3}"'] \arrow[u, phantom, shift left=4ex, "\curlywedge^{(f_1,f_2f_3)}"] & \mathcal{C}^{f(3)} \arrow[u, phantom, shift left=4ex, "\curlywedge^{(f_2,f_3)}"']
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{f(1)} \arrow[r, "\otimes^{f_2}"] & \mathcal{C}^{f(2)} \arrow[d, "\otimes^{f_3}"] \\
\mathcal{C}^{f(0)} \arrow[u, "\otimes^{f_1}"] \arrow[ur, "\otimes^{f_1f_2}" description] \arrow[r, "\otimes^{f_1f_2f_3}"'] \arrow[u, phantom, shift left=4ex, "\curlywedge^{(f_1,f_2)}"] & \mathcal{C}^{f(3)} \arrow[u, phantom, shift right=4ex, "\curlywedge^{(f_1f_2,f_3)}"]
\end{tikzcd}
\tag{2.14.1}
$$

Proof. We may replace $\mathcal{C}$ with ordinary Monoidal category $\overline{\mathcal{C}}$ due to Remark 2.9. So we reduce the question to $\mathcal{V} = \mathcal{S}et$ and a lax (symmetric) Monoidal category $\mathcal{C}$.

Let us put $I = f(0)$, $J = f(1)$, $K = f(2)$, $L = f(3)$, $f = f_1$, $g = f_2$, $h = f_3$ for brevity. Let us also introduce the following shorthands: for a fixed $l \in L$ put

$$
f_l = f|_{f^{-1}g^{-1}h^{-1}l} : f^{-1}g^{-1}h^{-1}l \to g^{-1}h^{-1}l, \qquad g_l = g|_{g^{-1}h^{-1}l} : g^{-1}h^{-1}l \to h^{-1}l.
$$

Then we have:

$$
\begin{aligned}
\leftthreetimes^{(f,gh)} &= \bigl(\lambda^{f_l} : \otimes^{i\in f^{-1}g^{-1}h^{-1}l} X_i \to \otimes^{j\in g^{-1}h^{-1}l} \otimes^{i\in f^{-1}j} X_i\bigr)_{l\in L},\\
\leftthreetimes^{(g,h)} &= \bigl(\lambda^{g_l} : \otimes^{j\in g^{-1}h^{-1}l} \otimes^{i\in f^{-1}j} X_i \to \otimes^{k\in h^{-1}l} \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i\bigr)_{l\in L},\\
\leftthreetimes^{(fg,h)} &= \bigl(\lambda^{f_l g_l} : \otimes^{i\in f^{-1}g^{-1}h^{-1}l} X_i \to \otimes^{k\in h^{-1}l} \otimes^{i\in f^{-1}g^{-1}k} X_i\bigr)_{l\in L},\\
\leftthreetimes^{(f,g)} &= \bigl(\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k} : \otimes^{i\in f^{-1}g^{-1}k} X_i \to \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i\bigr)_{k\in K}.
\end{aligned}
$$

The left hand side of equation (2.14.1) equals

$$
\leftthreetimes^{(f,gh)} \cdot \leftthreetimes^{(g,h)} = \bigl(\otimes^{i\in f^{-1}g^{-1}h^{-1}l} X_i \xrightarrow{\lambda^{f_l}} \otimes^{j\in g^{-1}h^{-1}l} \otimes^{i\in f^{-1}j} X_i \xrightarrow{\lambda^{g_l}} \otimes^{k\in h^{-1}l} \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i\bigr)_{l\in L}.
$$

Similarly, the right hand side of (2.14.1) is

$$
\leftthreetimes^{(fg,h)} \cdot (\leftthreetimes^{(f,g)} \cdot \otimes^h) = \bigl(\otimes^{i\in f^{-1}g^{-1}h^{-1}l} X_i \xrightarrow{\lambda^{f_l g_l}} \otimes^{k\in h^{-1}l} \otimes^{i\in f^{-1}g^{-1}k} X_i \xrightarrow{\otimes^{k\in h^{-1}l}\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k}} \otimes^{k\in h^{-1}l} \otimes^{j\in g^{-1}k} \otimes^{i\in f^{-1}j} X_i\bigr)_{l\in L}.
$$

The equation follows from (2.5.4) written for maps $f^{-1}g^{-1}h^{-1}l \xrightarrow{f_l} g^{-1}h^{-1}l \xrightarrow{g_l} h^{-1}l$. $\square$

For every element $(f_i)_{i\in\mathbf{n}}$ of height $n > 2$ in $\mathsf{N}_n\mathcal{O}$ (resp. in $\mathsf{N}_n\mathcal{S}$, resp. in $\mathsf{B}_n\mathcal{S}$) we define a morphism of $\mathcal{V}$-functors

$$
\begin{tikzcd}
\mathcal{C}^{f(1)} \arrow[r, "\otimes^{f_2}"] & \cdots \arrow[r, "\otimes^{f_{n-1}}"] & \mathcal{C}^{f(n-1)} \arrow[d, "\otimes^{f_n}"] \\
\mathcal{C}^{f(0)} \arrow[u, "\otimes^{f_1}"] \arrow[rr, "\otimes^{f_1\cdot\dots\cdot f_n}"'] \arrow[ur, phantom, "\Uparrow\leftthreetimes^{(f_i)_{i\in\mathbf{n}}}"] & & \mathcal{C}^{f(n)}
\end{tikzcd}
$$

as the composition of $(n-1)$ morphisms $\leftthreetimes^{(\text{-},\text{-})}$ corresponding to any triangulation of the above $(n+1)$-gon, in particular as the composition of $\leftthreetimes^{(f_1,f_2)}$, $\leftthreetimes^{(f_1f_2,f_3)}$, $\dots$, $\leftthreetimes^{(f_1\cdots f_{n-1},f_n)}$. According to equation (2.14.1) the result is independent of triangulation. For $n \leqslant 2$ the transformation $\leftthreetimes^{(f_i)_{i\in\mathbf{n}}}$ was defined above.

For a 2-morphism $(\pi : \mathbf{n} \to \mathbf{m}) = (\psi = [\pi] : [m] \to [n]) : (f_i)_{i\in\mathbf{n}} = f \to \psi\cdot f$ in $\mathsf{N}\mathcal{I} = \mathsf{N}\mathcal{O}, \mathsf{N}\mathcal{S}, \mathsf{B}\mathcal{S}$, we define a morphism of $\mathcal{V}$-functors ${}^{\pi}\leftthreetimes^{f} = {}_{\psi}\leftthreetimes^{f} : \otimes^{\psi\cdot f} \to \otimes^{f}$ as the pasting

$$
\begin{tikzcd}[column sep=huge]
\mathcal{C}^{f(0)} \arrow[r, bend left, "\otimes^{f_{0\to\psi(1)}}"] \arrow[r, bend right, "\otimes^{(f_i)_{i\in\pi^{-1}(1)}}"'] \arrow[r, phantom, "\Downarrow\leftthreetimes^{(f_i)_{i\in\pi^{-1}(1)}}"] & \mathcal{C}^{f(\psi(1))} \arrow[r, bend left, "\otimes^{f_{\psi(1)\to\psi(2)}}"] \arrow[r, bend right, "\otimes^{(f_i)_{i\in\pi^{-1}(2)}}"'] \arrow[r, phantom, "\Downarrow\leftthreetimes^{(f_i)_{i\in\pi^{-1}(2)}}"] & \cdots \arrow[r, bend left, "\otimes^{f_{\psi(m-1)\to n}}"] \arrow[r, bend right, "\otimes^{(f_i)_{i\in\pi^{-1}(m)}}"'] \arrow[r, phantom, "\Downarrow\leftthreetimes^{(f_i)_{i\in\pi^{-1}(m)}}"] & \mathcal{C}^{f(n)}.
\end{tikzcd}
$$

In particular,

$$
{}^{\mathbf{n}\to\mathbf{1}}\leftthreetimes^{f} = \leftthreetimes^{(f_i)_{i\in\mathbf{n}}} : \otimes^{f_1\cdots f_n} = \otimes^{f_{0\to n}} \to \otimes^{(f_i)_{i\in\mathbf{n}}} : \mathcal{C}^{f(0)} \to \mathcal{C}^{f(n)}.
$$

2.15 Proposition. *Any lax (symmetric) Monoidal $\mathcal{V}$-category $\mathcal{C}$ defines a strict 2-functor $\otimes_{\mathcal{C}} : \mathsf{N}\mathcal{O}^{\text{2-op}} \to \mathcal{V}\text{-}\mathcal{C}at$ (resp. $\otimes_{\mathcal{C}} : \mathsf{N}\mathcal{S}^{\text{2-op}} \to \mathcal{V}\text{-}\mathcal{C}at$) such that $I \mapsto \mathcal{C}^I$, $(f_i)_{i\in\mathbf{n}} \mapsto \otimes^{(f_i)_{i\in\mathbf{n}}}$, and a 2-morphism $(\psi = [\pi] : [m] \to [n]) : f \to \psi \cdot f$ is mapped to the natural transformation ${}_\psi\lambda^f : \otimes^{\psi\cdot f} \to \otimes^f : \mathcal{C}^{f(0)} \to \mathcal{C}^{f(n)}$.*

Proof. Since each 1-morphism of $\mathsf{N}\mathcal{I}$ can be represented as a product of generators in a unique way, it is obvious that $\otimes_{\mathcal{C}}$ preserves composition of 1-morphisms. It is also straightforward from the definition that $\otimes_{\mathcal{C}}$ preserves horizontal composition of 2-morphisms. To check that $\otimes_{\mathcal{C}}$ preserves vertical composition, it suffices to check that it preserves composition of 2-morphisms of the form $\mathbf{n} \xrightarrow{\pi} \mathbf{m} \to \mathbf{1}$. Indeed, by the interchange law, an arbitrary composite $\mathbf{n} \xrightarrow{\phi} \mathbf{m} \xrightarrow{\psi} \mathbf{k}$ can be computed by stacking horizontally the vertical composites $\phi^{-1}\psi^{-1}p \to \psi^{-1}p \to \{p\}$, obtained by restriction to $p \in \mathbf{k}$. Thus, we have to prove the equation

$$
{}_{\mathbf{n}\to\mathbf{1}}\lambda^f = \Bigl(\otimes^{f_{0\to n}} \xrightarrow{{}_{\mathbf{m}\to\mathbf{1}}\lambda^{[\pi]\cdot f}} \otimes^{[\pi]\cdot f} \xrightarrow{{}_\pi\lambda^f} \otimes^f\Bigr). \tag{2.15.1}
$$

If π is surjective, it holds true due to λ^f being well-defined and independent of triangulation of a polygon.

Suppose π is injective. The element $g = [\pi] \cdot f$ of $\mathsf{N}_m\mathcal{O}$ (resp. $\mathsf{N}_m\mathcal{S}$) is given by the sequence of morphisms

$$
J_0 \xrightarrow{g_1} J_1 \xrightarrow{g_2} \dots \xrightarrow{g_m} J_m,
$$

where $g_k = f_{[\pi](k)} : J_{k-1} = I_{[\pi](k)-1} \to I_{[\pi](k)} = J_k$ if k is in the image of π, and

$$
g_k = \mathrm{id}_{I_{[\pi](k)}} : J_{k-1} = I_{[\pi](k)} \to I_{[\pi](k)} = J_k
$$

otherwise. The source of the morphism ${}_{\mathbf{m}\to\mathbf{1}}\lambda^{[\pi]\cdot f}$ is the $\mathcal{V}$-functor $\otimes^{f_1,\dots,f_n} = \otimes^{g_1,\dots,g_m} : \mathcal{C}^{I_0} \to \mathcal{C}^{I_n}$; the target is the composite

$$
\mathcal{C}^{J_0} \xrightarrow{\otimes^{g_1}} \mathcal{C}^{J_1} \xrightarrow{\otimes^{g_2}} \dots \xrightarrow{\otimes^{g_m}} \mathcal{C}^{J_m}.
$$

The composite in the right hand side of equation (2.15.1) is given by the pasting diagram obtained by attaching the cell

$$
\begin{tikzcd}
\mathcal{C}^{I_i} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "\otimes^{\mathrm{id}_{I_i}} = \otimes^{g_k}"'] \arrow[r, phantom, "\Uparrow \lambda^{(I_i)}"] & \mathcal{C}^{I_i}
\end{tikzcd}
$$

to the $(m+1)$-gon representing the morphism ${}_{\mathbf{m}\to\mathbf{1}}\lambda^{[\pi]\cdot f}$, along the edge $\otimes^{g_k}$, for each $k \in \mathbf{m}$ which is not in the image of π; here $i = [\pi](k)$. By triangulating the $(m+1)$-gon suitably and using identities (2.10.1), we can pass from the described pasting diagram representing the right hand side of equation (2.15.1) to a triangulation of the $(n+1)$-gon representing the morphism ${}_{\mathbf{n}\to\mathbf{1}}\lambda^f$.

General case follows from the two above particular cases. Indeed, an arbitrary (non-decreasing) map π decomposes into a (non-decreasing) surjection and a (non-decreasing) injection, $\pi = \big(\mathbf{n} \xrightarrow{\sigma} \mathbf{k} \xhookrightarrow{\iota} \mathbf{m}\big)$. Therefore, by the previous cases

$$
\begin{aligned}
{}^{\mathbf{n}\to\mathbf{1}}\leftthreetimes^{f} &= \big(\otimes^{f_{0\to n}} \xrightarrow{{}^{\mathbf{k}\to\mathbf{1}}\leftthreetimes^{[\sigma]\cdot f}} \otimes^{[\sigma]\cdot f} \xrightarrow{{}^{\sigma}\leftthreetimes^{f}} \otimes^{f}\big) \\
&= \big(\otimes^{f_{0\to n}} \xrightarrow{{}^{\mathbf{m}\to\mathbf{1}}\leftthreetimes^{[\iota]\cdot[\sigma]\cdot f}} \otimes^{[\iota]\cdot[\sigma]\cdot f} \xrightarrow{{}^{\iota}\leftthreetimes^{[\sigma]\cdot f}} \otimes^{[\sigma]\cdot f} \xrightarrow{{}^{\sigma}\leftthreetimes^{f}} \otimes^{f}\big).
\end{aligned}
$$

It remains to prove that ${}^{\iota}\leftthreetimes^{[\sigma]\cdot f} \cdot {}^{\sigma}\leftthreetimes^{f} = {}^{\pi}\leftthreetimes^{f}$. This is one of the equations claiming the compatibility of $\otimes_{\mathcal{C}}$ with vertical composition of 2-morphisms. Therefore, it follows from equation (2.15.1), written for the pairs $\mathbf{0} \to \mathbf{0} \to \mathbf{1}$ and $p \to \mathbf{1} \to \mathbf{1}$. These are particular cases of already proven equations. Hence equation (2.15.1) holds true for general π. $\square$

2.16 Remark. Applying this proposition to the sequence

$$\mathbf{n} = \mathbf{n}_0 \xrightarrow{\pi_1} \mathbf{n}_1 \xrightarrow{\pi_2} \mathbf{n}_2 \dots \xrightarrow{\pi_k} \mathbf{n}_k = \mathbf{1}$$

of composable 2-morphisms we get

$${}^{\pi_k}\leftthreetimes^{f} \cdot {}^{\pi_{k-1}}\leftthreetimes^{f} \cdot \dots \cdot {}^{\pi_1}\leftthreetimes^{f} = {}^{\mathbf{n}\to\mathbf{1}}\leftthreetimes^{f} = \leftthreetimes^{(f_i)_{i\in\mathbf{n}}} : \otimes^{f_1\cdots f_n} \to \otimes^{(f_i)_{i\in\mathbf{n}}} : \mathcal{C}^{f(0)} \to \mathcal{C}^{f(n)}.$$

The left hand side composite does not depend on $\pi_1, \dots, \pi_k$. Expressed in terms of λ's, this means that all pastings $\otimes^{f_1\cdots f_n} \to \otimes^{(f_i)_{i\in\mathbf{n}}} : \mathcal{C}^{f(0)} \to \mathcal{C}^{f(n)}$ constructed from λ's coincide.

Let us spell out the definition of lax Monoidal $\mathcal{V}$-functor given by Day and Street [DS03]. Given a functor $F : \mathcal{C} \to \mathcal{D}$ we define a functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$, $(X_i)_{i\in I} \mapsto (FX_i)_{i\in I}$ via the following morphism

$$
\begin{aligned}
\mathcal{C}^I((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i, Y_i) &\xrightarrow{\otimes_{\mathcal{V}}^{i\in I}(F)_i} \otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(FX_i, FY_i) \\
&= \mathcal{D}^I((FX_i)_{i\in I}, (FY_i)_{i\in I}).
\end{aligned}
$$

2.17 Definition. A *lax (symmetric, braided) Monoidal $\mathcal{V}$-functor* between lax (symmetric, braded) $\mathcal{V}$-categories

$$(F, \phi^I) : (\mathcal{C}, \otimes_{\mathcal{C}}^I, \lambda_{\mathcal{C}}^f) \to (\mathcal{D}, \otimes_{\mathcal{D}}^I, \lambda_{\mathcal{D}}^f)$$

consists of

i) a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$,

ii) a functorial morphism for each set $I \in \operatorname{Ob}\mathcal{S}$

$$
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "F^I"] \arrow[d, "\otimes^I"'] & \mathcal{D}^I \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\phi^I"'] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}
\tag{2.17.1}
$$

such that

$$\rho^L = \bigl[\otimes^L \circ F \xrightarrow{\phi^L} F \circ \otimes^L \xrightarrow{F\rho^L} F\bigr],$$

for each 1-element set L, and for every map $f : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) the following equation holds:

$$\begin{tikzcd}
& \mathcal{C}^I \arrow[rr, "F^I"] \arrow[dl, "\otimes^f_{\mathcal{C}}"'] & & \mathcal{D}^I \arrow[dl, "\otimes^f_{\mathcal{D}}"'] \arrow[dd, "\otimes^I_{\mathcal{D}}"] \arrow[dlll, Rightarrow, "\otimes^{j\in J}_{\mathcal{V}}\phi^{f^{-1}j}"'] \\
\mathcal{C}^J \arrow[rr, "F^J"'] \arrow[dr, "\otimes^J_{\mathcal{C}}"'] & & \mathcal{D}^J \arrow[dr, "\otimes^J_{\mathcal{D}}"] \arrow[dl, Rightarrow, "\phi^J"'] \arrow[r, phantom, "\Leftarrow\lambda^f_{\mathcal{D}}"] & {} \\
& \mathcal{C} \arrow[rr, "F"'] & & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C}^I \arrow[r, "F^I"] \arrow[dl, "\otimes^f_{\mathcal{C}}"'] \arrow[dd, "\otimes^I_{\mathcal{C}}"] & \mathcal{D}^I \arrow[dd, "\otimes^I_{\mathcal{D}}"] \arrow[ddl, Rightarrow, "\phi^I"'] \\
\mathcal{C}^J \arrow[dr, "\otimes^J_{\mathcal{C}}"'] \arrow[r, phantom, "\Leftarrow\lambda^f_{\mathcal{C}}"] & {} & \\
& \mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}
\tag{2.17.2}$$

Notice that given natural transformations $\phi^{f^{-1}j} : \mathbb{1} \to \mathcal{D}(\otimes^{i\in f^{-1}j}(FX_i), F(\otimes^{i\in f^{-1}j}X_i))$ induce for every map $f : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) the natural transformation

$$\begin{aligned}
\phi^f \overset{\text{def}}{=} \otimes^{j\in J}_{\mathcal{V}}\phi^{f^{-1}j} = \bigl(\mathbb{1} \xrightarrow{\lambda^{\varnothing\to J}_{\mathcal{V}}} \otimes^{j\in J}_{\mathcal{V}} \mathbb{1} &\xrightarrow{\otimes^{j\in J}_{\mathcal{V}}\phi^{f^{-1}j}} \otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j}FX_i, F\otimes^{i\in f^{-1}j}X_i) \\
&= \mathcal{D}^J\bigl((\otimes^{i\in f^{-1}j}FX_i)_{j\in J}, (F\otimes^{i\in f^{-1}j}X_i)_{j\in J}\bigr)\bigr),
\end{aligned}$$

used in the top square of left diagram above. In particular, $\phi^{I\to\mathbf{1}} = \phi^I$.

For any $f \in \mathsf{N}_n\mathcal{O}$ (resp. $f \in \mathsf{N}_n\mathcal{S}$, resp. $f \in \mathsf{B}_n\mathcal{S}$) we put $\phi^{(f_i)_{i\in\mathbf{n}}}$ to be the pasting

$$\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, "\otimes^{f_1}"] \arrow[d, "F^{f(0)}"'] & \mathcal{C}^{f(1)} \arrow[r, "\otimes^{f_2}"] \arrow[d, "F^{f(1)}"] & {} & \cdots & \mathcal{C}^{f(n-1)} \arrow[r, "\otimes^{f_n}"] \arrow[d, "F^{f(n-1)}"'] & \mathcal{C}^{f(n)} \arrow[d, "F^{f(n)}"] \\
\mathcal{D}^{f(0)} \arrow[r, "\otimes^{f_1}"'] \arrow[ur, Rightarrow, "\phi^{f_1}"'] & \mathcal{D}^{f(1)} \arrow[r, "\otimes^{f_2}"'] \arrow[ur, Rightarrow, "\phi^{f_2}"'] & {} & & \mathcal{D}^{f(n-1)} \arrow[r, "\otimes^{f_n}"'] \arrow[ur, Rightarrow, "\phi^{f_n}"'] & \mathcal{D}^{f(n)}
\end{tikzcd}$$

In particular, for the element $f = (I) \in \mathsf{N}_0\mathcal{I}$ we have $\phi^{(I)} = \mathrm{id} : F^I \to F^I$. For $n = 1$ we have $\phi^{(g)} = \phi^g$.

2.18 Proposition. *For any $(f_i)_{i\in\mathbf{n}}$ in $\mathsf{N}_n\mathcal{O}$ (resp. $\mathsf{N}_n\mathcal{S}$, $\mathsf{B}_n\mathcal{S}$) the following equation holds*

$$\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, bend left, "\otimes^{(f_i)_{i\in\mathbf{n}}}"{name=U}] \arrow[r, bend right, "\otimes^{f_1\cdots f_n}"'{name=L}] \arrow[d, "F^{f(0)}"'] & \mathcal{C}^{f(n)} \arrow[d, "F^{f(n)}"] \\
\mathcal{D}^{f(0)} \arrow[r, "\otimes^{f_1\cdots f_n}"'] \arrow[ur, Rightarrow, "\phi^{f_1\cdots f_n}"] & \mathcal{D}^{f(n)}
\arrow[from=L, to=U, phantom, "\Uparrow\lambda^{(f_i)_{i\in\mathbf{n}}}_{\mathcal{C}}"]
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, "\otimes^{(f_i)_{i\in\mathbf{n}}}"] \arrow[d, "F^{f(0)}"'] & \mathcal{D}^{f(n)} \arrow[d, "F^{f(n)}"] \\
\mathcal{D}^{f(0)} \arrow[r, bend left, "\otimes^{(f_i)_{i\in\mathbf{n}}}"{name=U}] \arrow[r, bend right, "\otimes^{f_1\cdots f_n}"'{name=L}] \arrow[ur, Rightarrow, "\phi^{(f_i)_{i\in\mathbf{n}}}"'] & \mathcal{D}^{f(n)}
\arrow[from=L, to=U, phantom, "\Uparrow\lambda^{(f_i)_{i\in\mathbf{n}}}_{\mathcal{D}}"]
\end{tikzcd}
. \tag{2.18.1}$$

For any $f = (f_i)_{i\in\mathbf{n}}$ in $\mathsf{N}_n\mathcal{I}$ and for any 2-morphism $(\psi = [\pi] : [m] \to [n]) : f \to \psi\cdot f$ the following equation holds

$$
\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, bend left, "\otimes^f"] \arrow[r, bend right, "\otimes^{\psi\cdot f}"'] \arrow[r, phantom, "\Uparrow\psi\lambda_{\mathcal{C}}^f"] \arrow[d, "F^{f(0)}"'] & \mathcal{C}^{f(n)} \arrow[d, "F^{f(n)}"] \\
\mathcal{D}^{f(0)} \arrow[r, "\otimes^{\psi\cdot f}"'] \arrow[ur, Rightarrow, "\phi^{\psi\cdot f}"] & \mathcal{D}^{f(n)}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, "\otimes^f"] \arrow[d, "F^{f(0)}"'] & \mathcal{D}^{f(n)} \arrow[d, "F^{f(n)}"] \\
\mathcal{D}^{f(0)} \arrow[r, bend left, "\otimes^f"] \arrow[r, bend right, "\otimes^{\psi\cdot f}"'] \arrow[r, phantom, "\Uparrow\psi\lambda_{\mathcal{D}}^f"] \arrow[ur, Rightarrow, "\phi^f"'] & \mathcal{D}^{f(n)}
\end{tikzcd}
. \tag{2.18.2}
$$

Proof. We may assume that $\mathcal{V} = \mathcal{S}\mathrm{et}$ due to Remark 2.9. For $n = 2$ and $\big(I \xrightarrow{f} J \xrightarrow{g} K\big) \in \mathsf{N}_2\mathcal{I}$ equation (2.18.1) takes the form

$$
\begin{tikzcd}
 & \mathcal{C}^I \arrow[rr, "F^I"] \arrow[dl, "\otimes_{\mathcal{C}}^f"'] & & \mathcal{D}^I \arrow[dl, "\otimes_{\mathcal{D}}^f"'] \arrow[dd, "\otimes_{\mathcal{D}}^{fg}"] \arrow[dlll, Rightarrow, "\phi^f"'] \\
\mathcal{C}^J \arrow[rr, "F^J"'] \arrow[dr, "\otimes_{\mathcal{C}}^g"'] & & \mathcal{D}^J \arrow[dr, "\otimes_{\mathcal{D}}^g"] \arrow[dl, Rightarrow, "\phi^g"'] \arrow[r, phantom, "\Leftarrow\lambda_{\mathcal{D}}^{(f,g)}"] & {} \\
 & \mathcal{C}^K \arrow[rr, "F^K"'] & & \mathcal{D}^K
\end{tikzcd}
=
\begin{tikzcd}
 & \mathcal{C}^I \arrow[r, "F^I"] \arrow[dl, "\otimes_{\mathcal{C}}^f"'] \arrow[dd, "\otimes_{\mathcal{C}}^{fg}"] & \mathcal{D}^I \arrow[dd, "\otimes_{\mathcal{D}}^{fg}"] \arrow[ddl, Rightarrow, "\phi^{fg}"'] \\
\mathcal{C}^J \arrow[dr, "\otimes_{\mathcal{C}}^g"'] \arrow[r, phantom, "\Leftarrow\lambda_{\mathcal{C}}^{(f,g)}"] & {} & \\
 & \mathcal{C}^K \arrow[r, "F^K"'] & \mathcal{D}^K
\end{tikzcd}
$$

Denote $f_k = f| : f^{-1}g^{-1}k \to g^{-1}k$ for $k \in K$. Explicitly this equation says that for all $k \in K$ and all families $(X_i)_{i\in I}$ of objects of $\mathcal{C}$

$$
\begin{tikzcd}
\otimes_{\mathcal{D}}^{i\in f^{-1}g^{-1}k} FX_i \arrow[rr, "\phi^{f^{-1}g^{-1}k}"] \arrow[d, "\lambda_{\mathcal{D}}^{f_k}"'] & & F\otimes_{\mathcal{C}}^{i\in f^{-1}g^{-1}k} X_i \arrow[d, "F\lambda_{\mathcal{C}}^{f_k}"] \\
\otimes_{\mathcal{D}}^{j\in g^{-1}k}\otimes_{\mathcal{D}}^{i\in f^{-1}j} FX_i \arrow[r, "\otimes_{\mathcal{D}}^{j\in g^{-1}k}\phi^{f^{-1}j}"] & \otimes_{\mathcal{D}}^{j\in g^{-1}k} F\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i \arrow[r, "\phi^{g^{-1}k}"] \arrow[u, phantom, "="] & F\otimes_{\mathcal{C}}^{j\in g^{-1}k}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i
\end{tikzcd}
$$

This is precisely equation (2.6.1), written for the map $f_k : f^{-1}g^{-1}k \to g^{-1}k$.

The general case of (2.18.1) follows from the case $n = 2$.

The proof of equation (2.18.2) is obtained by splitting it into parts of the form

$$
\begin{tikzcd}
\mathcal{C}^{f(\psi(j-1))} \arrow[r, shift left=2, "\otimes^{(f_i)_{i\in\pi^{-1}(j)}}"] \arrow[r, shift right=2, "\otimes^{f_{\psi(j-1)\to\psi(j)}}"'] \arrow[r, phantom, "\Uparrow\lambda_{\mathcal{C}}^{(f_i)_{i\in\pi^{-1}(j)}}"] \arrow[d, "F^{f(\psi(j-1))}"'] & \mathcal{C}^{f(\psi(j))} \arrow[d, "F^{f(\psi(j))}"] \\
\mathcal{D}^{f(\psi(j-1))} \arrow[r, "\otimes^{f_{\psi(j-1)\to\psi(j)}}"'] \arrow[ur, Rightarrow, "\phi^{f_{\psi(j-1)\to\psi(j)}}"] & \mathcal{D}^{f(\psi(j))}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{f(\psi(j-1))} \arrow[r, "\otimes^{(f_i)_{i\in\pi^{-1}(j)}}"] \arrow[d, "F^{f(\psi(j-1))}"'] & \mathcal{C}^{f(\psi(j))} \arrow[d, "F^{f(\psi(j))}"] \\
\mathcal{D}^{f(\psi(j-1))} \arrow[r, shift left=2, "\otimes^{(f_i)_{i\in\pi^{-1}(j)}}"] \arrow[r, shift right=2, "\otimes^{f_{\psi(j-1)\to\psi(j)}}"'] \arrow[r, phantom, "\Uparrow\lambda_{\mathcal{D}}^{(f_i)_{i\in\pi^{-1}(j)}}"] \arrow[ur, Rightarrow, "\phi^{(f_i)_{i\in\pi^{-1}(j)}}"'] & \mathcal{D}^{f(\psi(j))}
\end{tikzcd}
$$

for every $j \in \mathbf{m}$. These equations are proven above. □

2.19 Proposition. *A lax (symmetric) Monoidal $\mathcal{V}$-functor $(F, \phi^I) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$ gives rise to a lax 2-transformation between strict 2-functors $\otimes_F : \otimes_{\mathcal{C}} \to \otimes_{\mathcal{D}} : \mathsf{N}\mathcal{I}^{\text{2-op}} \to \mathcal{V}\text{-}\mathcal{C}at$ which assigns the $\mathcal{V}$-functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$ to each object I and the transformation $\phi^{(f_i)_{i\in\mathbf{n}}}$ to each 1-morphism $(f_i)_{i\in\mathbf{n}} : I \to J$.*

Proof. Naturality of the 2-morphism $\phi^{(f_i)_{i\in\mathbf{n}}}$ in our case coincides with equation (2.18.2). Relation involving ϕ^{id_I} is trivially satisfied, because functors we are dealing with are strict, and $\phi^{(I)} = \mathrm{id} : F^I \to F^I$ by the definition. It remains to verify equation

$$
\begin{tikzcd}
\mathcal{C}^I \arrow[rr, "\otimes^{fg}"] \arrow[d, "F^I"'] & & \mathcal{C}^K \arrow[d, "F^K"] \\
\mathcal{D}^I \arrow[rr, "\otimes^{fg}"] \arrow[urr, Rightarrow, "\phi_{fg}"] \arrow[dr, "\otimes^f"'] & \arrow[d, phantom, "="] & \mathcal{D}^K \\
& \mathcal{D}^J \arrow[ur, "\otimes^g"'] &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^I \arrow[rr, "\otimes^{fg}"] \arrow[d, "F^I"'] \arrow[dr, "\otimes^f"] & \arrow[d, phantom, "="] & \mathcal{C}^K \arrow[d, "F^K"] \\
\mathcal{D}^I \arrow[r, Rightarrow, "\phi_f"] \arrow[dr, "\otimes^f"'] & \mathcal{C}^J \arrow[ur, "\otimes^g"] \arrow[d, "F^J"] & \mathcal{D}^K \\
& \mathcal{D}^J \arrow[ur, "\otimes^g"'] \arrow[uur, Rightarrow, "\phi_g"] &
\end{tikzcd},
$$

where $f = (f_\nu)_{\nu\in\mathbf{n}} \in \mathsf{N}_n\mathcal{I}(I, J)$, $g = (g_\mu)_{\mu\in\mathbf{m}} \in \mathsf{N}_m\mathcal{I}(J, K)$. It is also satisfied, because by definition both sides are equal to the pasting

$$
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "\otimes^{f_1}"] \arrow[d, "F^I"'] & \mathcal{C}^{f(1)} \arrow[d, "F^{f(1)}"] & \dots & \mathcal{C}^{f(n-1)} \arrow[r, "\otimes^{f_n}"] \arrow[d, "F^{f(n-1)}"'] & \mathcal{C}^J \arrow[r, "\otimes^{g_1}"] \arrow[d, "F^J"] & \mathcal{C}^{g(1)} \arrow[d, "F^{g(1)}"] & \dots & \mathcal{C}^{g(m-1)} \arrow[r, "\otimes^{g_m}"] \arrow[d, "F^{g(m-1)}"'] & \mathcal{C}^K \arrow[d, "F^K"] \\
\mathcal{D}^I \arrow[r, "\otimes^{f_1}"'] \arrow[ur, Rightarrow, "\phi^{f_1}"] & \mathcal{D}^{f(1)} & \dots & \mathcal{D}^{f(n-1)} \arrow[r, "\otimes^{f_n}"'] \arrow[ur, Rightarrow, "\phi^{f_n}"'] & \mathcal{D}^J \arrow[r, "\otimes^{g_1}"'] \arrow[ur, Rightarrow, "\phi^{g_1}"'] & \mathcal{D}^{g(1)} & \dots & \mathcal{D}^{g(m-1)} \arrow[r, "\otimes^{g_m}"'] \arrow[ur, Rightarrow, "\phi^{g_m}"'] & \mathcal{D}^K
\end{tikzcd}
$$

The claim is proven. □

An arbitrary natural transformation $t : F \to G : \mathcal{C} \to \mathcal{D}$, represented by a family of morphisms $t(X) : \mathbb{1} \to \mathcal{D}(FX, GX)$, $X \in \operatorname{Ob}\mathcal{C}$ induces a natural transformation $t^I : F^I \to G^I : \mathcal{C}^I \to \mathcal{D}^I$, represented by

$$t^I[(X_i)_{i\in I}]$$
$$= \bigl(\mathbb{1} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to I}} \otimes_{\mathcal{V}}^{i\in I} \mathbb{1} \xrightarrow{\otimes_{\mathcal{V}}^{i\in I} t(X_i)} \otimes_{\mathcal{V}}^{i\in I} \mathcal{D}(FX_i, GX_i) = \mathcal{D}^I((FX_i)_{i\in I}, (GX_i)_{i\in I})\bigr).$$

2.20 Definition. A *Monoidal transformation* (morphism of lax (symmetric, braided) Monoidal $\mathcal{V}$-functors)

$$t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes_{\mathcal{C}}^I, \lambda_{\mathcal{C}}^f) \to (\mathcal{D}, \otimes_{\mathcal{D}}^I, \lambda_{\mathcal{D}}^f)$$

is a natural transformation $t : F \to G$ such that for every $I \in \operatorname{Ob}\mathcal{S}$

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, shift left=1.5, "F^I"] \arrow[r, shift right=1.5, "G^I"'] \arrow[r, phantom, "t^I\Downarrow"] \arrow[d, "\otimes^I"'] & \mathcal{D}^I \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\psi^I"'] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "F^I"] \arrow[d, "\otimes^I"'] & \mathcal{D}^I \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\phi^I"] \\
\mathcal{C} \arrow[r, shift left=1.5, "F"] \arrow[r, shift right=1.5, "G"'] \arrow[r, phantom, "t\Downarrow"] & \mathcal{D}
\end{tikzcd}
. \tag{2.20.1}$$

2.21 Proposition. *For any $(f_i)_{i\in\mathbf{n}}$ in $\mathsf{N}_n\mathcal{O}$ (resp. $\mathsf{N}_n\mathcal{S}$, $\mathsf{B}_n\mathcal{S}$) the following equation holds*

$$\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, shift left=1.5, "F^{f(0)}"] \arrow[r, shift right=1.5, "G^{f(0)}"'] \arrow[r, phantom, "\Downarrow t^{f(0)}"] \arrow[d, "\otimes^{(f_i)_{i\in\mathbf{n}}}"'] & \mathcal{D}^{f(0)} \arrow[d, "\otimes^{(f_i)_{i\in\mathbf{n}}}"] \arrow[dl, Rightarrow, "\psi^{(f_i)_{i\in\mathbf{n}}}"'] \\
\mathcal{C}^{f(n)} \arrow[r, "G^{f(n)}"'] & \mathcal{D}^{f(n)}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^{f(0)} \arrow[r, "F^{f(0)}"] \arrow[d, "\otimes^{(f_i)_{i\in\mathbf{n}}}"'] & \mathcal{D}^{f(0)} \arrow[d, "\otimes^{(f_i)_{i\in\mathbf{n}}}"] \arrow[dl, Rightarrow, "\phi^{(f_i)_{i\in\mathbf{n}}}"] \\
\mathcal{C}^{f(n)} \arrow[r, shift left=1.5, "F^{f(n)}"] \arrow[r, shift right=1.5, "G^{f(n)}"'] \arrow[r, phantom, "\Downarrow t^{f(n)}"] & \mathcal{D}^{f(n)}
\end{tikzcd}
.$$

Proof. We may assume that $\mathcal{V} = \mathcal{S}\mathrm{et}$. For $n = 1$ and $(I \xrightarrow{f} J) \in \mathsf{N}_1\mathcal{I}$ the equation takes the form

$$\begin{tikzcd}
\mathcal{C}^I \arrow[r, shift left=1.5, "F^I"] \arrow[r, shift right=1.5, "G^I"'] \arrow[r, phantom, "\Downarrow t^I"] \arrow[d, "\otimes_{\mathcal{C}}^f"'] & \mathcal{D}^I \arrow[d, "\otimes_{\mathcal{D}}^f"] \arrow[dl, Rightarrow, "\psi^f"'] \\
\mathcal{C}^J \arrow[r, "G^J"'] & \mathcal{D}^J
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C}^I \arrow[r, "F^I"] \arrow[d, "\otimes_{\mathcal{C}}^f"'] & \mathcal{D}^I \arrow[d, "\otimes_{\mathcal{D}}^f"] \arrow[dl, Rightarrow, "\phi^f"] \\
\mathcal{C}^J \arrow[r, shift left=1.5, "F^J"] \arrow[r, shift right=1.5, "G^J"'] \arrow[r, phantom, "\Downarrow t^J"] & \mathcal{D}^J
\end{tikzcd}
.$$

Explicitly this equation says that for all $j \in J$ and all families $(X_i)_{i\in I}$ of objects in $\mathcal{C}$

$$\begin{tikzcd}
\otimes_{\mathcal{D}}^{i\in f^{-1}j} FX_i \arrow[r, "\phi^{f^{-1}j}"] \arrow[d, "\otimes_{\mathcal{D}}^{f^{-1}j} t(X_i)"'] \arrow[dr, phantom, "="] & F \otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i \arrow[d, "t(\otimes_{\mathcal{C}}^{f^{-1}j} X_i)"] \\
\otimes_{\mathcal{D}}^{i\in f^{-1}j} GX_i \arrow[r, "\psi^{f^{-1}j}"] & G \otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i
\end{tikzcd}$$

This is precisely equation (2.7.1), written for the set $f^{-1}j$.

The general case follows from the case $n = 1$. □

2.22 Proposition. *A Monoidal transformation $t : (F, \phi^I) \to (G, \psi^I)$ is precisely a modification $\otimes_F \to \otimes_G : \otimes_{\mathcal{C}} \to \otimes_{\mathcal{D}} : \mathsf{N}\mathcal{I}^{\text{2-op}} \to \mathcal{V}\text{-}\mathcal{C}at$, which assigns the natural transformation $t^I : F^I \to G^I : \mathcal{C}^I \to \mathcal{D}^I$ to each object I of $\mathsf{N}\mathcal{I}$, where $t : F \to G : \mathcal{C} \to \mathcal{D}$ is a fixed transformation.*

Proof. The required equation is exactly the statement of Proposition 2.21. □

2.23 Example. Denote by $\mathbb{1}$ the symmetric Monoidal $\mathcal{V}$-category with one object $*$; the endomorphisms object $\mathbb{1}(*, *) = \mathbb{1}_{\mathcal{V}}$ is the unit object of $\mathcal{V}$; the $\mathcal{V}$-functor $\otimes^I : \mathbb{1}^I \to \mathbb{1}$, $(I \to \{*\}) \mapsto *$ is specified on morphisms by the structure isomorphisms $(\lambda_{\mathcal{V}}^{\varnothing\to I})^{-1} : \otimes_{\mathcal{V}}^{i\in I}(\mathbb{1}_{\mathcal{V}})_i \to \mathbb{1}_{\mathcal{V}}$ of the symmetric Monoidal category $\mathcal{V}$; the identity morphism of $\mathcal{V}$-functors for every map $f : I \to J$ in $\operatorname{Mor}\mathcal{S}$:

$$\begin{tikzcd}
\mathbb{1}^I \arrow[rr, "\otimes^f"] \arrow[drr, "\otimes^I"'] & & \mathbb{1}^J \arrow[d, "\otimes^J"] \arrow[dl, phantom, "="] \\
& & \mathbb{1}
\end{tikzcd}$$

specified by $\mathrm{id} : \mathbb{1}_{\mathcal{V}} \to \mathbb{1}_{\mathcal{V}} \in \mathcal{V}$. The equation between two $\mathcal{V}$-functors above follows from equation (2.10.3) for $\mathcal{V}$, written for maps $\varnothing \to I \xrightarrow{f} J$.

2.24 Algebras. We are going to study 'associative' algebras in lax Monoidal categories.

2.25 Definition. A (*commutative*) *algebra* A in a lax (symmetric, braided) Monoidal $\mathcal{V}$-category $\mathcal{C}$ is a lax (symmetric, braided) Monoidal $\mathcal{V}$-functor $A : \mathbb{1} \to \mathcal{C}$. A *morphism* $g : A \to B$ of such algebras is a Monoidal transformation $g : A \to B : \mathbb{1} \to \mathcal{C}$.

Equivalently, an algebra (resp. commutative algebra) A in $\mathcal{C}$ is

i) an object A of $\mathcal{C}$,

ii) a morphism $\mu^I : \otimes^I A \to A$ in $\mathcal{C}$, that is, $\mu^I : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(\otimes^I A, A)$, for each set $I \in \operatorname{Ob} \mathcal{S}$, such that $\mu^L = \rho^L$ for each 1-element set L, and for every map $f : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) the following equation holds:

$$\mu^I = \bigl(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu^{f^{-1}j}} \otimes^J A \xrightarrow{\mu^J} A\bigr). \tag{2.25.1}$$

A morphism $g : A \to B$ of such algebras is a morphism $g : A \to B$ in $\mathcal{C}$, that is, $g : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(A, B)$, such that for every set I the following equation holds:

$$\bigl(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\mu^I_B} B\bigr) = \bigl(\otimes^I A \xrightarrow{\mu^I_A} A \xrightarrow{g} B\bigr). \tag{2.25.2}$$

Equations (2.25.1), (2.25.2) are shorthand for longer equations involving $\mathcal{C}$ and $\mathcal{V}$. They are equivalent to the same equations in $\overline{\mathcal{C}}$ that hold on the nose (μ^I are natural transformations), and define algebras in the lax Monoidal category $\overline{\mathcal{C}}$.

Category $\mathcal{O}$, equipped with disjoint union $\sqcup_{i\in I} : \mathcal{O}^I \to \mathcal{O}$, $(X_i)_{i\in I} \mapsto \sqcup_{i\in I} X_i$, given by (2.1.1), is an example of a Monoidal category. To an isotonic map $f : I \to J$ the only isotonic bijection $\lambda^f_{\mathcal{O}} : \sqcup_{i\in I} X_i \to \sqcup_{j\in J} \sqcup_{i\in f^{-1}j} X_i$ is associated.

2.26 Definition. A *colax (symmetric, braided) Monoidal functor* between lax (symmetric, braided) Monoidal categories

$$(F, \phi^I) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$$

consists of

i) a functor $F : \mathcal{C} \to \mathcal{D}$,

ii) a functorial morphism for each set $I \in \operatorname{Ob} \mathcal{O}$

$$\phi^I : F \circ \otimes^I_{\mathcal{C}} \to \otimes^I_{\mathcal{D}} \circ F^I : \mathcal{C}^I \to \mathcal{D}, \quad \phi^I : F \otimes^{i\in I}_{\mathcal{C}} X_i \to \otimes^{i\in I}_{\mathcal{D}} F X_i$$

such that $\phi^I = \mathrm{id}_F$ for each 1-element set I, and for every map $f : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) and all families $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ the following equation holds:

$$\begin{tikzcd}
\otimes^{i\in I}_{\mathcal{D}} F X_i \arrow[d, "\lambda^f_{\mathcal{D}}"'] & & & F \otimes^{i\in I}_{\mathcal{C}} X_i \arrow[lll, "\phi^I"'] \arrow[d, "F\lambda^f_{\mathcal{C}}"] \\
\otimes^{j\in J}_{\mathcal{D}} \otimes^{i\in f^{-1}j}_{\mathcal{D}} F X_i & & \otimes^{j\in J}_{\mathcal{D}} F \otimes^{i\in f^{-1}j}_{\mathcal{C}} X_i \arrow[ll, "\otimes^{j\in J}_{\mathcal{D}} \phi^{f^{-1}j}"'] & F \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} X_i \arrow[l, "\phi^J"']
\arrow[from=1-1, to=2-4, phantom, "="]
\end{tikzcd}$$

2.27 Proposition. *An algebra A in a lax Monoidal $\mathcal{V}$-category $\mathcal{C}$ defines a colax Monoidal functor*

$$(F, \phi^I) : (\mathcal{O}, \sqcup_I, \lambda^f_{\mathcal{O}}) \to (\overline{\mathcal{C}}, \otimes^I_{\overline{\mathcal{C}}}, \lambda^f_{\mathcal{C}}), \qquad FX = \otimes^X A,$$

$$(f : X \to Y) \longmapsto \mu^f_A = \bigl(\otimes^X A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{y\in Y} \otimes^{f^{-1}y} A \xrightarrow{\otimes^{y\in Y}\mu^{f^{-1}y}_A} \otimes^Y A\bigr),$$
$$\phi^I = \lambda^{\mathrm{pr}:\sqcup_{i\in I} X_i \to I} : \otimes^{\sqcup_{i\in I} X_i} A \to \otimes^{i\in I} \otimes^{X_i} A.$$

Proof. Clearly, $\mu_A^{\mathrm{id}_I} = \mathrm{id}_{\otimes^I A}$. To show that F is a functor, we have to prove for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ in $\mathcal{O}$ the following equations in $\overline{\mathcal{C}}$:

$$
\begin{aligned}
\mu_A^f \cdot \mu_A^g = \bigl(&\otimes^I A \xrightarrow{\lambda_{\mathcal{C}}^f} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu_A^{f^{-1}j}} \otimes^J A \\
&\xrightarrow{\lambda_{\mathcal{C}}^g} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu_A^{g^{-1}k}} \otimes^K A\bigr) \\
= \bigl(&\otimes^I A \xrightarrow{\lambda_{\mathcal{C}}^f} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\lambda_{\mathcal{C}}^g} \\
&\otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{k\in K}\otimes^{j\in g^{-1}k}\mu_A^{f^{-1}j}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu_A^{g^{-1}k}} \otimes^K A\bigr) \\
= \bigl(&\otimes^I A \xrightarrow{\lambda_{\mathcal{C}}^{fg}} \otimes^{k\in K} \otimes^{f^{-1}g^{-1}k} A \xrightarrow{\otimes^{k\in K}\lambda_{\mathcal{C}}^{f|:f^{-1}g^{-1}k\to g^{-1}k}} \\
&\otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{k\in K}\otimes^{j\in g^{-1}k}\mu_A^{f^{-1}j}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu_A^{g^{-1}k}} \otimes^K A\bigr) \\
= \bigl(&\otimes^I A \xrightarrow{\lambda_{\mathcal{C}}^{fg}} \otimes^{k\in K} \otimes^{f^{-1}g^{-1}k} A \xrightarrow{\mu_A^{f^{-1}g^{-1}k}} \otimes^K A\bigr) = \mu_A^{fg}.
\end{aligned}
\tag{2.27.1}
$$

To show that transformation ϕ^I is natural, we notice that the following diagram is commutative:

$$
\begin{tikzcd}
\otimes^{\sqcup_{i\in I} X_i} A \arrow[r, "\lambda^{\sqcup_{i\in I} X_i \to I}"] \arrow[d, "\lambda^{\sqcup_{i\in I} f_i}"'] \arrow[dd, bend right=60, "\mu^{\sqcup_{i\in I} f_i}"'] \arrow[dr, phantom, "="] & \otimes^{i\in I} \otimes^{X_i} A \arrow[d, "\otimes^{i\in I}\lambda^{f_i}"] \arrow[dd, bend left=60, "\otimes^{i\in I}\mu^{f_i}"] \\
\otimes^{(i,y_i)\in\sqcup_{i\in I} Y_i} \otimes^{f_i^{-1} y_i} A \arrow[r, "\lambda^{\sqcup_{i\in I} Y_i \to I}"] \arrow[d, "\otimes^{(i,y_i)\in\sqcup_{i\in I} Y_i}\mu^{f_i^{-1} y_i}"'] \arrow[dr, phantom, "="] & \otimes^{i\in I} \otimes^{y_i\in Y_i} \otimes^{f_i^{-1} y_i} A \arrow[d, "\otimes^{i\in I}\otimes^{y_i\in Y_i}\mu^{f_i^{-1} y_i}"] \\
\otimes^{\sqcup_{i\in I} Y_i} A \arrow[r, "\lambda^{\sqcup_{i\in I} Y_i \to I}"] & \otimes^{i\in I} \otimes^{Y_i} A
\end{tikzcd}
\tag{2.27.2}
$$

The pair (F, ϕ^I) is a colax Monoidal functor, since the diagram

$$
\begin{tikzcd}
\otimes^{\sqcup_{i\in I} X_i} A \arrow[rr, "\lambda^{\sqcup_{i\in I} X_i \to I}"] \arrow[d, "\lambda_{\mathcal{C}}^{\lambda_{\mathcal{O}}^f}"'] & & \otimes^{i\in I} \otimes^{X_i} A \arrow[dd, "\lambda^{f:I\to J}"] \\
\otimes^{\sqcup_{j\in J}\sqcup_{i\in f^{-1}j} X_i} A \arrow[d, "\lambda^{\sqcup_{j\in J}\sqcup_{i\in f^{-1}j} X_i \to J}"'] & = & \\
\otimes^{j\in J} \otimes^{\sqcup_{i\in f^{-1}j} X_i} A \arrow[rr, "\otimes^{j\in J}\lambda^{\sqcup_{i\in f^{-1}j} X_i \to f^{-1}j}"] & & \otimes^{j\in J} \otimes^{i\in f^{-1}j} \otimes^{X_i} A
\end{tikzcd}
\tag{2.27.3}
$$

commutes being equation (2.5.4) for maps $\sqcup_{i\in I} X_i \xrightarrow{\mathrm{pr}} I \xrightarrow{f} J$. □

Let us single out familiar pieces of structure of an algebra $(A, \mu^I) : (\mathbb{1}, \otimes_{\mathbb{1}}^I, \mathrm{id}) \to (\mathcal{C}, \otimes_{\mathcal{C}}^I, \lambda_{\mathcal{C}}^f)$. It determines an object $A(*)$ of $\mathcal{C}$, denoted also A. Functorial morphism (2.17.1) for $I = \varnothing$ gives the unit $\eta_A = \mu^\varnothing : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(\mathbb{1}_{\mathcal{C}}, A)$ of A, shortened

to $\eta_A : \mathbb{1}_{\mathcal{C}} \to A$. For $I = \mathbf{2}$ functorial morphism (2.17.1) gives the multiplication $\mu_A = \mu^{\mathbf{2}} : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(A \otimes A, A)$ of A, shortened to $\mu_A : A \otimes A \to A$, where $A \otimes A = \otimes^{\mathbf{2}}(A, A)$.

The morphism η_A is, indeed, the unit of μ_A as the following equations show:

$$\big(\otimes^{\mathbf{1}} A \xrightarrow{\lambda_{\mathcal{C}}^{\mathsf{I}\cdot}} (\otimes^{\mathbf{1}} A) \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{\rho^{\mathbf{1}} \otimes \eta_A} A \otimes A \xrightarrow{\mu_A} A\big) = \rho^{\mathbf{1}}, \tag{2.27.4}$$

$$\big(\otimes^{\mathbf{1}} A \xrightarrow{\lambda_{\mathcal{C}}^{\cdot\mathsf{I}}} \mathbb{1}_{\mathcal{C}} \otimes (\otimes^{\mathbf{1}} A) \xrightarrow{\eta_A \otimes \rho^{\mathbf{1}}} A \otimes A \xrightarrow{\mu_A} A\big) = \rho^{\mathbf{1}}. \tag{2.27.5}$$

They follow from equation (2.25.1) written for the maps

$$\begin{aligned} \mathsf{I}\,. &= e_1 : \mathbf{1} \to \mathbf{2}, \qquad && e_1(1) = 1, \\ .\,\mathsf{I} &= e_2 : \mathbf{1} \to \mathbf{2}, \qquad && e_2(1) = 2, \end{aligned} \tag{2.27.6}$$

respectively, and from the condition $\mu^{\mathbf{1}} = \rho^{\mathbf{1}}$. The associativity of multiplication is expressed by the equations

$$\begin{aligned} \big(A \otimes A \otimes A \xrightarrow{\lambda_{\mathcal{C}}^{\mathsf{IV}}} (\otimes^{\mathbf{1}} A) \otimes (\otimes^{\{2,3\}}(A, A)) \xrightarrow{\rho^{\mathbf{1}} \otimes \ell^h} A \otimes (A \otimes A) & \\ \xrightarrow{1 \otimes \mu_A} A \otimes A \xrightarrow{\mu_A} A\big) &= \mu^{\mathbf{3}}, \\ \big(A \otimes A \otimes A \xrightarrow{\lambda_{\mathcal{C}}^{\mathsf{VI}}} (A \otimes A) \otimes (\otimes^{\{3\}} A) \xrightarrow{\mu_A \otimes \rho^{\{3\}}} A \otimes A \xrightarrow{\mu_A} A\big) &= \mu^{\mathbf{3}}, \end{aligned}$$

where $A \otimes A \otimes A = \otimes^{i \in \mathbf{3}}(A)_i$, and $h : \{2, 3\} \to \mathbf{2}$ is the only non-decreasing bijection. These equations are nothing else but equation (2.25.1) written for the maps

$$\begin{aligned} \mathsf{IV} &= f : \mathbf{3} \to \mathbf{2}, \qquad && f(1) = 1, \quad f(2) = f(3) = 2, \\ \mathsf{VI} &= g : \mathbf{3} \to \mathbf{2}, \qquad && f(1) = f(2) = 1, \quad f(3) = 2, \end{aligned} \tag{2.27.7}$$

respectively. Also, equation (2.25.1) written for the map h implies that

$$\mu^{\{2,3\}} = \big[\otimes^{\{2,3\}}(A, A) \xrightarrow{\lambda_{\mathcal{C}}^h} (\otimes^{\{2\}} A) \otimes (\otimes^{\{3\}} A) \xrightarrow{\rho^{\{2\}} \otimes \rho^{\{3\}}} A \otimes A \xrightarrow{\mu^{\mathbf{2}}} A\big] = \ell^h \cdot \mu_A.$$

The following description of algebras in a lax Monoidal category will be used in the next chapter in the characterization of multicategory.

2.28 Proposition. *An object A of a lax Monoidal $\mathcal{V}$-category $\mathcal{C}$, equipped with morphisms $\eta_A : \mathbb{1}_{\mathcal{C}} \to A$, $\mu_A : A \otimes A \to A$ such that equations* (2.27.4), (2.27.5) *and*

$$\begin{aligned} \big(A \otimes A \otimes A \xrightarrow{\lambda_{\mathcal{C}}^{\mathsf{IV}}} (\otimes^{\mathbf{1}} A) \otimes (\otimes^{\{2,3\}}(A, A)) \xrightarrow{\rho^{\mathbf{1}} \otimes \ell^h} A \otimes (A \otimes A) \xrightarrow{1 \otimes \mu_A} A \otimes A \xrightarrow{\mu_A} A\big) & \\ = \big(A \otimes A \otimes A \xrightarrow{\lambda_{\mathcal{C}}^{\mathsf{VI}}} (A \otimes A) \otimes (\otimes^{\{3\}} A) \xrightarrow{\mu_A \otimes \rho^{\{3\}}} A \otimes A \xrightarrow{\mu_A} A\big)&, \end{aligned} \tag{2.28.1}$$

hold, admits a unique structure (A, μ^I) of an algebra in $\mathcal{C}$ such that $\mu^{\varnothing} = \eta_A$, $\mu^{\mathbf{2}} = \mu_A$.

Proof. We may and we shall assume that $\mathcal{V} = \mathcal{S}\mathrm{et}$ and $\mathcal{C}$ is a lax Monoidal category. The morphisms $\mu^I : \otimes^I A \to A$, $I \in \mathrm{Ob}\,\mathcal{O}$, are constructed by induction on cardinality of I. Consider the following property $\mathscr{P}_n$:

> μ^I is defined for all $I \in \mathrm{Ob}\,\mathcal{O}$ with $|I| < n$ and for all $f : I \to J$ in $\mathrm{Mor}\,\mathcal{O}$ with $|I| < n$, $|J| < n$ equation (2.25.1) holds. For all $f : I \to J$ in $\mathrm{Mor}\,\mathcal{O}$ with $|I| = n$ and $1 < |\operatorname{Im} f| \leqslant |J| < n$ denote
>
> $$\mu^I_f = \bigl(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu^{f^{-1}j}} \otimes^J A \xrightarrow{\mu^J} A\bigr). \tag{2.28.2}$$
>
> At last, it is required that for all $I \in \mathrm{Ob}\,\mathcal{O}$ with $|I| = n$ and all $f : I \to J$, $g : I \to K$ in $\mathrm{Mor}\,\mathcal{O}$ with $1 < |\operatorname{Im} f| \leqslant |J| < n$, $1 < |\operatorname{Im} g| \leqslant |K| < n$ the compositions μ^I_f and μ^I_g coincide.

For each 1-element set L, define $\mu^L : \otimes^L A \to A$ simply as $\rho^L : \otimes^L A \to A$. For each 2-element set I, define

$$\mu^I = \bigl[\otimes^I(A, A) \xrightarrow{\ell^g} A \otimes A \xrightarrow{\mu_A} A\bigr],$$

where $g : I \to \mathbf{2}$ is a unique order-preserving bijection. Thus μ^I is defined for each $I \in \mathrm{Ob}\,\mathcal{O}$ with $|I| < 3$. It follows immediately from the assumptions that property $\mathscr{P}_3$ is satisfied. Suppose the property $\mathscr{P}_n$ holds for $n \geqslant 3$. We claim that property $\mathscr{P}_{n+1}$ is satisfied. Let us define μ^I for $|I| = n$ as μ^I_f, where $f : I \to J$ is an arbitrary map in $\mathrm{Mor}\,\mathcal{O}$ with $1 < |\operatorname{Im} f| \leqslant |J| < n$. We use the notation μ^I_f for the right hand side of (2.28.2) also for arbitrary $f : I \to J$ in $\mathrm{Mor}\,\mathcal{O}$ with $|I|, |J| \leqslant n$, since it is well-defined now.

2.29 Lemma. *Let the property $\mathscr{P}_n$ hold. Let $h = \bigl(I \xrightarrow{f} J \xrightarrow{g} K\bigr)$ be maps in $\mathcal{O}$ with $|I|, |J| \leqslant n$, $|K| < n$. Assume that $|g^{-1}k| < n$, $|h^{-1}k| < n$ for all $k \in K$. Then $\mu^I_f = \mu^I_h = \mu^I$.*

Proof. We have

$$\mu^J = \bigl(\otimes^J A \xrightarrow{\lambda^g_{\mathcal{C}}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu^{g^{-1}k}} \otimes^K A \xrightarrow{\mu^K} A\bigr), \tag{2.29.1}$$

$$\mu^{h^{-1}k} = \bigl(\otimes^{h^{-1}k} A \xrightarrow{\lambda^{f_k}_{\mathcal{C}}} \otimes^{j\in g^{-1}k} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in g^{-1}k}\mu^{f^{-1}j}} \otimes^{j\in g^{-1}k} A \xrightarrow{\mu^{g^{-1}k}} A\bigr), \tag{2.29.2}$$

where $f_k = f| : f^{-1}g^{-1}k \to g^{-1}k$ for all $k \in K$. Therefore,

$$\begin{aligned}
\mu^I_f &= \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu^{f^{-1}j}} \otimes^J A \xrightarrow{\mu^J} A\big) \\
&= \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu^{f^{-1}j}} \otimes^J A \\
&\qquad \xrightarrow{\lambda^g_{\mathcal{C}}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu^{g^{-1}k}} \otimes^K A \xrightarrow{\mu^K} A\big), \\
&= \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \otimes^{f^{-1}j} A \xrightarrow{\lambda^g_{\mathcal{C}}} \otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{f^{-1}j} A \\
&\qquad \xrightarrow{\otimes^{k\in K}\otimes^{j\in g^{-1}k}\mu^{f^{-1}j}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu^{g^{-1}k}} \otimes^K A \xrightarrow{\mu^K} A\big), \\
&= \big(\otimes^I A \xrightarrow{\lambda^{fg}_{\mathcal{C}}} \otimes^{k\in K} \otimes^{f^{-1}g^{-1}k} A \xrightarrow{\otimes^{k\in K}\lambda^{f_k}_{\mathcal{C}}} \otimes^{k\in K} \otimes^{j\in g^{-1}k} \otimes^{f^{-1}j} A \\
&\qquad \xrightarrow{\otimes^{k\in K}\otimes^{j\in g^{-1}k}\mu^{f^{-1}j}} \otimes^{k\in K} \otimes^{g^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu^{g^{-1}k}} \otimes^K A \xrightarrow{\mu^K} A\big), \\
&= \big(\otimes^I A \xrightarrow{\lambda^h_{\mathcal{C}}} \otimes^{k\in K} \otimes^{h^{-1}k} A \xrightarrow{\otimes^{k\in K}\mu^{h^{-1}k}} \otimes^K A \xrightarrow{\mu^K} A\big) = \mu^I_h = \mu^I,
\end{aligned} \tag{2.29.3}$$

due to property $\mathscr{P}_n$. □

Let us check that equation $\mu^I_f = \mu^I$ holds for an arbitrary $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$ with $|I|, |J| \leqslant n$. If $|\operatorname{Im} f| > 1$, then there is an isotonic surjection $g : J \longrightarrow\!\!\triangleright\, \mathbf{2} = K$ such that $h = fg : I \to \mathbf{2}$ is a surjection. Such data satisfy assumptions of Lemma 2.29, which gives $\mu^I_f = \mu^I$. Suppose $I = \varnothing$. If $J = \varnothing$ or J is a 1-element set, then the equation $\mu^I_f = \mu^I$ is part of property $\mathscr{P}_3$, which is satisfied by assumption. If J contains at least 2 elements, choose an arbitrary surjection $g : J \longrightarrow\!\!\triangleright\, \mathbf{2}$. Lemma 2.29 applied to the maps $h = \big(\varnothing \xrightarrow{f} J \xrightarrow{g}\!\!\triangleright\, \mathbf{2}\big)$ implies that $\mu^I_f = \mu^I$.

It remains to consider the case when $|\operatorname{Im} f| = 1$. Suppose that $f(i) = k$, for each $i \in I$, i.e., f factorizes as

$$f = (I \xrightarrow{p}\!\!\triangleright\, \{k\} \xhookrightarrow{e} J).$$

Using the second of two equations (2.10.1), we find that

$$\begin{aligned}
\mu^I &= \big[\otimes^I A \xrightarrow{\lambda^p_{\mathcal{C}}} \otimes^{\{k\}} \otimes^I A \xrightarrow{\rho^{\{k\}}} \otimes^I A \xrightarrow{\mu^I} A\big] \\
&= \big[\otimes^I A \xrightarrow{\lambda^p_{\mathcal{C}}} \otimes^{\{k\}} \otimes^I A \xrightarrow{\otimes^{\{k\}}\mu^I} \otimes^{\{k\}} A \xrightarrow{\rho^{\{k\}}} A\big] \\
&= \big[\otimes^I A \xrightarrow{\lambda^p_{\mathcal{C}}} \otimes^{\{k\}} \otimes^I A \xrightarrow{\otimes^{\{k\}}\mu^I} \otimes^{\{k\}} A \xrightarrow{\mu^{\{k\}}} A\big],
\end{aligned} \tag{2.29.4}$$

where the second equality is due to the naturality of $\rho^{\{k\}}$. Therefore,

$$\mu_f^I = \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^J \big((\mathbb{1})_{j<k}, \otimes^I A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^J((\eta_A)_{j<k},\mu^I,(\eta_A)_{j>k})} \otimes^J A \xrightarrow{\mu^J} A\big)$$
$$= \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^I A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((1)_{j<k},\lambda^p_{\mathcal{C}},(1)_{j>k})}$$
$$\otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^{\{k\}} \otimes^I A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((1)_{j<k},\otimes^{\{k\}}\mu^I,(1)_{j>k})}$$
$$\otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^{\{k\}} A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((\eta_A)_{j<k},\mu^{\{k\}},(\eta_A)_{j>k})} \otimes^J A \xrightarrow{\mu^J} A\big).$$

Using equation (2.10.3) for the pair of maps p and e, we replace the last expression by the composite

$$\big(\otimes^I A \xrightarrow{\lambda^p_{\mathcal{C}}} \otimes^{\{k\}} \otimes^I A \xrightarrow{\lambda^e_{\mathcal{C}}} \otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^{\{k\}} \otimes^I A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((1)_{j<k},\otimes^{\{k\}}\mu^I,(1)_{j>k})}$$
$$\otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^{\{k\}} A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((\eta_A)_{j<k},\mu^{\{k\}},(\eta_A)_{j>k})} \otimes^J A \xrightarrow{\mu^J} A\big)$$
$$= \big(\otimes^I A \xrightarrow{\lambda^p_{\mathcal{C}}} \otimes^{\{k\}} \otimes^I A \xrightarrow{\otimes^{\{k\}}\mu^I} \otimes^{\{k\}} A \xrightarrow{\lambda^e_{\mathcal{C}}}$$
$$\otimes^{j\in J} \big((\mathbb{1})_{j<k}, \otimes^{\{k\}} A, (\mathbb{1})_{j>k}\big) \xrightarrow{\otimes^{j\in J}((\eta_A)_{j<k},\mu^{\{k\}},(\eta_A)_{j>k})} \otimes^J A \xrightarrow{\mu^J} A\big).$$

The last three arrows compose to $\mu_e^{\{k\}} = \mu^{\{k\}}$ by Lemma 2.29, applied to some pair $\{k\} \xrightarrow{e} J \xrightarrow{g} \mathbf{2}$. Hence the whole composite equals

$$\big(\otimes^I A \xrightarrow{\lambda^p} \otimes^{\{k\}} \otimes^I A \xrightarrow{\otimes^{\{k\}}\mu^I} \otimes^{\{k\}} A \xrightarrow{\mu^{\{k\}}} A\big),$$

which is simply μ^I by (2.29.4).

We extend the notation μ_f^I to the right hand side of (2.28.2) for $f : I \to J$ with $|I| = n+1$ and $1 < |\operatorname{Im} f| \leqslant |J| \leqslant n$.

2.30 Lemma. *Let $f : I \to J$ be a map in $\mathcal{O}$ with $|I| = n+1$ and $1 < |\operatorname{Im} f| \leqslant |J| \leqslant n$. Suppose that f is factorized as*

$$f = (I \xrightarrow{p} N \xrightarrow{e} J)$$

with $|N| = n$. Then $\mu_p^I = \mu_f^I$.

Proof. Indeed, since $|J| \leqslant |N| = n$, we have $\mu^N = \mu_e^N$, which plays the part of (2.29.1). Due to $|f^{-1}j|, |e^{-1}j| \leqslant n$ we have $\mu^{f^{-1}j} = \mu^{f^{-1}j}_{p|:f^{-1}j\to e^{-1}j}$, which is used in place of (2.29.2). Using (2.29.3) as in Lemma 2.29, we get $\mu_p^I = \mu_f^I$. □

Finally, let us show that if I is an $(n+1)$-element set and $f : I \to J$ and $g : I \to K$ are maps with $1 < |\operatorname{Im} f| \leqslant |J| \leqslant n$ and $1 < |\operatorname{Im} g| \leqslant |K| \leqslant n$, then $\mu_f^I = \mu_g^I$. Factorize f and g as

$$f = (I \xrightarrow{p} N \xrightarrow{e} J), \quad g = (I \xrightarrow{q} M \xrightarrow{d} K),$$

where $|N| = n$ and $|M| = n$. By Lemma 2.30, $\mu^I_f = \mu^I_p$ and $\mu^I_g = \mu^I_q$. There exist surjective maps $\phi : N \to L$ and $\psi : M \to L$ with $|L| = n-1$ such that $p\phi = q\psi$. Applying Lemma 2.30 to the factorizations

$$p\phi = \bigl[I \xrightarrow{p} N \xrightarrow{\phi} L\bigr], \quad q\psi = \bigl[I \xrightarrow{q} M \xrightarrow{\psi} L\bigr],$$

we obtain $\mu^I_p = \mu^I_{p\phi}$ and $\mu^I_q = \mu^I_{q\psi}$. Hence

$$\mu^I_f = \mu^I_p = \mu^I_{p\phi} = \mu^I_{q\psi} = \mu^I_q = \mu^I_g,$$

property $\mathscr{P}_{n+1}$ is satisfied, and the induction goes through. □

2.31 Proposition. *Let (A, μ^I_A), (B, μ^I_B) be algebras in a lax Monoidal $\mathcal{V}$-category $\mathcal{C}$. A morphism $g : A \to B$ in $\mathcal{C}$ is a morphism of algebras if and only if*

$$\eta_B = \bigl(\mathbb{1}_{\mathcal{C}} \xrightarrow{\eta_A} A \xrightarrow{g} B\bigr), \tag{2.31.1}$$

$$\bigl(A \otimes A \xrightarrow{g\otimes g} B \otimes B \xrightarrow{\mu_B} B\bigr) = \bigl(A \otimes A \xrightarrow{\mu_A} A \xrightarrow{g} B\bigr). \tag{2.31.2}$$

In this case for any set $I \in \operatorname{Ob} \mathcal{O}$ denote by

$$\mu^I_g = \bigl(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\mu^I_B} B\bigr) = \bigl(\otimes^I A \xrightarrow{\mu^I_A} A \xrightarrow{g} B\bigr). \tag{2.31.3}$$

the value of both sides of (2.25.2). Then for each map $\phi : I \to J$ in $\mathcal{O}$ the following equation holds:

$$\bigl(\otimes^I A \xrightarrow{\lambda^\phi_{\mathcal{C}}} \otimes^{j\in J} \otimes^{\phi^{-1}j} A \xrightarrow{\otimes^{j\in J}\mu_g^{\phi^{-1}j}} \otimes^{j\in J} B \xrightarrow{\mu^J_B} B\bigr) = \mu^I_g. \tag{2.31.4}$$

Proof. If $g : A \to B$ in $\mathcal{C}$ is a morphism of algebras, then (2.31.1), (2.31.2) obviously hold. Assume that these equations hold. We have to prove equation (2.25.2). We do this by induction on $|I|$. For $|I| = 0, 1, 2$ the equation holds by assumption. Let $|I| \geqslant 3$ and let $f : I \to \mathbf{2}$ be an arbitrary surjection. Then since (A, μ^I_A) and (B, μ^I_B) are algebras in $\mathcal{C}$, we have:

$$\mu^I_A = (\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1} A) \otimes (\otimes^{f^{-1}2} A) \xrightarrow{\mu_A^{f^{-1}1}\otimes\mu_A^{f^{-1}2}} A \otimes A \xrightarrow{\mu_A} A),$$
$$\mu^I_B = (\otimes^I B \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1} B) \otimes (\otimes^{f^{-1}2} B) \xrightarrow{\mu_B^{f^{-1}1}\otimes\mu_B^{f^{-1}2}} B \otimes B \xrightarrow{\mu_B} B).$$

Therefore,

$$\begin{aligned}
&\bigl(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\mu^I_B} B\bigr) \\
&= \bigl(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1} B) \otimes (\otimes^{f^{-1}2} B) \xrightarrow{\mu_B^{f^{-1}1}\otimes\mu_B^{f^{-1}2}} B \otimes B \xrightarrow{\mu_B} B\bigr) \\
&= \bigl(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1} A) \otimes (\otimes^{f^{-1}2} A) \xrightarrow{(\otimes^{f^{-1}1} g)\otimes(\otimes^{f^{-1}2} g)} (\otimes^{f^{-1}1} B) \otimes (\otimes^{f^{-1}2} B) \\
&\qquad\qquad \xrightarrow{\mu_B^{f^{-1}1}\otimes\mu_B^{f^{-1}2}} B \otimes B \xrightarrow{\mu_B} B\bigr).
\end{aligned} \tag{2.31.5}$$

Since f is surjective, $|f^{-1}1|$, $|f^{-1}2| < |I|$, hence by induction hypothesis

$$\big(\otimes^{f^{-1}1}A \xrightarrow{\otimes^{f^{-1}1}g} \otimes^{f^{-1}1} B \xrightarrow{\mu_B^{f^{-1}1}} B\big) = \big(\otimes^{f^{-1}1}A \xrightarrow{\mu_A^{f^{-1}1}} A \xrightarrow{g} B\big),$$
$$\big(\otimes^{f^{-1}2}A \xrightarrow{\otimes^{f^{-1}2}g} \otimes^{f^{-1}2} B \xrightarrow{\mu_B^{f^{-1}2}} B\big) = \big(\otimes^{f^{-1}2}A \xrightarrow{\mu_A^{f^{-1}2}} A \xrightarrow{g} B\big).$$

Hence expression (2.31.5) can be rewritten as follows:

$$\begin{aligned}
&\big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1}A) \otimes (\otimes^{f^{-1}2}A) \xrightarrow{\mu_A^{f^{-1}1}\otimes\mu_A^{f^{-1}2}} A\otimes A \xrightarrow{g\otimes g} B\otimes B \xrightarrow{\mu_B} B\big) \\
&= \big(\otimes^I A \xrightarrow{\lambda^f_{\mathcal{C}}} (\otimes^{f^{-1}1}A) \otimes (\otimes^{f^{-1}2}A) \xrightarrow{\mu_A^{f^{-1}1}\otimes\mu_A^{f^{-1}2}} A\otimes A \xrightarrow{\mu_A} A \xrightarrow{g} B\big) \\
&= \big(\otimes^I A \xrightarrow{\mu_A^I} A \xrightarrow{g} B\big),
\end{aligned}$$

and equation (2.25.2) is proven.

The last statement is proved as follows. Substituting definition (2.31.3) into the left hand side of (2.31.4) we get, due to $\lambda^\phi_{\mathcal{C}}$ being a natural transformation, the equation

$$\begin{aligned}
&\big(\otimes^I A \xrightarrow{\lambda^\phi_{\mathcal{C}}} \otimes^{j\in J} \otimes^{\phi^{-1}j} A \xrightarrow{\otimes^{j\in J}\otimes^{\phi^{-1}j}g} \otimes^{j\in J} \otimes^{\phi^{-1}j} B \xrightarrow{\otimes^{j\in J}\mu_B^{\phi^{-1}j}} \otimes^{j\in J} B \xrightarrow{\mu_B^I} B\big) \\
&= \big(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\lambda^\phi_{\mathcal{C}}} \otimes^{j\in J} \otimes^{\phi^{-1}j} B \xrightarrow{\otimes^{j\in J}\mu_B^{\phi^{-1}j}} \otimes^{j\in J} B \xrightarrow{\mu_B^I} B\big) \\
&= \big(\otimes^I A \xrightarrow{\otimes^I g} \otimes^I B \xrightarrow{\mu_B^I} B\big) = \mu_g^I,
\end{aligned}$$

since μ_B is associative. □

Commutativity of an algebra A in a lax symmetric Monoidal category $(\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}})$ is expressed by equation

$$\big(A\otimes A \xrightarrow{\ell^{\mathsf{X}}_{\mathcal{C}}} A\otimes A \xrightarrow{\mu_A} A\big) = \mu_A, \tag{2.31.6}$$

where the permutation

$$\mathsf{X} = \sigma : \mathbf{2} \to \mathbf{2}, \qquad \sigma(1) = 2, \quad \sigma(2) = 1 \tag{2.31.7}$$

implies the symmetry $\ell^{\mathsf{X}}_{\mathcal{C}} : X\otimes Y \to Y \otimes X$ of the category $\mathcal{C}$. The above equation is equation (2.25.1), written for the map σ.

When $\mathcal{C}$ is a Monoidal category, Definition 2.25 of an algebra agrees with the usual one. Indeed, in the strict Monoidal case all higher multiplications $\mu^{\mathbf{k}} : \otimes^{i\in\mathbf{k}}(A)_i \to A$, $k > 2$, in algebra A are iterations of $\mu_A : A\otimes A \to A$. Therefore, the category of algebras in a strict Monoidal category is isomorphic to the category of usual algebras (A, μ_A, η_A) in a strict monoidal category.

2.32 Coherence principle. We are going to formulate an observation which shortens up significantly verification of many statements.

2.33 Lemma. *An equation between isomorphisms of functors, constructed from arbitrary plain (resp. symmetric, resp. braided) Monoidal $\mathcal{V}$-category data holds, if it holds for arbitrary plain (resp. symmetric, resp. braided) strict Monoidal categories.*

Proof. A $\mathcal{V}$-category $\mathcal{C}$ has an underlying category $\overline{\mathcal{C}}$ with the same set of objects and with the sets of morphisms $\overline{\mathcal{C}}(X,Y) = \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{C}(X,Y))$. A $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ has an underlying functor $\overline{F} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ with $\operatorname{Ob} \overline{F} = \operatorname{Ob} F$. A natural $\mathcal{V}$-transformation $t : F \to G : \mathcal{C} \to \mathcal{D}$ between $\mathcal{V}$-functors F, G is a family $t_X \in \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{D}(FX, GX))$, $X \in \operatorname{Ob} \mathcal{C}$ that satisfies (2.8.1). In particular, it is a natural transformation $\overline{t} : \overline{F} \to \overline{G} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ between the underlying functors. An equation between compositions of natural $\mathcal{V}$-transformations t_i means precisely an equation between compositions of underlying natural transformations $\overline{t_i}$, see Remark 2.9.

For example, a plain (resp. symmetric, resp. braided) Monoidal $\mathcal{V}$-category $\mathcal{C}$ has an underlying plain (resp. symmetric, resp. braided) Monoidal category $\overline{\mathcal{C}}$. The meaning of the phrase 'the considered equation holds in $\mathcal{C}$' is that it holds in $\overline{\mathcal{C}}$. The Monoidal category $\overline{\mathcal{C}}$ is Monoidally equivalent to a strict Monoidal category $\mathcal{A}$ by Leinster [Lei03, Theorem 3.1.6]. The equation we consider holds in $\mathcal{A}$ by assumption. A Monoidal equivalence $(F, \phi^I) : \overline{\mathcal{C}} \to \mathcal{A}$ gives rise to a prism (in which edges are isomorphisms) with commutative walls, whose bottom is the considered equation in $\mathcal{A}$. Therefore, its top, which is the required equation in $\overline{\mathcal{C}}$, also commutes. □

2.34 Remark. All isomorphisms of functors which can be constructed for arbitrary symmetric strictly monoidal category data, coincide, if their source and target coincide. Therefore, the same property holds for isomorphisms of functors which can be constructed for arbitrary symmetric Monoidal $\mathcal{V}$-category data. Similarly, all isomorphisms of functors which can be constructed from λ^f with isotonic f and from ρ^L for arbitrary Monoidal $\mathcal{V}$-category data, coincide, if their source and target coincide. Two functorial isomorphisms, constructed from generic braided Monoidal $\mathcal{V}$-category data, coincide, if their source and target coincide and they determine the same element of the braid group.

Let $(\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric Monoidal $\mathcal{V}$-category. Let $I_1, \dots, I_k$ be totally ordered finite sets. Consider the sequential tree $f = (f_p)_{p \in \mathbf{k}} \in \mathsf{N}_k \mathcal{S}$, where $f(p) = I_1 \times \dots \times I_{k-p}$, $p \in \mathbf{k}$, $f(k) = \mathbf{1}$, f_p is the projection $I_1 \times \dots \times I_{k-p+1} \to I_1 \times \dots \times I_{k-p}$, $p \in \mathbf{k}$. The tree f gives rise to an isomorphism of functors $\lambda^f : \otimes^{I_1 \times \dots \times I_k} \to \otimes^{f_k} \circ \dots \circ \otimes^{f_1}$. Note that the functor $\otimes^{f_p} : \mathcal{C}^{I_1 \times \dots \times I_{k-p+1}} \to \mathcal{C}^{I_1 \times \dots \times I_{k-p}}$ maps a family

$$I_1 \times \dots \times I_{k-p+1} \ni (i_1, \dots, i_{k-p+1}) \mapsto X_{i_1, \dots, i_{k-p+1}}$$

of objects of $\mathcal{C}$ to the family

$$I_1 \times \dots \times I_{k-p} \ni (i_1, \dots, i_{k-p}) \mapsto \otimes^{(i_1, \dots, i_{k-p}, i_{k-p+1}) \in \{i_1\} \times \dots \times \{i_{k-p}\} \times I_{k-p+1}} X_{i_1, \dots, i_{k-p+1}}.$$

For each element $(i_1, \dots, i_{k-p}) \in I_1 \times \cdots \times I_{k-p}$, denote by g_p the obvious bijection $\{i_1\} \times \cdots \times \{i_{k-p}\} \times I_{k-p+1} \to I_{k-p+1}$. Define an isomorphisms of functors $\nu^f : \otimes^{I_1 \times \cdots \times I_k} \to \otimes^{I_1} \circ \cdots \circ \otimes^{I_k}$ by the composite

$$\nu^f = \bigl[\otimes^{I_1 \times \cdots \times I_k} \xrightarrow{\curlywedge^f} \otimes^{f_k} \circ \cdots \circ \otimes^{f_1} \xrightarrow{\ell^{g_k} \circ \cdots \circ \ell^{g_1}} \otimes^{I_1} \circ \cdots \circ \otimes^{I_k}\bigr],$$

where the last arrow is the horizontal composition of 2-isomorphisms $\ell^{g_p} : \otimes^{f_p} \to \otimes^{I_{k-p}}$, $p \in \mathbf{k}$. An arbitrary permutation $\pi \in \mathfrak{S}_k$ induces the sequential tree f^π, where $f^\pi(i) = I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}i}$, $i \in \mathbf{k}$, f_i^π is the projection $I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}(k-i+1)} \to I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}(k-i)}$. Respectively, we get isomorphisms of functors $\curlywedge^{f^\pi} : \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \to \otimes^{f_k^\pi} \circ \cdots \circ \otimes^{f_1^\pi}$ and $\nu^{f^\pi} : \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \to \otimes^{I_{\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\pi^{-1}k}}$. Denote by σ_π the composite

$$\sigma_\pi = \bigl[\otimes^{I_1} \circ \cdots \circ \otimes^{I_k} \xrightarrow{(\nu^f)^{-1}} \otimes^{I_1 \times \cdots \times I_k} \xrightarrow{\ell^{P_\pi}} \\ \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi} \xrightarrow{\nu^{f^\pi} \circ \mathcal{C}^{P_\pi}} \otimes^{I_{\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi}\bigr],$$

where $P_\pi : I_1 \times \cdots \times I_k \to I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}$ is the obvious bijection induced by π.

2.35 Proposition. *For each pair of permutations $\pi, \tau \in \mathfrak{S}_k$, the equation*

$$\bigl[\otimes^{I_1} \circ \cdots \circ \otimes^{I_k} \xrightarrow{\sigma_\pi} \otimes^{I_{\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi} \\ \xrightarrow{\sigma_\tau \circ \mathcal{C}^{P_\pi}} \otimes^{I_{\tau^{-1}\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_{\tau\circ\pi}}\bigr] = \sigma_{\tau\circ\pi}$$

holds true. Furthermore, $\sigma_{\mathrm{id}} = \mathrm{id}$.

Proof. Note that $(f^\pi)^\tau = f^{\tau\circ\pi}$, in particular σ_τ in the above equation is equal to the composite

$$\sigma_\tau = \bigl[\otimes^{I_{\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\pi^{-1}k}} \xrightarrow{(\nu^{f^\pi})^{-1}} \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \xrightarrow{\ell^{P_\tau}} \\ \otimes^{I_{\tau^{-1}\pi^{-1}1} \times \cdots \times I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_\tau} \xrightarrow{\nu^{f^{\tau\circ\pi}} \circ \mathcal{C}^{P_\tau}} \otimes^{I_{\tau^{-1}\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_\tau}\bigr],$$

therefore

$$\sigma_\tau \circ \mathcal{C}^{P_\pi} = \bigl[\otimes^{I_{\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi} \xrightarrow{(\nu^{f^\pi})^{-1} \circ \mathcal{C}^{P_\pi}} \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi} \xrightarrow{\ell^{P_\tau} \circ \mathcal{C}^{P_\pi}} \\ \otimes^{I_{\tau^{-1}\pi^{-1}1} \times \cdots \times I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_\tau} \circ \mathcal{C}^{P_\pi} \xrightarrow{\nu^{f^{\tau\circ\pi}} \circ \mathcal{C}^{P_\tau} \circ \mathcal{C}^{P_\pi}} \otimes^{I_{\tau^{-1}\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_\tau} \circ \mathcal{C}^{P_\pi}\bigr].$$

It follows that

$$\sigma_\pi \cdot (\sigma_\tau \circ \mathcal{C}^{P_\pi}) = \bigl[\otimes^{I_1} \circ \cdots \circ \otimes^{I_k} \xrightarrow{(\nu^f)^{-1}} \otimes^{I_1 \times \cdots \times I_k} \xrightarrow{\ell^{P_\pi}} \otimes^{I_{\pi^{-1}1} \times \cdots \times I_{\pi^{-1}k}} \circ \mathcal{C}^{P_\pi} \xrightarrow{\ell^{P_\tau} \circ \mathcal{C}^{P_\pi}} \\ \otimes^{I_{\tau^{-1}\pi^{-1}1} \times \cdots \times I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_\tau} \circ \mathcal{C}^{P_\pi} \xrightarrow{\nu^{f^{\tau\circ\pi}} \circ \mathcal{C}^{P_{\tau\circ\pi}}} \otimes^{I_{\tau^{-1}\pi^{-1}1}} \circ \cdots \circ \otimes^{I_{\tau^{-1}\pi^{-1}k}} \circ \mathcal{C}^{P_{\tau\circ\pi}}\bigr].$$

The assertion of the proposition follows now from Proposition 2.12. $\square$

Note that if $\pi = \pi' \times \pi''$ for $\pi' \in \mathfrak{S}_p$, $\pi'' \in \mathfrak{S}_q$, $p+q=k$, then $\sigma_\pi = (\otimes^{I_1} \cdots \otimes^{I_p} \sigma_{\pi''}) \cdot \sigma_{\pi'}$. Indeed, for symmetric strict Monoidal categories the equation in question reduces to an equation between two permutations. The general case follows by Lemma 2.33.

Assume that $\mathcal{C}$ is a symmetric Monoidal $\mathcal{V}$-category. Let I, J be finite sets from $\mathcal{O}$, let $X : I \times J \to \operatorname{Ob} \mathcal{C}$, $(i,j) \mapsto X_{ij}$ be an object of $\mathcal{C}^{I\times J}$. The morphism $\sigma_{(12)} : \otimes^{i\in I} \otimes^{j\in J} X_{ij} \to \otimes^{j\in J} \otimes^{i\in I} X_{ij}$ is given explicitly by

$$\begin{aligned}\sigma_{(12)} = \Big[\otimes^{i\in I} \otimes^{j\in J} X_{ij} &\xrightarrow{(\otimes^{i\in I}\ell^{\{i\}\times J\to J})^{-1}} \otimes^{i\in I} \otimes^{(i,j)\in\{i\}\times J} X_{ij} \\ &\xrightarrow{(\lambda^{\mathrm{pr}:I\times J\to I})^{-1}} \otimes^{(i,j)\in I\times J} X_{ij} \xrightarrow{\ell^{P_{(12)}:I\times J\to J\times I}} \otimes^{(j,i)\in J\times I} X_{ij} \\ &\xrightarrow{\lambda^{\mathrm{pr}:J\times I\to J}} \otimes^{j\in J} \otimes^{(j,i)\in\{j\}\times I} X_{ij} \xrightarrow{\otimes^{j\in J}\ell^{\{j\}\times I\to I}} \otimes^{j\in J} \otimes^{i\in I} X_{ij}\Big]. \end{aligned} \tag{2.35.1}$$

2.36 Proposition. *Let $\mathcal{C}$ be a symmetric Monoidal $\mathcal{V}$-category. Then the category* $\operatorname{Alg}(\mathcal{C})$ *of algebras in $\mathcal{C}$ has the following structure of a symmetric Monoidal category: a functor $\otimes^K_{\operatorname{Alg}\mathcal{C}} : \operatorname{Alg}(\mathcal{C})^K \to \operatorname{Alg}(\mathcal{C})$, given on objects by $((A_k, \mu^I_k))_{k\in K} \mapsto (\otimes^{k\in K}_{\mathcal{C}} A_k, \mu^I)$,*

$$\mu^I = \big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^I_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\mu^I_k} \otimes^{k\in K}_{\mathcal{C}} A_k\big],$$

and by $(f_k)_{k\in K} \mapsto \otimes^{k\in K}_{\mathcal{C}} f_k$ on morphisms. The algebra isomorphisms $\lambda^f_{\operatorname{Alg}\mathcal{C}} : \otimes^I_{\operatorname{Alg}\mathcal{C}} \xrightarrow{\sim} \otimes^f \otimes^J_{\operatorname{Alg}\mathcal{C}}$ and $\rho^L_{\operatorname{Alg}\mathcal{C}} : \otimes^L_{\operatorname{Alg}\mathcal{C}} \to \mathrm{Id}$ are defined as $\lambda^f_{\mathcal{C}}$ and $\rho^L_{\mathcal{C}}$ respectively.

The same statement holds for the category of commutative algebras $\operatorname{ComAlg}(\mathcal{C})$.

Proof. We must check the following statements:

(i) Let (A_k, μ^I_k), $k \in K$, be a family of algebras in $\mathcal{C}$. For every set $I \in \operatorname{Ob}\mathcal{O}$ define μ^I as

$$\mu^I = \big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^I_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\mu^I_k} \otimes^{k\in K}_{\mathcal{C}} A_k\big].$$

Then $(\otimes^{k\in K}_{\mathcal{C}} A_k, \mu^I)$ is an algebra in $\mathcal{C}$.

(ii) Let $f_k : (A_k, \mu^I_{A_k}) \to (B_k, \mu^I_{B_k})$, $k \in K$, be a family of algebra morphisms. Then $\otimes^{k\in K} f_k : \otimes^{k\in K}_{\mathcal{C}} A_k \to \otimes^{k\in K}_{\mathcal{C}} B_k$ is an algebra morphism.

(iii) Let (A_i, μ^K_i), $i \in I$, be a family of algebras in $\mathcal{C}$. Then for every map $f : I \to J$ in $\operatorname{Mor}\mathcal{S}$ the morphism $\lambda^f_{\mathcal{C}} : \otimes^{i\in I} A_i \to \otimes^{j\in J} \otimes^{i\in f^{-1}j} A_i$ is an algebra morphism.

(iv) For each 1-element set L and an algebra (A, μ^I), the morphism $\rho^L : \otimes^L A \to A$ is an algebra morphism.

Let us prove (i). Clearly, $\mu^I = \mathrm{id}$ for every 1-element set I. Let us show that equation (2.25.1) holds for every map $f : I \to J$ in $\operatorname{Mor}\mathcal{O}$. We have

$$\begin{aligned}
&\big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\otimes^{j\in J}\sigma_{(12)}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k \\
&\qquad \xrightarrow{\otimes^{j\in J}\otimes^{k\in K}\mu^{f^{-1}j}_k} \otimes^J_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^J_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\mu^J_k} \otimes^{k\in K}_{\mathcal{C}} A_k\big] \\
&= \big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\otimes^{j\in J}\sigma_{(12)}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k \\
&\qquad \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\otimes^{j\in J}\mu^{f^{-1}j}_k} \otimes^{k\in K}_{\mathcal{C}} \otimes^J_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\mu^J_k} \otimes^{k\in K}_{\mathcal{C}} A_k\big].
\end{aligned}$$

By Lemma 2.33 and Remark 2.34

$$\begin{aligned}
&\big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\lambda^f_{\mathcal{C}}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\otimes^{j\in J}\sigma_{(12)}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k \\
&\qquad \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k\big] \\
&= \big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^I_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\lambda^f_{\mathcal{C}}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k\big],
\end{aligned}$$

therefore we can write the previous expression as follows:

$$\begin{aligned}
&\big[\otimes^I_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \xrightarrow{\sigma_{(12)}} \otimes^{k\in K}_{\mathcal{C}} \otimes^I_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\lambda^f_{\mathcal{C}}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{f^{-1}j}_{\mathcal{C}} A_k \\
&\qquad \xrightarrow{\otimes^{k\in K}\otimes^{j\in J}\mu^{f^{-1}j}_k} \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} A_k \xrightarrow{\otimes^{k\in K}\mu^J_k} \otimes^{k\in K}_{\mathcal{C}} A_k\big].
\end{aligned}$$

The last three arrows compose to $\otimes^{k\in K}\mu^I_k$ by (2.25.1), hence the assertion.

Statement (ii) follows from the diagram below:

$$\begin{tikzcd}
\otimes^{i\in I}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_k \arrow[r, "\otimes^{i\in I}\otimes^{k\in K} f_k"] \arrow[d, "\sigma_{(12)}"'] & \otimes^{i\in I}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} B_k \arrow[d, "\sigma_{(12)}"] \\
\otimes^{k\in K}_{\mathcal{C}} \otimes^{i\in I}_{\mathcal{C}} A_k \arrow[r, "\otimes^{k\in K}\otimes^{i\in I} f_k"] \arrow[d, "\otimes^{k\in K}\mu^I_{A_k}"'] & \otimes^{k\in K}_{\mathcal{C}} \otimes^{i\in I}_{\mathcal{C}} B_k \arrow[d, "\otimes^{k\in K}\mu^I_{B_k}"] \\
\otimes^{k\in K}_{\mathcal{C}} A_k \arrow[r, "\otimes^{k\in K} f_k"] & \otimes^{k\in K}_{\mathcal{C}} B_k
\end{tikzcd}$$

The upper rectangle is commutative due to naturality of $\sigma_{(12)}$, the lower rectangle commutes by (2.25.2).

Statement (iii) follows from the following diagram:

$$\begin{tikzcd}
\otimes^{k\in K}_{\mathcal{C}} \otimes^{i\in I}_{\mathcal{C}} A_i \arrow[r, "\otimes^{k\in K}\lambda^f_{\mathcal{C}}"] \arrow[dd, "\sigma_{(12)}"'] & \otimes^{k\in K}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} A_i \arrow[d, "\sigma_{(12)}"] \\
 & \otimes^{j\in J}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} A_i \arrow[d, "\otimes^{j\in J}\sigma_{(12)}"] \\
\otimes^{i\in I}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_i \arrow[r, "\lambda^f_{\mathcal{C}}"] \arrow[d, "\otimes^{i\in I}\mu^K_i"'] & \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} \otimes^{k\in K}_{\mathcal{C}} A_i \arrow[d, "\otimes^{j\in J}\otimes^{f^{-1}j}\mu^K_i"] \\
\otimes^{i\in I}_{\mathcal{C}} A_i \arrow[r, "\lambda^f_{\mathcal{C}}"] & \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in f^{-1}j}_{\mathcal{C}} A_i
\end{tikzcd}$$

Commutativity of the pentagon is a consequence of Lemma 2.33 and Remark 2.34. The lower rectangle commutes by naturality of $\lambda^f_{\mathcal{C}}$.

Finally, statement (iv) follows from the commutative diagram:

$$\begin{tikzcd}
\otimes^I \otimes^L \arrow[rr, "\otimes^I \rho^L"] \arrow[d, "\sigma_{(12)}"'] & & \otimes^I A \arrow[d, "\mu^I"] \\
\otimes^L \otimes^I A \arrow[r, "\otimes^L \otimes \mu^I"'] \arrow[urr, "\rho^L"] & \otimes^L A \arrow[r, "\rho^L"'] & A
\end{tikzcd}$$

The quadrilateral commutes by the naturality of ρ^L, the triangle commutes by Lemma 2.33 and Remark 2.34. □

2.37 Example. Let us look at the particular case of a symmetric Monoidal category $\mathcal{V} = (\mathcal{C}at, \times, \mathbb{1})$, the category of small categories. In this book, we encounter only examples of symmetric Monoidal $\mathcal{C}at$-categories in which the isomorphisms ρ^L can be chosen to be identity morphisms. With this additional assumption, Definition 2.10 turns into the following.

A symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ consists of

1. A strict 2-category $\mathfrak{C}$.

2. A strict 2-functor $\boxtimes^I : \mathfrak{C}^I \to \mathfrak{C}$, for every set $I \in \operatorname{Ob} \mathcal{S}$, such that $\boxtimes^I = \mathrm{Id}_{\mathfrak{C}}$ for each 1-element set I. In particular, a functor $\boxtimes^I : \prod_{i\in I} \mathfrak{C}(X_i, Y_i) \to \mathfrak{C}(\boxtimes^I(X_i), \boxtimes^I(Y_i))$ is given.

   For a map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$ consider the 2-functor $\boxtimes^f = \boxtimes^f_{\mathfrak{C}} : \mathfrak{C}^I \to \mathfrak{C}^J$ which to a function $X : I \to \operatorname{Ob} \mathfrak{C}$, $i \mapsto X_i$ assigns the function $J \to \operatorname{Ob} \mathfrak{C}$, $j \mapsto \boxtimes^{i\in f^{-1}(j)}(X_i)_i$.

It acts on categories of morphisms via the functor

$$\prod_{i\in I}\mathfrak{C}(X_i,Y_i)\xrightarrow{\sim}\prod_{j\in J}\prod_{i\in f^{-1}j}\mathcal{C}(X_i,Y_i)\xrightarrow{\prod_{j\in J}\boxtimes^{f^{-1}j}}\prod_{j\in J}\mathfrak{C}(\boxtimes^{i\in f^{-1}j}(X_i),\boxtimes^{i\in f^{-1}j}(Y_i)).$$

3. An invertible strict 2-transformation Λ^f for every map $f:I\to J$ in $\operatorname{Mor}\mathcal{S}$:

$$\begin{tikzcd}
\mathfrak{C}^I \arrow[rr, "\boxtimes^f_{\mathfrak{C}}"] \arrow[rrd, "\boxtimes^I"'] & \arrow[d, phantom, "\Lambda^f"] & \mathfrak{C}^J \arrow[d, "\boxtimes^J"] \\
 & & \mathfrak{C}
\end{tikzcd} \tag{2.37.1}$$

(a family of invertible 1-morphisms $\Lambda^f:\boxtimes^{i\in I}X_i\to\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}X_i$, $X_i\in\operatorname{Ob}\mathfrak{C}$) such that

(i) for all sets $I\in\operatorname{Ob}\mathcal{O}$, for all 1-element sets J

$$\Lambda^{\mathrm{id}_I}=\mathrm{id},\qquad \Lambda^{I\to J}=\mathrm{id};$$

(ii) for any pair of composable maps $I\xrightarrow{f}J\xrightarrow{g}K$ from $\mathcal{S}$ this equation holds:

$$\begin{tikzcd}
\mathfrak{C}^J \arrow[r, "\boxtimes^g"] \arrow[rd, "\boxtimes^J"] & \mathfrak{C}^K \arrow[d, "\boxtimes^K"] \arrow[ld, phantom, "\Lambda^g"] \\
\mathfrak{C}^I \arrow[u, "\boxtimes^f"] \arrow[r, "\boxtimes^I"'] \arrow[u, phantom, shift left=4, "\Lambda^f"] & \mathfrak{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathfrak{C}^J \arrow[r, "\boxtimes^g"] \arrow[r, phantom, shift right=4, "\prod_{k\in K}\Lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] & \mathfrak{C}^K \arrow[d, "\boxtimes^K"] \\
\mathfrak{C}^I \arrow[u, "\boxtimes^f"] \arrow[ru, "\boxtimes^{g\circ f}"] \arrow[r, "\boxtimes^I"'] \arrow[r, phantom, shift left=4, "\Lambda^{g\circ f}"] & \mathfrak{C}
\end{tikzcd} \tag{2.37.2}$$

A symmetric Monoidal $\mathcal{C}at$-category is a particular case of a symmetric monoidal 2-category, as defined e.g. in [Lyu99, Definition A.6.1], which in turn is a very particular case of a weak 6-category.

Let $\mathcal{V}=(\mathcal{V},\otimes^I_{\mathcal{V}},\lambda^f_{\mathcal{V}})$ be a symmetric Monoidal category. Then the 2-category $\mathfrak{C}$ of $\mathcal{V}$-categories, $\mathcal{V}$-functors and their natural transformations is a symmetric Monoidal $\mathcal{C}at$-category. Indeed, a strict 2-functor $\boxtimes^I:\mathfrak{C}^I\to\mathfrak{C}$ is defined as follows. To a family of $\mathcal{V}$-categories $(\mathcal{C}_i)_{i\in I}$ it assigns their product $\mathcal{C}=\boxtimes^{i\in I}\mathcal{C}_i$ with $\operatorname{Ob}\mathcal{C}=\prod_{i\in I}\operatorname{Ob}\mathcal{C}_i$ and $\mathcal{C}((X_i)_{i\in I},(Y_i)_{i\in I})=\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i,Y_i)$. To a family of $\mathcal{V}$-functors $(F_i:\mathcal{C}_i\to\mathcal{D}_i)_{i\in I}$ it assigns their product $F=\boxtimes^{i\in I}F_i$ with $\operatorname{Ob}F=\prod_{i\in I}\operatorname{Ob}F_i$ and

$$F=\otimes^{i\in I}_{\mathcal{V}}F_i:\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}_i(X_i,Y_i)\to\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}_i(F_iX_i,F_iY_i).$$

To a family of natural $\mathcal{V}$-transformations $r_i : F_i \to G_i : \mathcal{C}_i \to \mathcal{D}_i$, which are morphisms $r_i : \mathbb{1} \to \mathcal{D}_i(F_iX_i, G_iX_i)$, the alleged 2-functor assigns the natural $\mathcal{V}$-transformation $r : F \to G : \mathcal{C} \to \mathcal{D}$ which equals $r = \bigl(\mathbb{1} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing\to I}} \otimes_{\mathcal{V}}^{I} \mathbb{1} \xrightarrow{\otimes_{\mathcal{V}}^{i\in I} r_i} \otimes_{\mathcal{V}}^{i\in I} \mathcal{D}_i(F_iX_i, G_iX_i)\bigr)$. Coherence with compositions is rather clear, so $\boxtimes^I$ is, indeed, a strict 2-functor.

2-transformation (2.37.1) is given by the family of $\mathcal{V}$-functors $\Lambda_{\mathfrak{C}}^f : \boxtimes^{i\in I}\mathcal{C}_i \to \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{C}_i$, defined as $\Lambda_{\mathfrak{C}}^f : (X_i)_{i\in I} \mapsto ((X_i)_{i\in f^{-1}j})_{j\in J}$ on objects, and by

$$\Lambda_{\mathfrak{C}}^f = \lambda_{\mathcal{V}}^f : \otimes_{\mathcal{V}}^{i\in I}\mathcal{C}_i(X_i, Y_i) \xrightarrow{\sim} \otimes_{\mathcal{V}}^{j\in J} \otimes_{\mathcal{V}}^{i\in f^{-1}j}\mathcal{C}_i(X_i, Y_i) \tag{2.37.3}$$

on morphisms. One can check that this 2-transformation is strict. Equation (2.37.2) for Λ^f follows from similar equation (2.10.3) for λ^f. Therefore, $\mathfrak{C} = \mathcal{V}\text{-}\mathcal{C}at$ is a symmetric Monoidal $\mathcal{C}at$-category.

Chapter 3

Multicategories

Multicategories came into usage with the work of Lambek [Lam69]. Various versions of them were used since then under various names [Bor98, Soi99b, DS03, BD04]. The idea of using morphisms with several sources and one target instead of tensor products seems to penetrate into mathematics deeper and deeper. One may view multicategories as generalizations of lax Monoidal categories. The opposite point of view is also fruitful: a lax Monoidal category is a particular case of a multicategory. The relationship between $\Bbbk$-polylinear maps and tensor products over $\Bbbk$ convinces in the latter approach. We prove in this chapter that the 2-category of lax (symmetric) Monoidal categories is equivalent to the 2-category of lax representable (symmetric) multicategories.

Actually, we define and work with multicategories enriched in a symmetric Monoidal category $\mathcal{V}$. This is a warm-up before defining multicategories enriched in symmetric multicategories in the next chapter. $\mathcal{V}$-multicategories are defined as algebras in a certain Monoidal category.

3.1 Multiquivers. Let $\mathcal{V}$ be a symmetric Monoidal $\mathscr{U}$-category. In the simplest case $\mathcal{V} = \mathcal{S}\mathrm{et}$ is the category of $\mathscr{U}$-small sets. It can be also the category $\mathcal{V} = \mathbf{gr} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$ (resp. $\mathcal{V} = \mathbf{dg} = \mathbf{dg}(\Bbbk\text{-}\mathbf{Mod})$) of (resp. differential) graded $\mathscr{U}$-small $\Bbbk$-modules.

3.2 Definition. A $\mathcal{V}$-quiver C consists of a $\mathscr{U}$-small set $\operatorname{Ob}\mathsf{C}$ of *objects*, and objects of *morphisms*

$$\mathsf{C}(X,Y) = \operatorname{Hom}_{\mathsf{C}}(X,Y) \in \operatorname{Ob}\mathcal{V},$$

given for each pair of objects $X, Y \in \operatorname{Ob}\mathsf{C}$. A *symmetric* (resp. *plain*) $\mathcal{V}$*-multiquiver* C consists of a $\mathscr{U}$-small set $\operatorname{Ob}\mathsf{C}$ of *objects*, objects of *multimorphisms*

$$\mathsf{C}((X_i)_{i\in I};Y) = \operatorname{Hom}_{\mathsf{C}}((X_i)_{i\in I};Y) \in \operatorname{Ob}\mathcal{V},$$

assigned to each map $I \sqcup \{*\} \to \operatorname{Ob}\mathsf{C}$, $i \mapsto X_i$, $* \mapsto Y$, where I is a finite totally ordered set.

In particular, for $I = \mathbf{n} = \{1, 2, \dots, n\}$, $n \geqslant 0$, the objects of $\mathcal{V}$

$$\mathsf{C}(X_1,\dots,X_n;Y) = \operatorname{Hom}_{\mathsf{C}}(X_1,\dots,X_n;Y) \in \operatorname{Ob}\mathcal{V}$$

are given for all sequences of objects $X_1, \dots, X_n, Y \in \operatorname{Ob}\mathsf{C}$.

Let ${}^{\mathcal{V}}\mathcal{Q}$, $\mathcal{MQ}_{\mathcal{V}}$ be categories, whose objects are $\mathcal{V}$-(multi)quivers. A morphism $f : \mathsf{C} \to \mathsf{C}'$ is a map between the sets of objects $\operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{C}'$, $X \mapsto Xf$, and a family of morphisms in $\mathcal{V}$ between objects of multimorphisms:

$$_{(X_i)_i;Y}f : \mathsf{C}((X_i)_{i\in I}; Y) \to \mathsf{C}'((X_i f)_{i\in I}; Yf),$$

($I = \{*\}$ for ${}^{\mathcal{V}}\mathcal{Q}$).

The categories ${}^{\mathcal{V}}\mathcal{Q}$, $\mathcal{MQ}_{\mathcal{V}}$ are symmetric Monoidal. To a family of $\mathcal{V}$-(multi)quivers $(\mathcal{Q}_i)_{i\in I}$ their product $\mathcal{Q} = \boxtimes^{i\in I}\mathcal{Q}_i$ is assigned with $\operatorname{Ob}\mathcal{Q} = \prod_{i\in I}\operatorname{Ob}\mathcal{Q}_i$ and

$$\mathcal{Q}(((X_i^k)_{k\in K})_{i\in I}; (Y_i)_{i\in I}) = \otimes_{\mathcal{V}}^{i\in I}\mathcal{Q}_i((X_i^k)_{k\in K}; Y_i)$$

($K = \{*\}$ for ${}^{\mathcal{V}}\mathcal{Q}$). To a family of morphisms of (multi)quivers $(F_i : \mathcal{P}_i \to \mathcal{Q}_i)_{i\in I}$ their product $F = \boxtimes^{i\in I}F_i$ is assigned with $\operatorname{Ob}F = \prod_{i\in I}\operatorname{Ob}F_i$ and

$$F = \otimes_{\mathcal{V}}^{i\in I}F_i : \otimes_{\mathcal{V}}^{i\in I}\mathcal{P}_i((X_i^k)_{k\in K}; Y_i) \to \otimes_{\mathcal{V}}^{i\in I}\mathcal{Q}_i((F_iX_i^k)_{k\in K}; F_iY_i).$$

This defines a $\mathcal{V}$-functor $\boxtimes^I : \mathcal{MQ}^I \to \mathcal{MQ}$. Corresponding isomorphisms $\lambda^f_{\mathcal{MQ}}$ are determined by $\lambda^f_{\mathcal{V}}$ of $\mathcal{V}$.

3.3 Notation for sequential trees. We just summarize below notation that we use when dealing with trees and forests. The reader may get familiar with the beginning of the section and skip the rest until necessity arises.

Let S be a set (of labels).

3.4 Definition. A *sequential* (resp. *symmetric sequential*, resp. *braided sequential*) *S-labeled forest* t of height $n \geqslant 0$ is an element of the nerve $\mathsf{N}_n(\mathcal{O}(S))$, (resp. $\mathsf{N}_n(\mathcal{S}(S))$, resp. $\mathsf{B}_n(\mathcal{S}(S))$), that is, a functor $t : [n] \to \mathcal{O}$ (resp. $t : [n] \to \mathcal{S}$, resp. $t : [n] \to \mathcal{S}$ which satisfies (2.3.1)), $(i \to j) \mapsto (t_{i\to j} : t(i) \to t(j))$, together with label maps $\ell^t_m : t(m) \to S$ for $0 \leqslant m \leqslant n$. A sequential forest t of height $n \geqslant 0$ is a *sequential tree*, if $|t(n)| = 1$.

Clearly, a (symmetric, braided) sequential forest t is unambiguously specified by a composable sequence

$$t = \bigl(t(0) \xrightarrow{t_1} t(1) \xrightarrow{t_2} \dots\ t(n-1) \xrightarrow{t_n} t(n) \mid (t(m) \ni i \xmapsto{\ell^t_m} X^i_m \in S)_{m=0}^n\bigr) \qquad (3.4.1)$$

of morphisms of $\mathcal{O}$ (resp. $\mathcal{S}$) and by elements $X^i_m \in S$. Here $t_m = t_{m-1\to m}$. Denote also $I_m = t(m)$. The number n of morphisms in the sequence is called the *height* of t. The totally ordered set

$$\mathrm{v}(t) = \sqcup_{m\in\mathbf{n}} t(m) = t(1) \sqcup t(2) \sqcup \dots \sqcup t(n)$$

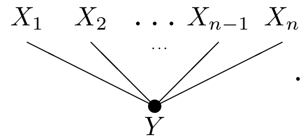
is called the *set of internal vertices* of t. A (symmetric, braided) sequential tree is often written as follows

$$t = \bigl(I_0 \xrightarrow{t_1} I_1 \xrightarrow{t_2} I_2 \dots \xrightarrow{t_{n-1}} I_{n-1} \xrightarrow{t_n} I_n \mid (\ell^t_m : I_m \to S)_{m=0}^n\bigr), \qquad (3.4.2)$$

where $|I_n| = 1$.

We draw (symmetric, braided) sequential trees in the plane placing points of $t(m)$ in the given order at the line $y = \text{const} - m$, $0 \leqslant m \leqslant n$, connected by edges (segments of straight lines) accordingly to maps t_m, $0 < m \leqslant n$. Edges are labeled by elements $X^i_{m-1} \in \ell^t_{m-1}(t(m-1)) \subset S$. One more label $X_n = \ell^t_n(*) \subset S$ is placed below the root vertex, the image of $* \in t(n)$, which is located on the line $y = \text{const} - n$.

For instance, the tree $t = (\mathbf{1} \mid X)$ of height 0 is specified by an element $X \in S$ and is drawn as $\underset{X}{\bullet}$. This tree has no internal vertices. The tree $t = (I \to \mathbf{1} \mid (X_i)_{i \in I}; Y)$ of height 1 is specified by a finite totally ordered set I and by elements X_i, $i \in I$, and Y of S. When $I \simeq \mathbf{n}$, this *elementary tree* is drawn as



. (3.4.3)

This tree has one internal vertex, $\mathrm{v}(t) = \mathbf{1}$. In the case $I = \varnothing$, the tree of height 1

$$t_1 = (\varnothing \to \mathbf{1} \mid (); Y) = \underset{Y}{\overset{\varnothing}{\bullet}}$$

differs from the tree $t_0 = (\mathbf{1} \mid Y) = \underset{Y}{\bullet}$ of height 0.

For any finite totally ordered set I we define $\mathsf{N}_I \mathcal{I}$ as $\mathsf{N}_n \mathcal{I}$, where $\mathbf{n}$ is isomorphic to I in $\mathcal{O}$. A map $f : I \to J \in \operatorname{Mor} \mathcal{O}$ induces the functor $[f] : [J] \to [I]$. For any sequential forest $q : [I] \to \mathcal{I}$ there is another sequential forest $q = [f] \cdot q : [J] \to \mathcal{I}$, and the map f determines a 2-morphism $f : q \to q^f$.

Let $\psi : [r] \to [n]$ be a functor, that is, an isotonic map of totally ordered sets, which we write down also as a non-decreasing sequence of integers $\psi = (\psi(0), \psi(1), \dots, \psi(r))$. Let t be a (symmetric, braided) sequential tree described via data (3.4.1). Denote $p = \psi(r)$, and assume that $j \in I_p = t(p)$. Define a new sequential tree $t^{|j}_\psi$ of height r restricting the composite functor $[r] \xrightarrow{\psi} [n] \xrightarrow{t} \mathcal{I}$, $\mathcal{I} = \mathcal{O}, \mathcal{S}$ and functions $\ell^t_{\psi(k)} : t(\psi(k)) \to S$, $0 \leqslant k \leqslant r$ as follows. Let $t^{|j}_\psi(k) = t^{-1}_{\psi(k) \to p}(j)$. Here the map $t_{q \to p} : t(q) \to t(p)$ is the value of the functor t on the arrow $q \to p$. The mapping $t^{|j}_\psi(k \to l)$ is the restriction of the mapping $t_{\psi k \to \psi l}$. The function $\ell^{t_\psi}_k$ is the restriction of the function $\ell^t_{\psi(k)} : t(\psi(k)) \to S$. If t is a sequential forest and J is a subset of $t(\psi(r))$, we define a new sequential forest t^J_ψ of height r by restricting $\psi \cdot t$ and ℓ^t to $t^{|j}_\psi(k) = t^{-1}_{\psi(k) \to p}(j)$.

For example, integers $0 < p \leqslant n$ define a functor $\psi : [1] \to [n]$, $0 \mapsto p - 1$, $1 \mapsto p$. A choice of $j \in I_p = t(p)$ determines a sequential tree

$$t^{|j}_{\rangle p} \overset{\text{def}}{=} t^{|j}_\psi = t^{|j}_{p-1,p} = \left(t^{-1}_p j \to \{j\} \mid (X^i_{p-1})_{i \in t^{-1}_p j}, X^j_p\right)$$

of height 1. The data it contains amount precisely to the list of labels of incoming edges for the vertex j and the label of the outgoing edge.

If $\psi(r) = n$, then j is the only element of $t(n)$, and we do not have to restrict the mappings. In this case we may shorten $t^{|1}_\psi = \psi \cdot t$ to t_ψ. If the isotonic map $\psi : [r] \to [n]$ is an injection, we shall denote $t^{|j}_\psi$ also as $t^{|j}_Q$, where the subset $Q = \psi[r] \subset [n]$ is the image of ψ. For instance, Q might be an interval $[k, p] = \{m \in \mathbb{Z} \mid k \leqslant m \leqslant p\}$. When both above conditions are fulfilled, we may write t_Q in place of $t^{|j}_\psi$. For example, $Q = \{0, n\}$ determines the tree of height 1

$$^{<}t \overset{\text{def}}{=} t_{0n} = \bigl(t(0) \to t(n) \mid (X^i_0)_{i\in t(0)}, X_n\bigr) \tag{3.4.4}$$

which keeps information about labels on leaves and the root label of the tree t. The same definition applies to the case of tree $t = (J \mid X)$ of height $n = 0$, where $|J| = 1$. It produces the tree of height 1

$$^{<}t \overset{\text{def}}{=} t_{00} = \bigl(J \to J \mid X, X\bigr).$$

With each sequential forest $(t : [n] \to \mathcal{I} \mid (\ell^t_m : t(m) \to S)^n_{m=0})$ is associated a staged forest $t : [n] \to \mathcal{S}\mathrm{et}$, $m \mapsto t(m) = I_m$, which we view as a partially ordered set, or as the associated category $\mathrm{Tree}(t)$. Its object set $\overline{\mathrm{v}}(t) \overset{\text{def}}{=} \mathrm{Ob}\,\mathrm{Tree}(t) = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_n$ is totally ordered. There is a unique morphism between objects (k, j) and (m, l), where $k, m \in [n]$, $j \in I_k$, $l \in I_m$, if $k \leqslant m$ and $t_{k\to m} : j \mapsto l$. Otherwise, $\mathrm{Tree}(t)((k, j), (m, l))$ is empty.

Any functor $\psi : [r] \to [n]$ induces the sequential forest $t_\psi = t^{I_{\psi(r)}}_\psi$ via composition $[r] \xrightarrow{\psi} [n] \xrightarrow{t} \mathcal{I}(S)$. Composition gives also a functor $\mathrm{Tree}(\psi) : \mathrm{Tree}(t_\psi) \to \mathrm{Tree}(t)$. On objects (p, j), $p \in [r]$ it acts as "identity" map: $\mathrm{Tree}(\psi) = \mathrm{id} : t_\psi(p) \to t(\psi p)$. The functor $\mathrm{Tree}(\psi)$ takes a morphism $(k, j) \to (m, (t_\psi)_{k\to m}(j))$ to the morphism $(\psi k, j) \to (\psi m, t_{\psi k\to\psi m}(j))$.

If $\psi(r) = n$, there is a left adjoint functor $\mathrm{Tree}(\psi)^* : \mathrm{Tree}(t) \to \mathrm{Tree}(t_\psi)$. Thus,

$$\mathrm{Tree}(t_\psi)\bigl(\mathrm{Tree}(\psi)^*(k, j), (m, l)\bigr) = \mathrm{Tree}(t)\bigl((k, j), \mathrm{Tree}(\psi)(m, l)\bigr).$$

It is given on the object (k, j), $k \in \mathbf{n}$, $j \in I_k = t(k)$, by the following formula:

$$\mathrm{Tree}(\psi)^*(k, j) = (p, t_{k\to\psi(p)}(j)), \tag{3.4.5}$$

where $p \in [r]$ satisfies $\psi(p - 1) < k \leqslant \psi(p)$ (with convention $\psi(-1) = -1$). The meaning of this formula is that a vertex of t slides to the right along the edges of t until it encounters a level which belongs to the image of the map $\psi : [r] \to [n]$.

We view $\overline{\mathrm{v}}_\psi \overset{\text{def}}{=} \mathrm{Ob}(\mathrm{Tree}(\psi)^*)$ as the mapping of sets of objects $\overline{\mathrm{v}}(t) = \mathrm{Ob}\,\mathrm{Tree}(t) = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_n \to I_{\psi 0} \sqcup I_{\psi 1} \sqcup \cdots \sqcup I_{\psi r} = \mathrm{Ob}\,\mathrm{Tree}(t_\psi) = \overline{\mathrm{v}}(t_\psi)$. If moreover, $\psi(0) = 0$, then inequalities $0 < k \leqslant \psi(p)$ imply that $p > 0$. In this case $\overline{\mathrm{v}}_\psi = \mathrm{Ob}(\mathrm{Tree}(\psi)^*)$ restricts to a map $\mathrm{v}_\psi \overset{\text{def}}{=} \overline{\mathrm{v}}_\psi| : \mathrm{v}(t) \to \mathrm{v}(t_\psi)$, which is also given by (3.4.5). It is defined only if $\psi(0) = 0$ and $\psi(r) = n$.

If $f : \mathbf{n} \to \mathbf{r}$ is a non-decreasing map, there is the associated functor $\psi = [f] : [r] \to [n]$ such that $\psi(0) = 0$ and $\psi(r) = n$. The sequential forest t_ψ is denoted also t^f and the map $\mathrm{v}_\psi : \mathrm{v}(t) \to \mathrm{v}(t^f)$ is denoted also v^f.

If S is a set, S^* denotes the set of finite totally ordered families $(X_i)_{i\in I}$ of elements X_i of S.

3.5 Spans. The purpose of this section and Propositions 3.6, 3.8 is to present $\mathcal{V}$-multicategories as algebras in a certain lax Monoidal category. With this goal in mind we introduce $\mathcal{V}$-spans, parameterized $\mathcal{V}$-multiquivers, etc. Direct Definition 3.7 of $\mathcal{V}$-multicategories is more important for practice, and the reader is advised to concentrate on it at the first reading. The subject of $\mathcal{V}$-spans and $\mathcal{V}$-multiquivers can be covered after the initial acquaintance with the book.

Recall that a *span* C is a pair of maps with a common source, which we denote $\mathrm{Ob}_s \mathsf{C} \xleftarrow{\mathrm{src}} \mathrm{Par}\,\mathsf{C} \xrightarrow{\mathrm{tgt}} \mathrm{Ob}_t \mathsf{C}$. We say that $\mathrm{Ob}_s \mathsf{C}$ is the set of source objects, $\mathrm{Par}\,\mathsf{C}$ is the set of parameters, and $\mathrm{Ob}_t \mathsf{C}$ is the set of target objects. A *morphism of spans* $F : \mathsf{C} \to \mathsf{D}$ is a triple of maps

$$\begin{aligned} \mathrm{Ob}_s F : \mathrm{Ob}_s \mathsf{C} \to \mathrm{Ob}_s \mathsf{D}, &\qquad X \mapsto FX, \\ \mathrm{Par}\, F : \mathrm{Par}\,\mathsf{C} \to \mathrm{Par}\,\mathsf{D}, &\qquad p \mapsto Fp, \\ \mathrm{Ob}_t F : \mathrm{Ob}_t \mathsf{C} \to \mathrm{Ob}_t \mathsf{D}, &\qquad Y \mapsto FY, \end{aligned}$$

compatible with the source mapping src and the target mapping tgt.

Let $\mathcal{V}$ be a symmetric Monoidal category. A *$\mathcal{V}$-span* C is a pair consisting of a span $\mathrm{Ob}_s \mathsf{C} \xleftarrow{\mathrm{src}} \mathrm{Par}\,\mathsf{C} \xrightarrow{\mathrm{tgt}} \mathrm{Ob}_t \mathsf{C}$ and a function

$$\mathrm{Par}\,\mathsf{C} \ni p \mapsto \mathsf{C}^p = \mathsf{C}^p(\mathrm{src}\, p, \mathrm{tgt}\, p) \in \mathrm{Ob}\,\mathcal{V}.$$

A *morphism of $\mathcal{V}$-spans* $F : \mathsf{C} \to \mathsf{D}$ is a morphism of spans $(\mathrm{Ob}_s F, \mathrm{Par}\, F, \mathrm{Ob}_t F)$ and for each $p \in \mathrm{Par}\,\mathsf{C}$ a morphism in $\mathcal{V}$

$$F^p : \mathsf{C}^p = \mathsf{C}^p(\mathrm{src}\, p, \mathrm{tgt}\, p) \to \mathsf{D}^{Fp}(F\,\mathrm{src}\, p, F\,\mathrm{tgt}\, p) = \mathsf{D}^{Fp}.$$

For example, a $\mathcal{V}$-quiver C is a $\mathcal{V}$-span with $\mathrm{Ob}\,\mathsf{C} = \mathrm{Ob}_s \mathsf{C} = \mathrm{Ob}_t \mathsf{C}$, $\mathrm{Par}\,\mathsf{C} = \mathrm{Ob}_s \mathsf{C} \times \mathrm{Ob}_t \mathsf{C}$, $\mathrm{src} = \mathrm{pr}_1$, $\mathrm{tgt} = \mathrm{pr}_2$. A morphism of $\mathcal{V}$-quivers $F : \mathsf{C} \to \mathsf{D}$ is a morphism of $\mathcal{V}$-spans with $\mathrm{Ob}\, F = \mathrm{Ob}_s F = \mathrm{Ob}_t F$. A $\mathcal{V}$-multiquiver C is a $\mathcal{V}$-span with $\mathrm{Ob}_s \mathsf{C} = (\mathrm{Ob}\,\mathsf{C})^*$ – the set of finite totally ordered families of elements of $\mathrm{Ob}\,\mathsf{C} = \mathrm{Ob}_t \mathsf{C}$, $\mathrm{Par}\,\mathsf{C} = \mathrm{Ob}_s \mathsf{C} \times \mathrm{Ob}_t \mathsf{C}$, $\mathrm{src} = \mathrm{pr}_1$, $\mathrm{tgt} = \mathrm{pr}_2$. A morphism of $\mathcal{V}$-multiquivers $F : \mathsf{C} \to \mathsf{D}$ is a morphism of $\mathcal{V}$-spans with $\mathrm{Ob}_s F = (\mathrm{Ob}\, F)^* : (X_i)_{i\in I} \mapsto (FX_i)_{i\in I}$, determined by $\mathrm{Ob}\, F = \mathrm{Ob}_t F : Y \mapsto FY$.

Consider *parameterized $\mathcal{V}$-multiquivers*, which are $\mathcal{V}$-spans C with $\mathrm{Ob}_s \mathsf{C} = (\mathrm{Ob}_t \mathsf{C})^*$. An element $X \in \mathrm{Ob}_s \mathsf{C}$ is a function $I \to \mathrm{Ob}_t \mathsf{C}$, $i \mapsto X_i$. Denote by $\mathrm{dom}\, X$ its domain I. For each $i \in \mathrm{dom}\, X$ there is an element $\mathrm{src}_i\, X = X_i \in \mathrm{Ob}_t \mathsf{C}$. *Morphisms of parameterized $\mathcal{V}$-multiquivers* $F : \mathsf{C} \to \mathsf{D}$ are morphisms of $\mathcal{V}$-spans such that $\mathrm{Ob}_s F = (\mathrm{Ob}_t F)^*$. The

set $\mathrm{Ob}_t\,\mathsf{C}$ is often abbreviated to $\mathrm{Ob}\,\mathsf{C}$; the function $\mathrm{Ob}_t\,F$ is abbreviated to $\mathrm{Ob}\,F$. Denote this category $\mathcal{PMQ}_{\mathcal{V}}$.

Fix a set S. Consider the subcategory ${}^S\mathcal{PMQ}$ of the category $\mathcal{PMQ}$ whose objects are parameterized $\mathcal{V}$-multiquivers C with $\mathrm{Ob}_t\,\mathsf{C} = S$ and whose morphisms are morphisms of parametrized $\mathcal{V}$-multiquivers $F : \mathsf{C} \to \mathsf{D}$ with $\mathrm{Ob}_t\,F = \mathrm{id}_S$. It follows that $\mathrm{Ob}_s\,\mathsf{C} = S^*$, for each object C of ${}^S\mathcal{PMQ}$, and $\mathrm{Ob}_s\,F = \mathrm{id}_{S^*}$, for each morphism F of ${}^S\mathcal{PMQ}$. We are going to turn ${}^S\mathcal{PMQ}$ into a lax Monoidal category with the tensor product $\odot$, defined for each finite totally ordered set I by the formula $\odot^{i\in I}\mathsf{C}_i = \odot^{k\in\mathbf{n}}\mathsf{C}_{i_k}$, where $\mathbf{n} \ni k \mapsto i_k \in I$ is the unique isotonic isomorphism for $n = |I|$. The functor $\odot^{\mathbf{n}}$ will be defined now. (One expects that $\mathcal{PMQ}$ is a lax bicategory as defined by [Lei03, Definition 3.4.1]. However, we do not pursue this subject here.)

Denote by $\mathsf{N}'_n\mathcal{J}(S) \subset \mathsf{N}_n\mathcal{J}(S)$ one of the sets of sequential trees $\mathsf{N}'_n\mathcal{O}(S) \subset \mathsf{N}_n\mathcal{O}(S)$, $\mathsf{N}'_n\mathcal{S}(S) \subset \mathsf{N}_n\mathcal{S}(S)$, $\mathsf{B}'_n\mathcal{S}(S) \subset \mathsf{B}_n\mathcal{S}(S)$, $\mathsf{N}'_n\mathcal{O}_s(S) \subset \mathsf{N}_n\mathcal{O}_s(S)$, $\mathsf{N}'_n\mathcal{S}_s(S) \subset \mathsf{N}_n\mathcal{S}_s(S)$, $\mathsf{B}'_n\mathcal{S}_s(S) \subset \mathsf{B}_n\mathcal{S}_s(S)$, relatively to the plain, symmetric or braided case. Recall that objects of $\mathcal{O}_s$ and $\mathcal{S}_s$ are sets $\mathbf{n}$, $n \in \mathbb{Z}_{\geqslant 0}$. Thus, elements $t \in \mathsf{N}'_n\mathcal{J}(S)$ are distinguished between elements of $\mathsf{N}_n\mathcal{J}(S)$ by condition $|t(n)| = 1$, as in (3.4.2). Let C_i, $i \in \mathbf{n}$, be parameterized $\mathcal{V}$-multiquivers with $\mathrm{Ob}_t\,\mathsf{C}_i = S$. A parameterized $\mathcal{V}$-multiquiver $\mathsf{C} = \odot^{i\in\mathbf{n}}\mathsf{C}_i$ with $\mathrm{Ob}_t\,\mathsf{C} = S$ is defined by the following data. The set of parameters of C is defined to be

$$\mathrm{Par}\,\odot^{i\in\mathbf{n}}\mathsf{C}_i = \Big\{(t,p) \mid t \in \mathsf{N}'_n\mathcal{J}(S),\ p = (p^j_m)^{j\in I_m}_{m\in\mathbf{n}} \in \prod_{m\in\mathbf{n}} (\mathrm{Par}\,\mathsf{C}_m)^{I_m},$$
$$\forall m \in \mathbf{n}\ \forall j \in I_m\ \ \mathrm{tgt}\,p^j_m = X^j_m,\ \mathrm{dom}\,\mathrm{src}\,p^j_m = t^{-1}_m j,\ \forall i \in t^{-1}_m j\ \ \mathrm{src}_i\,p^j_m = X^i_{m-1}\Big\},$$

where, as usually, $I_m = t(m)$, $t_m = t_{m-1\to m}$, $X^j_m = \ell^t_m(j) \in S$. The target mapping for C is given by $\mathrm{tgt}(t,p) = X^1_n \in S$. Depending on the plain, symmetric or braided case, we denote $\odot$ as $\odot_{\mathcal{PMQ}^{\mathcal{O}}}$, $\odot_{\mathcal{PMQ}^{\mathcal{S}}}$ or $\odot_{\mathcal{PMQ}^{\mathcal{B}}}$. The source mapping is $\mathrm{src} : \mathrm{Par}\,\mathsf{C} \to S^*$, $(t,p) \mapsto (\ell^t_0 : I_0 \to S,\ i \mapsto X^i_0)$. Thus, $\mathrm{dom}\,\mathrm{src}(t,p) = I_0 = t(0)$, $\mathrm{src}_i(t,p) = X^i_0$. The object of $\mathcal{V}$ associated to (t,p) is

$$\mathsf{C}^{(t,p)} = \otimes^{(m,j)\in\mathrm{v}(t)}_{\mathcal{V}} \mathsf{C}^{p^j_m}_m.$$

For example, if $n = 0$ we get the unit parameterized $\mathcal{V}$-multiquiver $\odot^0$ with

$$\mathrm{Par}\,\odot^0 = \{(X,L) \in S \times \mathrm{Ob}\,\mathcal{O} \mid |L| = 1\},$$

$\mathrm{tgt} = \mathrm{pr}_1 : \mathrm{Par}\,\odot^0 \to S = \mathrm{Ob}_t\,\odot^0$, $(X,L) \mapsto X$, $\mathrm{src} : \mathrm{Par}\,\odot^0 \hookrightarrow S^* = \mathrm{Ob}_s\,\odot^0$, $(X,L) \mapsto (L \ni l \mapsto X)$ is the natural embedding, and $(\odot^0)^{(X,L)} = \mathbb{1}_{\mathcal{V}}$, $(X,L) \in \mathrm{Par}\,\odot^0$.

Let us describe associativity morphism $\lambda^\phi_{\mathcal{PMQ}} : \odot^{l\in\mathbf{n}}\mathsf{C}_l \to \odot^{m\in\mathbf{k}}\odot^{l\in\phi^{-1}m}\mathsf{C}_l$, corresponding to an isotonic map $\phi : \mathbf{n} \to \mathbf{k}$. First of all, the parameter set of the source can be embedded into the parameter set of the target. Indeed, the 2-morphism $(\phi : \mathbf{n} \to \mathbf{k}, \psi = [\phi] : [k] \to [n])$ and a sequential tree $t \in \mathsf{N}'_n\mathcal{J}(S)$ induce trees

$$t_\psi = \big([k] \xrightarrow{\psi} [n] \xrightarrow{t} \mathcal{J}(S)\big) \in \mathsf{N}'_k\mathcal{J}(S)$$

and $t^{|j}_{[\psi(m-1),\psi(m)]} \in \mathsf{N}'_{[\psi(m-1),\psi(m)]}\mathcal{I}(S)$ for all $m \in \mathbf{k}$, $j \in t(\psi(m)) = I_{\psi(m)}$. This gives the mapping

$$\operatorname{Par} \lambda^{\phi}_{\mathcal{PMQ}} : \operatorname{Par} \odot^{l\in\mathbf{n}} \mathsf{C}_l \longrightarrow \operatorname{Par} \odot^{m\in\mathbf{k}} \odot^{l\in\phi^{-1}m} \mathsf{C}_l,$$
$$(t, (p^j_l)^{j\in I_l}_{l\in\mathbf{n}}) \longmapsto (t_\psi, (t^{|j}_{[\psi(m-1),\psi(m)]}, (p^i_l)^{i\in t^{|j}_{[\psi(m-1),\psi(m)]}(l)}_{l\in\phi^{-1}m})^{j\in I_{\psi(m)}}_{m\in\mathbf{k}}),$$

which is clearly an injection. There is also a mapping in the inverse direction, constructed via the disjoint union of a finite totally ordered family of finite totally ordered sets. The union is again a finite totally ordered set. In the skeletal case $\mathcal{I} = \mathcal{O}_s$ or $\mathcal{S}_s$, the said mapping is inverse to $\operatorname{Par} \lambda^{\phi}_{\mathcal{PMQ}}$. For parameter (t, p) we define the morphism in $\mathcal{V}$

$$\lambda^{\phi}_{\mathcal{PMQ}} = \lambda^{\mathrm{v}_\psi:\mathrm{v}(t)\to\mathrm{v}(t_\psi)}_{\mathcal{V}} : \otimes^{(l,i)\in\mathrm{v}(t)}_{\mathcal{V}} \mathsf{C}^{p^i_l}_l \longrightarrow \otimes^{(m,j)\in\mathrm{v}(t_\psi)}_{\mathcal{V}} \otimes^{(l,i)\in\mathrm{v}(t^{|j}_{[\psi(m-1),\psi(m)]})}_{\mathcal{V}} \mathsf{C}^{p^i_l}_l.$$

Notice that $\mathrm{v}(t^{|j}_{[\psi(m-1),\psi(m)]}) = \mathrm{v}^{-1}_\psi(m, j)$.

In the case $n = 1$, the set of parameters of the parameterized $\mathcal{V}$-multiquiver $\odot^{\mathbf{1}}\mathsf{C}$ consists of pairs (t, p), where t is a tree of height 1 of the form $(I \xrightarrow{\triangleright} L \mid I \ni i \mapsto X_i, L \ni l \mapsto Y)$ with a 1-element set L, and p is an element of $\operatorname{Par} \mathsf{C}$ subject to the conditions $\operatorname{tgt} p = Y$, $\operatorname{dom}\operatorname{src} p = I$, and $\operatorname{src}_i p = X_i$, for each $i \in I$. For each parameter (t, p), the object $(\odot^{\mathbf{1}}\mathsf{C})^{(t,p)}$ is equal to C^p. Define a morphism $\rho^{\mathbf{1}} : \odot^{\mathbf{1}}\mathsf{C} \to \mathsf{C}$ by the map

$$\operatorname{Par} \rho^{\mathbf{1}} : \operatorname{Par} \odot^{\mathbf{1}}\mathsf{C} \to \operatorname{Par} \mathsf{C}, \quad (t, p) \mapsto p,$$

on parameters and by the morphisms in $\mathcal{V}$

$$\rho^{\mathbf{1}} = \mathrm{id} : (\odot^{\mathbf{1}}\mathsf{C})^{(t,p)} \to \mathsf{C}^p.$$

In the skeletal case, $\operatorname{Par} \rho^{\mathbf{1}}$ is a bijection and $\rho^{\mathbf{1}}$ is an isomorphism.

3.6 Proposition. *In all three cases (plain, symmetric or braided) the so defined triple* $({}^S\mathcal{PMQ}, \odot^I, \lambda^{\phi}_{\mathcal{PMQ}}, \rho^{\mathbf{1}})$ *is a lax Monoidal category, which we denote* $\mathcal{PMQ}^{\mathcal{O}}_{\mathcal{V}}$, $\mathcal{PMQ}^{\mathcal{S}}_{\mathcal{V}}$ *or* $\mathcal{PMQ}^{\mathcal{B}}_{\mathcal{V}}$*. In the skeletal case,* $({}^S\mathcal{PMQ}, \odot^I, \lambda^{\phi}_{\mathcal{PMQ}}, \rho^{\mathbf{1}})$ *is a Monoidal category.*

Proof. Equations (2.5.1) and (2.5.2) are to be checked on parameters only, since $\lambda^{\mathrm{id}_I}_{\mathcal{PMQ}}$, $\lambda^{I\to J}_{\mathcal{PMQ}}$, and $\rho^{\mathbf{1}}$ are identities on objects of morphisms. The corresponding equations between mappings of parameters are straightforward from the definitions.

Let $\mathbf{a} \xrightarrow{f} \mathbf{b} \xrightarrow{g} \mathbf{c}$ be non-decreasing maps. Denote the associated functors by

$[a] \xleftarrow[{[f]}]{\phi} [b] \xleftarrow[{[g]}]{\psi} [c]$. Consider a sequential tree $t \in \mathsf{N}'_a\mathcal{I}(S)$. We have to prove the identity

$$\begin{aligned}
&\Big[\otimes^{\mathrm{v}(t)}_{\mathcal{V}} \xrightarrow{\lambda^{\mathrm{v}_\phi:\mathrm{v}(t)\to\mathrm{v}(t_\phi)}} \otimes^{(m,j)\in\mathrm{v}(t_\phi)}_{\mathcal{V}} \otimes^{\mathrm{v}_\phi^{-1}(m,j)}_{\mathcal{V}} \\
&\qquad\qquad \xrightarrow{\lambda^{\mathrm{v}_\psi:\mathrm{v}(t_\phi)\to\mathrm{v}(t_{\phi\circ\psi})}} \otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \otimes^{(m,j)\in\mathrm{v}_\psi^{-1}(l,i)}_{\mathcal{V}} \otimes^{\mathrm{v}_\phi^{-1}(m,j)}_{\mathcal{V}}\Big] \\
= &\Big[\otimes^{\mathrm{v}(t)}_{\mathcal{V}} \xrightarrow{\lambda^{\mathrm{v}_{\phi\circ\psi}:\mathrm{v}(t)\to\mathrm{v}(t_{\phi\circ\psi})}} \otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \otimes^{\mathrm{v}(t|^i_{[\phi\psi(l-1),\phi\psi l]})}_{\mathcal{V}} \\
&\qquad\qquad \xrightarrow{\otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \lambda^{\mathrm{v}(\phi|):\mathrm{v}(t|^i_{[\phi\psi(l-1),\phi\psi l]})\to\mathrm{v}(t|^i_{[\psi(l-1),\psi l]})}} \\
&\qquad\qquad\qquad \otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \otimes^{(m,j)\in\mathrm{v}(t|^i_{[\psi(l-1),\psi l]})}_{\mathcal{V}} \otimes^{\mathrm{v}_\phi^{-1}(m,j)}_{\mathcal{V}}\Big].
\end{aligned}$$

Here we use the identifications $\mathrm{v}_\psi^{-1}(l,i) = \mathrm{v}(t|^i_{[\psi(l-1),\psi l]})$ and $\mathrm{v}_{\phi\circ\psi}^{-1}(l,i) = \mathrm{v}(t|^i_{[\phi\psi(l-1),\phi\psi l]})$. Using the equations $\mathrm{v}_{\phi\circ\psi} = \mathrm{v}_\phi \circ \mathrm{v}_\psi$ and

$$\mathrm{v}_{\phi|:[\psi(l-1),\psi l]\to[\phi\psi(l-1),\phi\psi l]} = \mathrm{v}_\phi\big| : \mathrm{v}(t|^i_{[\phi\psi(l-1),\phi\psi l]}) \to \mathrm{v}(t|^i_{[\psi(l-1),\psi l]}),$$

we transform the right hand side to

$$\begin{aligned}
&\Big[\otimes^{\mathrm{v}(t)}_{\mathcal{V}} \xrightarrow{\lambda^{\mathrm{v}_\phi\circ\mathrm{v}_\psi:\mathrm{v}(t)\to\mathrm{v}(t_{\phi\circ\psi})}} \otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \otimes^{\mathrm{v}_\phi^{-1}\mathrm{v}_\psi^{-1}(l,i)}_{\mathcal{V}} \\
&\qquad \xrightarrow{\otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \lambda^{\mathrm{v}_\phi|:\mathrm{v}_\phi^{-1}\mathrm{v}_\psi^{-1}(l,i)\to\mathrm{v}_\psi^{-1}(l,i)}} \otimes^{(l,i)\in\mathrm{v}(t_{\phi\circ\psi})}_{\mathcal{V}} \otimes^{(m,j)\in\mathrm{v}_\psi^{-1}(l,i)}_{\mathcal{V}} \otimes^{\mathrm{v}_\phi^{-1}(m,j)}_{\mathcal{V}}\Big].
\end{aligned}$$

This equals the left hand side due to axiom (2.10.3), which holds for $\mathcal{V}$. □

$\mathcal{V}$-multicategories, which we are going to define now, are examples of algebras in $\mathcal{PMQ}_{\mathcal{V}}$.

3.7 Definition. Let $\mathcal{V}$ be a symmetric Monoidal category. A *plain* (resp. *symmetric*, *braided*) $\mathcal{V}$*-multicategory* consists of the following data:

- a $\mathcal{V}$-multiquiver C;
- for each map $\phi : I \to J$ from $\mathcal{O}$ (resp. $\mathcal{S}$) and objects $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, a morphism in $\mathcal{V}$

  $$\mu_\phi : \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}(j)}; Y_j)\big)_{j\in J}, \mathsf{C}((Y_j)_{j\in J}; Z)\big] \to \mathsf{C}((X_i)_{i\in I}; Z),$$

  called *composition*;
- for each object $X \in \operatorname{Ob}\mathsf{C}$ and 1-element set L, a morphism $\eta_{X,L} : \mathbb{1} \to \mathsf{C}((X)_L; X)$, called the *identity* of X, also denoted by $1_{X,L}$.

$$
\begin{tikzcd}
{} & \bigotimes^{J\sqcup 1}\Big[\big(\mathbf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\bigotimes^{K\sqcup 1}\Big(\big(\mathbf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\mathbf{C}((Z_k)_{k\in K};W)\Big)\Big] \arrow[d, "\otimes^{J\sqcup 1}((1)_{j\in J},\mu_\psi)"] \\
\bigotimes^{J\sqcup K\sqcup 1}\Big[\big(\mathbf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\big(\mathbf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\mathbf{C}((Z_k)_{k\in K};W)\Big] \arrow[ur, "\lambda^{\alpha}_{\mathcal{V}}"] \arrow[d, "\lambda^{\beta}_{\mathcal{V}}"'] & \bigotimes^{J\sqcup 1}\Big[\big(\mathbf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathbf{C}((Y_j)_{j\in J};W)\Big] \arrow[d, "\mu_\phi"] \\
\bigotimes^{K\sqcup 1}\Big[\Big(\bigotimes^{\psi^{-1}k\sqcup 1}\big[\big(\mathbf{C}((X_i)_{i\in\phi_k^{-1}j};Y_j)\big)_{j\in\psi^{-1}k},\mathbf{C}((Y_j)_{j\in\psi^{-1}k};Z_k)\big]\Big)_{k\in K},\mathbf{C}((Z_k)_{k\in K};W)\Big] \arrow[d, "\otimes^{K\sqcup 1}((\mu_{\phi_k})_{k\in K},1)"'] & \mathbf{C}((X_i)_{i\in I};W) \\
\bigotimes^{K\sqcup 1}\big[\big(\mathbf{C}((X_i)_{i\in(\phi\psi)^{-1}(k)};Z_k)\big)_{k\in K},\mathbf{C}((Z_k)_{k\in K};W)\big] \arrow[ur, "\mu_{\phi\psi}"] &
\end{tikzcd}
$$

Figure 3.1: Associativity in $\mathcal{V}$-multicategories

These data are required to satisfy the following axioms.

- (Associativity) For each pair of composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ from $\mathcal{O}$ (resp. each pair from $\mathcal{S}$, resp. each pair from $\mathcal{S}$ that satisfies condition (2.5.3)), the diagram shown on the preceding page commutes. Here $\phi_k = \phi|_{(\phi\psi)^{-1}(k)} : (\phi\psi)^{-1}(k) \to \psi^{-1}(k)$, $k \in K$. The other maps involved in the diagram are $\alpha = \mathrm{id}_J \sqcup \triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$, where $\triangleright : K \sqcup \mathbf{1} \to \mathbf{1}$ is the only map, and $\beta : J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}$ given by
$$\beta\big|_{K\sqcup\mathbf{1}} = \mathrm{id} : K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}, \qquad \beta\big|_J = \big(J \xrightarrow{\psi} K \hookrightarrow K \sqcup \mathbf{1}\big).$$
Note that α preserves the order, while β in general does not.

- (Identity) If J is a 1-element set and $Y_j = Z$ for the only $j \in J$, then for $\phi : I \to J$ the equation
$$\big[\mathsf{C}((X_i)_{i\in I}; Z) \xrightarrow{\lambda^{J\hookrightarrow J\sqcup\mathbf{1}}} \otimes^{J\sqcup\mathbf{1}} (\mathsf{C}((X_i)_{i\in I}; Z), \mathbb{1}_{\mathcal{V}}) \xrightarrow{\otimes^{J\sqcup\mathbf{1}}(1,\eta_{Z,J})} \otimes^{J\sqcup\mathbf{1}} (\mathsf{C}((X_i)_{i\in I}; Z), \mathsf{C}((Z)_J; Z)) \xrightarrow{\mu_{\phi:I\to J}} \mathsf{C}((X_i)_{i\in I}; Z)\big] = \mathrm{id}. \quad (3.7.1)$$
holds true. If $\phi = \mathrm{id} : I \to I$ and $X_i = Y_i$ for all $i \in I$, then the equation
$$\big[\mathsf{C}((X_i)_{i\in I}; Z) \xrightarrow{\lambda^{\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}} \otimes^{I\sqcup\mathbf{1}} \big((\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathsf{C}((X_i)_{i\in I}; Z)\big) \xrightarrow{\otimes^{I\sqcup\mathbf{1}}((\eta_{X_i,\{i\}})_{i\in I},1)} \otimes^{I\sqcup\mathbf{1}} \big[\big(\mathsf{C}((X_i)_{\{i\}}; X_i)\big)_{i\in I}, \mathsf{C}((X_i)_{i\in I}; Z)\big] \xrightarrow{\mu_{\mathrm{id}_I}} \mathsf{C}((X_i)_{i\in I}; Z)\big] = \mathrm{id}. \quad (3.7.2)$$
holds true.

Restricting this definition to sets $I = \mathbf{n}$ we find that a multicategory is the same as a substitude $\mathcal{A}$ of Day and Street [DS03] with the unit $\eta : \mathcal{A}(A, B) \to P_1(A; B)$ equal to the identity map. In the case $\mathcal{V} = \mathcal{S}et$, our notion of symmetric multicategory is precisely the same as that of fat symmetric multicategory introduced by Leinster [Lei03, Definition A.2.1]. Also a symmetric multicategory is nearly the same as a pseudo-tensor category of Beilinson and Drinfeld [BD04, Definition 1.1.1]. The difference is that they use only surjective maps $f : I \to J \in \mathcal{S}$, which is analogous to not having a unit object in a monoidal category. Pseudo-tensor categories are essentially the same as the multilinear categories of Borcherds [Bor98]. A braided multicategory is close to a pseudo-braided category of Soibelman [Soi99b, Soi99a].

Similarly to Leinster [Lei03, Lemma A.2.2], one can prove that, for each (resp. order preserving) bijection $\phi : I \to J$ such that $X_i = Y_{\phi(i)}$, $i \in I$, there is an isomorphism $\mathsf{C}(\phi; Z) : \mathsf{C}((Y_j)_{j\in J}; Z) \to \mathsf{C}((X_i)_{i\in I}; Z)$ given by the composite
$$\mathsf{C}(\phi; Z) = \big[\mathsf{C}((Y_j)_{j\in J}; Z) \xrightarrow{\lambda_{\mathcal{V}}^{\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}} \otimes^{J\sqcup\mathbf{1}} [(\mathbb{1}_{\mathcal{V}})_{j\in J}, \mathsf{C}((Y_j)_{j\in J}; Z)] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[(1^{\mathsf{C}}_{X_{\phi^{-1}j},\{\phi^{-1}j\}})_{j\in J},1]} \otimes^{J\sqcup\mathbf{1}} [(\mathsf{C}((X_{\phi^{-1}j})_{\{\phi^{-1}j\}}; Y_j))_{j\in J}, \mathsf{C}((Y_j)_{j\in J}; Z)] \xrightarrow{\mu^{\mathsf{C}}_{\phi}} \mathsf{C}((X_i)_{i\in I}; Z)\big]. \quad (3.7.3)$$

These isomorphisms satisfy the equation $\mathsf{C}(\psi;Z)\mathsf{C}(\phi;Z) = \mathsf{C}(\phi\psi;Z)$, whenever the left hand side is defined, and moreover $\mathsf{C}(\mathrm{id};Z) = \mathrm{id}$.

3.8 Proposition. *Let $\mathcal{V}$ be a symmetric Monoidal category. The structure of a plain (resp. symmetric, braided) $\mathcal{V}$-multicategory on a $\mathcal{V}$-multiquiver C is equivalent to an algebra structure $\mu^{\mathsf{C}}_{\mathbf{n}}$ on C in the lax Monoidal category $({}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}^{\mathcal{O}}_{\mathcal{V}}, \odot^I, \lambda^{\phi}_{\mathcal{PMQ}})$ (respectively in $({}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}^{\mathcal{S}}_{\mathcal{V}}, \odot^I, \lambda^{\phi}_{\mathcal{PMQ}})$, $({}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}^{\mathcal{B}}_{\mathcal{V}}, \odot^I, \lambda^{\phi}_{\mathcal{PMQ}})$) such that binary composition*

$$\mu^{\mathsf{C}}_{\mathbf{2}} : \otimes^{J\sqcup K}\big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}(j)};Y_j)\big)_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z)\big] \to \mathsf{C}((X_i)_{i\in I};Z),$$

indexed by the tree $(I \xrightarrow{\phi} J \xrightarrow{\triangleright} K \mid (X_i)_{i\in I}, (Y_j)_{j\in J}, (Z)_K)$ does not depend on the choice of 1-element set K, meaning that the diagram

$$\begin{tikzcd}
\otimes^{J\sqcup K}\big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}(j)};Y_j)\big)_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z)\big] \arrow[r, "\mu_{\mathbf{2}}^{I,\phi,J,\triangleright,K}"] \arrow[d, "\lambda_{\mathcal{V}}^{\mathrm{id}\sqcup\triangleright:J\sqcup K\to J\sqcup\mathbf{1}}"'] & \mathsf{C}((X_i)_{i\in I};Z) \\
\otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_i)_{i\in\phi^{-1}(j)};Y_j)\big)_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z)\big] \arrow[ur, "\mu_{\mathbf{2}}^{I,\phi,J,\triangleright,\mathbf{1}}"'] &
\end{tikzcd} \tag{3.8.1}$$

commutes.

Proof. Since C is a multiquiver, $\mathrm{Par}\,\mathsf{C} = \mathrm{Ob}_s\,\mathsf{C}\times\mathrm{Ob}_t\,\mathsf{C} = (\mathrm{Ob}\,\mathsf{C})^*\times\mathrm{Ob}\,\mathsf{C}$. The parameterized multiquiver $\odot^{\mathbf{n}}\mathsf{C}$ has the set of parameters $\mathrm{Par}\,\odot^{\mathbf{n}}\mathsf{C} = \mathsf{N}'_n\mathcal{I}(\mathrm{Ob}\,\mathsf{C})$. The source–target mapping $(\mathrm{src},\mathrm{tgt}) : \mathrm{Par}\,\odot^{\mathbf{n}}\mathsf{C} \to \mathrm{Ob}_s\,\mathsf{C}\times\mathrm{Ob}_t\,\mathsf{C} = \mathrm{Par}\,\mathsf{C}$, $t \mapsto \big((X^i_0)_{i\in I_0}, X^1_n\big)$ is identified with the mapping $t \mapsto {}^{<}t$, given by (3.4.4).

As Proposition 2.28 shows, in order to give an algebra structure on C, it suffices to specify a nullary multiplication $\mu^{\mathsf{C}}_{\mathbf{0}}$ and a binary multiplication $\mu^{\mathsf{C}}_{\mathbf{2}}$ subject to certain equations. The unit $\eta = \mu^{\mathsf{C}}_{\mathbf{0}} : \odot^{\mathbf{0}} \to \mathsf{C}$ of the algebra amounts to morphisms $\eta_{X,L} = 1_{X,L} : (\odot^{\mathbf{0}})^{(X,L)} = \mathbb{1}_{\mathcal{V}} \to \mathsf{C}((X)_L;X)$, for each $X\in\mathrm{Ob}\,\mathsf{C}$ and 1-element set $L\in\mathrm{Ob}\,\mathcal{O}$.

Since C is a $\mathcal{V}$-multiquiver, the map $\mathrm{Par}\,\mu^{\mathsf{C}}_{\mathbf{2}} : \mathrm{Par}(\mathsf{C}\odot\mathsf{C}) = \mathsf{N}_2\mathcal{I}(\mathrm{Ob}\,\mathsf{C}) \to \mathrm{Par}\,\mathsf{C} = \mathrm{Ob}_s\,\mathsf{C}\times\mathrm{Ob}_t\,\mathsf{C}$ is determined by the condition $\mathrm{Ob}_t\,\mu^{\mathsf{C}}_{\mathbf{2}} = \mathrm{id}$, which implies that

$$\mathrm{Par}\,\mu^{\mathsf{C}}_{\mathbf{2}} : (I \xrightarrow{\phi} J \xrightarrow{\triangleright} K \mid (X_i)_{i\in I}, (Y_j)_{j\in J}, (Z)_K) \mapsto ((X_i)_{i\in I};Z).$$

Under additional assumptions (3.8.1), the equations involving $\mu^{\mathsf{C}}_{\mathbf{0}}$ and $\mu^{\mathsf{C}}_{\mathbf{2}}$ take the form of Definition 3.7. The associativity is expressed by the equation for each pair of composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ from $\mathcal{O}$ (resp. each pair from $\mathcal{S}$, resp. each pair from $\mathcal{S}$ that satisfies condition (2.5.3)), shown on page 73, $\phi_k = \phi|_{(\phi\psi)^{-1}(k)} : (\phi\psi)^{-1}(k) \to \psi^{-1}(k)$. Notice that $\phi_k^{-1}(j) = \phi^{-1}(j)$ for any $j\in\psi^{-1}(k)$. The sequential tree $t = \big(I \xrightarrow{\phi} J \xrightarrow{\psi} K \xrightarrow{\triangleright} \mathbf{1} \mid (X_i)_{i\in I}, (Y_j)_{j\in J}, (Z_k)_{k\in K}, W\big)$ and the map $\pi = \mathsf{IV} : \mathbf{3} \to \mathbf{2}$, $[\pi] = \mathsf{II}\,.\,\mathsf{I} : [2] \to [3]$ produce the map

$$\alpha = \mathrm{v}([\pi]) = \mathrm{id}_J \sqcup \triangleright : \mathrm{v}(t) = J\sqcup K\sqcup\mathbf{1} \to J\sqcup\mathbf{1} = \mathrm{v}(t_{[\pi]}).$$

The map $\pi = \mathsf{VI} : \mathbf{3} \to \mathbf{2}$, $[\pi] = \mathsf{I}\,.\,\mathsf{II} : [2] \to [3]$ induces the map

$$\beta = \mathrm{v}([\pi]) : \mathrm{v}(t) = J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1} = \mathrm{v}(t_{[\pi]})$$

given by

$$\beta\big|_{K\sqcup\mathbf{1}} = \mathrm{id} : K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}, \qquad \beta\big|_J = \big(J \xrightarrow{\psi} K \hookrightarrow K \sqcup \mathbf{1}\big). \tag{3.8.2}$$

$\square$

Note that for $I = \varnothing$ we have

$$\mu_{\varnothing\to\varnothing} = \mathrm{id} : \mathsf{C}(;Z) \to \mathsf{C}(;Z).$$

3.9 Lax 2-categories and (co)lax 2-functors. The notion of lax Monoidal category can be easily generalized to the notion of lax 2-category. This was done by Leinster in [Lei03, Definition 3.4.1]. We recall his definition with the direction of arrows suitable for our applications. The definition is simplified by the requirement similar to the condition that ρ^L be the identity morphism in the case of lax Monoidal categories. We use the notions of lax 2-category and colax 2-functor mostly for illustrative purposes, see Section 3.12 and Exercise 3.13. Practically minded reader is advised to proceed to Definition 3.14 of a $\mathcal{V}$-multifunctor.

For $i \in I \in \mathrm{Ob}\,\mathcal{O}$ let $i - 1 \in [I] = \{0\} \sqcup I$ denote the preceding element.

3.10 Definition. A *lax 2-category* $\mathfrak{C}$ consists of the following data:

1. a class of objects $\mathrm{Ob}\,\mathfrak{C}$;
2. for every pair of objects X, $Y \in \mathrm{Ob}\,\mathfrak{C}$ a category $\mathfrak{C}(X, Y)$;
3. for every set $I \in \mathrm{Ob}\,\mathcal{O}$ and every collection $(X_i)_{i\in[I]}$ of objects of $\mathfrak{C}$ a functor

$$\bullet^I : \prod_{i\in I} \mathfrak{C}(X_{i-1}, X_i) \to \mathfrak{C}(X_0, X_{\max I})$$

such that $\bullet^I = \mathrm{Id}$ for each 1-element set I. In particular, a map

$$\bullet^I : \prod_{i\in I} \mathfrak{C}(X_{i-1}, X_i)(F_i, G_i) \to \mathfrak{C}(X_0, X_{\max I})(\bullet^{i\in I} F_i, \bullet^{i\in I} G_i)$$

is given.

For a map $f : I \to J$ in $\mathrm{Mor}\,\mathcal{O}$ introduce a functor

$$\bullet^f : \prod_{i\in I} \mathfrak{C}(X_{i-1}, X_i) \to \prod_{j\in J} \mathfrak{C}(X_{[f](j-1)}, X_{[f](j)})$$

which to a collection $(F_i)_{i\in I}$ assigns a collection $(\bullet^{i\in f^{-1}j}F_i)_{j\in J}$. The action on morphisms is given by the map

$$\prod_{i\in I}\mathfrak{C}(X_{i-1},X_i)(F_i,G_i)\xrightarrow{\sim}\prod_{j\in J}\prod_{i\in f^{-1}j}\mathfrak{C}(X_{i-1},X_i)(F_i,G_i)$$
$$\xrightarrow{\prod_{j\in J}\bullet^{f^{-1}j}}\prod_{j\in J}\mathfrak{C}(X_{[f](j-1)},X_{[f](j)})(\bullet^{i\in f^{-1}j}F_i,\bullet^{i\in f^{-1}j}G_i).$$

4. a morphism of functors

$$\lambda^f:\bullet^I\to\bullet^J\circ\bullet^f:\prod_{i\in I}\mathfrak{C}(X_{i-1},X_i)\to\mathfrak{C}(X_0,X_{\max I}),\qquad \lambda^f:\bullet^{i\in I}F_i\to\bullet^{j\in J}\bullet^{i\in f^{-1}j}F_i$$

for every map $f:I\to J$ in $\operatorname{Mor}\mathcal{O}$

such that

(i) for all sets $I\in\operatorname{Ob}\mathcal{O}$, for all 1-element sets J

$$\lambda^{\mathrm{id}_I}=\mathrm{id},\qquad \lambda^{I\to J}=\mathrm{id};$$

(ii) for any pair of composable maps $I\xrightarrow{f}J\xrightarrow{g}K$ from $\mathcal{O}$ the following equation holds:

$$\begin{tikzcd}
\bullet^{i\in I}F_i \arrow[rr, "\lambda^f"] \arrow[d, "\lambda^{fg}"'] & {} \arrow[d, phantom, "="] & \bullet^{j\in J}\bullet^{i\in f^{-1}j}F_i \arrow[d, "\lambda^g"] \\
\bullet^{k\in K}\bullet^{i\in f^{-1}g^{-1}k}F_i \arrow[rr, "\bullet^{k\in K}\lambda^{f|:f^{-1}g^{-1}k\to g^{-1}k}"] & {} & \bullet^{k\in K}\bullet^{j\in g^{-1}k}\bullet^{i\in f^{-1}j}F_i
\end{tikzcd}$$

3.11 Definition. A *colax 2-functor* between lax 2-categories

$$(F,\phi^I):(\mathfrak{C},\bullet^I_{\mathfrak{C}},\lambda^f_{\mathfrak{C}})\to(\mathfrak{D},\bullet^I_{\mathfrak{D}},\lambda^f_{\mathfrak{D}})$$

consists of

1. a function $F:\operatorname{Ob}\mathfrak{C}\to\operatorname{Ob}\mathfrak{D}$;

2. for every pair of objects $X,Y\in\operatorname{Ob}\mathfrak{C}$ a functor $F_{X,Y}:\mathfrak{C}(X,Y)\to\mathfrak{D}(FX,FY)$;

3. a functorial morphism for each set $I\in\operatorname{Ob}\mathcal{O}$

$$\phi^I:F\circ\bullet^I_{\mathfrak{C}}\to\bullet^I_{\mathfrak{D}}\circ\prod_{i\in I}F_{X_{i-1},X_i}:\prod_{i\in I}\mathfrak{C}(X_{i-1},X_i)\to\mathfrak{D}(FX_0,FX_{\max I})$$
$$\phi^I:F\bullet^{i\in I}_{\mathfrak{C}}K_i\to\bullet^{i\in I}_{\mathfrak{D}}FK_i$$

such that $\phi^I = \mathrm{id}$ for each one element set I, and for every map $f : I \to J$ of $\mathcal{O}$ the following equation holds:

$$\begin{tikzcd}
F\bullet_{\mathfrak{C}}^{i\in I} K_i \arrow[rr, "\phi^I"] \arrow[d, "F\lambda^f_{\mathfrak{C}}"'] & & \bullet_{\mathfrak{D}}^{i\in I} FK_i \arrow[d, "\lambda^f_{\mathfrak{D}}"] \\
F\bullet_{\mathfrak{C}}^{j\in J}\bullet_{\mathfrak{C}}^{i\in f^{-1}j} K_i \arrow[r, "\phi^J"] & \bullet_{\mathfrak{D}}^{j\in J} F\bullet_{\mathfrak{C}}^{i\in f^{-1}j} K_i \arrow[r, "\bullet_{\mathfrak{D}}^{j\in J}\phi^{f^{-1}j}"] \arrow[u, phantom, "="] & \bullet_{\mathfrak{D}}^{j\in J}\bullet_{\mathfrak{D}}^{i\in f^{-1}j} FK_i
\end{tikzcd}$$

3.12 $\mathcal{V}$-multicategory as an algebra. Due to Proposition 2.27 a (symmetric, braided) $\mathcal{V}$-multicategory C has well-defined multiplications $\mu^g_{\mathsf{C}} : \odot^I\mathsf{C} \to \odot^J\mathsf{C}$ for each isotonic map $g : I \to J$. They define a colax Monoidal functor $\mathcal{O} \to \overline{{}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}^{\mathcal{I}}_{\mathcal{V}}}$. In details, $\mathrm{Par}\odot^I\mathsf{C} = \mathsf{N}'_I\mathcal{I}(\mathrm{Ob}\,\mathsf{C})$ and

$$\odot^I \mathsf{C}(q) = \mu^q_{\mathsf{C}} \overset{\text{def}}{=} \otimes^{(i,a)\in\mathrm{v}(q)}_{\mathcal{V}}\mathsf{C}(q^{|a}_{\rangle i}) \tag{3.12.1}$$

is associated to any sequential tree $q \in \mathsf{N}'_I\mathcal{I}(\mathrm{Ob}\,\mathsf{C})$. Also there is multiplication with I entries: $\mu^{\mathsf{C}}_q = \mu^{\mathsf{C}}_I : \odot^I\mathsf{C}(q) \to \mathsf{C}({}^<q)$.

Each isotonic map $f : I \to J$ gives a morphism $f : q \to q^f$ in $\mathsf{N}'\mathcal{I}(\mathrm{Ob}\,\mathsf{C})$, whose source is q. According to Proposition 2.27 $\mu^f_{\mathsf{C}} : \odot^I\mathsf{C} \to \odot^J\mathsf{C}$ is the composition

$$\begin{aligned}
\mu^{f:q\to q^f}_{\mathsf{C}} = \Big[(\odot^I\mathsf{C})(q) &= \otimes^{(i,a)\in\mathrm{v}(q)}_{\mathcal{V}}\mathsf{C}(q^{|a}_{\rangle i}) \\
&\xrightarrow{\lambda^{\mathrm{v}^f:\mathrm{v}(q)\to\mathrm{v}(q^f)}} \otimes^{(j,b)\in\mathrm{v}(q^f)}_{\mathcal{V}} \otimes^{(i,a)\in(\mathrm{v}^f)^{-1}(j,b)}_{\mathcal{V}}\mathsf{C}(q^{|a}_{\rangle i}) \\
&\xrightarrow{\otimes^{(j,b)\in\mathrm{v}(q^f)}_{\mathcal{V}}\mu^{\mathsf{C}}_{q^{|b}_{[[f](j-1),[f]j]}}} \otimes^{(j,b)\in\mathrm{v}(q^f)}_{\mathcal{V}} \mathsf{C}(q^{f|b}_{\rangle j}) = (\odot^J\mathsf{C})(q^f)\Big].
\end{aligned} \tag{3.12.2}$$

Notice that $(\mathrm{v}^f)^{-1}(j,b) = \mathrm{v}(q^{|b}_{[[f](j-1),[f]j]})$. So defined functor $\mathcal{O} \to \overline{{}^{\mathrm{Ob}\,\mathsf{C}}\mathcal{PMQ}^{\mathcal{I}}_{\mathcal{V}}}$ can be also written as a functor $\mathsf{N}'\mathcal{I}(\mathrm{Ob}\,\mathsf{C}) \to \mathcal{V}$, $\mathsf{N}'_I\mathcal{I}(\mathrm{Ob}\,\mathsf{C}) \ni q \mapsto (\odot^I\mathsf{C})(q)$, $(f : q \to q^f) \mapsto \mu^{f:q\to q^f}_{\mathsf{C}}$.

3.13 Exercise. Let C be a (symmetric) $\mathcal{V}$-multicategory. The functor $\mathsf{N}'\mathcal{I}(\mathrm{Ob}\,\mathsf{C}) \to \mathcal{V}$ extends by (3.12.1) and (3.12.2) to a weak colax 2-functor $(\mu_{\mathsf{C}}, \phi^I) : \mathsf{N}\mathcal{I}(\mathrm{Ob}\,\mathsf{C}) \to \mathcal{V}$, where $\mathcal{V}$ is viewed as a lax 2-category with one object,

$$\begin{aligned}
\mu_{\mathsf{C}} : q &\longmapsto \mu^q_{\mathsf{C}} \overset{\text{def}}{=} \otimes^{(x,a)\in\mathrm{v}(q)}_{\mathcal{V}}\mathsf{C}(q^{|a}_{\rangle x}), \\
f : q \to q^f &\longmapsto \mu^{f:q\to q^f}_{\mathsf{C}} : \mu^q_{\mathsf{C}} \to \mu^{q^f}_{\mathsf{C}}, \\
\phi^I = \lambda^{\mathrm{v}(\bullet^{i\in I}q_i)\to I}_{\mathcal{V}} &: \mu^{\bullet^{i\in I}q_i}_{\mathsf{C}} \longrightarrow \otimes^{i\in I}\mu^{q_i}_{\mathsf{C}},
\end{aligned}$$

where $\bullet^{i\in I}q_i$ is the composition of 1-morphisms q_i. A family of sequential forests $p^k \in \mathsf{N}_I\mathcal{I}(\mathrm{Ob}\,\mathsf{C})$, $k \in K \in \mathrm{Ob}\,\mathcal{O}$, and an isotonic map $f : I \to J$ induce the family $q^k = (p^k)^f \in$

$\mathsf{N}_J\mathfrak{I}(\operatorname{Ob}\mathsf{C})$ and the following equation holds:

$$\begin{tikzcd}
\mu_{\mathsf{C}}^{\sqcup_{k\in K}p^k} \arrow[r, "\lambda^{\mathrm{v}(\sqcup_{k\in K}p^k)\to K}"] \arrow[d, "\mu_{\mathsf{C}}^f"'] \arrow[dr, phantom, "="] & \otimes_{\mathcal{V}}^{k\in K}\mu_{\mathsf{C}}^{p^k} \arrow[d, "\otimes_{\mathcal{V}}^{k\in K}\mu_{\mathsf{C}}^f"] \\
\mu_{\mathsf{C}}^{\sqcup_{k\in K}q^k} \arrow[r, "\lambda^{\mathrm{v}(\sqcup_{k\in K}q^k)\to K}"] & \otimes_{\mathcal{V}}^{k\in K}\mu_{\mathsf{C}}^{q^k}
\end{tikzcd}$$

Notice that colax 2-functors whose transformation ϕ^I is invertible are equivalent to lax 2-functors with invertible transformation component. The bijection is given by inverting the transformation.

3.14 Definition. Let C, D be plain (resp. symmetric, resp. braided) $\mathcal{V}$-multicategories. A plain (resp. symmetric, resp. braided) $\mathcal{V}$-*multifunctor* $F : \mathsf{C} \to \mathsf{D}$ is a mapping of objects $\operatorname{Ob} F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{D}$, $X \mapsto FX$, together with morphisms

$$F_{(X_i)_{i\in I};Y} : \mathsf{C}((X_i)_{i\in I};Y) \to \mathsf{D}((FX_i)_{i\in I};FY)$$

of $\mathcal{V}$, given for each function $I \sqcup \mathbf{1} \to \operatorname{Ob}\mathsf{C}$, $i \mapsto X_i$, $1 \mapsto Y$, such that for each $X \in \operatorname{Ob}\mathsf{C}$ and 1-element set J

$$\bigl(\mathbb{1}_{\mathcal{V}} \xrightarrow{1^{\mathsf{C}}_{X,J}} \mathsf{C}((X)_J;X) \xrightarrow{F_{(X)_J;X}} \mathsf{D}((FX)_J;FX)\bigr) = 1^{\mathsf{D}}_{FX,J},$$

and for an order-preserving (resp. arbitrary) map $\phi : I \to J$ together with a map $I \sqcup J \sqcup \mathbf{1} \to \operatorname{Ob}\mathsf{C}$, $i \mapsto X_i$, $j \mapsto Y_j$, $1 \mapsto Z$, we have

$$\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\bigr)_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z)\bigr] \arrow[r, "\mu^{\mathsf{C}}_\phi"] \arrow[d, "\otimes^{J\sqcup\mathbf{1}}[(F_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},F_{(Y_j)_{j\in J};Z}]"'] \arrow[dr, phantom, "="] & \mathsf{C}((X_i)_{i\in I};Z) \arrow[d, "F_{(X_i)_{i\in I};Z}"] \\
\otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\mathsf{D}((FX_i)_{i\in\phi^{-1}j};FY_j)\bigr)_{j\in J}, \mathsf{D}((FY_j)_{j\in J};FZ)\bigr] \arrow[r, "\mu^{\mathsf{D}}_\phi"] & \mathsf{D}((FX_i)_{i\in I};FZ)
\end{tikzcd}$$

A $\mathcal{V}$-multicategory C gives rise to a $\mathcal{V}$-category $\mathcal{C}$ with the same set of objects and $\mathcal{C}(X,Y) = \mathsf{C}((X)_{\mathbf{1}};Y)$, for each $X,Y \in \operatorname{Ob}\mathsf{C}$. Composition in $\mathcal{C}$ is given by $\mu^{\mathsf{C}}_{\mathbf{1}\to\mathbf{1}}$, and for each $X \in \operatorname{Ob}\mathsf{C}$, the element $1^{\mathsf{C}}_{X,\mathbf{1}} : \mathbb{1}_{\mathcal{V}} \to \mathsf{C}((X)_{\mathbf{1}};X) = \mathcal{C}(X,X)$ is the identity of X. The $\mathcal{V}$-category $\mathcal{C}$ is called the *underlying* $\mathcal{V}$-*category* of the multicategory C and is often denoted by the same symbol. A multifunctor $F : \mathsf{C} \to \mathsf{D}$ induces a $\mathcal{V}$-functor between the underlying $\mathcal{V}$-categories, an *underlying* $\mathcal{V}$-*functor* of F. It acts on objects like F, and the action on morphisms is given by $F_{(X)_{\mathbf{1}};Y} : \mathsf{C}((X)_{\mathbf{1}};Y) \to \mathsf{D}((FX)_{\mathbf{1}};FY)$, for each $X,Y \in \operatorname{Ob}\mathsf{C}$.

3.15 Definition. A *multinatural transformation* of $\mathcal{V}$-multifunctors $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a collection of elements $r_{X,L} \in \mathsf{D}((FX)_L;GX)$, for each object $X \in \operatorname{Ob}\mathsf{C}$ and 1-element

set L, such that

$$\begin{aligned}
&\big[\mathsf{C}((X_i)_{i\in I};Y) \xrightarrow{\lambda_{\mathcal{V}}^{J\hookrightarrow J\sqcup\mathbf{1}}} \otimes^{J\sqcup\mathbf{1}}\big(\mathsf{C}((X_i)_{i\in I};Y),\mathbb{1}_{\mathcal{V}}\big) \xrightarrow{\otimes^{J\sqcup\mathbf{1}}(F_{(X_i)_{i\in I};Y},r_{Y,J})} \\
&\qquad \otimes^{J\sqcup\mathbf{1}}\big(\mathsf{D}((FX_i)_{i\in I};FY),\mathsf{D}((FY)_J;GY)\big) \xrightarrow{\mu^{\mathsf{D}}_{I\to J}} \mathsf{D}((FX_i)_{i\in I};GY)\big] \\
&= \big[\mathsf{C}((X_i)_{i\in I};Y) \xrightarrow{\lambda_{\mathcal{V}}^{J\to I\sqcup\mathbf{1}}} \otimes^{I\sqcup\mathbf{1}}\big((\mathbb{1}_{\mathcal{V}})_{i\in I},\mathsf{C}((X_i)_{i\in I};Y)\big) \xrightarrow{\otimes^{I\sqcup\mathbf{1}}((r_{X_i,\{i\}})_{i\in I},G_{(X_i)_{i\in I};Y})} \\
&\qquad \otimes^{I\sqcup\mathbf{1}}\big((\mathsf{D}((FX_i)_{\{i\}};GX_i))_{i\in I},\mathsf{D}((GX_i)_{i\in I};GY)\big) \xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_I}} \mathsf{D}((FX_i)_{i\in I};GY)\big]
\end{aligned} \tag{3.15.1}$$

for each family $(X_i)_{i\in I},Y$ of objects of C and 1-element set J. Here the map $J\to I\sqcup\mathbf{1}$ is the composite $\big(J\xrightarrow{\triangleright}\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}\big)$. A *natural transformation* of $\mathcal{V}$-multifunctors $r:F\to G:\mathsf{C}\to\mathsf{D}$ is a natural transformation of underlying $\mathcal{V}$-functors. It satisfies the above equation for one-element sets I.

Suppose $r:F\to G:\mathsf{C}\to\mathsf{D}$ is a multinatural transformation. Then for two 1-element sets I and J, the elements $r_{X,I}\in\mathsf{D}((FX)_I;GX)$ and $r_{X,J}\in\mathsf{D}((FX)_J;GX)$ are related by the formula

$$r_{X,I} = \big[\mathbb{1}_{\mathcal{V}} \xrightarrow{r_{X,J}} \mathsf{D}((FX)_J;GX) \xrightarrow{\mathsf{D}(\triangleright;GX)} \mathsf{D}((FX)_I;GX)\big], \tag{3.15.2}$$

where $\triangleright:I\to J$ is the only map and $\mathsf{D}(\triangleright;GX)$ is given by (3.7.3).

A multinatural transformation of multifunctors $r:F\to G:\mathsf{C}\to\mathsf{D}$ admits another equivalent description in terms of maps $\mathsf{C}(X_1,\dots,X_n;Y)\to\mathsf{D}(FX_1,\dots,FX_n;GY)$. Plain (resp. symmetric) multicategories, multifunctors and their natural transformations form a 2-category $\mathcal{MC}at$ (resp. $\mathcal{SMC}at$). Multinatural transformations are distinguished between all natural transformations and they give a 2-subcategory $\mathcal{MC}atm$ of $\mathcal{MC}at$ (resp. $\mathcal{SMC}atm$ of $\mathcal{SMC}at$).

Recall that multiquivers form a symmetric Monoidal category by Section 3.1. The 2-categories $\mathcal{MC}at$, $\mathcal{MC}atm$ (resp. $\mathcal{SMC}at$, $\mathcal{SMC}atm$) of plain (resp. symmetric) $\mathcal{V}$-multicategories, $\mathcal{V}$-multifunctors and their (multi)natural transformations are symmetric Monoidal $\mathcal{C}at$-categories. The strict 2-functor $\boxtimes^I:\mathcal{MC}at^I\to\mathcal{MC}at$ and the 2-transformation $\Lambda^f_{\mathcal{MC}at}$ come from the corresponding multiquiver notions, see also (2.37.3) for $\mathcal{V}$-category case.

3.16 Example. Consider the symmetric multicategory $\widehat{\Bbbk\text{-}\mathbf{Mod}}$ over $\mathcal{S}et$. Its objects are $\mathscr{U}$-small $\Bbbk$-modules. Multimaps $f\in\widehat{\Bbbk\text{-}\mathbf{Mod}}((X_i)_{i\in I};Y)$ are $\Bbbk$-polylinear maps $f:\prod_{i\in I}X_i\to Y$. The composition $\mu_{\phi:I\to J}$ assigns to $\Bbbk$-polylinear maps $f_j:\prod_{i\in\phi^{-1}j}X_i\to Y_j$ and $g:\prod_{j\in J}Y_j\to Z$ the $\Bbbk$-polylinear map

$$\prod_{i\in I}X_i \simeq \prod_{j\in J}\prod_{i\in\phi^{-1}j}X_i \xrightarrow{\prod_{j\in J}f_j} \prod_{j\in J}Y_j \xrightarrow{g} Z.$$

3.17 Example. Consider the symmetric multicategory $\widehat{\mathbf{gr}}$ over $\mathcal{S}et$. Its objects are $\mathbb{Z}$-graded $\mathscr{U}$-small $\Bbbk$-modules, that is, functions $X : \mathbb{Z} \to \operatorname{Ob}\Bbbk\text{-}\mathbf{Mod}$, $n \mapsto X^n$. Multimaps $f \in \widehat{\mathbf{gr}}((X_i)_{i\in I};Y)$ (of degree 0) are families of $\Bbbk$-polylinear maps $(f^{(n_i)_{i\in I}} : \prod_{i\in I} X_i^{n_i} \to Y^{\sum_{i\in I} n_i})_{(n_i)\in\mathbb{Z}^I}$. The composition$\mu_{\phi:I\to J}$ assigns to multimaps $f_j = (f_j^{(n_i)_{i\in\phi^{-1}j}} : \prod_{i\in\phi^{-1}j} X_i^{n_i} \to Y_j^{\sum_{i\in\phi^{-1}j} n_i})_{(n_i)\in\mathbb{Z}^{\phi^{-1}j}}$ and $g = (g^{(m_j)_{j\in J}} : \prod_{j\in J} Y_j^{m_j} \to Z^{\sum_{j\in J} m_j})_{(m_j)\in\mathbb{Z}^J}$ the $\Bbbk$-polylinear map

$$\prod_{i\in I} X_i^{n_i} \simeq \prod_{j\in J}\prod_{i\in\phi^{-1}j} X_i^{n_i} \xrightarrow{\prod_{j\in J} f_j^{(n_i)}} \prod_{j\in J} Y_j^{\sum_{i\in\phi^{-1}j} n_i}$$
$$\xrightarrow{g^{(\sum_{i\in\phi^{-1}j} n_i)}} Z^{\sum_{j\in J}\sum_{i\in\phi^{-1}j} n_i} \xrightarrow{(-1)^\sigma} Z^{\sum_{i\in I} n_i},$$

where

$$\sigma = \sum_{i<p,\ \phi(i)>\phi(p)}^{i,p\in I} n_i n_p. \tag{3.17.1}$$

The above sign is prescribed by the Koszul sign rule. Equation at Fig. 3.1 for composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ follows from identities

$$\begin{aligned}
\sum_{k\in K} \sum_{i<p,\ \phi(i)>\phi(p)}^{i,p\in\phi^{-1}\psi^{-1}k} n_i n_p + \sum_{i<p,\ \psi\phi(i)>\psi\phi(p)}^{i,p\in I} n_i n_p
&= \sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)=\psi\phi(p)}^{i,p\in I} n_i n_p + \sum_{i<p,\ \phi(i)<\phi(p),\ \psi\phi(i)>\psi\phi(p)}^{i,p\in I} n_i n_p \\
&\quad + \sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)>\psi\phi(p)}^{i,p\in I} n_i n_p,
\end{aligned}$$

$$\begin{aligned}
\sum_{j<q,\ \psi(j)>\psi(q)}^{j,q\in J} \Big(\sum_{i\in\phi^{-1}j} n_i\Big)\Big(\sum_{p\in\phi^{-1}q} n_p\Big) + \sum_{i<p,\ \phi(i)>\phi(p)}^{i,p\in I} n_i n_p
&= \sum_{\phi(i)<\phi(p),\ \psi\phi(i)>\psi\phi(p)}^{i,p\in I} n_i n_p + \sum_{i<p,\ \phi(i)>\phi(p)}^{i,p\in I} n_i n_p \\
&= \sum_{i<p,\ \phi(i)<\phi(p),\ \psi\phi(i)>\psi\phi(p)}^{i,p\in I} n_i n_p + \sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)<\psi\phi(p)}^{i,p\in I} n_p n_i \\
&\quad + \sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)<\psi\phi(p)}^{i,p\in I} n_i n_p + \sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)\geqslant\psi\phi(p)}^{i,p\in I} n_i n_p.
\end{aligned}$$

In fact, these expressions differ by $2\sum_{i<p,\ \phi(i)>\phi(p),\ \psi\phi(i)<\psi\phi(p)}^{i,p\in I} n_i n_p$, which proves Fig. 3.1.

3.18 Example. Consider the symmetric multicategory $\widehat{\mathbf{dg}}$ over $\mathcal{S}\mathrm{et}$. Its objects are $\mathbb{Z}$-graded $\mathscr{U}$-small $\Bbbk$-modules X equipped with a differential, a family of $\Bbbk$-linear maps $(d : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ such that $d^2 = 0$. Morphisms $f \in \widehat{\mathbf{dg}}((X_i)_{i\in I}; Y)$ are chain multimaps, that is, elements $f \in \widehat{\mathbf{gr}}((X_i)_{i\in I}; Y)$ such that

$$
\begin{aligned}
&\Bigl[\prod_{i\in I} X_i^{n_i} \xrightarrow{f^{(n_i)_{i\in I}}} Y^{\sum_{i\in I} n_i} \xrightarrow{d} Y^{1+\sum_{i\in I} n_i}\Bigr] \\
&\qquad = \sum_{q\in I} (-1)^{\sum_{i>q} n_i} \Bigl[\prod_{i\in I} X_i^{n_i} \xrightarrow{\prod_{i\in I}[(1)_{i<q}, d, (1)_{i>q}]} \prod_{i\in I} X_i^{n_i+\delta_{iq}} \xrightarrow{f^{(n_i+\delta_{iq})_{i\in I}}} Y^{1+\sum_{i\in I} n_i}\Bigr]
\end{aligned}
$$

for all $(n_i)_{i\in I} \in \mathbb{Z}^I$. Here $\delta_{iq} = 1$ if $i = q$ and $\delta_{iq} = 0$ otherwise. Composition $\mu_{\phi:I\to J}$ in $\widehat{\mathbf{gr}}$ of chain multimaps is again a chain multimap. Indeed, this follows from the identity

$$
\begin{aligned}
&\sum_{i<p,\ \phi i>\phi p}^{i,p\in I} n_i n_p + \sum_{\phi i>\phi q}^{i\in I} n_i + \sum_{i>q,\ \phi i=\phi q}^{i\in I} n_i + 2\sum_{i>q,\ \phi i<\phi q}^{i\in I} n_i \\
&\qquad = \sum_{i<p,\ \phi i>\phi p}^{i,p\in I} (n_i + \delta_{iq})(n_p + \delta_{pq}) + \sum_{i>q}^{i\in I} n_i
\end{aligned}
$$

which holds for all $q \in I$. Therefore, there is a faithful multifunctor $\widehat{\mathbf{dg}} \to \widehat{\mathbf{gr}}$ which forgets the differential.

3.19 Example. Denote by $\mathbb{1}$ the symmetric $\mathcal{V}$-multicategory with one object $*$; the multimorphism object $\mathbb{1}((*)_{i\in I}; *) = \mathbb{1}_{\mathcal{V}}$ is the unit object of $\mathcal{V}$; the identity morphism is $1_* = \mathrm{id} : \mathbb{1}_{\mathcal{V}} \to \mathbb{1}_{\mathcal{V}} = \mathbb{1}(*; *)$, multiplication is $\mu_{\phi:I\to J} = (\lambda_{\mathcal{V}}^{\varnothing\to J\sqcup\mathbf{1}})^{-1} : \otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}} \mathbb{1}_{\mathcal{V}} \to \mathbb{1}_{\mathcal{V}}$.

We have to verify equation at Fig. 3.1, which is the exterior of the diagram with invertible arrows

$$
\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}\bigl[(\mathbb{1}_{\mathcal{V}})_{j\in J}, \otimes^{K\sqcup\mathbf{1}}\bigl((\mathbb{1}_{\mathcal{V}})_{k\in K}, \mathbb{1}_{\mathcal{V}}\bigr)\bigr] \arrow[dr, "\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},(\lambda_{\mathcal{V}}^{\varnothing\to K\sqcup\mathbf{1}})^{-1})"] & \\
\otimes^{J\sqcup K\sqcup\mathbf{1}}\bigl[(\mathbb{1}_{\mathcal{V}})_{j\in J}, (\mathbb{1}_{\mathcal{V}})_{k\in K}, \mathbb{1}_{\mathcal{V}}\bigr] \arrow[u, "\lambda_{\mathcal{V}}^{\alpha}"] \arrow[d, "\lambda_{\mathcal{V}}^{\beta}"'] \arrow[dr, "(\lambda_{\mathcal{V}}^{\varnothing\to J\sqcup K\sqcup\mathbf{1}})^{-1}"] & \otimes^{J\sqcup\mathbf{1}}\bigl[(\mathbb{1}_{\mathcal{V}})_{j\in J}, \mathbb{1}_{\mathcal{V}}\bigr] \arrow[d, "(\lambda_{\mathcal{V}}^{\varnothing\to J\sqcup\mathbf{1}})^{-1}"] \\
\otimes^{K\sqcup\mathbf{1}}\bigl[\bigl(\otimes^{\psi^{-1}k\sqcup\mathbf{1}}[(\mathbb{1}_{\mathcal{V}})_{j\in\psi^{-1}k}, \mathbb{1}_{\mathcal{V}}]\bigr)_{k\in K}, \mathbb{1}_{\mathcal{V}}\bigr] \arrow[dr, "\otimes^{K\sqcup\mathbf{1}}(((\lambda_{\mathcal{V}}^{\varnothing\to\psi^{-1}k\sqcup\mathbf{1}})^{-1})_{k\in K},1)"'] & \mathbb{1}_{\mathcal{V}} \\
& \otimes^{K\sqcup\mathbf{1}}\bigl[(\mathbb{1}_{\mathcal{V}})_{k\in K}, \mathbb{1}_{\mathcal{V}}\bigr] \arrow[u, "(\lambda_{\mathcal{V}}^{\varnothing\to K\sqcup\mathbf{1}})^{-1}"']
\end{tikzcd}
$$

where

$$\alpha = \mathrm{id}_J \sqcup \triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1},$$

and

$$\beta = \mathrm{v}^\pi : \mathrm{v}(t) = J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1} = \mathrm{v}(t^\pi)$$

is induced by the map $\pi = \mathsf{VI} : \mathbf{3} \to \mathbf{2}$, $[\pi] = \mathsf{I}\,.\,\mathsf{II} : [2] \to [3]$. Inverting the arrows which contain $\lambda_{\mathcal{V}}^{-1}$, we find two squares. They commute due to equation (2.5.4), written for pairs of maps $\varnothing \to J \sqcup K \sqcup \mathbf{1} \xrightarrow{\alpha} J \sqcup \mathbf{1}$ and $\varnothing \to J \sqcup K \sqcup \mathbf{1} \xrightarrow{\beta} K \sqcup \mathbf{1}$. Therefore, the exterior of the above diagram commutes.

3.20 Definition (Algebras in multicategories)**.** A (*commutative*) *algebra* A in a (symmetric, braided) $\mathcal{V}$-multicategory C is a (symmetric, braded) $\mathcal{V}$-multifunctor $A : \mathbb{1} \to \mathsf{C}$. A *morphism* $g : A \to B$ of such algebras is a multinatural $\mathcal{V}$-transformation $g : A \to B : \mathbb{1} \to \mathsf{C}$.

Equivalently, an algebra (resp. commutative algebra) A in C is

i) an object A of C,

ii) a multimorphism $\mu_A^I \in \mathsf{C}((A)_{i\in I}; A)$, that is,$\mu_A^I : \mathbb{1}_{\mathcal{V}} \to \mathsf{C}((A)_{i\in I}; A)$, for each set $I \in \mathrm{Ob}\,\mathcal{S}$, such that $\mu_A^I = 1_{A,I}^{\mathsf{C}}$ for each 1-element set I, and for every map $\phi : I \to J$ of $\mathcal{O}$ (resp. $\mathcal{S}$) the following equation holds:

$$\mu_A^I = \Big[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{\varnothing \to J \sqcup \mathbf{1}}} \otimes^{J\sqcup\mathbf{1}} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[(\mu_A^{\phi^{-1}j})_{j\in J}, \mu_A^J]} \otimes^{J\sqcup\mathbf{1}} \Big[\big(\mathsf{C}((A)_{i\in\phi^{-1}j}; A)\big)_{j\in J}, \mathsf{C}((A)_{j\in J}; A)\Big] \xrightarrow{\mu_\phi^{\mathsf{C}}} \mathsf{C}((A)_{i\in I}; A)\Big]. \tag{3.20.1}$$

A morphism $g : A \to B$ of such algebras consists of morphisms $g_L : (A)_L \to B$ in C, that is, $g_L : \mathbb{1}_{\mathcal{V}} \to \mathsf{C}((A)_L; B)$, for each 1-element set L, such that for each set $I \in \mathrm{Ob}\,\mathcal{O}$ and 1-element set J the following equation holds:

$$\begin{aligned}
&\Big[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{J\hookrightarrow J\sqcup\mathbf{1}}} \otimes^{J\sqcup\mathbf{1}} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{J\sqcup\mathbf{1}}(\mu_A^I, g_J)} \otimes^{J\sqcup\mathbf{1}} \big(\mathsf{C}((A)_{i\in I}; A), \mathsf{C}((A)_J; B)\big) \xrightarrow{\mu_{I\to J}^{\mathsf{C}}} \mathsf{C}((A)_{i\in I}; B)\Big] \\
&= \Big[\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda_{\mathcal{V}}^{J\to I\sqcup\mathbf{1}}} \otimes^{I\sqcup\mathbf{1}} \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes^{I\sqcup\mathbf{1}}[(g_{\{i\}})_{i\in I}, \mu_B^I]} \otimes^{I\sqcup\mathbf{1}} \Big[\big(\mathsf{C}((A)_{\{i\}}; B)\big)_{i\in I}, \mathsf{C}((B)_{i\in I}; B)\Big] \xrightarrow{\mu_{\mathrm{id}_I}^{\mathsf{C}}} \mathsf{C}((A)_{i\in I}; B)\Big].
\end{aligned} \tag{3.20.2}$$

By formula (3.15.2), each of the morphisms $g_L : \mathbb{1}_{\mathcal{V}} \to \mathsf{C}((A)_L; B)$ determines all the other morphisms unambiguously.

3.21 From Monoidal categories to multicategories and back. In the remaining part of the chapter we consider only lax Monoidal categories in which $\otimes^L = \mathrm{Id}$ and $\rho^L : \otimes^L \to \mathrm{Id}$ is the identity morphism, for each 1-element set L.

Versions of the following statements have appeared without a proof in many articles, e.g. [Lam69], [DS03].

3.22 Proposition. *A plain (resp. symmetric, braided) lax Monoidal $\mathcal{V}$-category $\mathcal{C}$ gives rise to a plain (resp. symmetric, braided) $\mathcal{V}$-multicategory $\widehat{\mathcal{C}}$ with*

- *the set of objects* $\operatorname{Ob}\widehat{\mathcal{C}} = \operatorname{Ob}\mathcal{C}$,
- *the objects of morphisms* $\widehat{\mathcal{C}}((X_i)_{i\in I}; Y) = \mathcal{C}(\otimes^I(X_i), Y)$,
- *the unit* $\eta_X^{\widehat{\mathcal{C}}} = \eta_X^{\mathcal{C}} : \mathbb{1}_{\mathcal{V}} \to \mathcal{C}(X, X)$,
- *the multiplication morphism for each map* $f : I \to J$

$$\mu_f : \otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}\bigl[\bigl(\widehat{\mathcal{C}}((X_i)_{i\in f^{-1}j}; Y_j)\bigr)_{j\in J}, \widehat{\mathcal{C}}((Y_j)_{j\in J}; Z)\bigr] \xrightarrow{\lambda^{\gamma:J\sqcup\mathbf{1}\to\mathbf{2}}} \bigl(\otimes_{\mathcal{V}}^{j\in J}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\bigr) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, Z) \xrightarrow{\otimes^J\otimes 1}$$
$$\mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{j\in J}Y_j) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, Z) \xrightarrow{\lambda^{f\cdot\_\cdot\_}} \mathcal{C}(\otimes^{i\in I}X_i, Z) = \widehat{\mathcal{C}}((X_i)_{i\in I}; Z),$$

where $\gamma\big|_J(j) = 1$, $\gamma\big|_{\mathbf{1}}(1) = 2$.

Proof. Substituting the definitions of unit and multiplication in $\widehat{\mathcal{C}}$ into unitality equations we get:

$$\bigl[\mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\lambda_{\mathcal{V}}^{\mathsf{I}\cdot}} \mathcal{C}(\otimes^{i\in I}X_i, Y) \otimes \mathbb{1} \xrightarrow{1\otimes\eta} \mathcal{C}(\otimes^{i\in I}X_i, Y) \otimes \mathcal{C}(Y, Y) \xrightarrow{\text{comp}} \mathcal{C}(\otimes^{i\in I}X_i, Y)\bigr] = \mathrm{id},$$

$$\bigl[\mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\lambda_{\mathcal{V}}^{\cdot\cdot\cdot\mathsf{I}}} \mathbb{1}^{\otimes I} \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\eta^{\otimes I}\otimes 1} \otimes^{i\in I}\mathcal{C}(X_i, X_i) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y)$$
$$\xrightarrow{\otimes^I\otimes 1} \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\text{composition}} \mathcal{C}(\otimes^{i\in I}X_i, Y)\bigr]$$
$$= \bigl[\mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\lambda_{\mathcal{V}}^{\cdot\mathsf{I}}} \mathbb{1} \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\eta\otimes 1} \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes \mathcal{C}(\otimes^{i\in I}X_i, Y) \xrightarrow{\text{comp}} \mathcal{C}(\otimes^{i\in I}X_i, Y)\bigr] = \mathrm{id},$$

as required.

In order to prove the associativity equation for a composable pair $I \xrightarrow{f} J \xrightarrow{g} K$ we consider objects X_i, Y_j, Z_k, W of $\mathcal{C}$ and substitute the definition of multiplication in $\widehat{\mathcal{C}}$. The last factor $\mathcal{C}(\otimes^{k\in K}Z_k, W)$ splits out and the equation takes the form of exterior of left diagram on the facing page. Here f_k denotes the map $f|_{(fg)^{-1}(k)} : (fg)^{-1}(k) \to$

$$
\begin{tikzcd}
{\begin{matrix}\otimes^{j\in J}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\\ \otimes\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)\end{matrix}} \arrow[rr, "\otimes^J\otimes\otimes^K"] \arrow[d, "\wr"'] \arrow[drr, phantom, "\boxed{3}"] & & {\begin{matrix}\mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{j\in J}Y_j)\\ \otimes\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j, \otimes^{k\in K}Z_k)\end{matrix}} \arrow[d, "(-\cdot\lambda^g)\otimes 1"] \\
{\begin{matrix}\otimes^{k\in K}[\otimes^{j\in g^{-1}k}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\\ \otimes\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)]\end{matrix}} \arrow[d, "\otimes^{k\in K}[\otimes^{g^{-1}k}\otimes 1]"'] & \otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, Z_k) \arrow[d, "\otimes^K"] & {\begin{matrix}\mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j)\\ \otimes\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j, \otimes^{k\in K}Z_k)\end{matrix}} \arrow[d, "\text{composition}"] \\
{\begin{matrix}\otimes^{k\in K}[\mathcal{C}(\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, \otimes^{j\in g^{-1}k}Y_j)\\ \otimes\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)]\end{matrix}} \arrow[ur, "\otimes^{k\in K}\text{comp}"] \arrow[d, "\otimes^{k\in K}[\lambda^{f_k}\cdot-\cdot-]"'] \arrow[dr, phantom, "\boxed{1}"] & \mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}Z_k) \arrow[r, "\lambda^g\cdot-"] \arrow[d, "(\otimes^{k\in K}\lambda^{f_k})\cdot-"] \arrow[dr, phantom, "\boxed{2}"] & \mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}Z_k) \arrow[d, "\lambda^f\cdot-"] \\
\otimes^{k\in K}\mathcal{C}(\otimes^{i\in f^{-1}g^{-1}k}X_i, Z_k) \arrow[r, "\otimes^K"] & \mathcal{C}(\otimes^{k\in K}\otimes^{i\in f^{-1}g^{-1}k}X_i, \otimes^{k\in K}Z_k) \arrow[r, "\lambda^{fg}\cdot-"] & \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{k\in K}Z_k)
\end{tikzcd}
$$

$$
\begin{tikzcd}
{\begin{matrix}\otimes^{j\in J}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\\ \otimes\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)\end{matrix}} \arrow[rr, "\otimes^J\otimes\otimes^K"] \arrow[d, "\wr"'] \arrow[dr, "\sim"] & & {\begin{matrix}\mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{j\in J}Y_j)\\ \otimes\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j, \otimes^{k\in K}Z_k)\end{matrix}} \arrow[dd, "(-\cdot\lambda^g)\otimes 1"] \\
{\begin{matrix}\otimes^{k\in K}[\otimes^{j\in g^{-1}k}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\\ \otimes\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)]\end{matrix}} \arrow[r, "\sim"] \arrow[d, "\otimes^{k\in K}[\otimes^{g^{-1}k}\otimes 1]"'] \arrow[dr, phantom, "\boxed{5}"] & {\begin{matrix}\otimes^{k\in K}\otimes^{j\in g^{-1}k}\mathcal{C}(\otimes^{i\in f^{-1}j}X_i, Y_j)\\ \otimes\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)\end{matrix}} \arrow[d, "\otimes^g_{\mathcal{C}}\otimes 1"] \arrow[dr, phantom, "\boxed{7}"] & \\
{\begin{matrix}\otimes^{k\in K}[\mathcal{C}(\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i,\\ \otimes^{j\in g^{-1}k}Y_j)\otimes\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)]\end{matrix}} \arrow[r, "\sim"] \arrow[dd, "\otimes^{k\in K}\text{composition}"'] & {\begin{matrix}\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, \otimes^{j\in g^{-1}k}Y_j)\\ \otimes\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}Y_j, Z_k)\end{matrix}} \arrow[d, "\otimes^K\otimes\otimes^K"] & {\begin{matrix}\mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j)\\ \otimes\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j, \otimes^{k\in K}Z_k)\end{matrix}} \arrow[dd, "\text{composition}"] \\
 & {\begin{matrix}\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j)\\ \otimes\mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}Y_j, \otimes^{k\in K}Z_k)\end{matrix}} \arrow[ur, "(\lambda^g\cdot-)\otimes 1"] \arrow[d, "\text{composition}"'] \arrow[dl, phantom, "\boxed{4}"] \arrow[dr, phantom, "\boxed{6}"] & \\
\otimes^{k\in K}\mathcal{C}(\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, Z_k) \arrow[r, "\otimes^K"] & \mathcal{C}(\otimes^{k\in K}\otimes^{j\in g^{-1}k}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}Z_k) \arrow[r, "\lambda^g\cdot-"] & \mathcal{C}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i, \otimes^{k\in K}Z_k)
\end{tikzcd}
$$

$g^{-1}(k)$. Pentagon 1 commutes due to $\otimes^K$ being a functor. Square 2 commutes due to equation (2.10.3). Polygon 3 in this diagram can be rewritten as the right diagram on the previous page. Here pentagon 4 commutes due to $\otimes^K$ being a functor. Square 5 is the definition of $\otimes^g_{\mathcal{C}}$. Quadrilateral 6 follows from associativity of composition in $\mathcal{C}$. Polygon 7 expresses naturality of transformation λ^g. Therefore, associativity of multiplication in $\widehat{\mathcal{C}}$ is proven. □

3.23 Definition. A plain (resp. symmetric, braided) $\mathcal{V}$-multicategory C is *lax representable* if the $\mathcal{V}$-functors $\mathsf{C}((X_i)_{i\in I};-):\mathsf{C}\to\mathcal{V}$ are representable for all families $(X_i)_{i\in I}$ of objects of C, that is, there exists an object X of C and an element $\tau:\mathbb{1}_{\mathcal{V}}\to\mathsf{C}((X_i)_{i\in I};X)$ such that the composition

$$\rho=\bigl[\mathsf{C}(X;Y)\xrightarrow{\lambda_{\mathcal{V}}^{\cdot\,\mathsf{I}}}\mathbb{1}_{\mathcal{V}}\otimes\mathsf{C}(X;Y)\xrightarrow{\tau\otimes1}\mathsf{C}((X_i)_{i\in I};X)\otimes\mathsf{C}(X;Y)\xrightarrow{\mu_{I\to\mathbf{1}}}\mathsf{C}((X_i)_{i\in I};Y)\bigr]$$

is an isomorphism.

We may, and always shall, assume that for any 1-element set I the chosen X coincides with X_i for the only $i\in I$, and $\tau=\eta:\mathbb{1}_{\mathcal{V}}\to\mathsf{C}(X;X)$ is chosen to be the unit. The corresponding map ρ is the identity map $\mathrm{id}_{\mathsf{C}(X;Y)}$.

Strong notion of representability is given by Hermida [Her00, Definition 8.3]. The above condition of lax representability is taken from definition 8.1(1) of [Her00]. Day and Street make in [DS03] remarks similar to the following statement.

3.24 Theorem. *A plain (resp. symmetric, braided) $\mathcal{V}$-multicategory C is lax representable if and only if it is isomorphic to $\widehat{\mathcal{C}}$ for some lax plain (resp. symmetric, braided) Monoidal $\mathcal{V}$-category $\mathcal{C}$.*

Proof. Let C be a plain (resp. symmetric, braided) $\mathcal{V}$-multicategory. We claim that category $\mathcal{C}$ with $\mathrm{Ob}\,\mathcal{C}=\mathrm{Ob}\,\mathsf{C}$, $\mathcal{C}(X,Y)=\mathsf{C}(X;Y)$ has the following lax plain (resp. symmetric, braided) Monoidal structure. The quiver map $\otimes^I_{\mathcal{C}}:\mathcal{C}^I\to\mathcal{C}$ which takes a family $(X_i)_{i\in I}$ to an object $X=\otimes^{i\in I}X_i$, representing the functor $\mathsf{C}((X_i)_{i\in I};-)$, and which is given on morphisms by

$$\begin{aligned}\otimes^I_{\mathcal{C}}=\bigl[\otimes^{i\in I}_{\mathcal{V}}\mathsf{C}(X_i;Y_i)&\xrightarrow{\lambda_{\mathcal{V}}^{I\hookrightarrow I\sqcup\mathbf{1}}\cdot\otimes^{I\sqcup\mathbf{1}}((1)_{i\in I},\tau)}\\&\otimes^{I\sqcup\mathbf{1}}_{\mathcal{V}}\bigl[\bigl(\mathsf{C}(X_i;Y_i)\bigr)_{i\in I},\mathsf{C}((Y_i)_{i\in I};\otimes^{i\in I}Y_i)\bigr]\\&\xrightarrow{\mu_{\mathrm{id}_I}}\mathsf{C}((X_i)_{i\in I};\otimes^{i\in I}Y_i)\xrightarrow[\sim]{\rho^{-1}}\mathsf{C}(\otimes^{i\in I}X_i;\otimes^{i\in I}Y_i)\bigr],\end{aligned}$$

is claimed to be a $\mathcal{V}$-functor. The collection

$$\begin{aligned}\lambda^f_{\mathcal{C}}=\bigl[\mathbb{1}_{\mathcal{V}}&\xrightarrow{\lambda^{\varnothing\to J\sqcup\mathbf{1}}}\otimes^{I\sqcup\mathbf{1}}_{\mathcal{V}}(\mathbb{1}_{\mathcal{V}})\xrightarrow{\otimes^{J\sqcup\mathbf{1}}\tau}\\&\otimes^{I\sqcup\mathbf{1}}_{\mathcal{V}}\bigl[\bigl(\mathsf{C}((X_i)_{i\in f^{-1}j};\otimes^{i\in f^{-1}j}X_i)\bigr)_{j\in J},\mathsf{C}((\otimes^{i\in f^{-1}j}X_i)_{j\in J};\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\bigr]\\&\xrightarrow{\mu_f}\mathsf{C}((X_i)_{i\in I};\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\xrightarrow[\sim]{\rho^{-1}}\mathsf{C}(\otimes^{i\in I}X_i;\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\bigr]\end{aligned}$$

is claimed to be a natural transformation $\lambda^f_{\mathcal{C}} : \otimes^I_{\mathcal{C}} \to \otimes^J_{\mathcal{C}} \otimes^f_{\mathcal{C}}$. The triple $(\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}})$ is claimed to be a lax (symmetric, braided) Monoidal category.

To prove all this, we consider an isomorphism $\alpha : \mathsf{D} \to \mathsf{C}$ of $\mathcal{V}$-multiquivers. Then there is a unique multicategory structure on D such that α is a multifunctor. If C is lax representable, then D is lax representable as well. We shall illustrate this statement in the particular case, when $\operatorname{Ob}\mathsf{D} = \operatorname{Ob}\mathsf{C}$, $\operatorname{Ob}\alpha = \mathrm{id}$. The element τ_{D} is found as

$$\tau_{\mathsf{D}} = \bigl[\mathbb{1}_{\mathcal{V}} \xrightarrow{\tau_{\mathsf{C}}} \mathsf{C}((X_i)_{i\in I}; \otimes^{i\in I} X_i) \xrightarrow[\sim]{\alpha^{-1}} \mathsf{D}((X_i)_{i\in I}; \otimes^{i\in I} X_i)\bigr].$$

All claims of the above paragraph are equivalent to similar claims for D, where $(\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$ is constructed from D exactly by the same formulae as given for C. Therefore, we may choose D and α at our convenience and to proceed to work with D.

We set $\mathsf{D}((X_i)_{i\in I}; Y) = \mathsf{C}(\otimes^{i\in I} X_i; Y)$, $\alpha = \rho_{\mathsf{C}} : \mathsf{D}((X_i)_{i\in I}; Y) = \mathsf{C}(\otimes^{i\in I} X_i; Y) \to \mathsf{C}((X_i)_{i\in I}; Y)$. In particular, for 1-element set I we have $\alpha = \mathrm{id} : \mathsf{D}(X; Y) \to \mathsf{C}(X; Y)$. Thus, the category $\mathcal{D}$, constructed from D, coincides with $\mathcal{C}$. From commutative square

$$\begin{tikzcd}
\mathsf{D}(\otimes^{i\in I} X_i; Y) \arrow[r, "\rho_{\mathsf{D}}"] \arrow[d, "\alpha"', "=\mathrm{id}"] & \mathsf{D}((X_i)_{i\in I}; Y) \arrow[d, "\alpha"', "=\rho_{\mathsf{C}}"] \\
\mathsf{C}(\otimes^{i\in I} X_i; Y) \arrow[r, "\rho_{\mathsf{C}}"] & \mathsf{C}((X_i)_{i\in I}; Y)
\end{tikzcd}$$

we find that $\rho_{\mathsf{D}} = \mathrm{id} : \mathsf{D}(\otimes^{i\in I} X_i; Y) \to \mathsf{D}((X_i)_{i\in I}; Y)$. Therefore, for each family $(X_i)_{i\in I}$ there exists an element $\tau_{\mathsf{D}} \in \mathsf{D}((X_i)_{i\in I}; \otimes^{i\in I} X_i)$ such that

$$\rho_{\mathsf{D}} = \bigl[\mathsf{D}(\otimes^{i\in I} X_i; Y) \xrightarrow{\lambda_{\mathcal{V}}^{\cdot\,\mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \mathsf{D}(\otimes^{i\in I} X_i; Y) \xrightarrow{\tau_{\mathsf{D}}\otimes 1} \mathsf{D}((X_i)_{i\in I}; \otimes^{i\in I} X_i) \otimes \mathsf{D}(\otimes^{i\in I} X_i; Y) \xrightarrow{\mu^{\mathsf{D}}_{I\to\mathbf{1}}} \mathsf{D}((X_i)_{i\in I}; Y)\bigr] = \mathrm{id} \quad (3.24.1)$$

for all objects Y. Considering $Y = \otimes^{i\in I} X_i$ and composing the above identity map with $\eta^{\mathsf{D}}_Y : \mathbb{1}_{\mathcal{V}} \to \mathsf{D}(\otimes^{i\in I} X_i; Y)$, we get by axiom (3.7.1) that $\tau_{\mathsf{D}} = \eta^{\mathsf{D}}_{\otimes^I X_i} \in \mathsf{D}(\otimes^{i\in I} X_i; \otimes^{i\in I} X_i)$. Thus, τ_{D} is the unit element.

The following diagram

$$\begin{tikzcd}
\mathsf{D}(\otimes^{i\in I} X_i; Y) \otimes \mathsf{D}(Y; Z) \arrow[d, "\lambda_{\mathcal{V}}^{\cdot\,\mathsf{II}}\cdot(\tau_{\mathsf{D}}\otimes1\otimes1)"'] & \\
\mathsf{D}((X_i)_{i\in I}; \otimes^{i\in I} X_i) \otimes \mathsf{D}(\otimes^{i\in I} X_i; Y) \otimes \mathsf{D}(Y; Z) \arrow[r, "\mu_{I\to\mathbf{1}}\otimes1"] \arrow[d, "1\otimes\mu_{\mathbf{1}\to\mathbf{1}}"'] & \mathsf{D}((X_i)_{i\in I}; Y) \otimes \mathsf{D}(Y; Z) \arrow[d, "\mu_{I\to\mathbf{1}}"] \\
\mathsf{D}((X_i)_{i\in I}; \otimes^{i\in I} X_i) \otimes \mathsf{D}(\otimes^{i\in I} X_i; Z) \arrow[r, "\mu_{I\to\mathbf{1}}"] & \mathsf{D}((X_i)_{i\in I}; Z)
\end{tikzcd}$$

commutes because the square in it is nothing else but particular case of the commutative diagram at Fig. 3.1, written for maps $I \to \mathbf{1} \to \mathbf{1}$. Using equation (3.24.1) we deduce that the multiplications in D

$$\begin{tikzcd}
\mathsf{D}(\otimes^{i\in I}X_i;Y)\otimes\mathsf{D}(Y;Z) \arrow[r, "\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}}", "\mu^{\mathcal{D}}"'] \arrow[d, equal] & \mathsf{D}(\otimes^{i\in I}X_i;Z) \arrow[d, equal] \\
\mathsf{D}((X_i)_{i\in I};Y)\otimes\mathsf{D}(Y;Z) \arrow[r, "\mu_{I\to\mathbf{1}}"] & \mathsf{D}((X_i)_{i\in I};Z)
\end{tikzcd}$$

coincide.

(a) Let us prove all the above claims for such D. In the following computations many isomorphisms $\lambda_{\mathcal{V}}$ and their inverses are neglected and dropped. The expressions can be written in standard form without parenthesized tensor signs, however, this would not improve the readability. Let us prove that

$$\otimes^I_{\mathcal{D}} = \bigl[\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i) \xrightarrow{\lambda^{\mathsf{I}\cdot}_{\mathcal{V}}\cdot(1\otimes\eta)} [\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Y_i) \xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_I}} \mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i)\bigr]$$

is a $\mathcal{V}$-functor, that is, diagram

$$\begin{tikzcd}
{[\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(Y_i,Z_i)} \arrow[r, "\otimes^I_{\mathcal{D}}\otimes\otimes^I_{\mathcal{D}}"] \arrow[d, "\sigma_{(12)}"'] & \mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i)\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Z_i) \arrow[dd, "\mu^{\mathcal{D}}"] \\
\otimes^{i\in I}_{\mathcal{V}}[\mathcal{D}(X_i,Y_i)\otimes\mathcal{D}(Y_i,Z_i)] \arrow[d, "\otimes^{i\in I}_{\mathcal{V}}\mu^{\mathcal{D}}"'] & \\
\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Z_i) \arrow[r, "\otimes^I_{\mathcal{D}}"] & \mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Z_i)
\end{tikzcd} \tag{3.24.2}$$

commutes for all families $(X_i)_{i\in I}$, $(Y_i)_{i\in I}$, $(Z_i)_{i\in I}$ of objects of D.

The top–right path equals

$$\begin{aligned}
&\bigl[[\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(Y_i,Z_i) \\
&\qquad\xrightarrow{\lambda^{\mathsf{II}\cdot}_{\mathcal{V}}\cdot(1\otimes1\otimes\eta)} [\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes[\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(Y_i,Z_i)]\otimes\mathcal{D}(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i) \\
&\qquad\xrightarrow{\lambda^{\mathsf{I}\cdot}_{\mathcal{V}}\otimes\mu^{\mathsf{D}}_{\mathrm{id}_I}} [\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\mathbb{1}\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Z_i) \\
&\qquad\xrightarrow{1\otimes\eta\otimes1} [\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Y_i)\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Z_i) \\
&\qquad\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_I}\otimes1} \mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i)\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Z_i) \xrightarrow{\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}}} \mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Z_i)\bigr] \\
&= \bigl[[\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(X_i,Y_i)]\otimes\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(Y_i,Z_i)
\end{aligned}$$

$$\begin{aligned}
&\xrightarrow{\lambda_{\mathcal V}^{\mathsf{I}\mathsf{I}\bullet}\cdot(1\otimes1\otimes\eta)} [\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)]\otimes[\otimes_{\mathcal V}^{i\in I}\mathcal D(Y_i,Z_i)]\otimes\mathcal D(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i)\\
&\xrightarrow{1\otimes\mu^{\mathsf D}_{\mathrm{id}_I}} [\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)]\otimes\mathcal D(\otimes^{i\in I}Y_i,\otimes^{i\in I}Z_i)\xrightarrow{\mu^{\mathsf D}_{\mathrm{id}_I}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{i\in I}Z_i)\big].
\end{aligned}$$

The left–bottom path of (3.24.2) equals

$$\begin{aligned}
&\big[[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)]\otimes\otimes_{\mathcal V}^{i\in I}\mathcal D(Y_i,Z_i)\xrightarrow{\sigma_{(12)}}\otimes_{\mathcal V}^{i\in I}[\mathcal D(X_i,Y_i)\otimes\mathcal D(Y_i,Z_i)]\\
&\quad\xrightarrow{\otimes_{\mathcal V}^{I}\mu^{\mathcal D}}\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Z_i)\xrightarrow{\lambda_{\mathcal V}^{\mathsf{I}\bullet}\cdot(1\otimes\eta)}[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Z_i)]\otimes\mathcal D(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i)\\
&\quad\xrightarrow{\mu^{\mathsf D}_{\mathrm{id}_I}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{i\in I}Z_i)\big]\\
&=\big[[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)]\otimes\otimes_{\mathcal V}^{i\in I}\mathcal D(Y_i,Z_i)\\
&\quad\xrightarrow{\lambda_{\mathcal V}^{\mathsf{I}\mathsf{I}\bullet}\cdot(1\otimes1\otimes\eta)}[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)]\otimes[\otimes_{\mathcal V}^{i\in I}\mathcal D(Y_i,Z_i)]\otimes\mathcal D(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i)\\
&\quad\xrightarrow{\sigma_{(12)}\otimes1}\otimes_{\mathcal V}^{i\in I}[\mathcal D(X_i,Y_i)\otimes\mathcal D(Y_i,Z_i)]\otimes\mathcal D(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i)\\
&\quad\xrightarrow{(\otimes_{\mathcal V}^{I}\mu^{\mathsf D}_{\mathbf 1\to\mathbf 1})\otimes1}[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Z_i)]\otimes\mathcal D(\otimes^{i\in I}Z_i,\otimes^{i\in I}Z_i)\xrightarrow{\mu^{\mathsf D}_{\mathrm{id}_I}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{i\in I}Z_i)\big].
\end{aligned}$$

The both paths are equal to each other because they coincide with the corresponding paths of commutative diagram at Fig. 3.1, written for maps $I\xrightarrow{\mathrm{id}}I\xrightarrow{\mathrm{id}}I$ and object $W=\otimes^{i\in I}Z_i$, multiplied with the first factor $\lambda_{\mathcal V}^{\mathsf{I}\mathsf{I}\bullet}\cdot(1\otimes1\otimes\eta)$. Therefore, $\otimes_{\mathcal D}^{I}$ is a $\mathcal V$-functor.

(b) Let us prove that

$$\begin{aligned}
\lambda_{\mathcal D}^{f}=\big[\mathbb 1&\xrightarrow{\lambda^{\varnothing\to J\sqcup\mathbf 1}\cdot\otimes^{J\sqcup\mathbf 1}\eta}\\
&\otimes_{\mathcal V}^{j\in J}\mathcal D(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}X_i)\otimes\mathcal D(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\\
&\xrightarrow{\mu^{\mathsf D}_{f}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\big]
\end{aligned}$$

is a natural transformation $\lambda_{\mathcal D}^{f}:\otimes_{\mathcal D}^{I}\to\otimes_{\mathcal D}^{J}\otimes^{f}$. This is expressed by equation

$$\begin{aligned}
&\big[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)\xrightarrow{\otimes_{\mathcal D}^{I}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i)\\
&\qquad\xrightarrow{\lambda_{\mathcal V}^{\mathsf I\bullet}\cdot(1\otimes\lambda_{\mathcal D}^{f})\cdot\mu^{\mathcal D}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big]\\
&=\big[\otimes_{\mathcal V}^{i\in I}\mathcal D(X_i,Y_i)\xrightarrow{\lambda_{\mathcal V}^{f}}\otimes_{\mathcal V}^{j\in J}\otimes_{\mathcal V}^{i\in f^{-1}j}\mathcal D(X_i,Y_i)\xrightarrow{\otimes_{\mathcal V}^{j\in J}\otimes_{\mathcal D}^{f^{-1}j}}\\
&\qquad\otimes_{\mathcal V}^{j\in J}\mathcal D(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}Y_i)\xrightarrow{\otimes_{\mathcal D}^{J}}\mathcal D(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\\
&\qquad\xrightarrow{\lambda_{\mathcal V}^{\bullet\mathsf I}\cdot(\lambda_{\mathcal D}^{f}\otimes1)\cdot\mu^{\mathcal D}}\mathcal D(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big].
\end{aligned}$$

The left hand side is

$$
\begin{aligned}
&\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i) \xrightarrow{\lambda^{\mathsf{I}\,\bullet\bullet\bullet\,:\mathbf{1}\hookrightarrow\mathbf{2}\sqcup J\sqcup\mathbf{1}}\cdot[1\otimes\eta\otimes(\otimes^J\eta)\otimes\eta]} \\
&[\otimes_{\mathcal{V}}^{I}\mathcal{D}(X_i,Y_i)]\otimes\mathcal{D}(\otimes^I Y_i,\otimes^I Y_i)\otimes[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{f^{-1}j}Y_i,\otimes^{f^{-1}j}Y_i)] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{f^{-1}j}Y_i,\otimes^{j\in J}\otimes^{f^{-1}j}Y_i) \\
&\qquad\qquad\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_I}\otimes\mu^{\mathsf{D}}_f}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i)\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\xrightarrow{\mu^{\mathcal{D}}}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big] \\
&=\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i)\xrightarrow{\lambda^{\mathsf{I}\,\bullet\bullet\,:\mathbf{1}\hookrightarrow\mathbf{1}\sqcup J\sqcup\mathbf{1}}\cdot[1\otimes(\otimes^J\eta)\otimes\eta]} \\
&[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i)]\otimes[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{i\in f^{-1}j}Y_i,\otimes^{i\in f^{-1}j}Y_i)]\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\xrightarrow{1\otimes\mu^{\mathsf{D}}_f}[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i)]\otimes\mathcal{D}(\otimes^{i\in I}Y_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_I}}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big].
\end{aligned}
\tag{3.24.3}
$$

The right hand side is

$$
\begin{aligned}
&\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i)\xrightarrow{\lambda^f_{\mathcal{V}}}\otimes_{\mathcal{V}}^{j\in J}\otimes_{\mathcal{V}}^{i\in f^{-1}j}\mathcal{D}(X_i,Y_i)\xrightarrow{\otimes_{\mathcal{V}}^{j\in J}[\lambda^{\mathsf{I}\,\bullet}\cdot(1\otimes\eta)]} \\
&\qquad\qquad\otimes_{\mathcal{V}}^{j\in J}\{[\otimes_{\mathcal{V}}^{i\in f^{-1}j}\mathcal{D}(X_i,Y_i)]\otimes\mathcal{D}(\otimes^{i\in f^{-1}j}Y_i,\otimes^{i\in f^{-1}j}Y_i)\} \\
&\qquad\qquad\xrightarrow{\otimes_{\mathcal{V}}^{j\in J}\mu^{\mathsf{D}}_{\mathrm{id}_{f^{-1}j}}}\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}Y_i)\xrightarrow{\lambda^{\mathsf{I}\,\bullet}\cdot(1\otimes\eta)} \\
&\qquad[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}Y_i)]\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\qquad\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}}\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\xrightarrow{\lambda^{\bullet\bullet\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{2}}\cdot[(\otimes^J\eta)\otimes\eta\otimes1]} \\
&[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{f^{-1}j}X_i,\otimes^{f^{-1}j}X_i)]\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{f^{-1}j}X_i,\otimes^{j\in J}\otimes^{f^{-1}j}X_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{f^{-1}j}X_i,\otimes^{j\in J}\otimes^{f^{-1}j}Y_i) \\
&\qquad\xrightarrow{\mu^{\mathsf{D}}_f\otimes1}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i)\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\xrightarrow{\mu^{\mathcal{D}}}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big] \\
&=\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{D}(X_i,Y_i)\xrightarrow{S} \\
&\qquad[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}Y_i)]\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\qquad\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}}\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\xrightarrow{\lambda^{\bullet\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}\cdot[(\otimes^J\eta)\otimes1]} \\
&\qquad[\otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes^{i\in f^{-1}j}X_i,\otimes^{i\in f^{-1}j}X_i)]\otimes\mathcal{D}(\otimes^{j\in J}\otimes^{i\in f^{-1}j}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\xrightarrow{\mu^{\mathsf{D}}_f}\mathcal{D}(\otimes^{i\in I}X_i,\otimes^{j\in J}\otimes^{i\in f^{-1}j}Y_i)\big],
\end{aligned}
$$

where S denotes the composition

$$S = \big[\otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(X_i, Y_i) \xrightarrow{\lambda^{\mathsf{I}\,\cdot\,\cdot\,:\mathbf{1}\hookrightarrow\mathbf{1}\sqcup J\sqcup\mathbf{1}}\cdot[1\otimes(\otimes^J\eta)\otimes\eta]}$$

$$[\otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(X_i, Y_i)] \otimes [\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j} Y_i, \otimes^{i\in f^{-1}j} Y_i)] \otimes \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)$$

$$\xrightarrow{[(\lambda^f_{\mathcal{V}}\otimes 1)\cdot\sigma_{(12)}]\otimes 1}$$

$$\otimes^{j\in J}_{\mathcal{V}} \{[\otimes^{i\in f^{-1}j}_{\mathcal{V}} \mathcal{D}(X_i, Y_i)] \otimes \mathcal{D}(\otimes^{i\in f^{-1}j} Y_i, \otimes^{i\in f^{-1}j} Y_i)\}$$

$$\otimes\, \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)$$

$$\xrightarrow{\otimes^{j\in J}_{\mathcal{V}} \mu^{\mathsf{D}}_{\mathrm{id}_{f^{-1}j}} \otimes 1} [\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i)] \otimes \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)\big].$$

Thus, the right hand side can be rewritten as

$$\big[\otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(X_i, Y_i) \xrightarrow{S}$$

$$[\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i)] \otimes \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)$$

$$\xrightarrow{\lambda^{\cdot\,\mathsf{II}:J\sqcup\mathbf{1}\hookrightarrow J\sqcup J\sqcup\mathbf{1}}\cdot[(\otimes^J\eta)\otimes 1\otimes 1]}$$

$$[\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{f^{-1}j} X_i, \otimes^{f^{-1}j} X_i)] \otimes [\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{f^{-1}j} X_i, \otimes^{f^{-1}j} Y_i)]$$

$$\otimes\, \mathcal{D}(\otimes^{j\in J} \otimes^{f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{f^{-1}j} Y_i)$$

$$\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}\cdot\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^{i\in I} X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)\big].$$

Applying axiom at Fig. 3.1, written for maps $I \xrightarrow{f} J \xrightarrow{\mathrm{id}} J$ and objects $Z_j = \otimes^{i\in f^{-1}j} Y_i$, $W = \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i$, we may replace this composition with

$$\big[\otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(X_i, Y_i) \xrightarrow{S}$$

$$[\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i)] \otimes \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)$$

$$\xrightarrow{\lambda^{\cdot\,\mathsf{II}:J\sqcup\mathbf{1}\hookrightarrow J\sqcup J\sqcup\mathbf{1}}\cdot[(\otimes^J\eta)\otimes 1\otimes 1]}$$

$$[\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{f^{-1}j} X_i, \otimes^{f^{-1}j} X_i)] \otimes [\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{f^{-1}j} X_i, \otimes^{f^{-1}j} Y_i)]$$

$$\otimes\, \mathcal{D}(\otimes^{j\in J} \otimes^{f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{f^{-1}j} Y_i)$$

$$\xrightarrow{(\sigma_{(12)}\otimes 1)\cdot(\otimes^{j\in J}\mu^{\mathsf{D}}_{f^{-1}j\to\{j\}}\otimes 1)\cdot\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^{i\in I} X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)\big]$$

$$= \big[\otimes^{i\in I}_{\mathcal{V}} \mathcal{D}(X_i, Y_i) \xrightarrow{S}$$

$$[\otimes^{j\in J}_{\mathcal{V}} \mathcal{D}(\otimes^{i\in f^{-1}j} X_i, \otimes^{i\in f^{-1}j} Y_i)] \otimes \mathcal{D}(\otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)$$

$$\xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^{i\in I} X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i)\big].$$

Using axiom at Fig. 3.1, written for maps $I \xrightarrow{\mathrm{id}} I \xrightarrow{f} J$ and objects $Z_j = \otimes^{i\in f^{-1}j} Y_i$, $W = \otimes^{j\in J} \otimes^{i\in f^{-1}j} Y_i$, we deduce that the above expression equals (3.24.3). Therefore, $\lambda^f_{\mathcal{D}} : \otimes^I_{\mathcal{D}} \to \otimes^J_{\mathcal{D}} \otimes^f$ is a natural transformation.

(c) To prove that $(\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$ satisfies equation (2.10.3) between natural $\mathcal{V}$-transformations, we may restrict the considerations to $\mathcal{V} = \mathcal{S}\mathrm{et}$. Indeed, the functor $\mathcal{MQ}_{\mathcal{V}} \to \mathcal{MQ}_{\mathcal{S}\mathrm{et}}$, $\mathcal{Q} \mapsto \mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{Q})$ takes $\mathcal{V}$-multicategories C, D and $\mathcal{V}$-category $\mathcal{D}$ to multicategories $\overline{\mathsf{C}}$, $\overline{\mathsf{D}}$ and category $\overline{\mathcal{D}}$, related in the same way. The $\mathcal{V}$-functor $\otimes^I_{\mathcal{D}}$ is taken to the functor $\overline{\otimes^I_{\mathcal{D}}} = \otimes^I_{\overline{\mathcal{D}}}$ and the natural $\mathcal{V}$-transformation $\lambda^f_{\mathcal{D}}$ is taken to the natural transformation $\lambda^f_{\overline{\mathcal{D}}}$. Thereby, we give the proof for $\mathcal{V} = \mathcal{S}\mathrm{et}$, in other words, in non-enriched version.

We have to show that for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ and for any family of objects $(X_i)_{i\in I}$ of $\mathcal{D}$ equation (2.5.4) holds. The morphism $\lambda^f_{\mathcal{D}} \cdot \lambda^g_{\mathcal{D}} : \otimes^I X_i \to \otimes^{(f,g,\triangleright)} X_i$ is obtained by applying the composition

$$
\begin{aligned}
&\Big[\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,\triangleright)} X_i) \\
&\qquad\qquad \times \Big[\prod_{k\in K} \mathcal{D}(\otimes^{(f,g|k)} X_i, \otimes^{(f,g|k)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \\
&\xrightarrow{\mu^{\mathsf{D}}_f \times \mu^{\mathsf{D}}_g} \mathcal{D}(\otimes^I X_i, \otimes^{(f,\triangleright)} X_i) \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)} X_i)
\end{aligned}
$$

to the family of morphisms $\eta : \otimes^{(f|j)} X_i \to \otimes^{(f|j)} X_i$, $j \in J$, $\eta : \otimes^{(f,\triangleright)} X_i \to \otimes^{(f,\triangleright)} X_i$, $\eta : \otimes^{(f,g|k)} X_i \to \otimes^{(f,g|k)} X_i$, $k \in K$, $\eta : \otimes^{(f,g,\triangleright)} X_i \to \otimes^{(f,g,\triangleright)} X_i$. Applying axiom at Fig. 3.1, written for maps $I \xrightarrow{f} J \to \mathbf{1}$ and objects $Y_j = \otimes^{(f|j)} X_i$, $Z = \otimes^{(f,\triangleright)} X_i$, $W = \otimes^{(f,g,\triangleright)} X_i$ we may replace this composition with

$$
\begin{aligned}
&\Big[\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,\triangleright)} X_i) \\
&\qquad\qquad \times \Big[\prod_{k\in K} \mathcal{D}(\otimes^{(f,g|k)} X_i, \otimes^{(f,g|k)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \\
&\qquad\qquad\qquad \xrightarrow{\prod_{j\in J} 1 \times 1 \times \mu^{\mathsf{D}}_g} \\
&\quad \Big[\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,\triangleright)} X_i) \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \\
&\xrightarrow{\prod_{j\in J} 1 \times \mu^{\mathsf{D}}_{J\to\mathbf{1}}} \Big[\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i)\Big] \times \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)} X_i).
\end{aligned}
$$

Note that by the axiom of units

$$(\eta, 1)\mu^{\mathsf{D}}_J = (\eta, 1)\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}} = \mathrm{id} : \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \to \mathcal{D}(\otimes^{(f,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i).$$

Therefore the morphism $\lambda^f_{\mathcal{D}} \cdot \lambda^g_{\mathcal{D}}$ is obtained by applying the composition

$$\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i) \times \prod_{k\in K} \mathcal{D}(\otimes^{(f,g|k)}X_i, \otimes^{(f,g|k)}X_i)$$
$$\times\, \mathcal{D}(\otimes^{(f,g,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i) \xrightarrow{\prod_{j\in J} 1\times\mu^{\mathsf{D}}_g}$$
$$\Big[\prod_{j\in J} \mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i)\Big] \times \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i) \xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)}X_i) \quad (3.24.4)$$

to the family of morphisms $\eta : \otimes^{(f|j)}X_i \to \otimes^{(f|j)}X_i$, $j \in J$, $\eta : \otimes^{(f,g|k)}X_i \to \otimes^{(f,g|k)}X_i$, $k \in K$, $\eta : \otimes^{(f,g,\triangleright)}X_i \to \otimes^{(f,g,\triangleright)}X_i$.

The morphism $\lambda^{fg}_{\mathcal{D}} \cdot \otimes^{k\in K}_{\mathcal{D}} \lambda^{f_k}_{\mathcal{D}}$, where $f_k = f|_{(fg)^{-1}k} : (fg)^{-1}k \to g^{-1}k$, is obtained by applying the composition

$$\Big[\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(fg|k)}X_i)\Big] \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(fg,\triangleright)}X_i)$$
$$\times \prod_{k\in K}\Big(\prod_{j\in g^{-1}k} \mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i) \times \mathcal{D}(\otimes^{(f,g|k)}X_i, \otimes^{(f,g|k)}X_i)\Big)$$
$$\times\, \mathcal{D}(\otimes^{(f,g,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i)$$
$$\xrightarrow{\mu^{\mathsf{D}}_{fg}\times(\prod_{k\in K}\mu^{\mathsf{D}}_{f_k}\times 1)\mu^{\mathsf{D}}_{\mathrm{id}_K}} \mathcal{D}(\otimes^I X_i, \otimes^{(fg,\triangleright)}X_i) \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i)$$
$$\xrightarrow{\mu^{\mathsf{D}}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)}X_i)$$

to the family of morphisms $\eta : \otimes^{(fg|k)}X_i \to \otimes^{(fg|k)}X_i$, $k \in K$, $\eta : \otimes^{(fg,\triangleright)}X_i \to \otimes^{(fg,\triangleright)}X_i$, $\eta : \otimes^{(f|j)}X_i \to \otimes^{(f|j)}X_i$, $j \in J$, $\eta : \otimes^{(f,g|k)}X_i \to \otimes^{(f,g|k)}X_i$, $k \in K$, $\eta : \otimes^{(f,g,\triangleright)}X_i \to \otimes^{(f,g,\triangleright)}X_i$. Applying axiom at Fig. 3.1, written for maps $I \xrightarrow{fg} K \to \mathbf{1}$ and objects $Y_k = \otimes^{(fg|k)}X_i$, $Z = \otimes^{(fg,\triangleright)}X_i$, $W = \otimes^{(f,g,\triangleright)}X_i$ we may replace this composition with

$$\Big[\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(fg|k)}X_i)\Big] \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(fg,\triangleright)}X_i)$$
$$\times \prod_{k\in K}\Big(\prod_{j\in g^{-1}k} \mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i) \times \mathcal{D}(\otimes^{(f,g|k)}X_i, \otimes^{(f,g|k)}X_i)\Big)$$
$$\times\, \mathcal{D}(\otimes^{(f,g,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i)$$
$$\xrightarrow{\prod_{k\in K} 1\times 1\times(\prod_{k\in K}\mu^{\mathsf{D}}_{f_k}\times 1)\mu^{\mathsf{D}}_{\mathrm{id}_K}}$$
$$\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(fg|k)}X_i) \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(fg,\triangleright)}X_i) \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i)$$
$$\xrightarrow{\prod_{k\in K} 1\times\mu^{\mathsf{D}}_{K\to\mathbf{1}}} \Big[\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(fg|k)}X_i)\Big] \times \mathcal{D}(\otimes^{(fg,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i)$$

$$\xrightarrow{\mu^{\mathsf{D}}_{fg}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)} X_i).$$

As above,

$$(\eta, 1)\mu^{\mathsf{D}}_{K\to\mathbf{1}} = (\eta, 1)\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}} = \mathrm{id} : \mathcal{D}(\otimes^{(fg,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \to \mathcal{D}(\otimes^{(fg,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i),$$

therefore the morphism $\lambda^{fg}_{\mathcal{D}} \cdot \otimes^{k\in K}_{\mathcal{D}} \lambda^{f_k}_{\mathcal{D}}$ is obtained by applying the composition

$$\begin{aligned}
&\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(fg|k)} X_i) \\
&\qquad \times \prod_{k\in K} \Big(\prod_{j\in g^{-1}k} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i) \times \mathcal{D}(\otimes^{(f,g|k)} X_i, \otimes^{(f,g|k)} X_i) \Big) \\
&\qquad\qquad \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\prod_{k\in K} 1 \times (\prod_{k\in K} \mu^{\mathsf{D}}_{f_k} \times 1)\mu^{\mathsf{D}}_{\mathrm{id}_K}} \\
&\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(fg|k)} X_i) \times \mathcal{D}(\otimes^{(fg,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\mu^{\mathsf{D}}_{fg}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)} X_i)
\end{aligned} \tag{3.24.5}$$

to the family of morphisms $\eta : \otimes^{(fg|k)} X_i \to \otimes^{(fg|k)} X_i$, $k \in K$, $\eta : \otimes^{(f|j)} X_i \to \otimes^{(f|j)} X_i$, $j \in J$, $\eta : \otimes^{(f,g|k)} X_i \to \otimes^{(f,g|k)} X_i$, $k \in K$, $\eta : \otimes^{(f,g,\triangleright)} X_i \to \otimes^{(f,g,\triangleright)} X_i$. Applying axiom at Fig. 3.1, written for maps $I \xrightarrow{fg} K \xrightarrow{\mathrm{id}_K} K$ and objects $Y_k = \otimes^{(fg|k)} X_i$, $Z_k = \otimes^{(f,g|k)} X_i$, $W = \otimes^{(f,g,\triangleright)} X_i$ we may replace (3.24.5) with

$$\begin{aligned}
&\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(fg|k)} X_i) \\
&\qquad \times \prod_{k\in K} \Big(\prod_{j\in g^{-1}k} \mathcal{D}(\otimes^{(f|j)} X_i, \otimes^{(f|j)} X_i) \times \mathcal{D}(\otimes^{(f,g|k)} X_i, \otimes^{(f,g|k)} X_i) \Big) \\
&\qquad\qquad \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\prod_{k\in K} 1 \times \prod_{k\in K} \mu^{\mathsf{D}}_{f_k} \times 1} \\
&\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(fg|k)} X_i) \times \prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(f,g|k)} X_i) \\
&\qquad\qquad\qquad \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\sim} \\
&\prod_{k\in K} \Big(\mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(fg|k)} X_i) \times \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(f,g|k)} X_i) \Big) \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \\
&\qquad \xrightarrow{\prod_{k\in K} \mu^{\mathsf{D}}_{(fg)^{-1}k\to\{k\}} \times 1} \\
&\Big[\prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)} X_i, \otimes^{(f,g|k)} X_i) \Big] \times \mathcal{D}(\otimes^{(f,g,\triangleright)} X_i, \otimes^{(f,g,\triangleright)} X_i) \xrightarrow{\mu^{\mathsf{D}}_{fg}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)} X_i).
\end{aligned}$$

By the axiom of units

$$(\eta,1)\mu^{\mathsf{D}}_{(fg)^{-1}k\to\{k\}} = (\eta,1)\mu^{\mathsf{D}}_{\mathbf{1}\to\mathbf{1}} = \mathrm{id} : \\ \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(f,g|k)}X_i) \to \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(f,g|k)}X_i).$$

Therefore the morphism $\lambda^{fg}_{\mathcal{D}} \cdot \otimes^{k\in K}_{\mathcal{D}} \lambda^{f_k}_{\mathcal{D}}$ is obtained by applying the composition

$$\prod_{k\in K}\Bigl(\prod_{j\in g^{-1}k} \mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i) \times \mathcal{D}(\otimes^{(f,g|k)}X_i, \otimes^{(f,g|k)}X_i)\Bigr) \\ \times \mathcal{D}(\otimes^{(f,g,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i) \\ \xrightarrow{\prod_{k\in K}\mu^{\mathcal{D}}_{f_k}\times 1} \prod_{k\in K} \mathcal{D}(\otimes^{(fg|k)}X_i, \otimes^{(f,g|k)}X_i) \times \mathcal{D}(\otimes^{(f,g,\triangleright)}X_i, \otimes^{(f,g,\triangleright)}X_i) \\ \xrightarrow{\mu^{\mathsf{D}}_{fg}} \mathcal{D}(\otimes^I X_i, \otimes^{(f,g,\triangleright)}X_i) \quad (3.24.6)$$

to the family of morphisms $\eta : \otimes^{(f|j)}X_i \to \otimes^{(f|j)}X_i$, $j \in J$, $\eta : \otimes^{(f,g|k)}X_i \to \otimes^{(f,g|k)}X_i$, $k \in K$, $\eta : \otimes^{(f,g,\triangleright)}X_i \to \otimes^{(f,g,\triangleright)}X_i$.

Expressions (3.24.4) and (3.24.6) coincide by axiom at Fig. 3.1, written for maps $I \xrightarrow{f} J \xrightarrow{g} K$ and objects $Y_j = \otimes^{(f|j)}X_i$, $j \in J$, $Z_k = \otimes^{(f,g|k)}X_i$, $k \in K$, $W = \otimes^{(f,g,\triangleright)}X_i$. Thus, we have proved that $\mathcal{D}$ is a lax (symmetric, braided) Monoidal category. As a corollary, we have proved in the enriched setting that $\mathcal{D}$ is a lax (symmetric, braided) Monoidal $\mathcal{V}$-category.

(d) We continue to work with $\mathcal{V}$-categories and $\mathcal{V}$-multicategories for the given symmetric Monoidal category $\mathcal{V}$. Let us prove that $\widehat{\mathcal{D}} = \mathsf{D}$. The main ingredient is the equality $\mu^{\widehat{\mathcal{D}}}_f = \mu^{\mathsf{D}}_f$ for every $f \in \operatorname{Mor}\mathcal{S}$. We have:

$$\mu^{\widehat{\mathcal{D}}}_f = \bigl[\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \\ \xrightarrow{\lambda^{\gamma:J\sqcup\mathbf{1}\to\mathbf{2}}} \bigl(\otimes^{j\in J}_{\mathcal{V}}\mathcal{D}(\otimes^{(f|j)}X_i, Y_j)\bigr) \otimes \mathcal{D}(\otimes^J Y_j, Z) \\ \xrightarrow{\otimes^J_{\mathcal{D}}\otimes 1} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^J Y_j) \otimes \mathcal{D}(\otimes^J Y_j, Z) \xrightarrow{\lambda^f_{\mathcal{D}}\cdot - \cdot -} \mathcal{D}(\otimes^I X_i, Z)\bigr] \\ = \bigl[\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \\ \xrightarrow{\lambda^{\gamma:J\sqcup\mathbf{1}\to\mathbf{2}}} \bigl(\otimes^{j\in J}_{\mathcal{V}}\mathcal{D}(\otimes^{(f|j)}X_i, Y_j)\bigr) \otimes \mathcal{D}(\otimes^J Y_j, Z) \\ \xrightarrow{[\lambda^{J\hookrightarrow J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\eta)]\otimes 1} \otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}} [(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, \otimes^J Y_j)] \otimes \mathcal{D}(\otimes^J Y_j, Z) \\ \xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}\otimes 1} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^J Y_j) \otimes \mathcal{D}(\otimes^J Y_j, Z) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \\ \xrightarrow{\lambda^{\cdot \mathrm{I}}} \mathbb{1} \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{[\lambda^{\varnothing\to J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}\eta]\otimes 1} \\ \otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}} [(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^{(f,\triangleright)}X_i)] \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \\ \xrightarrow{\mu^{\mathsf{D}}_f\otimes 1} \mathcal{D}(\otimes^I X_i, \otimes^{(f,\triangleright)}X_i) \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes^I X_i, Z)\bigr], \quad (3.24.7)$$

where $\gamma = \triangleright \sqcup \mathrm{id} : J \sqcup \mathbf{1} \to \mathbf{2}$. Let us split this expression into composition of two morphisms:

$$\begin{aligned}
\xi = \Big[&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \\
&\xrightarrow{\lambda^{\gamma:J\sqcup\mathbf{1}\to\mathbf{2}}} \big(\otimes^{j\in J}_{\mathcal{V}}\mathcal{D}(\otimes^{(f|j)}X_i, Y_j)\big) \otimes \mathcal{D}(\otimes^J Y_j, Z) \xrightarrow{[\lambda^{J\hookrightarrow J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\eta)]\otimes 1} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, \otimes^J Y_j)] \otimes \mathcal{D}(\otimes^J Y_j, Z) \\
&\xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}\otimes 1} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^J Y_j) \otimes \mathcal{D}(\otimes^J Y_j, Z) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)\Big], \\
\zeta = \Big[&\mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{\lambda^{\cdot\,\mathsf{I}}} \mathbb{1} \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{[\lambda^{\varnothing\to J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}\eta]\otimes 1} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, \otimes^{(f,\triangleright)}X_i)] \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \\
&\xrightarrow{\mu^{\mathsf{D}}_f\otimes 1} \mathcal{D}(\otimes^I X_i, \otimes^{(f,\triangleright)}X_i) \otimes \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes^I X_i, Z)\Big].
\end{aligned}$$

Applying axiom at Fig. 3.1, written for maps $J \xrightarrow{\mathrm{id}} J \to \mathbf{1}$ and objects $\otimes^J Y_j$ in place of Z and Z in place of W we find that

$$\begin{aligned}
\xi = \Big[&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\lambda^{\cdot\,\mathsf{I}}\cdot(\eta\otimes 1)\cdot\mu^{\mathsf{D}}_{J\to\mathbf{1}})} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)\Big] = \mu^{\mathsf{D}}_{\mathrm{id}_J}.
\end{aligned}$$

Applying axiom at Fig. 3.1, written for maps $I \xrightarrow{f} J \to \mathbf{1}$ and objects $\otimes^{(f|j)}X_i$ in place of Y_j, $\otimes^{(f,\triangleright)}X_i$ in place of Z, and Z in place of W we may write:

$$\begin{aligned}
\zeta = \Big[&\mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{\lambda^{\cdots\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}((\eta)_{j\in J},1)} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\lambda^{\cdot\,\mathsf{I}}\cdot(\eta\otimes 1)\cdot\mu^{\mathsf{D}}_{J\to\mathbf{1}})} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)] \xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^I X_i, Z)\Big] \\
= \Big[&\mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \xrightarrow{\lambda^{\cdots\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}\cdot\otimes^{J\sqcup\mathbf{1}}((\eta)_{j\in J},1)} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)] \xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^I X_i, Z)\Big].
\end{aligned}$$

The morphism $\mu^{\widehat{\mathcal{D}}}_f = \xi \cdot \zeta$ may be rewritten:

$$\begin{aligned}
\mu^{\widehat{\mathcal{D}}}_f = \Big[&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \xrightarrow{\mu^{\mathsf{D}}_{\mathrm{id}_J}} \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z) \\
&\xrightarrow{\lambda^{\cdots\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}} \otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathbb{1})_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}((\eta)_{j\in J},1)} \\
&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\,[(\mathcal{D}(\otimes^{(f|j)}X_i, \otimes^{(f|j)}X_i))_{j\in J}, \mathcal{D}(\otimes^{(f,\triangleright)}X_i, Z)] \xrightarrow{\mu^{\mathsf{D}}_f} \mathcal{D}(\otimes^I X_i, Z)\Big] \\
= \Big[&\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\mathcal{D}(\otimes^{(f|j)}X_i, Y_j))_{j\in J}, \mathcal{D}(\otimes^J Y_j, Z)] \xrightarrow{\lambda^{\cdots\,\mathsf{I}:\mathbf{1}\hookrightarrow J\sqcup\mathbf{1}}}
\end{aligned}$$

$$\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathbb{1})_{j\in J},\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,Y_j))_{j\in J},\mathcal{D}(\otimes^J Y_j,Z)]]\xrightarrow{\otimes^{J\sqcup\mathbf{1}}((\eta)_{j\in J},1)}$$
$$\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,\otimes^{(f|j)}X_i))_{j\in J},\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,Y_j))_{j\in J},\mathcal{D}(\otimes^J Y_j,Z)]]$$
$$\xrightarrow{\otimes^{J\sqcup\mathbf{1}}((1)_{j\in J},\mu^{\mathsf{D}}_{\mathrm{id}_J})}\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,\otimes^{(f|j)}X_i))_{j\in J},\mathcal{D}(\otimes^{(f,\triangleright)}X_i,Z)]$$
$$\xrightarrow{\mu^{\mathsf{D}}_f}\mathcal{D}(\otimes^I X_i,Z)\big].$$

Applying axiom at Fig. 3.1, written for maps $I\xrightarrow{f}J\xrightarrow{\mathrm{id}_J}J$ and objects $\otimes^{(f|j)}X_i$ in place of Y_j, Y_j in place of Z_j, and Z in place of W, we get:

$$\mu_f^{\widehat{\mathcal{D}}}=\big[\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,Y_j))_{j\in J},\mathcal{D}(\otimes^J Y_j,Z)]\xrightarrow{\otimes^{J\sqcup\mathbf{1}}(\lambda^{\cdot\;\mathsf{I}\cdot}(\eta\otimes 1)\cdot\mu^{\mathsf{D}}_{f^{-1}j\to\{j\}},1)}$$
$$\otimes_{\mathcal{V}}^{J\sqcup\mathbf{1}}[(\mathcal{D}(\otimes^{(f|j)}X_i,Y_j))_{j\in J},\mathcal{D}(\otimes^J Y_j,Z)]\xrightarrow{\mu^{\mathsf{D}}_f}\mathcal{D}(\otimes^I X_i,Z)\big]=\mu^{\mathsf{D}}_f.$$

For an arbitrary lax representable $\mathcal{V}$-multicategory C we have constructed an isomorphism $\widehat{\mathcal{D}}=\mathsf{D}\xrightarrow{\alpha}\mathsf{C}$ with the multicategory $\widehat{\mathcal{D}}$, coming from a lax (symmetric, braided) Monoidal $\mathcal{V}$-category $\mathcal{D}$. The converse statement is obvious. □

3.25 Example. The symmetric multicategory $\widehat{\Bbbk\text{-}\mathbf{Mod}}$ from Example 3.16 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\Bbbk$-**Mod**. For instance, we shall define $\otimes_{\Bbbk}^{i\in I}X_i$ to be the free $\Bbbk$-module, generated by the set $\prod_{i\in I}X_i$, divided by $\Bbbk$-polylinearity relations. The tautological map $\tau:\prod_{i\in I}X_i\to\otimes_{\Bbbk}^{i\in I}X_i$ determines the isomorphisms $\lambda_{\Bbbk}^f$.

3.26 Example. The symmetric multicategory $\widehat{\mathbf{gr}}$ from Example 3.17 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\mathcal{V}=\mathbf{gr}=\mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$. We define $(\otimes_{\mathbf{gr}}^{i\in I}X_i)^n=\oplus_{\sum_i n_i=n}\otimes_{\Bbbk}^{i\in I}X_i^{n_i}$. The isomorphism $\lambda_{\mathbf{gr}}^f$ is $\lambda_{\Bbbk}^f$, extended additively to direct sums, multiplied with the sign $(-1)^\sigma$, where σ is given by Koszul sign rule (3.17.1).

3.27 Example. The symmetric multicategory $\widehat{\mathbf{dg}}$ from Example 3.18 is representable. Thus, it comes from some symmetric Monoidal structure of the category $\mathcal{V}=\mathbf{dg}=\mathbf{dg}(\Bbbk\text{-}\mathbf{Mod})$. We define $\otimes_{\mathbf{dg}}^{i\in I}X_i$ as the graded $\Bbbk$-module $\otimes_{\mathbf{gr}}^{i\in I}X_i$ equipped with the differential given by the formula $d=\sum_{j\in I}\underline{\otimes}^{i\in I}[(1)_{i<j},d,(1)_{i>j}]$. The notation $\underline{\otimes}$ allows to omit the permutation signs while we consider d as a degree 1 map of graded $\Bbbk$-modules. Occasionally we abuse the notation employing the usual tensor symbol $\otimes$ in place of $\underline{\otimes}$. In our sign conventions the matrix elements of the j-th summand are

$$(-1)^{\sum_{i>j}n_i}1\otimes\cdots\otimes d\otimes\cdots\otimes 1:\otimes_{\Bbbk}^{i\in I}X_i^{n_i}\to\otimes_{\Bbbk}^{i\in I}X_i^{n_i+\delta_{ij}}.$$

Here d is considered as $\Bbbk$-module map. The isomorphism $\lambda_{\mathbf{dg}}^f$ coincides with $\lambda_{\mathbf{gr}}^f$.

3.28 Proposition. *Any lax (symmetric) Monoidal $\mathcal{V}$-functor $(F,\phi^I) : (\mathcal{C},\otimes^I,\lambda^f_{\mathcal{C}}) \to (\mathcal{D},\otimes^I,\lambda^f_{\mathcal{D}})$ between lax (symmetric) Monoidal $\mathcal{V}$-categories gives rise to a (symmetric) $\mathcal{V}$-multifunctor $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ with*

- *the mapping of objects* $\operatorname{Ob}\widehat{F} = \operatorname{Ob} F$,
- *the morphism of objects of morphisms*

$$\widehat{F}_{(X_i);Y} = \bigl[\widehat{\mathcal{C}}((X_i)_{i\in I};Y) = \mathcal{C}(\otimes^{i\in I}X_i,Y) \xrightarrow{F_{\otimes^{i\in I}X_i,Y}} \mathcal{D}(F(\otimes^{i\in I}X_i),FY) \\ \xrightarrow{\mathcal{D}(\phi^I,FY)} \mathcal{D}(\otimes^{i\in I}(FX_i),FY) = \widehat{\mathcal{D}}((FX_i)_{i\in I};FY)\bigr]. \quad (3.28.1)$$

Here

$$\mathcal{D}(\phi^I,FY) = \phi^I\cdot - = \bigl[\mathcal{D}(F(\otimes^{i\in I}X_i),FY) \xrightarrow{\lambda_{\mathcal{V}}^{\cdot\to 1}} \mathbb{1}_{\mathcal{V}}\otimes\mathcal{D}(F(\otimes^{i\in I}X_i),FY) \xrightarrow{\phi^I\otimes 1} \\ \mathcal{D}(\otimes^{i\in I}(FX_i),F(\otimes^{i\in I}X_i))\otimes\mathcal{D}(F(\otimes^{i\in I}X_i),FY) \xrightarrow{\text{composition}} \mathcal{D}(\otimes^{i\in I}(FX_i),FY)\bigr].$$

Proof. When $I = \mathbf{1}$ and $X_1 = Y$, then $\phi^I = \mathrm{id}$ and the map

$$\widehat{F}_{Y;Y} = F_{Y,Y} : \mathcal{C}(Y,Y) \to \mathcal{D}(FY,FY)$$

takes units to units.

Compatibility of $\widehat{F}$ with multiplication that corresponds to a map $f : I \to J$ is expressed by equation

$$\begin{tikzcd}
\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\bigl[\bigl(\widehat{\mathcal{C}}((X_i)_{i\in f^{-1}j};Y_j)\bigr)_{j\in J},\widehat{\mathcal{C}}((Y_j)_{j\in J};Z)\bigr] \arrow[r, "\mu^{\widehat{\mathcal{C}}}_f"] \arrow[d, "\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}[(\widehat{F}_{(X_i)_{i\in f^{-1}j};Y_j})_{j\in J},\widehat{F}_{(Y_j)_{j\in J};Z}]"'] \arrow[dr, phantom, "="] & \widehat{\mathcal{C}}((X_i)_{i\in I};Z) \arrow[d, "\widehat{F}_{(X_i)_{i\in I};Z}"] \\
\otimes^{J\sqcup\mathbf{1}}_{\mathcal{V}}\bigl[\bigl(\widehat{\mathcal{D}}((FX_i)_{i\in f^{-1}j};FY_j)\bigr)_{j\in J},\widehat{\mathcal{D}}((FY_j)_{j\in J};FZ)\bigr] \arrow[r, "\mu^{\widehat{\mathcal{D}}}_f"] & \widehat{\mathcal{D}}((FX_i)_{i\in I};FZ)
\end{tikzcd}$$

It coincides with exterior of diagram on the facing page. Here square 1 commutes due to $\otimes^J$ being a functor. Quadrilateral 2 follows from associativity of composition in $\mathcal{D}$. Quadrilateral 3 commutes due to F being a functor. The remaining polygon 4 is the exterior of diagram 100. In this diagram square 5 is due to F being a functor. Triangle 6 commutes, as equation (2.17.2) shows. Hexagon 7 follows from naturality of transformation ϕ^J. Therefore, the whole diagram commutes, and $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ is a multifunctor. $\square$

3.29 Proposition. *A Monoidal transformation $r : (F,\phi^I) \to (G,\psi^I) : \mathcal{C} \to \mathcal{D}$ gives rise to a multinatural transformation of $\mathcal{V}$-multifunctors $\widehat{r} : \widehat{F} \to \widehat{G} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$, determined by the morphisms*

$$\widehat{r}_X = r_X \in \mathcal{D}(FX,GX).$$

$$
\begin{tikzcd}[column sep=small]
\bigotimes_{j\in J}\mathcal{C}(\bigotimes_{i\in f^{-1}j}X_i, Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J}Y_j, Z) \arrow[rr, "\otimes^J\otimes 1"] \arrow[dddd, "\bigotimes_{j\in J}F_{\bigotimes_{i\in f^{-1}j}X_i,Y_j}\otimes F_{\bigotimes_{j\in J}Y_j,Z}"'] & & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i, \bigotimes_{j\in J}Y_j)\otimes\mathcal{C}(\bigotimes_{j\in J}Y_j, Z) \arrow[rr, "\lambda^f_{\mathcal{C}}\cdot-\cdot-"] \arrow[d, "\text{composition}"'] & & \mathcal{C}(\bigotimes_{i\in I}X_i, Z) \arrow[dd, "F_{\bigotimes_{i\in I}X_i,Z}"] \\
 & & \mathcal{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i, Z) \arrow[urr, "\lambda^f_{\mathcal{C}}\cdot-"'] \arrow[d, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i,Z}"'] \arrow[drr, phantom, "\boxed{3}"] & & \\
 & \boxed{4} & \mathcal{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}X_i), FZ) \arrow[rr, "(F\lambda^f_{\mathcal{C}})\cdot-"] \arrow[drr, "(\phi^I\cdot F\lambda^f_{\mathcal{C}})\cdot-"] & & \mathcal{D}(F(\bigotimes_{i\in I}X_i), FZ) \arrow[d, "\phi^I\cdot-"] \\
 & & \mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j}X_i)), F(\bigotimes_{j\in J}Y_j))\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j), FZ) \arrow[rr, "(\lambda^f_{\mathcal{D}}\cdot\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot-\cdot-"'] & & \mathcal{D}(\bigotimes_{i\in I}FX_i, FZ) \\
\bigotimes_{j\in J}\mathcal{D}(F(\bigotimes_{i\in f^{-1}j}X_i), FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j), FZ) \arrow[rr, "\otimes^J\otimes 1"] \arrow[d, "\bigotimes_{j\in J}(\phi^{f^{-1}j}\cdot-)\otimes(\phi^J\cdot-)"'] \arrow[drr, phantom, "\boxed{1}"] & & \mathcal{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j}X_i)), \bigotimes_{j\in J}FY_j)\otimes\mathcal{D}(F(\bigotimes_{j\in J}Y_j), FZ) \arrow[u, "(-\cdot\phi^J)\otimes 1"', "\boxed{2}"] \arrow[d, "{[(\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot-]\otimes(\phi^J\cdot-)}"'] & & \\
\bigotimes_{j\in J}\mathcal{D}(\bigotimes_{i\in f^{-1}j}FX_i, FY_j)\otimes\mathcal{D}(\bigotimes_{j\in J}FY_j, FZ) \arrow[rr, "\otimes^J\otimes 1"] & & \mathcal{D}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j}FX_i, \bigotimes_{j\in J}FY_j)\otimes\mathcal{D}(\bigotimes_{j\in J}FY_j, FZ) \arrow[uurr, "\lambda^f_{\mathcal{D}}\cdot-\cdot-"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
{\begin{array}{r}\bigotimes_{j\in J}\mathfrak{C}(\bigotimes_{i\in f^{-1}j} X_i, Y_j)\\ \otimes\mathfrak{C}(\bigotimes_{j\in J} Y_j, Z)\end{array}} \arrow[r, "\otimes^J\otimes 1"] \arrow[d, "\bigotimes_{j\in J} F_{\bigotimes_{i\in f^{-1}j} X_i, Y_j}\otimes F_{\bigotimes_{j\in J} Y_j, Z}"'] &
{\begin{array}{r}\mathfrak{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, \bigotimes_{j\in J} Y_j)\\ \otimes\mathfrak{C}(\bigotimes_{j\in J} Y_j, Z)\end{array}} \arrow[r, "\mathrm{comp}"] \arrow[d, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, \bigotimes_{j\in J} Y_j}\otimes F_{\bigotimes_{j\in J} Y_j, Z}"'] \arrow[dr, phantom, "\boxed{5}"] &
\mathfrak{C}(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, Z) \arrow[d, "F_{\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i, Z}"] \\
{\begin{array}{r}\bigotimes_{j\in J}\mathfrak{D}(F(\bigotimes_{i\in f^{-1}j} X_i), FY_j)\\ \otimes\mathfrak{D}(F(\bigotimes_{j\in J} Y_j), FZ)\end{array}} \arrow[d, "\otimes^J\otimes 1"'] \arrow[dr, phantom, "\boxed{7}"] &
{\begin{array}{r}\mathfrak{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i), F(\bigotimes_{j\in J} Y_j))\\ \otimes\mathfrak{D}(F(\bigotimes_{j\in J} Y_j), FZ)\end{array}} \arrow[r, "\mathrm{comp}"] \arrow[d, "(\phi^J\cdot -)\otimes 1"'] \arrow[dr, "(\phi^I\cdot F\lambda^f_{\mathfrak{C}})\cdot -\cdot -" near start, "\boxed{6}"'] &
\mathfrak{D}(F(\bigotimes_{j\in J}\bigotimes_{i\in f^{-1}j} X_i), FZ) \arrow[d, "(\phi^I\cdot F\lambda^f_{\mathfrak{C}})\cdot -"] \\
{\begin{array}{r}\mathfrak{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j} X_i)), \bigotimes_{j\in J} FY_j)\\ \otimes\mathfrak{D}(F(\bigotimes_{j\in J} Y_j), FZ)\end{array}} \arrow[r, "(-\cdot\phi^J)\otimes 1"] &
{\begin{array}{r}\mathfrak{D}(\bigotimes_{j\in J}(F(\bigotimes_{i\in f^{-1}j} X_i)), F(\bigotimes_{j\in J} Y_j))\\ \otimes\mathfrak{D}(F(\bigotimes_{j\in J} Y_j), FZ)\end{array}} \arrow[r, "(\lambda^f_{\mathfrak{D}}\cdot\bigotimes_{j\in J}\phi^{f^{-1}j})\cdot -\cdot -"'] &
\mathfrak{D}(\bigotimes_{i\in I} FX_i, FZ)
\end{tikzcd}
$$

Strictly speaking, r_X are elements of $\mathcal{V}(\mathbb{1}_{\mathcal{V}}, \mathcal{D}(FX, GX))$, but we shall not always care about this subtlety in notation.

Proof. We have to prove the following property of $\widehat{r}$:

$$\begin{tikzcd}
\widehat{\mathcal{C}}((X_i)_{i\in I}; Y) \arrow[r, "\widehat{G}_{(X_i);Y}"] \arrow[d, "\widehat{F}_{(X_i);Y}"'] \arrow[dr, phantom, "="] & \widehat{\mathcal{D}}((GX_i)_{i\in I}; GY) \arrow[d, "(\widehat{r}_{X_i})_{i\in I}\cdot -"] \\
\widehat{\mathcal{D}}((FX_i)_{i\in I}; FY) \arrow[r, "-\cdot\widehat{r}_Y"] & \widehat{\mathcal{D}}((FX_i)_{i\in I}; GY)
\end{tikzcd}$$

Plugging in definitions of $\widehat{F}$, $\widehat{G}$ and compositions we get the following equation to verify:

$$\begin{aligned}
&\big[\mathcal{C}(\otimes^{i\in I} X_i, Y) \xrightarrow{\lambda_{\mathcal{C}}^{\cdot\cdot\cdot\shortmid}} \mathbb{1}_{\mathcal{V}}^{\otimes I} \otimes \mathcal{C}(\otimes^{i\in I} X_i, Y) \\
&\qquad \xrightarrow{(\otimes^I r_{X_i})\otimes G_{\otimes^I X_i, Y}} [\otimes^{i\in I}\mathcal{D}(FX_i, GX_i)] \otimes \mathcal{D}(G(\otimes^{i\in I} X_i), GY) \\
&\qquad \xrightarrow{\otimes^I_{\mathcal{D}}\otimes(\psi^I\cdot -)} \mathcal{D}(\otimes^{i\in I} FX_i, \otimes^{i\in I} GX_i) \otimes \mathcal{D}(\otimes^{i\in I} GX_i, GY) \xrightarrow{\text{comp}} \mathcal{D}(\otimes^{i\in I} FX_i, GY)\big] \\
&= \big[\mathcal{C}(\otimes^{i\in I} X_i, Y) \xrightarrow{\lambda_{\mathcal{V}}^{\shortmid\cdot}} \mathcal{C}(\otimes^{i\in I} X_i, Y) \otimes \mathbb{1} \\
&\qquad \xrightarrow{F_{\otimes^I X_i, Y}\otimes r_Y} \mathcal{D}(F(\otimes^{i\in I} X_i), FY) \otimes \mathcal{D}(FY, GY) \\
&\qquad \xrightarrow{(\phi^I\cdot -)\otimes 1} \mathcal{D}(\otimes^{i\in I} FX_i, FY) \otimes \mathcal{D}(FY, GY) \xrightarrow{\text{comp}} \mathcal{D}(\otimes^{i\in I} FX_i, GY)\big].
\end{aligned} \tag{3.29.1}$$

Equation (2.20.1) means that elements

$$\begin{gathered}
\otimes^I_{\mathcal{D}}(r_{X_i}) \otimes \psi^I \in \mathcal{D}(\otimes^{i\in I} FX_i, \otimes^{i\in I} GX_i) \otimes \mathcal{D}(\otimes^{i\in I} GX_i, G(\otimes^{i\in I} X_i)), \\
\phi^I \otimes r_{\otimes^I X_i} \in \mathcal{D}(\otimes^{i\in I} FX_i, F(\otimes^{i\in I} X_i)) \otimes \mathcal{D}(F(\otimes^{i\in I} X_i), G(\otimes^{i\in I} X_i))
\end{gathered}$$

compose to the same element

$$(\otimes^I r_{X_i}) \cdot \psi^I = \phi^I \cdot r_{\otimes^I X_i} \in \mathcal{D}(\otimes^{i\in I} FX_i, G(\otimes^{i\in I} X_i)).$$

Therefore, in the following short form of (3.29.1)

$$\begin{tikzcd}
\mathcal{C}(\otimes^{i\in I} X_i, Y) \arrow[r, "G_{\otimes^I X_i, Y}"] \arrow[d, "F_{\otimes^I X_i, Y}"'] \arrow[dr, phantom, "="] & \mathcal{D}(G(\otimes^{i\in I} X_i), GY) \arrow[d, "(\otimes^I r_{X_i})\cdot\psi^I\cdot -"] \\
\mathcal{D}(F(\otimes^{i\in I} X_i), FY) \arrow[r, "\phi^I\cdot -\cdot r_Y"] & \mathcal{D}(\otimes^{i\in I} FX_i, GY)
\end{tikzcd}$$

we may replace the right vertical arrow with $\phi^I \cdot r_{\otimes^I X_i} \cdot -$. The equation reduces to

$$\begin{tikzcd}
\mathcal{C}(\otimes^{i\in I} X_i, Y) \arrow[r, "G_{\otimes^I X_i, Y}"] \arrow[d, "F_{\otimes^I X_i, Y}"'] \arrow[dr, phantom, "="] & \mathcal{D}(G(\otimes^{i\in I} X_i), GY) \arrow[d, "r_{\otimes^I X_i}\cdot -"] \\
\mathcal{D}(F(\otimes^{i\in I} X_i), FY) \arrow[r, "-\cdot r_Y"] & \mathcal{D}(F(\otimes^{i\in I} X_i), GY)
\end{tikzcd}$$

This is nothing else but naturality of the transformation $r : F \to G : \mathcal{C} \to \mathcal{D}$. $\square$

3.30 Proposition. *Let $\mathcal{C}$, $\mathcal{D}$ be lax (symmetric) Monoidal $\mathcal{V}$-categories. Then the maps* $\text{lax-(sym-)Mono}(\mathcal{C},\mathcal{D}) \to (\mathcal{S})\mathcal{MC}\text{atm}(\widehat{\mathcal{C}},\widehat{\mathcal{D}})$, $F \mapsto \widehat{F}$, $r \mapsto \widehat{r}$, *constructed in Propositions 3.28, 3.29 are bijective.*

Compare this statement with a result of Hermida [Her00, Theorem 9.8]. He constructs a 2-equivalence between 2-categories of monoidal categories and representable multicategories.

Proof. Let us write an inverse map to the map $F \mapsto \widehat{F}$. Let $G : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ be a (symmetric) $\mathcal{V}$-multifunctor. Define a $\mathcal{V}$-functor $F : \mathcal{C} \to \mathcal{D}$ with $\operatorname{Ob} F = \operatorname{Ob} G$,

$$F_{X,Y} : \mathcal{C}(X,Y) = \widehat{\mathcal{C}}(X;Y) \xrightarrow{G_{X;Y}} \widehat{\mathcal{D}}(GX;GY) = \mathcal{D}(GX,GY).$$

Define a family of morphisms

$$\begin{aligned}\phi^I = \bigl[\mathbb{1} \xrightarrow{\eta} \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) = \widehat{\mathcal{C}}((X_i)_{i\in I}; \otimes^{i\in I}X_i)&\\ \xrightarrow{G_{(X_i)_{i\in I};\otimes^{i\in I}X_i}} \widehat{\mathcal{D}}((GX_i)_{i\in I}; G(\otimes^{i\in I}X_i)) = \mathcal{D}(\otimes^{i\in I}GX_i, G(\otimes^{i\in I}X_i))\bigr]&.\end{aligned}$$

We claim that it is a natural transformation $\phi^I : \otimes^I_{\mathcal{D}} \circ F^I \to F \circ \otimes^I_{\mathcal{C}}$. This is expressed by commutativity of the diagram

$$\begin{tikzcd}
\mathcal{C}^I((X_i)_{i\in I},(Y_i)_{i\in I}) \arrow[r, "(\otimes^I_{\mathcal{D}}F^I)\otimes\phi^I"] \arrow[d, "\phi^I\otimes F\otimes^I_{\mathcal{C}}"'] & \mathcal{D}(\otimes^{i\in I}GX_i, \otimes^{i\in I}GY_i) \otimes \mathcal{D}(\otimes^{i\in I}GY_i, G\otimes^{i\in I}Y_i) \arrow[d, "\mu_{\mathcal{D}}"] \\
\mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}X_i) \otimes \mathcal{D}(G\otimes^{i\in I}X_i, G\otimes^{i\in I}Y_i) \arrow[r, "\mu_{\mathcal{D}}"] & \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i).
\end{tikzcd}$$

Expanded form of this equation is

$$\begin{aligned}
\bigl[\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}(X_i,Y_i) &\xrightarrow{(\otimes^{i\in I}_{\mathcal{V}}G_{X_i;Y_i})\otimes_{\mathcal{V}}\eta} [\otimes^{i\in I}_{\mathcal{V}}\mathcal{D}(GX_i,GY_i)] \otimes_{\mathcal{V}} \mathcal{C}(\otimes^{i\in I}Y_i, \otimes^{i\in I}Y_i)\\
&\xrightarrow{\otimes^I_{\mathcal{D}}\otimes_{\mathcal{V}}G_{(Y_i);\otimes^I Y_i}} \mathcal{D}(\otimes^{i\in I}GX_i, \otimes^{i\in I}GY_i) \otimes_{\mathcal{V}} \mathcal{D}(\otimes^{i\in I}GY_i, G\otimes^{i\in I}Y_i)\\
&\xrightarrow{\mu_{\mathcal{D}}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i)\bigr]\\
= \bigl[\otimes^{i\in I}_{\mathcal{V}}\mathcal{C}(X_i,Y_i) &\xrightarrow{\eta\otimes_{\mathcal{V}}\otimes^I_{\mathcal{C}}} \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes_{\mathcal{V}} \mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}Y_i)\\
&\xrightarrow{G_{(X_i);\otimes^I X_i}\otimes_{\mathcal{V}}G_{\otimes^I X_i;\otimes^I Y_i}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}X_i) \otimes_{\mathcal{V}} \mathcal{D}(G\otimes^{i\in I}X_i, G\otimes^{i\in I}Y_i)\\
&\xrightarrow{\mu_{\mathcal{D}}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i)\bigr].
\end{aligned}$$

It is equivalent to

$$
\begin{aligned}
&\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i,Y_i) \xrightarrow{1\otimes_{\mathcal{V}}\eta} [\otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i,Y_i)] \otimes_{\mathcal{V}} \mathcal{C}(\otimes^{i\in I}Y_i,\otimes^{i\in I}Y_i) \\
&\qquad \xrightarrow{(\otimes_{\mathcal{C}}^I\otimes_{\mathcal{V}}1)\cdot\mu_{\mathcal{C}}} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i) \xrightarrow{G_{(X_i);\otimes^I Y_i}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i)\big] \\
&= \big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i,Y_i) \xrightarrow{\eta\otimes_{\mathcal{V}}\otimes_{\mathcal{C}}^I} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}X_i) \otimes_{\mathcal{V}} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i) \\
&\qquad \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i) \xrightarrow{G_{(X_i);\otimes^I Y_i}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i)\big].
\end{aligned}
$$

This equation is satisfied because the both sides are equal to

$$
\big[\otimes_{\mathcal{V}}^{i\in I}\mathcal{C}(X_i,Y_i) \xrightarrow{\otimes_{\mathcal{C}}^I} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}Y_i) \xrightarrow{G_{(X_i);\otimes^I Y_i}} \mathcal{D}(\otimes^{i\in I}GX_i, G\otimes^{i\in I}Y_i)\big].
$$

We conclude that $\phi^I : \otimes_{\mathcal{D}}^I \circ F^I \to F \circ \otimes_{\mathcal{C}}^I$ is a natural transformation.

Let us prove that $(F,\phi^I) : \mathcal{C} \to \mathcal{D}$ is a lax (symmetric) Monoidal functor. If I is a 1-element, then $\phi^I = \eta$ defines the identity transformation id_F. We have to prove equation (2.17.2) for every map $f : I \to J$, namely,

$$
\begin{aligned}
&\big[\otimes_{\mathcal{D}}^{i\in I}GX_i \xrightarrow{\lambda_{\mathcal{D}}^f} \otimes_{\mathcal{D}}^{j\in J}\otimes_{\mathcal{D}}^{i\in f^{-1}j}GX_i \xrightarrow{\otimes_{\mathcal{D}}^{j\in J}\phi^{f^{-1}j}} \\
&\qquad\qquad \otimes_{\mathcal{D}}^{j\in J} G\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i \xrightarrow{\phi^J} G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i\big] \\
&= \big[\otimes_{\mathcal{D}}^{i\in I}GX_i \xrightarrow{\phi^I} G\otimes_{\mathcal{D}}^{i\in I}X_i \xrightarrow{G\lambda_{\mathcal{C}}^f} G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i\big]. \qquad (3.30.1)
\end{aligned}
$$

The left hand side is expanded below:

$$
\begin{aligned}
&\big[1\!\!1 \xrightarrow{\otimes^J\eta} \otimes_{\mathcal{V}}^{j\in J}\mathcal{C}(\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i,\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \xrightarrow{\otimes_{\mathcal{V}}^{j\in J}G_{(X_i)_{i\in f^{-1}j};\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i}\otimes\eta} \\
&\quad \otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes_{\mathcal{D}}^{i\in f^{-1}j}GX_i, G\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \otimes_{\mathcal{V}} \mathcal{C}(\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i,\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \\
&\qquad \xrightarrow{\lambda_{\mathcal{D}}^f\otimes\otimes_{\mathcal{D}}^J\otimes G_{(\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i)_{j\in J};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i}} \\
&\mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}GX_i,\otimes_{\mathcal{D}}^{j\in J}\otimes_{\mathcal{D}}^{i\in f^{-1}j}GX_i) \otimes \mathcal{D}(\otimes_{\mathcal{D}}^{j\in J}\otimes_{\mathcal{D}}^{i\in f^{-1}j}GX_i,\otimes_{\mathcal{D}}^{j\in J}G\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i)\otimes \\
&\qquad\qquad \otimes\,\mathcal{D}(\otimes_{\mathcal{D}}^{j\in J}G\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \\
&\qquad\qquad \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i)\big] \\
&= \big[1\!\!1 \xrightarrow{\otimes^J\eta\otimes\eta} \otimes_{\mathcal{V}}^{j\in J}\mathcal{C}(\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i,\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \otimes_{\mathcal{V}} \mathcal{C}(\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i,\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \\
&\qquad \xrightarrow{\otimes_{\mathcal{V}}^{j\in J}G_{(X_i)_{i\in f^{-1}j};\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i}\otimes G_{(\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i)_{j\in J};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i}} \\
&\quad \otimes_{\mathcal{V}}^{j\in J}\mathcal{D}(\otimes_{\mathcal{D}}^{i\in f^{-1}j}GX_i, G\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i) \otimes \mathcal{D}(\otimes_{\mathcal{D}}^{j\in J}G\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j}X_i)
\end{aligned}
$$

$$
\begin{aligned}
&\xrightarrow{\mu_f^{\widehat{\mathcal{D}}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big] \\
= \big[\mathbb{1} &\xrightarrow{\otimes^J\eta\otimes\eta} \otimes_{\mathcal{V}}^{j\in J}\,\mathcal{C}(\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i, \otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\otimes_{\mathcal{V}}\mathcal{C}(\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{\mu_f^{\widehat{\mathcal{C}}}} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big] \\
= \big[\mathbb{1} &\xrightarrow{\lambda_{\mathcal{C}}^f} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big].
\end{aligned}
$$

The right hand side of (3.30.1) is expanded below:

$$
\begin{aligned}
\big[\mathbb{1} &\xrightarrow{\eta\otimes\lambda_{\mathcal{C}}^f} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{i\in I} X_i)\otimes\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{i\in I} X_i}\otimes G_{\otimes_{\mathcal{C}}^{i\in I} X_i;\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i}} \\
&\mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{i\in I} X_i)\otimes\mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I} X_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{\mu_{I\to\mathbf{1}}^{\widehat{\mathcal{D}}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big] \\
= \big[\mathbb{1} &\xrightarrow{\eta\otimes\lambda_{\mathcal{C}}^f} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{i\in I} X_i)\otimes\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \xrightarrow{\mu_{I\to\mathbf{1}}^{\widehat{\mathcal{C}}}} \\
&\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big] \\
= \big[\mathbb{1} &\xrightarrow{\lambda_{\mathcal{C}}^f} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i) \\
&\xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{j\in J}\otimes_{\mathcal{C}}^{i\in f^{-1}j} X_i)\big].
\end{aligned}
$$

This coincides with the left hand side. Hence, equation (3.30.1) is proven. Therefore, $(F,\phi^I) : \mathcal{C}\to\mathcal{D}$ is a lax (symmetric) Monoidal functor.

Now we are going to prove that the two constructed maps are inverse to each other. Given a multifunctor G, we have produced a lax Monoidal functor (F,ϕ^I) out of it. Let us prove that $\widehat{F} = G$. Indeed, both multifunctors give $\operatorname{Ob} F = \operatorname{Ob} G$ on objects. Both give on morphisms $F_{X,Y} = G_{X;Y}$. Let us show that both $\widehat{F}$ and G coincide on multimorphisms. Indeed,

$$
\begin{aligned}
\widehat{F}_{(X_i)_{i\in I};Y} = \big[&\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, Y) \xrightarrow{G_{\otimes_{\mathcal{C}}^{i\in I} X_i;Y}} \mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I} X_i, GY) \xrightarrow{\lambda^{\cdot\,\mathsf{I}}} \\
&\mathbb{1}_{\mathcal{V}}\otimes\mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I} X_i, GY) \xrightarrow{\eta\otimes 1} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I} X_i, \otimes_{\mathcal{C}}^{i\in I} X_i)\otimes\mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I} X_i, GY) \xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{i\in I} X_i}\otimes 1} \\
&\mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, G\otimes_{\mathcal{C}}^{i\in I} X_i)\otimes\mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I} X_i, GY) \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I} GX_i, GY)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \big[\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{\lambda^{\cdot\,\mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{\eta\otimes 1} \\
&\qquad \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,\otimes_{\mathcal{C}}^{i\in I}X_i) \otimes \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{G_{(X_i)_{i\in I};\otimes_{\mathcal{C}}^{i\in I}X_i}\otimes G_{\otimes_{\mathcal{C}}^{i\in I}X_i;Y}} \\
&\qquad \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}GX_i,G\otimes_{\mathcal{C}}^{i\in I}X_i) \otimes \mathcal{D}(G\otimes_{\mathcal{C}}^{i\in I}X_i,GY) \xrightarrow{\mu_{I\to\mathbf{1}}^{\widehat{\mathcal{D}}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}GX_i,GY)\big] \\
&= \big[\mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{\lambda^{\cdot\,\mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{\eta\otimes 1} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,\otimes_{\mathcal{C}}^{i\in I}X_i) \otimes \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \\
&\qquad \xrightarrow{\mu_{I\to\mathbf{1}}^{\widehat{\mathcal{C}}}} \mathcal{C}(\otimes_{\mathcal{C}}^{i\in I}X_i,Y) \xrightarrow{G_{(X_i)_{i\in I};Y}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}GX_i,GY)\big] = G_{(X_i)_{i\in I};Y}.
\end{aligned}
$$

Therefore, $\widehat{F} = G$.

Given a lax (symmetric) Monoidal functor $(F,\phi^I) : \mathcal{C} \to \mathcal{D}$, we make a (symmetric) multifunctor $G = \widehat{F}$ out of it via (3.28.1). It gives rise to a lax (symmetric) Monoidal functor $(H,\psi^I) : \mathcal{C} \to \mathcal{D}$. Let us prove that $(H,\psi^I) = (F,\phi^I)$. Indeed, both functors give $\operatorname{Ob} F = \operatorname{Ob} G = \operatorname{Ob} H$ on objects. Both coincide on morphisms, $H_{X,Y} = G_{X;Y} = F_{X,Y}$. Let us show that $\psi = \phi$:

$$
\begin{aligned}
\psi^I &= \big[\mathbb{1} \xrightarrow{\eta} \mathcal{C}(\otimes^{i\in I}X_i,\otimes^{i\in I}X_i) \xrightarrow{F_{\otimes^{i\in I}X_i;\otimes^{i\in I}X_i}} \mathcal{D}(F\otimes^{i\in I}X_i,F\otimes^{i\in I}X_i) \\
&\qquad \xrightarrow{\lambda^{\cdot\,\mathsf{I}}\cdot(\phi^I\otimes 1)} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}FX_i,F\otimes_{\mathcal{C}}^{i\in I}X_i) \otimes \mathcal{D}(F\otimes_{\mathcal{C}}^{i\in I}X_i,F\otimes_{\mathcal{C}}^{i\in I}X_i) \\
&\qquad \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}FX_i,F\otimes_{\mathcal{C}}^{i\in I}X_i)\big] \\
&= \big[\mathbb{1} \xrightarrow{\phi^I} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}FX_i,F\otimes_{\mathcal{C}}^{i\in I}X_i) \xrightarrow{\lambda^{\mathsf{I}\,\cdot}\cdot(1\otimes\eta)} \\
&\qquad \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}FX_i,F\otimes_{\mathcal{C}}^{i\in I}X_i) \otimes \mathcal{D}(F\otimes_{\mathcal{C}}^{i\in I}X_i,F\otimes_{\mathcal{C}}^{i\in I}X_i) \\
&\qquad \xrightarrow{\mu^{\mathcal{D}}} \mathcal{D}(\otimes_{\mathcal{D}}^{i\in I}FX_i,F\otimes_{\mathcal{C}}^{i\in I}X_i)\big] = \phi^I.
\end{aligned}
$$

Therefore, $(H,\psi^I) = (F,\phi^I)$, and bijectivity on (multi)functors is proven.

Bijectivity on transformations is clear. Thus Proposition 3.30 is proven. □

Chapter 4

Closed multicategories

Closed monoidal categories are well-understood and widely used in mathematics. Definition of closedness transfers without significant changes from monoidal categories to multicategories. A_∞-categories form the main example of a closed multicategory for this book. Having a closed symmetric multicategory C we construct in this chapter a symmetric multicategory $\underline{\mathsf{C}}$ enriched in C. Objects of $\underline{\mathsf{C}}$ are objects of C and $\underline{\mathsf{C}}((X_i)_{i\in I};Z) \in \operatorname{Ob}\mathsf{C}$ are inner homomorphism objects of C. For any multifunctor between closed multicategories we construct its closing transformation. We also consider augmented multifunctors which are multifunctors $F : \mathsf{C} \to \mathsf{C}$ equipped with a multinatural transformation $u_F : \mathrm{Id}_{\mathsf{C}} \to F$. Some multifunctors used in this book are monads. In particular, they are augmented. Augmented comonad multifunctors occur as well.

4.1 Multicategories enriched in multicategories. According to classical picture, categories can be enriched in monoidal categories, and (symmetric) monoidal categories can be enriched in symmetric monoidal categories. As we have seen, (symmetric) multicategories are generalizations of (symmetric) Monoidal categories. Therefore, it is not surprising that categories can be enriched in multicategories, and (symmetric) multicategories can be enriched in symmetric multicategories. We shall encounter such situations further in this book. In this section we give definitions of (multi)categories, (multi)functors, and (multi)natural transformations enriched in a symmetric multicategory V. At the down-to-earth level, a multicategory enriched in V comes with multiplications indexed by trees of height 2 which satisfy associativity equations corresponding to trees of height 3. Similarly, V-multifunctors can be given by morphisms of V for trees of height 1, so that equations corresponding to trees of height 2 hold. Analogously, (multi)natural V-transformations are given by morphisms for trees of height 0 satisfying equations corresponding to trees of height 1. Altogether V-multicategories, V-multifunctors and their (multi)natural transformations form a 2-category.

4.2 Definition. Let V be a symmetric multicategory. A *plain* (resp. *symmetric*) V-*multicategory* C consists of the following data.

- A V-multiquiver C.
- For each map $\phi : I \to J$ in $\operatorname{Mor}\mathcal{O}$ (resp. $\operatorname{Mor}\mathcal{S}$) and $X_i, Y_j, Z \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, a morphism

$$\mu^{\mathsf{C}}_\phi : \big(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}, \mathsf{C}((Y_j)_{j\in J};Z) \to \mathsf{C}((X_i)_{i\in I};Z) \tag{4.2.1}$$

in V, called *composition*. Its source is indexed by the totally ordered set $J \sqcup \mathbf{1}$.

- For each $X \in \operatorname{Ob}\mathsf{C}$ and 1-element set L, a morphism $1^{\mathsf{C}}_{X,L} : () \to \mathsf{C}((X)_L; X)$ in V, called the *identity* of X.

These data are subject to the following axioms.

- Associativity: for each pair of composable maps $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ in $\operatorname{Mor}\mathcal{S}$ (resp. $\operatorname{Mor}\mathcal{O}$) and objects $X_i, Y_j, Z_k, W \in \operatorname{Ob}\mathsf{C}$, $i \in I$, $j \in J$, $k \in K$, the diagram

$$\begin{tikzcd}[column sep=large]
\begin{matrix}\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J},\\ \big(\mathsf{C}((Y_j)_{j\in\psi^{-1}k}; Z_k)\big)_{k\in K},\\ \mathsf{C}((Z_k)_{k\in K}; W)\end{matrix} \arrow[r, "(1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j),\{j\}})_{j\in J},\mu^{\mathsf{C}}_{\psi}"] \arrow[d, "(\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1^{\mathsf{V}}_{\mathsf{C}((Z_k)_{k\in K};W),\mathbf{1}}"'] & \begin{matrix}\big(\mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J},\\ \mathsf{C}((Y_j)_{j\in J}; W)\end{matrix} \arrow[d, "\mu^{\mathsf{C}}_{\phi}"] \\
\begin{matrix}\mathsf{C}((X_i)_{i\in(\phi\psi)^{-1}k}; Z_k)\big)_{k\in K},\\ \mathsf{C}((Z_k)_{k\in K}; W)\end{matrix} \arrow[r, "\mu^{\mathsf{C}}_{\phi\psi}"] & \mathsf{C}((X_i)_{i\in I}; W)
\end{tikzcd}$$

commutes; more precisely, denoting composition in V simply by dot $\cdot$, the equation

$$\big((1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j),\{j\}})_{j\in J}, \mu^{\mathsf{C}}_{\psi}\big) \cdot_{\alpha} \mu^{\mathsf{C}}_{\phi} = \big((\mu^{\mathsf{C}}_{\phi_k})_{k\in K}, 1^{\mathsf{V}}_{\mathsf{C}((Z_k)_{k\in K};W),\mathbf{1}}\big) \cdot_{\beta} \mu^{\mathsf{C}}_{\phi\psi} \tag{4.2.2}$$

holds true. The map ϕ_k is the restriction $\phi|_{\phi^{-1}\psi^{-1}k} : \phi^{-1}\psi^{-1}k \to \psi^{-1}k$, $k \in K$. The map $\alpha = \mathrm{id}_J \sqcup \triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$ preserves the order, while the map $\beta : J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}$ is not necessarily order-preserving; it is given by (3.8.2).

- Right identity: for each $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, and for each map $I \to J$ from I to a 1-element set J, the equation

$$\big[(\mathsf{C}((X_i)_{i\in I}; Y))_J \xrightarrow{(1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y)})_J, 1^{\mathsf{C}}_{Y,J}} (\mathsf{C}((X_i)_{i\in I}; Y))_J, \mathsf{C}((Y)_J; Y) \xrightarrow{\mu^{\mathsf{C}}_{I\to J}} \mathsf{C}((X_i)_{i\in I}; Y)\big] = 1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y),J}$$

holds true, where the composition in V is taken in the sense of the map $J \hookrightarrow J \sqcup \mathbf{1}$;

- Left identity: for each $X_i, Y \in \operatorname{Ob}\mathsf{C}$, $i \in I$, the equation

$$\big[\mathsf{C}((X_i)_{i\in I}; Y) \xrightarrow{(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I}, 1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y),\mathbf{1}}} \big(\mathsf{C}((X_i)_{\{i\}}; X_i)\big)_{i\in I}, \mathsf{C}((X_i)_{i\in I}; Y) \xrightarrow{\mu^{\mathsf{C}}_{\mathrm{id}_I}} \mathsf{C}((X_i)_{i\in I}; Y)\big] = 1^{\mathsf{V}}_{\mathsf{C}((X_i)_{i\in I};Y),\mathbf{1}}$$

holds true, where the composition in V is take in the sense of the map $\mathbf{1} \hookrightarrow I \sqcup \mathbf{1}$.

A V-*category* is like a V-multicategory, the difference being that the objects $\mathsf{C}((X_i)_{i\in I};Y)$ of morphisms are given only for the set $I=\mathbf{1}$ and composition (4.2.1) is given only for the map $\mathbf{1}\to\mathbf{1}$. The associativity and identity axioms retain their meaning. Obviously, an arbitrary V-multicategory has an underlying V-category.

As in the case of multicategories enriched in a symmetric Monoidal category considered in Chapter 3, for each (resp. order preserving) bijection $\phi:I\to J$ such that $X_i=Y_{\phi(i)}$, $i\in I$, there is an isomorphism $\mathsf{C}(\phi;Z):\mathsf{C}((Y_j)_{j\in J};Z)\to\mathsf{C}((X_i)_{i\in I};Z)$ in V given by the composite

$$\mathsf{C}(\phi;Z)=\bigl[\mathsf{C}((Y_j)_{j\in J};Z)\xrightarrow{(1^{\mathsf{C}}_{X_{\phi^{-1}j},\{\phi^{-1}j\}})_{j\in J},1^{\mathsf{V}}_{\mathsf{C}((Y_j)_{j\in J};Z)}} \\ (\mathsf{C}((X_{\phi^{-1}j})_{\{\phi^{-1}j\}};Y_j))_{j\in J},\mathsf{C}((Y_j)_{j\in J};Z)\xrightarrow{\mu^{\mathsf{C}}_\phi}\mathsf{C}((X_i)_{i\in I};Z)\bigr]. \tag{4.2.3}$$

The axioms imply that $\mathsf{C}(\mathrm{id};Z)=\mathrm{id}$ and $\mathsf{C}(\psi;Z)\mathsf{C}(\phi;Z)=\mathsf{C}(\phi\psi;Z)$, whenever the left hand side is defined.

Given a symmetric multicategory V we shall consider not only V-(multi)categories, but also V-(multi)functors.

4.3 Definition. Let C, D be symmetric (resp. plain) V-multicategories. A symmetric (resp. plain) V-*multifunctor* $F:\mathsf{C}\to\mathsf{D}$ is a mapping of objects $\operatorname{Ob}F:\operatorname{Ob}\mathsf{C}\to\operatorname{Ob}\mathsf{D}$, $X\mapsto FX$, together with morphisms

$$F^L_{(X_i)_{i\in I};Y}\in\mathsf{V}\bigl((\mathsf{C}((X_i)_{i\in I};Y))_L;\mathsf{D}((FX_i)_{i\in I};FY)\bigr),$$

for each $X_i,Y\in\operatorname{Ob}\mathsf{C}$, $i\in I$, and 1-element set L, such that identities and composition are preserved. The former means that

$$\bigl[()\xrightarrow{(1^{\mathsf{C}}_{X,L})_K}(\mathsf{C}((X)_L;X))_K\xrightarrow{F^K_{(X)_L;X}}\mathsf{D}((FX)_L;FX)\bigr]=1^{\mathsf{D}}_{FX,L}, \tag{4.3.1}$$

for each $X\in\operatorname{Ob}\mathsf{C}$ and 1-element sets K and L. The composite in the left hand side is in the sense of the map $\varnothing\to K$. The second condition means that for each map $\phi:I\to J$ in $\operatorname{Mor}\mathcal{S}$ (resp. $\operatorname{Mor}\mathcal{O}$), objects $X_i,Y_j,Z\in\operatorname{Ob}\mathsf{C}$, $i\in I$, $j\in J$, and 1-element set L, the diagram

$$\begin{tikzcd}
\bigl(\mathsf{C}((X_i)_{i\in\phi^{-1}j};Y_j)\bigr)_{j\in J},\mathsf{C}((Y_j)_{j\in J};Z) \arrow[r, "(\mu^{\mathsf{C}}_\phi)_L"] \arrow[d, "(F^{\{j\}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},F^{\mathbf{1}}_{(Y_j)_{j\in J};Z}"'] & (\mathsf{C}((X_i)_{i\in I};Z))_L \arrow[d, "F^L_{(X_i)_{i\in I};Z}"] \\
\bigl(\mathsf{D}((FX_i)_{i\in\phi^{-1}j};FY_j)\bigr)_{j\in J},\mathsf{D}((FY_j)_{j\in J};FZ) \arrow[r, "\mu^{\mathsf{D}}_\phi"] & \mathsf{D}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{4.3.2}$$

commutes. Here the top-right composite is calculated in the sense of the map $\triangleright : J \sqcup \mathbf{1} \to L$; the right-bottom composite is calculated in the sense of the map $\mathrm{id}_{J\sqcup\mathbf{1}} : J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$. For two 1-element sets K and L, the canonical isomorphism

$$\mathsf{V}(\triangleright; \mathsf{D}((FX_i)_{i\in I}; FY)) : \mathsf{V}((\mathsf{C}((X_i)_{i\in I}; Y))_L; \mathsf{D}((FX_i)_{i\in I}; FY)) \to \mathsf{V}((\mathsf{C}((X_i)_{i\in I}; Y))_K; \mathsf{D}((FX_i)_{i\in I}; FY)),$$

defined by (3.7.3), maps $F^L_{(X_i)_{i\in I};Y}$ to $F^K_{(X_i)_{i\in I};Y}$. Therefore, each of the morphisms $F^L_{(X_i)_{i\in I};Y}$ determines the other $F^K_{(X_i)_{i\in I};Y}$ unambiguously. For V-categories C and D, a V-*functor* $F : \mathsf{C} \to \mathsf{D}$ is a mapping of objects $\operatorname{Ob} F : \operatorname{Ob} \mathsf{C} \to \operatorname{Ob} \mathsf{D}$, $X \mapsto FX$, together with morphisms $F = F_{X;Y} \in \mathsf{V}\big((\mathsf{C}(X;Y))_{\mathbf{1}}; \mathsf{D}(FX;FY)\big)$ which satisfy equations (4.3.1) and (4.3.2) for $I = J = \mathbf{1}$. Obviously, a V-multifunctor induces a V-functor between the underlying V-categories.

We complete the V-multicategory picture for a given symmetric multicategory V by considering multinatural (resp. natural) transformations of V-multifunctors (resp. V-functors).

4.4 Definition. A *multinatural transformation* of V-multifunctors $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a family of elements $r_{X,L} \in \mathsf{V}(\,;\mathsf{D}((FX)_L; GX))$, for each $X \in \operatorname{Ob}\mathsf{C}$ and 1-element set L, such that for each $X_i, Y \in \operatorname{Ob}\mathsf{C}$ and 1-element sets J and L, the diagram

$$\begin{tikzcd}
(\mathsf{C}((X_i)_{i\in I}; Y))_L \arrow[r, "(F^L_{(X_i)_{i\in I};Y})_J, r_{Y,J}"] \arrow[d, "(r_{X_i,\{i\}})_{i\in I}, G^L_{(X_i)_{i\in I};Y}"'] & (\mathsf{D}((FX_i)_{i\in I}; FY))_J, \mathsf{D}((FY)_J; GY) \arrow[d, "\mu^{\mathsf{D}}_{I\to J}"] \\
\big(\mathsf{D}((FX_i)_{\{i\}}; GX_i)\big)_{i\in I}, \mathsf{D}((GX_i)_{i\in I}; GY) \arrow[r, "\mu^{\mathsf{D}}_{\mathrm{id}_I}"] & \mathsf{D}((FX_i)_{i\in I}; GY)
\end{tikzcd} \tag{4.4.1}$$

commutes. The top-right composite in V is taken in the sense of the map $L \xrightarrow{\triangleright} J \hookrightarrow J \sqcup \mathbf{1}$, while the right-bottom composite is taken in the sense of the map $L \xrightarrow{\triangleright} \mathbf{1} \hookrightarrow I \sqcup \mathbf{1}$. A *natural transformation* of V-functors $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a family of elements $r_X \in \mathsf{V}(\,;\mathsf{D}(FX;GX))$, $X \in \operatorname{Ob}\mathsf{C}$, such that diagram (4.4.1) commutes for $I = \mathbf{1}$.

If $r : F \to G : \mathsf{C} \to \mathsf{D}$ is a multinatural transformation of V-multifunctors, then the elements $r_{X,I} \in \mathsf{V}(\,;\mathsf{D}((FX)_I; GX))$ and $r_{X,J} \in \mathsf{V}(\,;\mathsf{D}((FX)_J; GX))$ for 1-element sets I and J are related by the formula

$$r_{X,I} = \big[() \xrightarrow{r_{X,J}} \mathsf{D}((FX)_J; GX) \xrightarrow{\mathsf{D}(\triangleright;GX)} \mathsf{D}((FX)_I; GX)\big],$$

where $\triangleright : I \to J$ is the only map and $\mathsf{D}(\triangleright; GX)$ is given by (4.2.3).

From now on we drop 1-element indexing sets from the notation, sometimes replacing them all with $*$ which might indicate different 1-element sets in the same formula. Sometimes instead of $*$ with use $\mathbf{1}$ with the same meaning. The reader can recover the exact

form of the expressions using the definitions given above. In practice, the dependence on a 1-element set is the same as the dependence of the product of a 1-element family $(S)_{i\in I}$ on the indexing set I. Formally, the product $\prod_{i\in I} S$ differs from the set S, but this difference will be ignored in any practical reasoning.

4.5 Proposition. *Let* V *be a symmetric multicategory. Then ($\mathscr{U}$-small) (possibly symmetric)* V*-(multi)categories,* V*-(multi)functors and their (multi)natural transformations form a 2-category.*

Proof. Given V-multifunctors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$ we see that the composition $H = F \cdot G : \mathsf{C} \to \mathsf{E}$ with $\operatorname{Ob} H = \operatorname{Ob} F \cdot \operatorname{Ob} G$, $H_{(X_i)_{i\in I};Y} = F_{(X_i)_{i\in I};Y} \cdot G_{(FX_i)_{i\in I};FY}$ is a V-multifunctor as well.

The 'vertical' composition $\kappa = \lambda{\cdot}\nu$ of (multi)natural transformations $F \xrightarrow{\lambda} G \xrightarrow{\nu} H : \mathsf{C} \to \mathsf{D}$ is given by morphisms

$$\kappa_X = \bigl[() \xrightarrow{\lambda_X,\nu_X} \mathsf{D}(FX;GX), \mathsf{D}(GX;HX) \xrightarrow{\mu^{\mathsf{D}}_{*\to*}} \mathsf{D}(FX;HX)\bigr].$$

It satisfies equation (4.4.1) as the following verification shows:

$$\begin{aligned}
&\bigl[\mathsf{C}((X_i)_{i\in I};Y) \xrightarrow{F,\lambda_Y,\nu_Y} \mathsf{D}((FX_i)_{i\in I};FY), \mathsf{D}(FY;GY), \mathsf{D}(GY;HY) \\
&\qquad\qquad \overset{(1,\mu^{\mathsf{D}}_{*\to*})\cdot\mu^{\mathsf{D}}_{I\to*}}{\underset{(\mu^{\mathsf{D}}_{I\to*},1)\cdot\mu^{\mathsf{D}}_{I\to*}}{\relbar\!\!\Vert\!\!\longrightarrow}} \mathsf{D}((FX_i)_{i\in I};HY)\bigr] \\
= {}&\bigl[\mathsf{C}((X_i)_{i\in I};Y) \xrightarrow{(\lambda_{X_i})_{i\in I},G,\nu_Y} (\mathsf{D}(FX_i;GX_i))_{i\in I}, \mathsf{D}((GX_i)_{i\in I};GY), \mathsf{D}(GY;HY) \\
&\qquad\qquad \overset{(\mu^{\mathsf{D}}_{\mathrm{id}_I},1)\cdot\mu^{\mathsf{D}}_{I\to*}}{\underset{((1)_I,\mu^{\mathsf{D}}_{I\to*})\cdot\mu^{\mathsf{D}}_{\mathrm{id}_I}}{\relbar\!\!\Vert\!\!\longrightarrow}} \mathsf{D}((FX_i)_{i\in I};HY)\bigr] \\
= {}&\bigl[\mathsf{C}((X_i)_{i\in I};Y) \xrightarrow{(\lambda_{X_i})_{i\in I},(\nu_{X_i})_{i\in I},H} \\
&\qquad (\mathsf{D}(FX_i;GX_i))_{i\in I}, (\mathsf{D}(GX_i;HX_i))_{i\in I}, \mathsf{D}((HX_i)_{i\in I};HY) \\
&\qquad\qquad \overset{((1)_I,\mu^{\mathsf{D}}_{\mathrm{id}_I})\cdot\mu^{\mathsf{D}}_{\mathrm{id}_I}}{\underset{((\mu^{\mathsf{D}}_{*\to*})_I,1)\cdot\mu^{\mathsf{D}}_{\mathrm{id}_I}}{\relbar\!\!\Vert\!\!\longrightarrow}} \mathsf{D}((FX_i)_{i\in I};HY)\bigr].
\end{aligned}$$

Composition $\nu = \Bigl(\mathsf{C} \underset{G}{\overset{F}{\underset{\longrightarrow}{\overset{\longrightarrow}{\lambda\Downarrow}}}} \mathsf{D} \xrightarrow{H} \mathsf{E}\Bigr)$ of a multinatural transformation λ and a V-multifunctor H is specified by the morphisms

$$\nu_X = \bigl[() \xrightarrow{\lambda_X} \mathsf{D}(FX;GX) \xrightarrow{H_{FX;GX}} \mathsf{E}(HFX;HGX)\bigr].$$

It satisfies equation (4.4.1) due to the same equation for λ composed with $H_{(FX_i)_{i\in I};GY}$.

Composition $\nu = \Big(\mathsf{B} \xrightarrow{H} \mathsf{C} \underset{G}{\overset{F}{\underset{\lambda\Downarrow}{\rightrightarrows}}} \mathsf{D}\Big)$ of a V-multifunctor H and a multinatural transformation λ is given by the morphisms $\nu_X = \lambda_{HX} : () \to \mathsf{D}(FHX; GHX)$. It satisfies equation (4.4.1) due to composition of $H_{(X_i)_{i\in I};Y} : \mathsf{B}((X_i)_{i\in I}; Y) \to \mathsf{C}((HX_i)_{i\in I}; HY)$ with the same equation for λ written for the objects $(HX_i)_{i\in I}$, HY.

Identity V-multifunctors and identity multinatural transformations are the obvious ones. Axioms of a 2-category are satisfied due to associativity of the composition in V and in V-multicategories. Also equation (4.4.1) is used. For instance, the axiom for 'horizontal' composition $\mathsf{C} \underset{G}{\overset{F}{\underset{\lambda\Downarrow}{\rightrightarrows}}} \mathsf{D} \underset{K}{\overset{H}{\underset{\nu\Downarrow}{\rightrightarrows}}} \mathsf{E}$ of multinatural transformations λ, ν of V-multifunctors reads:

$$
\begin{aligned}
&\big[() \xrightarrow{\lambda_X} \mathsf{D}(FX; GX) \xrightarrow{H_{FX;GX}, \nu_{GX}} \mathsf{E}(HFX; HGX), \mathsf{E}(HGX; KGX) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\mu^{\mathsf{E}}_{*\to *}} \mathsf{E}(HFX; KGX)\big] \\
&= \big[() \xrightarrow{\lambda_X} \mathsf{D}(FX; GX) \xrightarrow{\nu_{FX}, K_{FX;GX}} \\
&\qquad\qquad \mathsf{E}(HFX; KFX), \mathsf{E}(KFX; KGX) \xrightarrow{\mu^{\mathsf{E}}_{*\to *}} \mathsf{E}(HFX; KGX)\big].
\end{aligned}
$$

It follows from equation (4.4.1), written for ν, a 1-element set I, and objects FX, GX of D. □

4.6 Definition. A plain Monoidal $\mathcal{V}$-category $\mathcal{C}$ is *closed* if for each pair X, Z of objects of $\mathcal{C}$ there is an object $\underline{\mathcal{C}}(X, Z)$ of $\mathcal{C}$ and an evaluation element

$$\mathrm{ev}^{\mathcal{C}}_{X,Z} \in \mathcal{C}\big(X \otimes \underline{\mathcal{C}}(X, Z), Z\big)$$

such that the composition in $\mathcal{V}$

$$
\begin{aligned}
\varphi^{\mathcal{C}} = \big[&\mathcal{C}(Y, \underline{\mathcal{C}}(X, Z)) \xrightarrow{\lambda^{\cdot \mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(Y, \underline{\mathcal{C}}(X, Z)) \xrightarrow{1_X \otimes \mathrm{id}} \mathcal{C}(X, X) \otimes \mathcal{C}(Y, \underline{\mathcal{C}}(X, Z)) \\
&\xrightarrow{\otimes^{\mathbf{2}}_{\mathcal{C}}} \mathcal{C}(X \otimes Y, X \otimes \underline{\mathcal{C}}(X, Z)) \xrightarrow{\lambda^{\mathsf{I}\cdot}} \mathcal{C}(X \otimes Y, X \otimes \underline{\mathcal{C}}(X, Z)) \otimes \mathbb{1}_{\mathcal{V}} \\
&\xrightarrow{\mathrm{id} \otimes \mathrm{ev}^{\mathcal{C}}} \mathcal{C}(X \otimes Y, X \otimes \underline{\mathcal{C}}(X, Z)) \otimes \mathcal{C}(X \otimes \underline{\mathcal{C}}(X, Z), Z) \xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(X \otimes Y, Z)\big]
\end{aligned}
\tag{4.6.1}
$$

is an isomorphism, for arbitrary objects X, Y, Z of $\mathcal{C}$. A symmetric (resp. braided) Monoidal category is closed if its underlying plain Monoidal category is closed.

The definition of a closed $\mathcal{S}et$-multicategory was first given by Joachim Lambek [Lam69, p. 106] in an equivalent form to the following

4.7 Definition. A plain $\mathcal{V}$-multicategory C is *closed* if for any collection $((X_i)_{i\in I}, Z)$, $I \in \mathrm{Ob}\,\mathcal{S}$, of objects of C there is an object $\underline{\mathsf{C}}((X_i)_{i\in I}; Z)$ of C and an evaluation element

$$\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} \in \mathsf{C}\big((X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z); Z\big),$$

where the sequence of inputs is indexed by the totally ordered set $I \sqcup \mathbf{1}$, such that the composition in $\mathcal{V}$ with $\iota = \bigl(\mathbf{1} = \varnothing \sqcup \mathbf{1} \sqcup \varnothing \xrightarrow{\triangleleft \sqcup \mathrm{id} \sqcup \triangleleft} I \sqcup \mathbf{1} \sqcup \mathbf{1} = I \sqcup \mathbf{2}\bigr)$

$$\begin{aligned}
\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} = \bigl[&\mathsf{C}\bigl((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\bigr) \\
&\xrightarrow{\lambda^{\iota:\mathbf{1}\hookrightarrow I\sqcup\mathbf{2}}} \otimes^{I\sqcup\mathbf{2}}\bigl[(\mathbb{1}_{\mathcal{V}})_{i\in I},\mathsf{C}\bigl((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\bigr),\mathbb{1}_{\mathcal{V}}\bigr] \xrightarrow{\otimes^{I\sqcup\mathbf{2}}[(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},\mathrm{id},\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}]} \\
&\otimes^{I\sqcup\mathbf{2}}\bigl[\bigl(\mathsf{C}((X_i)_{\{i\}};X_i)\bigr)_{i\in I},\mathsf{C}\bigl((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\bigr),\mathsf{C}\bigl((X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Z);Z\bigr)\bigr] \\
&\xrightarrow{\mu^{\mathsf{C}}_{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}} \mathsf{C}\bigl((X_i)_{i\in I},(Y_j)_{j\in J};Z\bigr)\bigr]
\end{aligned} \tag{4.7.1}$$

is an isomorphism for an arbitrary sequence $(Y_j)_{j\in J}$, $J \in \operatorname{Ob}\mathcal{S}$, of objects of C. A symmetric (resp. braided) multicategory is closed if its underlying plain multicategory is closed.

Here $\triangleleft : \varnothing \to K$ for an arbitrary set K is the only map. Concatenation of sequences indexed by I and J is indexed by the disjoint union $I \sqcup J$, where $i < j$ for all $i \in I$, $j \in J$.

Notice that for $I = \varnothing$ an object $\underline{\mathsf{C}}(;Z)$ and an element $\mathrm{ev}^{\mathsf{C}}_{;Z}$ with the required property always exist. Namely, we shall always take $\underline{\mathsf{C}}(;Z) = Z$ and $\mathrm{ev}^{\mathsf{C}}_{;Z} = 1_Z : Z \to Z$. With this choice $\varphi_{(Y_j)_{j\in J};;Z} = \mathrm{id} : \mathsf{C}\bigl((Y_j)_{j\in J};Z\bigr) \to \mathsf{C}\bigl((Y_j)_{j\in J};Z\bigr)$ is the identity map.

The second of the above definitions is a generalization of the first, as the following result shows.

4.8 Proposition. *Let $\mathcal{C}$ be a (symmetric) closed Monoidal $\mathcal{V}$-category. Then $\widehat{\mathcal{C}}$ is closed, inner homomorphism objects are $\underline{\widehat{\mathcal{C}}}((X_i)_{i\in I};Z) = \underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)$ with evaluations represented by compositions in $\mathcal{C}$*

$$\mathrm{ev}^{\widehat{\mathcal{C}}} = \bigl[\otimes^{I\sqcup\mathbf{1}}((X_i)_{i\in I},\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)) \xrightarrow{\lambda^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}_{\mathcal{C}}} (\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\bigr].$$

Proof. The above formula of $\mathrm{ev}^{\widehat{\mathcal{C}}}$ is a short form of

$$\begin{aligned}
\mathrm{ev}^{\widehat{\mathcal{C}}} = \bigl[&\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\varnothing\to\mathbf{2}}} \mathbb{1}_{\mathcal{V}}\otimes\mathbb{1}_{\mathcal{V}} \xrightarrow{\lambda^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}_{\mathcal{C}}\otimes\mathrm{ev}^{\mathcal{C}}} \\
&\mathcal{C}\bigl(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)],(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)\bigr) \\
&\qquad\otimes\mathcal{C}\bigl((\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z),Z\bigr) \\
&\xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)],Z)\bigr].
\end{aligned}$$

Since $\mathcal{C}$ is a closed Monoidal $\mathcal{V}$-category, the morphism $\varphi^{\mathcal{C}}$ given by (4.6.1) is an isomor-

phism in $\mathcal{V}$. According to (4.7.1) we have:

$$
\begin{aligned}
\varphi^{\widehat{\mathcal{C}}} = \big[&\mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \xrightarrow{\lambda^{\iota:\mathbf{1}\hookrightarrow I\sqcup\mathbf{2}}} \\
&\otimes^{I\sqcup\mathbf{2}}\,[(\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)), \mathbb{1}_{\mathcal{V}}] \xrightarrow{\otimes^{I\sqcup\mathbf{2}}[(\mathrm{id})_{i\in I},\mathrm{id},\lambda^{\varnothing\to\mathbf{2}}]} \\
&\otimes^{I\sqcup\mathbf{2}}\,[(\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)), \mathbb{1}_{\mathcal{V}}\otimes\mathbb{1}_{\mathcal{V}}] \xrightarrow{\otimes^{I\sqcup\mathbf{2}}[(1_{X_i})_{i\in I},\mathrm{id},\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}\otimes\mathrm{ev}^{\mathcal{C}}]} \\
&\otimes^{I\sqcup\mathbf{2}}\,[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)), \\
&\qquad \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], (\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \\
&\qquad\qquad \otimes\mathcal{C}((\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z), Z)] \\
&\xrightarrow{\otimes^{I\sqcup\mathbf{2}}[(1_{X_i})_{i\in I},\mathrm{id},\mu^{\mathcal{C}}]} \\
&\otimes^{I\sqcup\mathbf{2}}\,[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)), \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], Z)] \\
&\xrightarrow{\lambda^{\gamma:I\sqcup\mathbf{2}\to\mathbf{2}}} \\
&\otimes^{I\sqcup\mathbf{1}}\,[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))]\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], Z) \\
&\xrightarrow{\otimes_{\mathcal{C}}^{I\sqcup\mathbf{1}}\otimes\mathrm{id}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], \otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)]) \\
&\qquad \otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], Z) \\
&\xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \xrightarrow{\lambda^{\cdot\shortmid}} \\
&\mathbb{1}_{\mathcal{V}}\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \xrightarrow{\lambda_{\mathcal{C}}^{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}\otimes\mathrm{id}} \\
&\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I}, (Y_j)_{j\in J}], \otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j])\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \\
&\qquad \xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I}, (Y_j)_{j\in J}], Z)\big],
\end{aligned}
$$

where

$$
\begin{aligned}
\iota &= \triangleleft\sqcup\mathrm{id}\sqcup\triangleleft : \mathbf{1} = \varnothing\sqcup\mathbf{1}\sqcup\varnothing \to I\sqcup\mathbf{1}\sqcup\mathbf{1} = I\sqcup\mathbf{2}, \\
\gamma &= \triangleright\sqcup\mathrm{id} : I\sqcup\mathbf{2} = (I\sqcup\mathbf{1})\sqcup\mathbf{1} \to \mathbf{1}\sqcup\mathbf{1} = \mathbf{2}.
\end{aligned}
$$

Using associativity of $\mu^{\mathcal{C}}$ we can transform this expression as follows:

$$
\begin{aligned}
\varphi^{\widehat{\mathcal{C}}} = \big[&\mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \xrightarrow{\lambda^{\triangleleft\sqcup\mathrm{id}:\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}} \\
&\otimes^{I\sqcup\mathbf{1}}\,[(\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\otimes^{I\sqcup\mathbf{1}}[(1_{X_i})_{i\in I},\mathrm{id}]} \otimes^{I\sqcup\mathbf{1}}\,[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\lambda^{\shortmid\cdot}} \otimes^{I\sqcup\mathbf{1}}\,[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))]\otimes\mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{C}}^{I\sqcup\mathbf{1}}\otimes\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], \otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)])\otimes
\end{aligned}
$$

$$
\begin{aligned}
&\otimes \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \\
&\xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \\
&\xrightarrow{\lambda^{\mathsf{I}\cdot}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \otimes \mathbb{1}_{\mathcal{V}} \xrightarrow{\mathrm{id}\otimes\mathrm{ev}^{\mathcal{C}}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \otimes \mathcal{C}((\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z), Z) \\
&\xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \xrightarrow{\lambda^{\cdot\mathsf{I}}} \\
&\mathbb{1}_{\mathcal{V}} \otimes \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \xrightarrow{\lambda_{\mathcal{C}}^{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}\otimes\mathrm{id}} \\
&\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I}, (Y_j)_{j\in J}], \otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j]) \otimes \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], Z) \\
&\xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I}, (Y_j)_{j\in J}], Z)\big].
\end{aligned}
$$

Naturality of $\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}$ is expressed by the following equation:

$$
\begin{aligned}
&\big[\otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \xrightarrow{\lambda^{\mathsf{I}\cdot}} \\
&\quad \otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \otimes \mathbb{1}_{\mathcal{V}} \xrightarrow{\otimes_{\mathcal{C}}^{I\sqcup\mathbf{1}}\otimes\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], \otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)])\otimes \\
&\quad \otimes \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \\
&\quad \xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))\big] \\
&= \big[\otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \xrightarrow{\lambda^{\cdot\mathsf{I}}} \\
&\quad \mathbb{1}_{\mathcal{V}} \otimes \otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \xrightarrow{\mathrm{id}\otimes\lambda^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}\cdot(\otimes_{\mathcal{C}}^{I}\otimes\mathrm{id})} \\
&\quad \mathbb{1}_{\mathcal{V}} \otimes [\mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \xrightarrow{\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}\otimes\otimes_{\mathcal{C}}^{\mathbf{2}}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes (\otimes^{j\in J}Y_j))\otimes \\
&\quad \otimes \mathcal{C}((\otimes^{i\in I}X_i) \otimes (\otimes^{j\in J}Y_j), (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \\
&\quad \xrightarrow{\mu^{\mathcal{C}}} \mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))\big].
\end{aligned}
$$

This allows to write

$$
\begin{aligned}
\varphi^{\widehat{\mathcal{C}}} = \big[&\mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z)) \xrightarrow{\lambda^{\triangleleft\sqcup\mathrm{id}:\mathbf{1}\hookrightarrow I\sqcup\mathbf{1}}} \otimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_{\mathcal{V}})_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\otimes^{I\sqcup\mathbf{1}}[(1_{X_i})_{i\in I},\mathrm{id}]} \otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\lambda^{\cdot\mathsf{I}}} \mathbb{1}_{\mathcal{V}} \otimes \otimes^{I\sqcup\mathbf{1}}[(\mathcal{C}(X_i,X_i))_{i\in I}, \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\mathrm{id}\otimes\lambda^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}} \mathbb{1}_{\mathcal{V}} \otimes [(\otimes^{i\in I}\mathcal{C}(X_i,X_i)) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \\
&\xrightarrow{\mathrm{id}\otimes(\otimes_{\mathcal{C}}^{I}\otimes\mathrm{id})} \mathbb{1}_{\mathcal{V}} \otimes [\mathcal{C}(\otimes^{i\in I}X_i, \otimes^{i\in I}X_i) \otimes \mathcal{C}(\otimes^{j\in J}Y_j, \underline{\mathcal{C}}(\otimes^{i\in I}X_i, Z))] \xrightarrow{\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}\otimes\otimes_{\mathcal{C}}^{\mathbf{2}}} \\
&\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I}, \otimes^{j\in J}Y_j], (\otimes^{i\in I}X_i) \otimes (\otimes^{j\in J}Y_j))
\end{aligned}
$$

$$\otimes\, \mathcal{C}((\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j),(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))$$
$$\xrightarrow{\mu^{\mathcal{C}}}\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))$$
$$\xrightarrow{\lambda^{\mathsf{I}\cdot}}\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))\otimes\mathbb{1}_{\mathcal{V}}\xrightarrow{\mathrm{id}\otimes\mathrm{ev}^{\mathcal{C}}}$$
$$\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))\otimes\mathcal{C}((\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z),Z)$$
$$\xrightarrow{\mu^{\mathcal{C}}}\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)\xrightarrow{\lambda^{\cdot\mathsf{I}}}$$
$$\mathbb{1}_{\mathcal{V}}\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)\xrightarrow{\lambda_{\mathcal{C}}^{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}\otimes\mathrm{id}}$$
$$\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I},(Y_j)_{j\in J}],\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j])\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)$$
$$\xrightarrow{\mu^{\mathcal{C}}}\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I},(Y_j)_{j\in J}],Z)\big].$$

Using associativity of $\mu^{\mathcal{C}}$ and the identity $\otimes_{\mathcal{C}}^{i\in I}1_{X_i}=1_{\otimes^{i\in I}X_i}$ implied by the fact that $\otimes_{\mathcal{C}}^{I}$ is a $\mathcal{V}$-functor, we obtain:

$$\varphi^{\widehat{\mathcal{C}}}=\big[\mathcal{C}(\otimes^{j\in J}Y_j,\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))\xrightarrow{\varphi^{\mathcal{C}}}\mathcal{C}((\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j),Z)$$
$$\xrightarrow{\lambda^{\cdot\mathsf{I}}}\mathbb{1}_{\mathcal{V}}\otimes\mathcal{C}((\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j),Z)\xrightarrow{\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}}\otimes\mathrm{id}}$$
$$\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],(\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j))\otimes\mathcal{C}((\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j),Z)\xrightarrow{\mu^{\mathcal{C}}}$$
$$\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)\xrightarrow{\lambda^{\cdot\mathsf{I}}}\mathbb{1}_{\mathcal{V}}\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)\xrightarrow{\lambda_{\mathcal{C}}^{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}\otimes\mathrm{id}}$$
$$\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I},(Y_j)_{j\in J}],\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j])\otimes\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)$$
$$\xrightarrow{\mu^{\mathcal{C}}}\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I},(Y_j)_{j\in J}],Z)\big],$$

that is,

$$\varphi^{\widehat{\mathcal{C}}}=\big[\mathcal{C}(\otimes^{j\in J}Y_j,\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z))\xrightarrow[\sim]{\varphi^{\mathcal{C}}}\mathcal{C}((\otimes^{i\in I}X_i)\otimes(\otimes^{j\in J}Y_j),Z)\xrightarrow[\sim]{\mathcal{C}(\lambda_{\mathcal{C}}^{\triangleright\sqcup\mathrm{id}:I\sqcup\mathbf{1}\to\mathbf{2}},1)}$$
$$\mathcal{C}(\otimes^{I\sqcup\mathbf{1}}[(X_i)_{i\in I},\otimes^{j\in J}Y_j],Z)\xrightarrow[\sim]{\mathcal{C}(\lambda_{\mathcal{C}}^{I\sqcup J\to I\sqcup\mathbf{1}},1)}\mathcal{C}(\otimes^{I\sqcup J}[(X_i)_{i\in I},(Y_j)_{j\in J}],Z)\big].$$

In particular, $\varphi^{\widehat{\mathcal{C}}}$ is an isomorphism. □

4.9 A closed multicategory gives an enriched multicategory. We show that a closed multicategory C gives rise to multicategory $\underline{\mathsf{C}}$ enriched in C. We begin by introducing some notation.

Let $t\in\mathsf{N}'_n\mathcal{S}$ be a symmetric sequential tree. We associate with it the symmetric sequential tree $\bar{t}\in\mathsf{N}'_n\mathcal{S}$ with ordered sets $\bar{t}(m)=t(m)\sqcup t(m+1)\sqcup\cdots\sqcup t(n)$, $m\in[n]$, and maps $\bar{t}_m:\bar{t}(m-1)\to\bar{t}(m)$, determined by

$$\bar{t}_m\big|_{t(m-1)}=\big(t(m-1)\xrightarrow{t_m}t(m)\hookrightarrow\bar{t}(m)\big),\qquad \bar{t}_m\big|_{t(m)\sqcup t(m+1)\sqcup\cdots\sqcup t(n)}=\mathrm{id}\,.$$

For example, a map $\phi : I \to J$ in $\operatorname{Mor} \mathcal{S}$ gives rise to a sequential tree $t = (I \xrightarrow{\phi} J \to \mathbf{1})$ of height 2. The induced tree $\bar{t}$ is of the form $(I \sqcup J \sqcup \mathbf{1} \xrightarrow{\overline{\phi}} J \sqcup \mathbf{1} \to \mathbf{1})$, where $\overline{\phi}$ is given by

$$\overline{\phi}|_I = \bigl(I \xrightarrow{\phi} J \hookrightarrow J \sqcup \mathbf{1}\bigr), \quad \overline{\phi}|_{J\sqcup\mathbf{1}} = \mathrm{id} : J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}. \tag{4.9.1}$$

In the sequel, composition

$$\mu^{\mathsf{C}}_{\phi} : \prod_{j\in J} \mathsf{C}((X_i)_{i\in\phi^{-1}j}; Y_j) \times \mathsf{C}((Y_j)_{j\in J}; Z) \to \mathsf{C}((X_i)_{i\in I}; Z)$$

in the multicategory C associated with a map $\phi : I \to J$ in $\operatorname{Mor} \mathcal{S}$ is written as $((f_j)_{j\in J}, g) \mapsto (f_j)_{j\in J} \cdot_{\phi} g$, or even as $((f_j)_{j\in J}, g) \mapsto (f_j)_{j\in J} \cdot g$ if $\phi : I \to J$ is an order-preserving map.

4.10 Proposition. *A closed symmetric multicategory* C *gives rise to a symmetric multicategory* $\underline{\mathsf{C}}$ *enriched in* C.

Proof. For each map $\phi : I \to J$ in $\operatorname{Mor} \mathcal{S}$ and $X_i, Y_j, Z \in \operatorname{Ob} \mathsf{C}$, $i \in I$, $j \in J$, there exists a unique morphism

$$\mu^{\underline{\mathsf{C}}}_{\phi} : \bigl(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\bigr)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$$

that makes the diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, \bigl(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\bigr)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},\mu^{\underline{\mathsf{C}}}_{\phi}"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};Z),\mathbf{1}}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}"] & Z
\end{tikzcd} \tag{4.10.1}$$

commute. More precisely, the commutativity in the above diagram means that the equation

$$\bigl((1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I}, \mu^{\underline{\mathsf{C}}}_{\phi}\bigr) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} = \bigl((\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};Z),\mathbf{1}}\bigr) \cdot_{\overline{\phi}} \mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}$$

holds true, where $\mathrm{id}_I \sqcup \triangleright : I \sqcup J \sqcup \mathbf{1} \to I \sqcup \mathbf{1}$, and $\overline{\phi} : I \sqcup J \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$ is given by (4.9.1). Furthermore, for each $X \in \operatorname{Ob} \mathsf{C}$ and a 1-element set L, there is a morphism

$$1^{\underline{\mathsf{C}}}_{X,L} \overset{\mathrm{def}}{=} \varphi^{-1}_{;(X)_L;X}(1^{\mathsf{C}}_{X,L}) \in \mathsf{C}(\,;\underline{\mathsf{C}}((X)_L; X)).$$

It is a unique solution to the equation

$$\bigl[(X)_L \xrightarrow{1^{\mathsf{C}}_{X,L},1^{\underline{\mathsf{C}}}_{X,L}} (X)_L, \underline{\mathsf{C}}((X)_L; X) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(X)_L;X}} X\bigr] = 1^{\mathsf{C}}_{X,L},$$

where the composition in V is taken in the sense of the map $L \hookrightarrow L \sqcup \mathbf{1}$. Let us check the conditions of Definition 4.2. Let $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ be a pair of composable maps, and let $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$, $(Z_k)_{k\in K}$, W be families of objects of C. Equation (4.2.2) reads

$$\big((1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j),\{j\}})_{j\in J}, \mu^{\mathsf{C}}_{\psi}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi} = \big((\mu^{\mathsf{C}}_{\phi_k})_{k\in K}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),\mathbf{1}}\big) \cdot_{\overline{\psi}} \mu^{\mathsf{C}}_{\phi\psi},$$

where $\mathrm{id}_J \sqcup \triangleright : J \sqcup K \sqcup \mathbf{1} \to J \sqcup \mathbf{1}$, and $\overline{\psi} : J \sqcup K \sqcup \mathbf{1} \to K \sqcup \mathbf{1}$ is given by (4.9.1). Applying the transformation $\varphi_{(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};Z);(X_i)_{i\in I};Z}$ one gets an equivalent equation

$$\begin{aligned}\Big((1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I}, \big[\big((1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j),\{j\}})_{j\in J}, \mu^{\mathsf{C}}_{\psi}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z} \\ = \Big((1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I}, \big[\big((\mu^{\mathsf{C}}_{\phi_k})_{k\in K}, 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),\mathbf{1}}\big) \cdot_{\overline{\psi}} \mu^{\mathsf{C}}_{\phi\psi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z},\end{aligned}$$

where $\mathrm{id}_I \sqcup \triangleright : I \sqcup J \sqcup \mathbf{1} \to I \sqcup \mathbf{1}$. It is proven as follows. To shorten the notation, we drop the subscripts of identities and of evaluation morphisms. The detailed form can be read from the diagrams on the next page. Since identities in C are idempotent, the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} I \sqcup J \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \triangleright} I \sqcup \mathbf{1}$$

yields

$$\begin{aligned}\Big((1)_{i\in I}, \big[\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Big((1)_{i\in I}, (1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\Big) \cdot_{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} \Big(\big((1)_{i\in I}, \mu^{\mathsf{C}}_{\phi}\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}\Big).\end{aligned}$$

By the definition of μ^{C}_{ϕ},

$$\big((1)_{i\in I}, \mu^{\mathsf{C}}_{\phi}\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} = \big((\mathrm{ev}^{\mathsf{C}})_{j\in J}, 1\big) \cdot_{\overline{\phi}} \mathrm{ev}^{\mathsf{C}},$$

which corresponds to commutative square $\boxed{2}$ of the diagram. The associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright} I \sqcup J \sqcup \mathbf{1} \xrightarrow{\overline{\phi}} J \sqcup \mathbf{1}$$

implies

$$\begin{aligned}\Big((1)_{i\in I}, \big[\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\psi}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\phi}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Big(\big(\big((1)_{i\in\phi^{-1}j}, 1\big) \cdot_{\mathrm{id}_{\phi^{-1}j\sqcup\mathbf{1}}} \mathrm{ev}^{\mathsf{C}}\big)_{j\in J}, \mu^{\mathsf{C}}_{\psi} \cdot_{K\sqcup\mathbf{1}\to\mathbf{1}} 1\Big) \cdot_{(\mathrm{id}_I \sqcup \mathrm{id}_J \sqcup \triangleright)\cdot\overline{\psi}} \mathrm{ev}^{\mathsf{C}}.\end{aligned}$$

By the identity axiom,

$$\big((1)_{i\in\phi^{-1}j}, 1\big) \cdot_{\mathrm{id}_{\phi^{-1}j\sqcup\mathbf{1}}} \mathrm{ev}^{\mathsf{C}} = \mathrm{ev}^{\mathsf{C}} \cdot_{\phi^{-1}j\sqcup\mathbf{1}\to\mathbf{1}} 1, \quad \mu^{\mathsf{C}}_{\psi} \cdot_{K\sqcup\mathbf{1}\to\mathbf{1}} 1 = \big((1)_{k\in K}, 1\big) \cdot_{\mathrm{id}_{K\sqcup\mathbf{1}}} \mu^{\mathsf{C}}_{\psi}.$$

$$
\begin{tikzcd}[column sep=large]
\begin{array}{l}(X_i)_{i\in I},\\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}\\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\\ \underline{\mathsf{C}}((Z_k)_{k\in K};W)\end{array}
\arrow[r, "(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},(1^{\mathsf{C}}_{\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)})_{j\in J},\mu^{\mathsf{C}}_{\psi}"]
\arrow[dd, "\begin{array}{l}(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},\\ (1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)})_{k\in K},\\ 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}\end{array}"]
\arrow[dr, phantom, "\boxed{1}"]
&
\begin{array}{l}(X_i)_{i\in I},\\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}\\ \underline{\mathsf{C}}((Y_j)_{j\in J};W)\end{array}
\arrow[r, "(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},\mu^{\mathsf{C}}_{\phi}"]
\arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in J};W),1}"']
\arrow[dr, phantom, "\boxed{2}"]
&
(X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};W)
\arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"]
\\
&
(Y_j)_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};W)
\arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};W}"]
\arrow[dr, phantom, "\boxed{3}"]
&
W
\\
\begin{array}{c}(Y_j)_{j\in J},\big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K}\\ \underline{\mathsf{C}}((Z_k)_{k\in K};W)\end{array}
\arrow[ur, "(1^{\mathsf{C}}_{Y_j,\{j\}})_{j\in J},\mu^{\mathsf{C}}_{\psi}"]
\arrow[rr, "(\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}"]
&
&
(Z_k)_{k\in K},\underline{\mathsf{C}}((Z_k)_{k\in K};W)
\arrow[u, "\mathrm{ev}^{\mathsf{C}}_{(Z_k)_{k\in K};W}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
\begin{array}{l}(X_i)_{i\in I},\\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J}\\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\\ \underline{\mathsf{C}}((Z_k)_{k\in K};W)\end{array}
\arrow[r, "(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},(\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}"]
\arrow[d, "\begin{array}{l}(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},\\ (1^{\mathsf{C}}_{\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)})_{k\in K},\\ 1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}\end{array}"]
\arrow[dr, phantom, "\boxed{4}"]
&
\begin{array}{l}(X_i)_{i\in I},\\ \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k)\big)_{k\in K},\\ \underline{\mathsf{C}}((Z_k)_{k\in K};W)\end{array}
\arrow[r, "(1^{\mathsf{C}}_{X_i,\{i\}})_{i\in I},\mu^{\mathsf{C}}_{\phi\psi}"]
\arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}"]
\arrow[dr, phantom, "\boxed{5}"]
&
(X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};W)
\arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(Z_k)_{k\in K};W}"]
\\
\begin{array}{l}(Y_j)_{j\in J},\\ \big(\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\big)_{k\in K},\\ \underline{\mathsf{C}}((Z_k)_{k\in K};W)\end{array}
\arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in\psi^{-1}k};Z_k})_{k\in K},1^{\mathsf{C}}_{\underline{\mathsf{C}}((Z_k)_{k\in K};W),1}"]
&
(Z_k)_{k\in K},\underline{\mathsf{C}}((Z_k)_{k\in K};W)
\arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"]
&
W
\end{tikzcd}
$$

Therefore, by the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\pi} J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_J \sqcup \triangleright} J \sqcup \mathbf{1},$$

where π is given by

$$\pi|_{J \sqcup K \sqcup \mathbf{1}} = \mathrm{id} : J \sqcup K \sqcup \mathbf{1} \to J \sqcup K \sqcup \mathbf{1}, \quad \pi|_I = \big(I \xrightarrow{\phi} J \hookrightarrow J \sqcup K \sqcup \mathbf{1}\big),$$

it follows that

$$\Big((1)_{i\in I}, \big[\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\overline{\psi}}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\overline{\phi}}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Big((\mathrm{ev}^{\mathsf{C}})_{j\in J}, (1)_{k\in K}, 1\Big) \cdot_\pi \Big(\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\overline{\psi}}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}\Big),$$

which means the commutativity of quadrilateral $\boxed{1}$. By the definition of $\mu^{\mathsf{C}}_{\overline{\psi}}$, the equation

$$\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\overline{\psi}}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} = \big((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\big) \cdot_{\overline{\psi}} \mathrm{ev}^{\mathsf{C}},$$

holds true, found in the diagram as commutative quadrilateral $\boxed{3}$. Therefore

$$\Big((1)_{i\in I}, \big[\big((1)_{j\in J}, \mu^{\mathsf{C}}_{\overline{\psi}}\big) \cdot_{\mathrm{id}_J \sqcup \triangleright} \mu^{\mathsf{C}}_{\overline{\phi}}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Big((\mathrm{ev}^{\mathsf{C}})_{j\in J}, (1)_{k\in K}, 1\Big) \cdot_\pi \Big(\big((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\big) \cdot_{\overline{\psi}} \mathrm{ev}^{\mathsf{C}}\Big).$$

Similarly, by the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \overline{\psi}} I \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \triangleright} I \sqcup \mathbf{1}$$

and the identity axiom,

$$\Big((1)_{i\in I}, \big[\big((\mu^{\mathsf{C}}_{\overline{\phi_k}})_{k\in K}, 1\big) \cdot_{\overline{\psi}} \mu^{\mathsf{C}}_{\overline{\phi\psi}}\big]\Big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} \\ = \Big((1)_{i\in I}, (\mu^{\mathsf{C}}_{\overline{\phi_k}})_{k\in K}, 1\Big) \cdot_{\mathrm{id}_I \sqcup \overline{\psi}} \Big(\big((1)_{i\in I}, \mu^{\mathsf{C}}_{\overline{\phi\psi}}\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}}\Big).$$

By the definition of $\mu^{\mathsf{C}}_{\overline{\phi\psi}}$,

$$\big((1)_{i\in I}, \mu^{\mathsf{C}}_{\overline{\phi\psi}}\big) \cdot_{\mathrm{id}_I \sqcup \triangleright} \mathrm{ev}^{\mathsf{C}} = \big((\mathrm{ev}^{\mathsf{C}})_{k\in K}, 1\big) \cdot_{\overline{\phi\psi}} \mathrm{ev}^{\mathsf{C}},$$

which corresponds to commutative square $\boxed{5}$ of the diagram. Applying the associativity axiom for the pair of maps

$$I \sqcup J \sqcup K \sqcup \mathbf{1} \xrightarrow{\mathrm{id}_I \sqcup \overline{\psi}} I \sqcup K \sqcup \mathbf{1} \xrightarrow{\overline{\phi\psi}} K \sqcup \mathbf{1}$$

leads to

$$\Bigl((1)_{i\in I},(\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1\Bigr)\cdot_{\mathrm{id}_I\sqcup\overline{\psi}}\Bigl(\bigl((1)_{i\in I},\mu^{\mathsf{C}}_{\phi\psi}\bigr)\cdot_{\mathrm{id}_I\sqcup\triangleright}\mathrm{ev}^{\mathsf{C}}\Bigr)$$
$$=\Bigl(\bigl(((1)_{i\in\phi^{-1}\psi^{-1}k},\mu^{\mathsf{C}}_{\phi_k})\cdot_{\mathrm{id}_{\phi^{-1}\psi^{-1}k}\sqcup\triangleright}\mathrm{ev}^{\mathsf{C}}\bigr)_{k\in K},1\Bigr)\cdot_{\overline{\psi}}\mathrm{ev}^{\mathsf{C}},$$

where $\mathrm{id}_{\phi^{-1}\psi^{-1}k}\sqcup\triangleright:\phi^{-1}\psi^{-1}k\sqcup\psi^{-1}k\sqcup\mathbf{1}\to\phi^{-1}\psi^{-1}k\sqcup\mathbf{1}$. Finally, since by the definition of $\mu^{\mathsf{C}}_{\phi_k}$ the equation

$$\bigl((1)_{i\in\phi^{-1}\psi^{-1}k},\mu^{\mathsf{C}}_{\phi_k}\bigr)\cdot_{\mathrm{id}_{\phi^{-1}\psi^{-1}k}\sqcup\triangleright}\mathrm{ev}^{\mathsf{C}}=\bigl((\mathrm{ev}^{\mathsf{C}})_{i\in\phi^{-1}\psi^{-1}k},1\bigr)\cdot_{\overline{\phi_k}}\mathrm{ev}^{\mathsf{C}}$$

holds true, found as commutative square $\boxed{4}$, it follows that

$$\Bigl((1)_{i\in I},\bigl[\bigl((\mu^{\mathsf{C}}_{\phi_k})_{k\in K},1\bigr)\cdot_{\overline{\psi}}\mu^{\mathsf{C}}_{\phi\psi}\bigr]\Bigr)\cdot_{\mathrm{id}_I\sqcup\triangleright}\mathrm{ev}^{\mathsf{C}}$$
$$=\Bigl(\Bigl(\bigl((\mathrm{ev}^{\mathsf{C}})_{i\in\phi^{-1}\psi^{-1}k},1\bigr)\cdot_{\overline{\phi_k}}\mathrm{ev}^{\mathsf{C}}\Bigr)_{k\in K},1\Bigr)\cdot_{(\mathrm{id}_I\sqcup\overline{\psi})\cdot\overline{\phi\psi}}\mathrm{ev}^{\mathsf{C}}.$$

The equation in question is a consequence of the associativity of composition in C, written for the pair of maps

$$I\sqcup J\sqcup K\sqcup\mathbf{1}\xrightarrow{\pi}J\sqcup K\sqcup\mathbf{1}\xrightarrow{\overline{\psi}}K\sqcup\mathbf{1}$$

and morphisms

$$\begin{aligned}
&\mathrm{ev}^{\mathsf{C}}:(X_i)_{i\in\phi^{-1}j},\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\to Y_j, && j\in J,\\
&\mathrm{ev}^{\mathsf{C}}:(Y_j)_{j\in\psi^{-1}k},\underline{\mathsf{C}}((Y_j)_{j\in\psi^{-1}k};Z_k)\to Z_k, && k\in K,\\
&\mathrm{ev}^{\mathsf{C}}:(Z_k)_{k\in K},\underline{\mathsf{C}}((Z_k)_{k\in K};W)\to W.
\end{aligned}$$

Note that $\pi\cdot\overline{\psi}=(\mathrm{id}_I\sqcup\overline{\psi})\cdot\overline{\phi\psi}$, and the map

$$\pi_k=\pi|_{\pi^{-1}\overline{\psi}^{-1}k}:\pi^{-1}\overline{\psi}^{-1}k=\phi^{-1}\psi^{-1}k\sqcup\psi^{-1}k\sqcup\mathbf{1}\to\psi^{-1}k\sqcup\mathbf{1}=\overline{\psi}^{-1}k$$

coincides with $\overline{\phi_k}$, $k\in K$. The verification of the identity axioms is left to the reader. □

4.11 Proposition. *For any symmetric (resp. plain) closed multicategory* C *the isomorphism* $\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$ *given by* (4.7.1) *is natural in* $(Y_j)_{j\in J}$. *That is, for an arbitrary (resp. isotonic) map* $f:K\to J$ *the following diagram commutes:*

$$\begin{tikzcd}
\prod_{j\in J}\mathsf{C}((W_k)_{k\in f^{-1}j};Y_j)\times\mathsf{C}\bigl((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\bigr) \arrow[r, "\mu_f"] \arrow[d, "\prod_i 1_{X_i}\times\mathrm{id}\times\varphi_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}"'] & \mathsf{C}\bigl((W_k)_{k\in K};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\bigr) \arrow[d, "\varphi_{(W_k)_{k\in K};(X_i)_{i\in I};Z}"] \\
\prod_{i\in I}\mathsf{C}(X_i;X_i)\times\prod_{j\in J}\mathsf{C}((W_k)_{k\in f^{-1}j};Y_j)\times\mathsf{C}\bigl((X_i)_I,(Y_j)_J;Z\bigr) \arrow[r, "\mu_{\mathrm{id}_I\sqcup f}"'] & \mathsf{C}\bigl((X_i)_{i\in I},(W_k)_{k\in K};Z\bigr)
\end{tikzcd}$$

Proof. Indeed, top square in the following diagram commutes since units 1_{X_i} are idempotents.

$$
\begin{tikzcd}
\begin{array}{c}\prod_{j\in J}\mathsf{C}((W_k)_{k\in f^{-1}j};Y_j)\\ \times\mathsf{C}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\end{array} \arrow[r, "\mu_f"] \arrow[d, "\prod_i 1_{X_i}\times\mathrm{id}\times\prod_i 1_{X_i}\times\mathrm{id}\times\mathrm{ev}_{(X_i);Z}"'] & \mathsf{C}\big((W_k)_K;\underline{\mathsf{C}}((X_i)_I;Z)\big) \arrow[d, "\prod_i 1_{X_i}\times\mathrm{id}\times\mathrm{ev}_{(X_i);Z}"] \\
\begin{array}{c}\prod_{i\in I}\mathsf{C}(X_i;X_i)\times\prod_{j\in J}\mathsf{C}((W_k)_{k\in f^{-1}j};Y_j)\times\\ \prod_{i\in I}\mathsf{C}(X_i;X_i)\times\mathsf{C}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\\ \times\mathsf{C}\big((X_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Z);Z\big)\end{array} \arrow[r, "\prod_i\mu_{\mathrm{id}_{\mathbf{1}}}", "\times\mu_f\times\mathrm{id}"'] \arrow[d, "\mathrm{id}\times\mathrm{id}\times\mu_{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}"'] & \begin{array}{c}\prod_{i\in I}\mathsf{C}(X_i;X_i)\\ \times\mathsf{C}\big((W_k)_K;\underline{\mathsf{C}}((X_i)_I;Z)\big)\\ \times\mathsf{C}\big((X_i)_I,\underline{\mathsf{C}}((X_i)_I;Z);Z\big)\end{array} \arrow[d, "\mu_{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}"] \\
\begin{array}{c}\prod_{i\in I}\mathsf{C}(X_i;X_i)\times\prod_{j\in J}\mathsf{C}((W_k)_{f^{-1}j};Y_j)\\ \times\mathsf{C}\big((X_i)_I,(Y_j)_J;Z\big)\end{array} \arrow[r, "\mu_{\mathrm{id}_I\sqcup f}"] & \mathsf{C}\big((X_i)_{i\in I},(W_k)_{k\in K};Z\big)
\end{tikzcd}
$$

The bottom commutative square is associativity property of multicategory C, written for mappings $I\sqcup K\xrightarrow{1\sqcup f}I\sqcup J\xrightarrow{1\sqcup\triangleright}I\sqcup\mathbf{1}$, see Fig. 3.1. Therefore, the exterior commutes and the proposition is proven. $\square$

4.12 Proposition. *The choice of evaluations $\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}$ for a closed multicategory C determines a unique isomorphism*

$$\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}:\underline{\mathsf{C}}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\to\underline{\mathsf{C}}\big((X_i)_{i\in I},(Y_j)_{j\in J};Z\big).$$

Proof. We have three isomorphisms natural in $(W_k)_{k\in K}$ by Proposition 4.11:

$$
\begin{aligned}
\psi_{(W_k)_{k\in K}}=\Big[&\mathsf{C}\Big((W_k)_{k\in K};\underline{\mathsf{C}}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\Big)\\
&\xrightarrow{\varphi_{(W_k)_{k\in K};(Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)}}\mathsf{C}\big((Y_j)_{j\in J},(W_k)_{k\in K};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)\\
&\xrightarrow{\varphi_{(Y_j)_{j\in J},(W_k)_{k\in K};(X_i)_{i\in I};Z}}\mathsf{C}\big((X_i)_{i\in I},(Y_j)_{j\in J},(W_k)_{k\in K};Z\big)\\
&\xrightarrow{\varphi^{-1}_{(W_k)_{k\in K};(X_i)_{i\in I},(Y_j)_{j\in J};Z}}\mathsf{C}\big((W_k)_{k\in K};\underline{\mathsf{C}}((X_i)_{i\in I},(Y_j)_{j\in J};Z)\big)\Big].
\end{aligned}
$$

Denote $A=\underline{\mathsf{C}}\big((Y_j)_{j\in J};\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big)$ and $B=\underline{\mathsf{C}}((X_i)_{i\in I},(Y_j)_{j\in J};Z)$. One deduces from Proposition 4.11 naturality of the composition ψ. Namely, for any map $f:K\to L$ in $\mathcal{O}$ we have the equation

$$
\begin{tikzcd}
\prod_{l\in L}\mathsf{C}((W_k)_{k\in f^{-1}l};U_l)\times\mathsf{C}((U_l)_{l\in L};A) \arrow[r, "\mu_f"] \arrow[d, "\mathrm{id}\times\psi_{(U_l)_{l\in L}}"'] \arrow[dr, phantom, "="] & \mathsf{C}((W_k)_{k\in K};A) \arrow[d, "\psi_{(W_k)_{k\in K}}"] \\
\prod_{l\in L}\mathsf{C}((W_k)_{k\in f^{-1}l};U_l)\times\mathsf{C}((U_l)_{l\in L};B) \arrow[r, "\mu_f"] & \mathsf{C}((W_k)_{k\in K};B)
\end{tikzcd}
\tag{4.12.1}
$$

Considering $K = L = \mathbf{1}$ we get an isomorphism of functors $W \mapsto \mathsf{C}(W; A)$ and $W \mapsto \mathsf{C}(W; B)$, which by ordinary Yoneda Lemma gives an isomorphism $\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z} : A \to B$ in C, the image of 1_A under the map ψ_A.

Notice that for arbitrary family $(W_k)_{k\in K}$ the isomorphism $\psi_{(W_k)_{k\in K}}$ is obtained by composition with $\underline{\varphi}_{(Y_j)_{j\in J};(X_i)_{i\in I};Z}$. Indeed, consider $f = \triangleright : K \to \mathbf{1}$, $U_1 = A$, an arbitrary morphism $g : (W_k)_{k\in K} \to A$ and apply equation (4.12.1) to the element $(g, 1_A)$. □

Let $g : (Y_j)_{j\in J} \to Z$ be a morphism in C, $X_i \in \operatorname{Ob} \mathsf{C}$, $i \in I$ a family of objects, and let $\phi : I \to J$ be a map in $\operatorname{Mor} \mathcal{S}$. The morphism g gives rise to a morphism

$$\underline{\mathsf{C}}(\phi; g) : (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; Z)$$

in C determined in a unique way via the diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \arrow[r, "(1)_I, \underline{\mathsf{C}}(\phi;g)"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] \\
(Y_j)_{j\in J} \arrow[r, "g"] & Z
\end{tikzcd} \tag{4.12.2}$$

Its existence and uniqueness follows from closedness of C.

Let $\psi : K \to I$ be a map in $\mathcal{S}$. Let $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$, $i \in I$, be morphisms in C. A morphism $\underline{\mathsf{C}}((f_i)_{i\in I}; 1) : \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \to \underline{\mathsf{C}}((W_k)_{k\in K}; Z)$ is defined as the only morphism that makes the following diagram commutative:

$$\begin{tikzcd}
(W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(1)_K, \underline{\mathsf{C}}((f_i)_{i\in I};1)"] \arrow[d, "(f_i)_{i\in I}, 1"'] & (W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(W_k);Z}"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] & Z
\end{tikzcd} \tag{4.12.3}$$

4.13 Lemma. *In the above assumptions the introduced morphisms satisfy the commutativity relation:*

$$\begin{aligned}
&\big[\big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{\underline{\mathsf{C}}(\phi;g)} \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\underline{\mathsf{C}}((f_i)_{i\in I};1)} \underline{\mathsf{C}}((W_k)_{k\in K}; Z)\big] \\
&= \big[\big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}} \big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j}; Y_j)\big)_{j\in J} \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{C}}(\psi\phi;g)} \underline{\mathsf{C}}((W_k)_{k\in K}; Z)\big].
\end{aligned}$$

Proof. We may rewrite the required equation in the form:

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(1)_K,\underline{\mathsf{C}}(\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \\
&\qquad \xrightarrow{(1)_K,\underline{\mathsf{C}}((f_i)_{i\in I};1)} (W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(1)_K,(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}} \\
&\qquad (W_k)_{k\in K}, \big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j};Y_j)\big)_{j\in J} \\
&\qquad \xrightarrow{(1)_K,\underline{\mathsf{C}}(\psi\cdot\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big].
\end{aligned}$$

The left hand side can be transformed as follows:

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(1)_K,\underline{\mathsf{C}}(\phi;g)} (W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \\
&\qquad \xrightarrow{(f_i)_{i\in I},1} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(f_i)_{i\in I},(1)_J} (X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \\
&\qquad \xrightarrow{(1)_I,\underline{\mathsf{C}}(\phi;g)} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} Z\big] \\
&= \big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(f_i)_{i\in I},(1)_J} (X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \\
&\qquad \xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}} (Y_j)_{j\in J} \xrightarrow{g} Z\big]. \qquad (4.13.1)
\end{aligned}$$

The right hand side can be transformed to:

$$\begin{aligned}
&\big[(W_k)_{k\in K}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(1)_K,(\underline{\mathsf{C}}((f_i)_{i\in\phi^{-1}j};1))_{j\in J}} \\
&\qquad (W_k)_{k\in K}, \big(\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}\phi^{-1}j};Y_j)\big)_{j\in J} \xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in\psi^{-1}\phi^{-1}j};Y_j})_{j\in J}} (Y_j)_{j\in J} \xrightarrow{g} Z\big],
\end{aligned}$$

which coincides with the last expression of (4.13.1). □

4.14 Lemma. *Let $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ be maps in $\mathcal{S}$, and let $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$, $(Z_k)_{k\in K}$, W be families of objects in a closed symmetric multicategory C. Let $f_j : (X_i)_{i\in\phi^{-1}j} \to Y_j$, $j \in J$, $g_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, be morphisms in C. Then*

$$\begin{aligned}
&\underline{\mathsf{C}}\big(((f_j)_{j\in\psi^{-1}k}\cdot g_k)_{k\in K};1\big) \\
&\qquad = \big[\underline{\mathsf{C}}((Z_k)_K;W) \xrightarrow{\underline{\mathsf{C}}((g_k)_{k\in K};1)} \underline{\mathsf{C}}((Y_j)_J;W) \xrightarrow{\underline{\mathsf{C}}((f_j)_{j\in J};1)} \underline{\mathsf{C}}((X_i)_I;W)\big].
\end{aligned}$$

Proof. The commutative diagram

$$
\begin{tikzcd}
 & & (X_i)_{i\in I}, \underline{\mathsf{C}}((Y_j)_J;W) \arrow[d, "(f_j)_{j\in J},1"'] \arrow[dr, "(1)_I,\underline{\mathsf{C}}((f_j)_J;1)"] & \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((Z_k)_K;W) \arrow[urr, "(1)_I,\underline{\mathsf{C}}((g_k)_K;1)"] \arrow[d, "(f_j)_{j\in J},1"'] & = & (Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_J;W) \arrow[dr, "\mathrm{ev}^{\mathsf{C}}_{(Y_j);W}"] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_I;W) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(X_i);W}"] \\
(Y_j)_{j\in J}, \underline{\mathsf{C}}((Z_k)_K;W) \arrow[urr, "(1)_J,\underline{\mathsf{C}}((g_k)_K;1)"] \arrow[rr, "(g_k)_{k\in K},1"'] & & (Z_k)_{k\in K}, \underline{\mathsf{C}}((Z_k)_K;W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(Z_k);W}"'] & W
\end{tikzcd}
$$

implies the lemma. ☐

4.15 Lemma. *Let $I \xrightarrow{\phi} J \xrightarrow{\psi} K$ be maps in $\mathcal{S}$, and let $(X_i)_{i\in I}$, $(Y_j)_{j\in J}$, $(Z_k)_{k\in K}$, W be families of objects in a closed symmetric multicategory C. Let $f_k : (Y_j)_{j\in\psi^{-1}k} \to Z_k$, $k \in K$, $g : (Z_k)_{k\in K} \to W$ be morphisms in C. Denote $\phi_k = \phi\big|: \phi^{-1}\psi^{-1}k \to \psi^{-1}k$. Then*

$$
\begin{aligned}
&\Big[\big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \xrightarrow{\underline{\mathsf{C}}(\phi;(f_k)\cdot g)} \underline{\mathsf{C}}((X_i)_{i\in I}; W)\Big] \\
&\quad = \Big[\big((\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in\psi^{-1}k}\big)_{k\in K} \xrightarrow{(\underline{\mathsf{C}}(\phi_k;f_k))_{k\in K}} \\
&\qquad\qquad\qquad \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k}; Z_k)\big)_{k\in K} \xrightarrow{\underline{\mathsf{C}}(\phi\psi;g)} \underline{\mathsf{C}}((X_i)_{i\in I}; W)\Big].
\end{aligned}
$$

Proof. Omitting the permutations of elements we write the following commutative diagram:

$$
\begin{tikzcd}
(X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j)\big)_{j\in J} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}"] \arrow[d, "(1)_I,(\underline{\mathsf{C}}(\phi_k;f_k))_{k\in K}"'] & (Y_j)_{j\in J} \arrow[d, "(f_k)_{k\in K}"] \\
(X_i)_{i\in I}, \big(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}\psi^{-1}k}; Z_k)\big)_{k\in K} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}\psi^{-1}k};Z_k})_{k\in K}"] \arrow[d, "(1)_I,\underline{\mathsf{C}}(\phi\psi;g)"'] & (Z_k)_{k\in K} \arrow[d, "g"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; W) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};W}"] & W
\end{tikzcd}
$$

Uniqueness implies the claim. ☐

Notation. Let $g : (Y_j)_{j\in J} \to Z$ be a morphism in a closed symmetric multicategory C. Denote by $\dot{g} : () \to \underline{\mathsf{C}}((Y_j)_{j\in J}; Z)$ the morphism $\varphi^{-1}_{();(Y_j)_{j\in J};Z}(g) \in \mathsf{C}(;\underline{\mathsf{C}}((Y_j)_{j\in J}; Z))$.

4.16 Lemma. *Let the above assumptions hold. Let $(X_i)_{i\in I}$ be a family of objects of C, and let $\phi : I \to J$ be a map in $\operatorname{Mor}\mathcal{S}$. Then*

$$\underline{\mathsf{C}}(\phi; g) = \bigl[(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(1)_J,\dot g} (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \xrightarrow{\mu^{\underline{\mathsf{C}}}_\phi} \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\bigr].$$

Proof. Plugging the right hand side into defining equation (4.12.2) for $\underline{\mathsf{C}}(\phi; g)$ we obtain the diagram in C

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}"] \arrow[d, "(1)_{I\sqcup J},\dot g"'] \arrow[dr, phantom, "="] & (Y_j)_{j\in J} \arrow[d, "(1)_J,\dot g"] \\
\begin{array}{c}(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J},\\ \underline{\mathsf{C}}((Y_j)_{j\in J}; Z)\end{array} \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J},1"] \arrow[d, "(1)_I,\mu^{\underline{\mathsf{C}}}_\phi"'] \arrow[dr, phantom, "="] & \begin{array}{c}(Y_j)_{j\in J},\\ \underline{\mathsf{C}}((Y_j)_{j\in J}; Z)\end{array} \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(Y_j)_{j\in J};Z}"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] & Z
\end{tikzcd}$$

The lower square is definition (4.10.1) of $\mu^{\underline{\mathsf{C}}}_\phi$. The right column composes to g due to the equation $\varphi_{();(Y_j)_{j\in J};Z}(\dot g) = g$. Commutativity of exterior of the diagram implies statement of the lemma. $\square$

4.17 Lemma. *Let $\psi : K \to I$ be a map in $\mathcal{S}$. Let $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$, $i \in I$, be morphisms in C. Then*

$$\underline{\mathsf{C}}((f_i)_{i\in I}; 1) = \bigl[\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{(\dot f_i)_{i\in I},1} (\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}i}; X_i))_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \xrightarrow{\mu^{\underline{\mathsf{C}}}_\psi} \underline{\mathsf{C}}((W_k)_{k\in K}; Z)\bigr].$$

Proof. Plug the right hand side into defining equation (4.12.3) for $\underline{\mathsf{C}}((f_i)_{i\in I}; 1)$. We obtain the diagram in C

$$\begin{tikzcd}
(W_k)_{k\in K}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[rr, "(f_i)_{i\in I},1"] \arrow[d, "(1)_K,(\dot f_i)_{i\in I},1"'] \arrow[dr, phantom, "="] & & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[dd, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] \\
(W_k)_{k\in K}, (\underline{\mathsf{C}}((W_k)_{k\in\psi^{-1}i}; X_i))_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[urr, "(\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in\psi^{-1}i};X_i})_{i\in I},1"'] \arrow[d, "(1)_K,\mu^{\underline{\mathsf{C}}}_\psi"'] \arrow[drr, phantom, "="] & & \\
(W_k)_{k\in K}, \underline{\mathsf{C}}((W_k)_{k\in K}; Z) \arrow[rr, "\mathrm{ev}^{\mathsf{C}}_{(W_k)_{k\in K};Z}"] & & Z
\end{tikzcd}$$

The quadrilateral is the definition of μ_ψ^{C}. The triangle follows from equations

$$\varphi_{();(W_k)_{k\in\psi^{-1}i};X_i}(\dot{f_i}) = f_i.$$

Commutativity of the exterior of the diagram implies statement of the lemma. $\square$

When C is a symmetric closed multicategory, it admits a transformation

$$\underline{\mathsf{C}}(\_;Z) : \underline{\mathsf{C}}(X;Y) \to \underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;Z);\underline{\mathsf{C}}(X;Z)), \tag{4.17.1}$$

natural in X, Y and dinatural in Z. It is determined unambiguously from the commutative diagram

$$\begin{tikzcd} \underline{\mathsf{C}}(Y;Z), \underline{\mathsf{C}}(X;Y) \arrow[r, "1{,}\underline{\mathsf{C}}(\_;Z)"] \arrow[dr, "\mu^{\underline{\mathsf{C}}^{\mathrm{op}}}"'] & \underline{\mathsf{C}}(\underline{\mathsf{C}}(Y;Z);\underline{\mathsf{C}}(X;Z)) \arrow[d, "\mathrm{ev}^{\mathsf{C}}"] \\ & \underline{\mathsf{C}}(X;Z) \end{tikzcd}$$

where $\mu^{\underline{\mathsf{C}}^{\mathrm{op}}}$ is the image of $\mu^{\underline{\mathsf{C}}}$ under the canonical isomorphism

$$\mathsf{C}((12);\underline{\mathsf{C}}(X;Z)) : \mathsf{C}(\underline{\mathsf{C}}(X;Y),\underline{\mathsf{C}}(Y;Z);\underline{\mathsf{C}}(X;Z)) \xrightarrow{\sim} \mathsf{C}(\underline{\mathsf{C}}(Y;Z),\underline{\mathsf{C}}(X,Y);\mathsf{C}(X;Z)).$$

4.18 Closing transformations. We begin by an easy motivational example. Given closed monoidal categories $\mathcal{C}, \mathcal{D}$ and a lax monoidal functor $(F, \phi) : \mathcal{C} \to \mathcal{D}$, there is a natural transformation $\underline{F}_{X,Y} : F\underline{\mathcal{C}}(X,Y) \to \underline{\mathcal{D}}(FX, FY)$ uniquely determined from the commutative diagram

$$\begin{tikzcd} FX \otimes F\underline{\mathcal{C}}(X,Y) \arrow[r, "1\otimes\underline{F}_{X,Y}"] \arrow[d, "\phi_{X,\underline{\mathcal{C}}(X,Y)}"'] & FX \otimes \underline{\mathcal{D}}(FX, FY) \arrow[d, "\mathrm{ev}_{FX,FY}"] \\ F(X \otimes \underline{\mathcal{C}}(X,Y)) \arrow[r, "F\,\mathrm{ev}_{X,Y}"] & FY \end{tikzcd}$$

It is called the closing transformation. The generalization to the case of a $\mathcal{V}$-multifunctor between closed $\mathcal{V}$-multicategories is straightforward. More specifically, let C, D be closed symmetric $\mathcal{V}$-multicategories. Let $F : \mathsf{C} \to \mathsf{D}$ be a (symmetric) $\mathcal{V}$-multifunctor. The symmetric multicategory $\underline{\mathsf{D}}$ is enriched in D. The symmetric multicategory $\underline{\mathsf{C}}$ enriched in C can be considered as a D-multicategory $\underline{\mathsf{C}}_F$ via the base change $F : \mathsf{C} \to \mathsf{D}$. Since D is closed, there is an isomorphism in $\mathcal{V}$

$$\begin{aligned} \varphi_{F\underline{\mathsf{C}}((X_i);Z);(FX_i)_i;FZ} = \Big[&\mathsf{D}\big(F\underline{\mathsf{C}}((X_i)_{i\in I};Z);\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\big) \\ &\xrightarrow{(\otimes^{i\in I}1_{FX_i})\otimes\mathrm{id}\otimes\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}} \\ \big(\otimes^{i\in I}\mathsf{D}(FX_i;FX_i)\big) \otimes \mathsf{D}\big(F\underline{\mathsf{C}}((X_i);Z);\underline{\mathsf{D}}((FX_i);FZ)\big) &\otimes \mathsf{D}\big((FX_i),\underline{\mathsf{D}}((FX_i);FZ);FZ\big) \\ &\xrightarrow{\mu} \mathsf{D}\big((FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z);FZ\big)\Big] \end{aligned} \tag{4.18.1}$$

(take $J = \mathbf{1}$ in Definition 4.7). Define a morphism in D

$$\underline{F}_{(X_i);Z} : F\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \to \underline{\mathsf{D}}((FX_i)_{i\in I}; FZ) \tag{4.18.2}$$

as the only element of the source of (4.18.1) which is mapped to the element $F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}$ of the target. It is uniquely determined by the commutative diagram

$$\begin{tikzcd}
(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \arrow[r, "(1)_I{,}\underline{F}_{(X_i);Y}"] \arrow[dr, "F\,\mathrm{ev}_{(X_i);Y}"'] & (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I}; FY) \arrow[d, "\mathrm{ev}_{(FX_i);FY}"] \\
& FY
\end{tikzcd} \tag{4.18.3}$$

The natural transformation (4.18.2) is called *closing transformation* of the $\mathcal{V}$-multifunctor F.

4.19 Lemma. *The following equation holds*

$$\begin{tikzcd}
\mathsf{C}\big((Y_j)_{j\in J}; \underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) \arrow[r, "F"] \arrow[dd, "\varphi_{(Y_j);(X_i);Z}"'] \arrow[ddr, phantom, "="] & \mathsf{D}\big((FY_j)_{j\in J}; F\underline{\mathsf{C}}((X_i)_{i\in I}; Z)\big) \arrow[d, "\mathsf{D}(1{,}\underline{F}_{(X_i);Z})"] \\
& \mathsf{D}\big((FY_j)_{j\in J}; \underline{\mathsf{D}}((FX_i)_{i\in I}; FZ)\big) \arrow[d, "\varphi_{(FY_j);(FX_i);FZ}"] \\
\mathsf{C}\big((X_i)_{i\in I}, (Y_j)_{j\in J}; Z\big) \arrow[r, "F"] & \mathsf{D}\big((FX_i)_{i\in I}, (FY_j)_{j\in J}; FZ\big)
\end{tikzcd} \tag{4.19.1}$$

for all $(Y_j)_{j\in J}$, $J \in \operatorname{Ob}\mathcal{S}$.

Proof. Associativity of μ allows to rewrite (4.19.1) as the exterior of the diagram

$$\begin{tikzcd}
\mathsf{C}\big((Y_j);\underline{\mathsf{C}}((X_i);Z)\big) \arrow[r, "F\otimes\underline{F}_{(X_i);Z}"] \arrow[dd, "(\otimes^{i\in I}1_{X_i})\otimes\mathrm{id}\otimes\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"'] \arrow[ddr, "(\otimes^{i\in I}1_{FX_i})\otimes F\otimes F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"'] & \begin{array}{c}\mathsf{D}\big((FY_j);F\underline{\mathsf{C}}((X_i);Z)\big)\\ \otimes\mathsf{D}\big(F\underline{\mathsf{C}}((X_i);Z);\underline{\mathsf{D}}((FX_i);FZ)\big)\end{array} \arrow[d, "\mathrm{id}\otimes(\otimes^{i\in I}1_{FX_i})\otimes\mathrm{id}\otimes\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}"] \\
 & \begin{array}{c}\mathsf{D}((FY_j);F\underline{\mathsf{C}}((X_i);Z))\otimes(\otimes^{i\in I}\mathsf{D}(FX_i;FX_i))\\ \otimes\mathsf{D}(F\underline{\mathsf{C}}((X_i);Z);\underline{\mathsf{D}}((FX_i);FZ))\\ \otimes\mathsf{D}((FX_i),\underline{\mathsf{D}}((FX_i);FZ);FZ)\end{array} \arrow[d, "(\otimes^{i\in I}1_{FX_i})\otimes\mathrm{id}\otimes\mu"] \\
\begin{array}{c}(\otimes^{I}\mathsf{C}(X_i;X_i))\\ \otimes\mathsf{C}((Y_j);\underline{\mathsf{C}}((X_i);Z))\\ \otimes\mathsf{C}((X_i),\underline{\mathsf{C}}((X_i);Z);Z)\end{array} \arrow[r, "(\otimes^{I}F)\otimes F\otimes F"] \arrow[d, "\mu"'] & \begin{array}{c}(\otimes^{i\in I}\mathsf{D}(FX_i;FX_i))\\ \otimes\mathsf{D}((FY_j);F\underline{\mathsf{C}}((X_i);Z))\\ \otimes\mathsf{D}((FX_i),F\underline{\mathsf{C}}((X_i);Z);FZ)\end{array} \arrow[d, "\mu"] \\
\mathsf{C}\big((X_i)_{i\in I},(Y_j)_{j\in J};Z\big) \arrow[r, "F"] & \mathsf{D}\big((FX_i)_{i\in I},(FY_j)_{j\in J};FZ\big)
\end{tikzcd}$$

which commutes due to F being a multifunctor. □

4.20 Corollary. *For $J=\varnothing$ we get the following relation between F and $\underline{F}$:*

$$\begin{tikzcd}
\mathsf{C}\big(;\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[r, "F"] \arrow[d, "\varphi_{();(X_i);Z}"'] & \mathsf{D}\big(;F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\big) \arrow[r, "\mathsf{D}(;\underline{F}_{(X_i);Z})"] & \mathsf{D}\big(;\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\big) \arrow[d, "\varphi_{();(FX_i);FZ}"] \\
\mathsf{C}\big((X_i)_{i\in I};Z\big) \arrow[rr, "F"] & & \mathsf{D}\big((FX_i)_{i\in I};FZ\big)
\end{tikzcd}$$

4.21 Proposition. *The following equation in D holds for an arbitrary map $\phi:I\to J$ in* $\operatorname{Mor}\mathcal{S}$ *and arbitrary symmetric $\mathcal{V}$-multifunctor $F:\mathsf{C}\to\mathsf{D}$:*

$$\begin{tikzcd}
(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[r, "F\mu^{\mathsf{C}}_{\phi}"] \arrow[d, "(\underline{F}_{(X_i);Y_j})_{j\in J},\underline{F}_{(Y_j);Z}"'] & F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\underline{F}_{(X_i);Z}"] \\
(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J},\underline{\mathsf{D}}((FY_j)_{j\in J};FZ) \arrow[r, "\mu^{\mathsf{D}}_{\phi}"] & \underline{\mathsf{D}}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{4.21.1}$$

Morphism (4.18.2) may be interpreted as a (symmetric) D-multifunctor $\underline{F}:\underline{\mathsf{C}}_F\to\underline{\mathsf{D}}$ such that $\operatorname{Ob}\underline{F}=\operatorname{Ob}F$.

Proof. Equation (4.21.1) is an equation between two elements of

$$\mathsf{D}((F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z);\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)).$$

We show that these elements are mapped to the same element by the isomorphism φ.

Applying equation (4.19.1) to the element

$$\mu_\phi^{\mathsf{C}}\in\mathsf{C}((\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};Z);\underline{\mathsf{C}}((X_i)_{i\in I};Z))$$

we conclude that the top-right path is mapped by φ to $F\varphi(\mu_\phi^{\mathsf{C}})$. Since

$$\varphi(\mu_\phi^{\mathsf{C}})=\bigl[(X_i)_{i\in I},(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};Z)\xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in J},1}$$
$$(Y_j)_{j\in J},\underline{\mathsf{C}}((Y_j)_{j\in J};Z)\xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(Y_j);Z}}Z\bigr]$$

by equation (4.10.1), it follows that

$$F\varphi(\mu_\phi^{\mathsf{C}})=\bigl[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z)$$
$$\xrightarrow{(F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in J},1}(Y_j)_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z)\xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}_{(Y_j);Z}}FZ\bigr]. \tag{4.21.2}$$

This coincides with the image of the left-bottom path. Indeed:

$$\bigl[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z)\xrightarrow{(1)_I,(\underline{F}_{(X_i);Y_j})_{j\in J},\underline{F}_{(Y_j);Z}}$$
$$(FX_i)_{i\in I},(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J},\underline{\mathsf{D}}((FY_j)_{j\in J};FZ)$$
$$\xrightarrow{(1)_I,\mu_\phi^{\mathsf{D}}}(FX_i)_{i\in I},\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}}FZ\bigr]$$
$$=\bigl[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},F\underline{\mathsf{C}}((Y_j)_{j\in J};Z)\xrightarrow{(1)_I,(\underline{F}_{(X_i);Y_j})_{j\in J},\underline{F}_{(Y_j);Z}}$$
$$(FX_i)_{i\in I},(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J},\underline{\mathsf{D}}((FY_j)_{j\in J};FZ)\xrightarrow{(\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY_j})_{j\in J},1}$$
$$(FY_j)_{j\in J},\underline{\mathsf{D}}((Y_j)_{j\in J};FZ)\xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FY_j);FZ}}FZ\bigr],$$

which equals (4.21.2) by (4.10.1) and by definition of $\underline{F}$. This proves the claim. □

4.22 Lemma. *Let $f:(Y_j)_{j\in J}\to Z$ be a morphism in C, $X_i\in\operatorname{Ob}\mathsf{C}$, $i\in I$ a family of objects, and $\phi:I\to J$ a map in* $\operatorname{Mor}\mathcal{S}$. *The following diagram commutes for an arbitrary symmetric $\mathcal{V}$-multifunctor $F:\mathsf{C}\to\mathsf{D}$:*

$$\begin{tikzcd}
(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J} \arrow[r, "F\underline{\mathsf{C}}(\phi;f)"] \arrow[d, "(\underline{F}_{(X_i);Y_j})_{j\in J}"'] & F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "\underline{F}_{(X_i);Z}"] \\
(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J} \arrow[r, "\underline{\mathsf{D}}(\phi;Ff)"] & \underline{\mathsf{D}}((FX_i)_{i\in I};FZ)
\end{tikzcd} \tag{4.22.1}$$

Proof. Commutativity of (4.22.1) is equivalent to the following equation in D:

$$\begin{aligned}
&\big[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}\xrightarrow{(1)_I,F\underline{\mathsf{C}}(\phi;f)}\\
&\qquad (FX_i)_{i\in I},F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\xrightarrow{(1)_I,\underline{F}_{(X_i);Z}}(FX_i)_{i\in I},\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FZ)\xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}}FZ\big]\\
&=\big[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}\xrightarrow{(1)_I,(\underline{F}_{(X_i);Y_j})_{j\in J}}\\
&\qquad (FX_i)_{i\in I},(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J}\xrightarrow{(1)_I,\underline{\mathsf{D}}(\phi;Ff)}\\
&\qquad\qquad (FX_i)_{i\in I},\underline{\mathsf{D}}((FX_i)_{i\in I};FZ)\xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}}FZ\big],
\end{aligned}\tag{4.22.2}$$

which we are going to prove. The left hand side of (4.22.2) equals

$$\begin{aligned}
&\big[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}\xrightarrow{(1)_I,F\underline{\mathsf{C}}(\phi;f)}\\
&\qquad (FX_i)_{i\in I},F\underline{\mathsf{C}}((X_i)_{i\in I};Z)\xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}}FZ\big]
\end{aligned}\tag{4.22.3}$$

by definition of $\underline{F}$. The right hand side of (4.22.2) equals

$$\begin{aligned}
&\big[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}\xrightarrow{(1)_I,(\underline{F}_{(X_i);Y_j})_{j\in J}}\\
&\qquad (FX_i)_{i\in I},(\underline{\mathsf{D}}((FX_i)_{i\in\phi^{-1}j};FY_j))_{j\in J}\xrightarrow{(\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY_j})_{j\in J}}(FY_i)_{j\in J}\xrightarrow{Ff}FZ\big]
\end{aligned}$$

by (4.12.2). Using definition of $\underline{F}$ we can write it as follows:

$$\big[(FX_i)_{i\in I},(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}\xrightarrow{(F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in J}}(FY_i)_{j\in J}\xrightarrow{Ff}FZ\big].\tag{4.22.4}$$

Expressions (4.22.3) and (4.22.4) coincide by (4.12.2) and due to F being a multifunctor. □

4.23 Lemma. *Let $\psi : K \to I$ be a map in $\mathcal{S}$. Let $f_i : (W_k)_{k\in\psi^{-1}i} \to X_i$, $i \in I$, be morphisms in C, and let $F : \mathsf{C} \to \mathsf{D}$ be a symmetric $\mathcal{V}$-multifunctor. The following diagram commutes:*

$$\begin{tikzcd}
F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "F\underline{\mathsf{C}}((f_i)_{i\in I};1)"] \arrow[d, "\underline{F}_{(X_i);Z}"'] & F\underline{\mathsf{C}}((W_k)_{k\in K};Z) \arrow[d, "\underline{F}_{(W_k);Z}"] \\
\underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \arrow[r, "\underline{\mathsf{D}}((Ff_i)_{i\in I};1)"] & \underline{\mathsf{D}}((FW_k)_{k\in K};FZ)
\end{tikzcd}\tag{4.23.1}$$

Proof. Commutativity of (4.23.1) is equivalent to the following equation in D:

$$\begin{aligned}
&\bigl[(FW_k)_{k\in K}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(1)_K,F\underline{\mathsf{C}}((f_i)_{i\in I};1)} (FW_k)_{k\in K}, F\underline{\mathsf{C}}((W_k)_{k\in K};Z) \\
&\qquad \xrightarrow{(1)_K,\underline{F}_{(W_k);Z}} (FW_k)_{k\in K}, \underline{\mathsf{D}}((FW_k)_{k\in K};FZ) \xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FW_k);FZ}} FZ\bigr] \\
&= \bigl[(FW_k)_{k\in K}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(1)_K,\underline{F}_{(X_i);Z}} (FW_k)_{k\in K}, \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \\
&\qquad \xrightarrow{(1)_K,\underline{\mathsf{D}}((Ff_i)_{i\in I};1)} (FW_k)_{k\in K}, \underline{\mathsf{D}}((FW_k)_{k\in K};FZ) \xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FW_k);FZ}} FZ\bigr].
\end{aligned} \tag{4.23.2}$$

The left hand side of (4.23.2) equals

$$\bigl[(FW_k)_{k\in K}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(1)_K,F\underline{\mathsf{C}}((f_i)_{i\in I};1)} (FW_k)_{k\in K}, F\underline{\mathsf{C}}((W_k)_{k\in K};Z) \xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}_{(W_k);Z}} FZ\bigr] \tag{4.23.3}$$

by definition of $\underline{F}$. The right hand side of (4.23.2) equals

$$\begin{aligned}
&\bigl[(FW_k)_{k\in K}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(1)_K,\underline{F}_{(X_i);Z}} (FW_k)_{k\in K}, \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \\
&\qquad \xrightarrow{(Ff_i)_{i\in I},1} (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};FZ) \xrightarrow{\mathrm{ev}^{\mathsf{D}}_{(FX_i);FZ}} FZ\bigr]
\end{aligned}$$

by equation (4.12.3). The latter can be written as

$$\bigl[(FW_k)_{k\in K}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{(Ff_i)_{i\in I},1} (FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}} FZ\bigr] \tag{4.23.4}$$

according to definition of $\underline{F}$. Expressions (4.23.3) and (4.23.4) coincide by (4.12.3) and due to F being a multifunctor. □

4.24 Lemma. *Given a multinatural transformation $\nu : F \to G : \mathsf{C} \to \mathsf{D}$, the following diagram is commutative*

$$\begin{tikzcd}
F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[rr, "\underline{F}_{(X_i);Y}"] \arrow[d, "\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}"] & & \underline{\mathsf{D}}((FX_i)_{i\in I};FY) \arrow[d, "\underline{\mathsf{D}}(\triangleright;\nu_Y)"'] \\
G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "\underline{G}_{(X_i);Y}"] & \underline{\mathsf{D}}((GX_i)_{\in I};GY) \arrow[r, "\underline{\mathsf{D}}((\nu_{X_i})_{i\in I};1)"] & \underline{\mathsf{D}}((FX_i)_{i\in I};GY)
\end{tikzcd} \tag{4.24.1}$$

Proof. Commutativity of (4.24.1) is equivalent to the following equation in D:

$$\begin{aligned}
&\bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\underline{F}_{(X_i);Y}} (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};FY) \\
&\qquad \xrightarrow{(1)_I,\underline{\mathsf{D}}(\triangleright;\nu_Y)} (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};GY) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} GY\bigr] \\
&= \bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}} (FX_i)_{i\in I}, G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\underline{G}_{(X_i);Y}} \\
&(FX_i)_{i\in I}, \underline{\mathsf{D}}((GX_i)_{i\in I};GY) \xrightarrow{(1)_I,\underline{\mathsf{D}}((\nu_{X_i})_{i\in I};1)} (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};GY) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} GY\bigr],
\end{aligned} \tag{4.24.2}$$

which we are going to prove now. By (4.12.2) the left hand side of (4.24.2) equals

$$\begin{aligned}
&\bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\underline{F}_{(X_i);Y}} \\
&\qquad (FX_i)_{i\in I}, \underline{\mathsf{D}}((FX_i)_{i\in I};FY) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} FY \xrightarrow{\nu_Y} GY\bigr] \\
&= \bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{F\,\mathrm{ev}^{\mathsf{C}}} FY \xrightarrow{\nu_Y} GY\bigr]. \qquad (4.24.3)
\end{aligned}$$

The last transformation is due to definition of $\underline{F}$. According to (4.12.3) the right hand side of (4.24.2) can be written as

$$\begin{aligned}
&\bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}} (FX_i)_{i\in I}, G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\underline{G}_{(X_i);Y}} \\
&\qquad (FX_i)_{i\in I}, \underline{\mathsf{D}}((GX_i)_{i\in I};GY) \xrightarrow{(\nu_{X_i})_{i\in I},1} (GX_i)_{i\in I}, \underline{\mathsf{D}}((GX_i)_{i\in I};GY) \xrightarrow{\mathrm{ev}^{\mathsf{D}}} GY\bigr] \\
&= \bigl[(FX_i)_{i\in I}, F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(\nu_{X_i})_{i\in I},\nu_{\underline{\mathsf{C}}((X_i)_{i\in I};Y)}} \\
&\qquad (GX_i)_{i\in I}, G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{G\,\mathrm{ev}^{\mathsf{C}}} GY\bigr]. \qquad (4.24.4)
\end{aligned}$$

The last transformation is due to definition of $\underline{G}$. Expressions (4.24.3) and (4.24.4) coincide by multinaturality of ν. □

4.25 Lemma. *Let C, D, E be closed (symmetric) $\mathcal{V}$-multicategories. Let $\mathsf{C} \xrightarrow{F} \mathsf{D} \xrightarrow{G} \mathsf{E}$ be (symmetric) $\mathcal{V}$-multifunctors. Then*

$$\begin{aligned}
\underline{G\circ F} = \bigl[GF\underline{\mathsf{C}}((X_i)_{i\in I};Y) &\xrightarrow{G\underline{F}_{(X_i);Y}} G\underline{\mathsf{D}}((FX_i)_{i\in I};FY) \\
&\xrightarrow{\underline{G}_{(FX_i);FY}} \underline{\mathsf{E}}((GFX_i)_{i\in I};GFY)\bigr].
\end{aligned}$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd}
(GFX_i)_{i\in I}, G\underline{\mathsf{D}}((FX_i)_{i\in I};FY) \arrow[r, "(1)_I,\underline{G}_{(FX_i);FY}"] \arrow[dr, "G\,\mathrm{ev}^{\mathsf{D}}_{(FX_i);FY}"] & (GFX_i)_{i\in I}, \underline{\mathsf{E}}((GFX_i)_{i\in I};GFY) \arrow[d, "\mathrm{ev}^{\mathsf{E}}_{(GFX_i);GFY}"] \\
(GFX_i)_{i\in I}, GF\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[u, "(1)_I,G\underline{F}_{(X_i);Y}"] \arrow[r, "GF\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Y}"] & GFY
\end{tikzcd}$$

Uniqueness of the closing transformation implies the result. □

4.26 Example. Let $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric Monoidal category. Then for each $J \in \operatorname{Ob}\mathcal{O}$ the category $\mathcal{C}^J$ has a natural symmetric Monoidal structure $(\mathcal{C}^J, \otimes^I_{\mathcal{C}^J}, \lambda^f_{\mathcal{C}^J})$. Here

$$\otimes^I_{\mathcal{C}^J} = \bigl[(\mathcal{C}^J)^I \xrightarrow{\sim} (\mathcal{C}^I)^J \xrightarrow{(\otimes^I)^J} \mathcal{C}^J\bigr],$$

and for a map $f : I \to K$ the isomorphism $\lambda^f_{\mathcal{C}^J}$ is given by the pasting

$$
\begin{tikzcd}
(\mathcal{C}^J)^I \arrow[r, "\sim"] \arrow[dd, "\wr"'] & \prod_{k\in K}(\mathcal{C}^J)^{f^{-1}k} \arrow[r, "\sim"] \arrow[d, phantom, "="] & \prod_{k\in K}(\mathcal{C}^{f^{-1}k})^J \arrow[r, "\prod_{k\in K}(\otimes^{f^{-1}k})^J"] \arrow[d, "\wr"] & \prod_{k\in K}\mathcal{C}^J \arrow[r, "\sim"] \arrow[d, phantom, "="] & (\mathcal{C}^J)^K \arrow[d, "\wr"] \\
& & \left(\prod_{k\in K}\mathcal{C}^{f^{-1}k}\right)^J \arrow[rr, "(\prod_{k\in K}\otimes^{f^{-1}k})^J"] & & (\mathcal{C}^K)^J \arrow[d, "(\otimes^K)^J"] \\
(\mathcal{C}^I)^J \arrow[urr, "\sim"] \arrow[urrrr, "(\otimes^f)^J"'] \arrow[rrrr, "(\otimes^I)^J"'] & \arrow[u, phantom, "="] & & \arrow[ur, Rightarrow, "(\lambda^f)^J"'] & \mathcal{C}^J
\end{tikzcd}
$$

Necessary equations (2.5.4) hold due to coherence principle of Lemma 2.33 and Remark 2.34.

The functor $\otimes^J : \mathcal{C}^J \to \mathcal{C}$, equipped with the functorial morphism

$$
\sigma_{(12)} = \quad
\begin{tikzcd}
\mathcal{C}^{J\times I} \arrow[d, "\wr"'] \arrow[drr, "\otimes^{J\times I}"] \arrow[dr, phantom, "\Leftarrow \lambda^{J\times I\to J}"'] & \mathcal{C}^{I\times J} \arrow[l, "\sim"'] \arrow[drr, "\otimes^{I\times J}"'] \arrow[d, phantom, "\Leftarrow \lambda^{I\times J\to J\times I}"] & (\mathcal{C}^J)^I \arrow[l, "\sim"'] \arrow[r, "(\otimes^J)^I"] \arrow[dr, phantom, "\Leftarrow (\lambda^{I\times J\to I})^{-1}"] & \mathcal{C}^I \arrow[d, "\otimes^I"] \\
(\mathcal{C}^I)^J \arrow[r, "(\otimes^I)^J"] & \mathcal{C}^J \arrow[r, "\otimes^J"] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
$$

(see (2.35.1)) is a symmetric Monoidal functor. Indeed, equation (2.6.1) holds due to the said coherence principle.

Assume that $\mathcal{C}$ is closed, then $\mathcal{C}^J$ is closed as well. Thus the symmetric Monoidal functor $(\otimes^J, \sigma_{(12)}) : (\mathcal{C}^J, \otimes^I_{\mathcal{C}^J}, \lambda^f_{\mathcal{C}^J}) \to (\mathcal{C}, \otimes^I, \lambda^f)$ determines the closing transformation

$$\underline{\otimes}^J : \otimes^{j\in J}\underline{\mathcal{C}}(X_j, Y_j) \to \underline{\mathcal{C}}(\otimes^{j\in J}X_j, \otimes^{j\in J}Y_j)$$

as at the beginning of Section 4.18. It is the only solution of the equation

$$
\begin{tikzcd}
(\otimes^{j\in J}X_j) \otimes (\otimes^{j\in J}\underline{\mathcal{C}}(X_j, Y_j)) \arrow[r, Rightarrow, "1\otimes\underline{\otimes}^J"] \arrow[d, "\sigma_{(12)}"'] \arrow[dr, phantom, "="] & (\otimes^{j\in J}X_j) \otimes \underline{\mathcal{C}}(\otimes^{j\in J}X_j, \otimes^{j\in J}Y_j) \arrow[d, "\mathrm{ev}^{\mathcal{C}}"] \\
\otimes^{j\in J}(X_j \otimes \underline{\mathcal{C}}(X_j, Y_j)) \arrow[r, "\otimes^J \mathrm{ev}^{\mathcal{C}}"] & \otimes^{j\in J}Y_j
\end{tikzcd}
\tag{4.26.1}
$$

These transformations turn $\underline{\mathcal{C}}$ into a symmetric Monoidal $\mathcal{C}$-category. For each $f : I \to J$, the isomorphism $\lambda^f_{\underline{\mathcal{C}}} : \mathbb{1}_{\mathcal{C}} \to \underline{\mathcal{C}}(\otimes^{i\in I}X_i, \otimes^{j\in J} \otimes^{i\in f^{-1}j} X_i)$ is the morphism $\dot{\lambda}^f$. Equation (2.10.3) for $\lambda^f_{\underline{\mathcal{C}}}$ follows from similar equation (2.5.4) for λ^f.

In the case of $\mathcal{C} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$ the transformation $\underline{\otimes^J}$ (or $\otimes^J$ by abuse of notation) serves as part of our conventions. The above diagram gives meaning to $\underline{\otimes^{j\in\mathbf{n}}} f_j = f_1 \otimes \cdots \otimes f_n$, where f_j are $\Bbbk$-linear mappings of various degrees.

Let $(F, \phi^J) : (\mathcal{C}, \otimes^I_{\mathcal{C}}, \lambda^f_{\mathcal{C}}) \to (\mathcal{D}, \otimes^I_{\mathcal{D}}, \lambda^f_{\mathcal{D}})$ be a lax symmetric Monoidal functor between symmetric Monoidal categories (not necessarily closed). Then $\phi^J : \otimes^J_{\mathcal{D}} \circ F^J \to F \circ \otimes^J_{\mathcal{C}} : \mathcal{C}^J \to \mathcal{D}$ is a Monoidal transformation. Indeed, the diagram

$$\begin{tikzcd}
\otimes^{i\in I}_{\mathcal{D}} \otimes^{j\in J}_{\mathcal{D}} FX_{ij} \arrow[r, "\sigma_{(12)}"] \arrow[dd, "\otimes^I_{\mathcal{D}}\phi^J"'] & \otimes^{j\in J}_{\mathcal{D}} \otimes^{i\in I}_{\mathcal{D}} FX_{ij} \arrow[rr, "\otimes^J_{\mathcal{D}}\phi^I"] & & \otimes^{j\in J}_{\mathcal{D}} F \otimes^{i\in I}_{\mathcal{C}} X_{ij} \arrow[dd, "\phi^J"] \\
& \otimes^{(i,j)\in I\times J}_{\mathcal{D}} FX_{ij} \arrow[ul, "\lambda^{I\times J\to I}"] \arrow[u, "\lambda^{I\times J\to J}"'] \arrow[r, "\phi^{I\times J}"] & F \otimes^{(i,j)\in I\times J}_{\mathcal{C}} X_{ij} \arrow[d, "F\lambda^{I\times J\to I}"'] \arrow[dr, "F\lambda^{I\times J\to J}"] & \\
\otimes^{i\in I}_{\mathcal{D}} F \otimes^{j\in J}_{\mathcal{C}} X_{ij} \arrow[rr, "\phi^I"] & & F \otimes^{i\in I}_{\mathcal{C}} \otimes^{j\in J}_{\mathcal{C}} X_{ij} \arrow[r, "F\sigma_{(12)}"] & F \otimes^{j\in J}_{\mathcal{C}} \otimes^{i\in I}_{\mathcal{C}} X_{ij}
\end{tikzcd}$$

commutes due to (2.6.1) and (2.35.1).

4.27 Lemma. *Let $\mathcal{C}$ be a symmetric closed Monoidal category. Let $\phi : I \to J$ be a map in $\mathcal{S}$, and let $g : (Y_j)_{j\in J} \to Z$ be a morphism in $\widehat{\mathcal{C}}$. The morphism $\underline{\widehat{\mathcal{C}}}(\phi, g) : (\underline{\widehat{\mathcal{C}}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \to \underline{\widehat{\mathcal{C}}}((X_i)_{i\in I}; Z)$ is given by the composition in $\mathcal{C}$*

$$\otimes^{j\in J} \underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j) \xrightarrow{\underline{\otimes^J}} \underline{\mathcal{C}}(\otimes^{j\in J} \otimes^{i\in\phi^{-1}j} X_i, \otimes^{j\in J} Y_j)$$
$$\xrightarrow{\underline{\mathcal{C}}(\lambda^\phi,1)} \underline{\mathcal{C}}(\otimes^{i\in I} X_i, \otimes^{j\in J} Y_j) \xrightarrow{\underline{\mathcal{C}}(1,g)} \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z).$$

Proof. The multicategory $\widehat{\mathcal{C}}$ is closed due to Proposition 4.8. The morphism $\underline{\widehat{\mathcal{C}}}(\phi, g)$ is determined by diagram (4.12.2) for $\mathsf{C} = \widehat{\mathcal{C}}$. This condition expands to the following equation in $\mathcal{C}$:

$$\big[\otimes^{I\sqcup J}((X_i)_{i\in I}, (\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j))_{j\in J}) \xrightarrow{\lambda^{1\sqcup\triangleright : I\sqcup J\to I\sqcup\mathbf{1}}}$$
$$\otimes^{I\sqcup\mathbf{1}} ((X_i)_{i\in I}, \otimes^{j\in J}\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j)) \xrightarrow{\otimes^{I\sqcup\mathbf{1}}((1)_I, \underline{\widehat{\mathcal{C}}}(\phi,g))}$$
$$\otimes^{I\sqcup\mathbf{1}} ((X_i)_{i\in I}, \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z)) \xrightarrow{\lambda^{\triangleright\sqcup 1 : I\sqcup\mathbf{1}\to\mathbf{2}}} (\otimes^{i\in I} X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Z) \xrightarrow{\mathrm{ev}^{\mathcal{C}}} Z\big]$$
$$= \big[\otimes^{I\sqcup J}((X_i)_{i\in I}, (\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j))_{j\in J}) \xrightarrow{\lambda^{(\phi,1) : I\sqcup J\to J}}$$
$$\otimes^{j\in J} \otimes^{\phi^{-1}j\sqcup\mathbf{1}}((X_i)_{i\in\phi^{-1}j}, \underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j)) \xrightarrow{\otimes^{j\in J}\lambda^{\triangleright\sqcup 1 : \phi^{-1}j\sqcup\mathbf{1}\to\mathbf{2}}}$$
$$\otimes^{j\in J} ((\otimes^{i\in\phi^{-1}j} X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j} X_i, Y_j)) \xrightarrow{\otimes^J \mathrm{ev}^{\mathcal{C}}} \otimes^{j\in J} Y_j \xrightarrow{g} Z\big].$$

Using diagram (4.26.1) and definition (4.12.2) of $\underline{\mathcal{C}}(1, g)$ we transform the above equation

to the following one

$$
\begin{aligned}
&\bigl[\otimes^{I\sqcup J}((X_i)_{i\in I},(\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j))_{j\in J})\xrightarrow{\lambda^{\triangleright\sqcup\triangleright:I\sqcup J\to\mathbf{2}}}\\
&\qquad(\otimes^{i\in I}X_i)\otimes\otimes^{j\in J}\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j)\xrightarrow{1\otimes\widehat{\underline{\mathcal{C}}}(\phi,g)}(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)\xrightarrow{\mathrm{ev}^{\mathcal{C}}}Z\bigr]\\
&=\bigl[\otimes^{I\sqcup J}((X_i)_{i\in I},(\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j))_{j\in J})\xrightarrow{\lambda^{\phi\sqcup 1:I\sqcup J\to J\sqcup J}}\\
&\qquad\otimes^{J\sqcup J}((\otimes^{i\in\phi^{-1}j}X_i)_{j\in J},(\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j))_{j\in J})\xrightarrow{\lambda^{\triangleright\sqcup\triangleright:J\sqcup J\to\mathbf{2}}}\\
&\qquad(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i)\otimes\otimes^{j\in J}\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j)\xrightarrow{1\otimes\underline{\otimes}^J}\\
&\qquad(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i)\otimes\underline{\mathcal{C}}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i,\otimes^{j\in J}Y_j)\\
&\qquad\xrightarrow{1\otimes\underline{\mathcal{C}}(1,g)}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i)\otimes\underline{\mathcal{C}}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i,Z)\xrightarrow{\mathrm{ev}^{\mathcal{C}}}Z\bigr].
\end{aligned}
$$

Using definition (4.12.3) of $\underline{\mathcal{C}}(\lambda^\phi,1)$ and Lemma 4.13 we may replace the right hand side with an equal expression that follows:

$$
\begin{aligned}
&\bigl[\otimes^{I\sqcup J}((X_i)_{i\in I},(\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j))_{j\in J})\xrightarrow{\lambda^{\triangleright\sqcup\triangleright:I\sqcup J\to\mathbf{2}}}\\
&\quad(\otimes^{i\in I}X_i)\otimes\otimes^{j\in J}\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j)\xrightarrow{1\otimes\underline{\otimes}^J}(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i,\otimes^{j\in J}Y_j)\\
&\qquad\xrightarrow{1\otimes\underline{\mathcal{C}}(\lambda^\phi,1)}(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,\otimes^{j\in J}Y_j)\\
&\qquad\xrightarrow{1\otimes\underline{\mathcal{C}}(1,g)}(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)\xrightarrow{\mathrm{ev}^{\mathcal{C}}}Z\bigr].
\end{aligned}
$$

Comparing this with the left hand side we come to conclusion of the lemma. □

4.28 Lemma. *Let $\mathcal{C}$ be a symmetric closed Monoidal category. Let $\phi : I \to J$ be a map in $\mathcal{S}$. Then multiplication in closed multicategory $\widehat{\underline{\mathcal{C}}}$ is given by the formula*

$$
\begin{aligned}
\mu_\phi^{\widehat{\underline{\mathcal{C}}}}&=\bigl[\otimes^{J\sqcup\mathbf{1}}[(\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j))_{j\in J},\underline{\mathcal{C}}(\otimes^{j\in J}Y_j,Z)]\xrightarrow{\lambda^{\triangleright\sqcup 1:I\sqcup\mathbf{1}\to\mathbf{2}}}\\
&\qquad[\otimes^{j\in J}\underline{\mathcal{C}}(\otimes^{i\in\phi^{-1}j}X_i,Y_j)]\otimes\underline{\mathcal{C}}(\otimes^{j\in J}Y_j,Z)\xrightarrow{\underline{\otimes}^J\otimes 1}\\
&\qquad\underline{\mathcal{C}}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i,\otimes^{j\in J}Y_j)\otimes\underline{\mathcal{C}}(\otimes^{j\in J}Y_j,Z)\\
&\qquad\xrightarrow{\mu^{\underline{\mathcal{C}}}}\underline{\mathcal{C}}(\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i,Z)\xrightarrow{\underline{\mathcal{C}}(\lambda^\phi,1)}\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Z)\bigr].
\end{aligned}
$$

Proof is similar to proof of the previous lemma.

4.29 Augmented multifunctors. An *augmented (symmetric) multifunctor* is a pair consisting of a (symmetric) multifunctor $F : \mathsf{C} \to \mathsf{C}$ from a closed symmetric multicategory C to itself and a multinatural transformation $u_F : \mathrm{Id} \to F : \mathsf{C} \to \mathsf{C}$.

4.30 Proposition. *An augmented (symmetric) multifunctor* $(F, u_F) : \mathsf{C} \to \mathsf{C}$ *provides a (symmetric)* C*-multifunctor* $F' : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ *such that* $\operatorname{Ob} F' = \operatorname{Ob} F$ *and*

$$F'_{(X_i);Y} = \bigl[\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{u_F} F\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{\underline{F}} \underline{\mathsf{C}}((FX_i)_{i\in I}; FY)\bigr].$$

Proof. Consider the diagram in C

$$\begin{tikzcd}
(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "\mu_\phi^{\underline{\mathsf{C}}}"] \arrow[d, "(u_F)_J, u_F"'] & \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "u_F"] \\
(F\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, F\underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "F\mu_\phi^{\underline{\mathsf{C}}}"] \arrow[d, "(\underline{F}_{(X_i);Y_j})_{j\in J}, \underline{F}_{(Y_j);Z}"'] & F\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "\underline{F}_{(X_i);Z}"] \\
(\underline{\mathsf{C}}((FX_i)_{i\in\phi^{-1}j}; FY_j))_{j\in J}, \underline{\mathsf{C}}((FY_j)_{j\in J}; FZ) \arrow[r, "\mu_\phi^{\underline{\mathsf{C}}}"] & \underline{\mathsf{C}}((FX_i)_{i\in I}; FZ)
\end{tikzcd}$$

The upper square commutes by multinaturality of u_F, the lower square is commutative due to (4.21.1). The exterior of this diagram is the following equation in C for an arbitrary map $\phi : I \to J$ in $\operatorname{Mor} \mathcal{O}$ (resp. in $\operatorname{Mor} \mathcal{S}$):

$$\begin{tikzcd}
(\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J}; Z) \arrow[r, "\mu_\phi^{\underline{\mathsf{C}}}"] \arrow[d, "(F'_{(X_i);Y_j})_{j\in J}, F'_{(Y_j);Z}"'] & \underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[d, "F'_{(X_i);Z}"] \\
(\underline{\mathsf{C}}((FX_i)_{i\in\phi^{-1}j}; FY_j))_{j\in J}, \underline{\mathsf{C}}((FY_j)_{j\in J}; FZ) \arrow[r, "\mu_\phi^{\underline{\mathsf{C}}}"] & \underline{\mathsf{C}}((FX_i)_{i\in I}; FZ)
\end{tikzcd} \tag{4.30.1}$$

This is condition (4.3.2) for F'.

For each object $X \in \operatorname{Ob} \mathsf{C}$ there is a distinguished unit element $\eta_X = \dot{1}_X \in \mathsf{C}(;\underline{\mathsf{C}}(X;X))$, the unique element that is taken to $1_X \in \mathsf{C}(X;X)$ by the bijection $\varphi : \mathsf{C}(;\underline{\mathsf{C}}(X;X)) \to \mathsf{C}(X;X)$. We claim that the morphism F' maps units to units:

$$\bigl[() \xrightarrow{\eta_X} \underline{\mathsf{C}}(X;X) \xrightarrow{F'} \underline{\mathsf{C}}(FX;FX)\bigr] = \eta_{FX}.$$

First of all, note that due to multinaturality condition (3.15.1)

$$- \cdot u_F = \mathsf{C}(1; u_F) = F : \mathsf{C}(;\underline{\mathsf{C}}(X;X)) \to \mathsf{C}(;F\underline{\mathsf{C}}(X;X)).$$

The following diagram commutes by Corollary 4.20:

$$\begin{tikzcd}
\mathsf{C}(;\underline{\mathsf{C}}(X;X)) \arrow[r, "-\cdot u_F", "F"'] \arrow[d, "\varphi_{();X;X}"'] & \mathsf{C}(;F\underline{\mathsf{C}}(X;X)) \arrow[r, "-\cdot\underline{F}_{X;X}", "\mathsf{C}(;\underline{F}_{X;X})"'] & \mathsf{C}(;\underline{\mathsf{C}}(FX;FX)) \arrow[d, "\varphi_{();FX;FX}"] \\
\mathsf{C}(X;X) \arrow[rr, "F"] & & \mathsf{C}(FX;FX)
\end{tikzcd}$$

The upper line composes to $-\cdot F'_{X;X} : \mathsf{C}(;\underline{\mathsf{C}}(X;X)) \to \mathsf{C}(;\underline{\mathsf{C}}(FX;FX))$. Starting with the element η_X of the source we get $\varphi_{();FX;FX}(\eta\cdot F'_{X;X}) = F\varphi_{();X;X}(\eta_X) = F1_X = 1_{FX}$, therefore $\eta\cdot F'_{X;X} = \eta_{FX}$. Thus, condition (4.3.1) holds for F'. $\square$

4.31 Definition. A *multinatural transformation of augmented multifunctors* $\lambda : (F,u_F) \to (G,u_G) : \mathsf{C} \to \mathsf{C}$ is a multinatural transformation $\lambda : F \to G : \mathsf{C} \to \mathsf{C}$ such that

$$u_G = \bigl(\mathrm{Id} \xrightarrow{u_F} F \xrightarrow{\lambda} G\bigr). \tag{4.31.1}$$

4.32 Proposition. *Suppose that $\lambda : (F,u_F) \to (G,u_G) : \mathsf{C} \to \mathsf{C}$ is a multinatural transformation of augmented multifunctors. Then $r = \dot{\lambda} : F' \to G' : \underline{\mathsf{C}} \to \underline{\mathsf{C}}$ is a multinatural transformation of C-multifunctors.*

Proof. The elements $r_X \in \mathsf{C}(;\underline{\mathsf{C}}(FX;GX))$ are mapped by bijection φ to the morphisms $\lambda_X \in \mathsf{C}(FX;GX)$, $X \in \operatorname{Ob}\mathsf{C}$.

The following diagram commutes due to (4.31.1) and multinaturality of λ:

$$\begin{tikzcd}
(FX_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "(1)_I{,}u_F"] \arrow[d, "(\lambda_{X_i})_{i\in I}{,}1"'] & (FX_i)_{i\in I},F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "F\,\mathrm{ev}^{\mathsf{C}}"] \arrow[d, "(\lambda_{X_i})_{i\in I}{,}\lambda_Y"'] & FY \arrow[d, "\lambda_Y"'] \\
(GX_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "(1)_I{,}u_G"] & (GX_i)_{i\in I},G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "G\,\mathrm{ev}^{\mathsf{C}}"] & GY
\end{tikzcd} \tag{4.32.1}$$

We claim that commutativity of its exterior is equivalent to r being a multinatural transformation of C-multifunctors. Indeed, the former is equivalent to commutativity of the exterior of the diagram:

$$\begin{tikzcd}
(FX_i)_{i\in I},\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "(1)_I{,}u_F"] \arrow[d, "(1)_I{,}u_G"'] & (FX_i)_{i\in I},F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[d, "(1)_I{,}\underline{F}"'] \arrow[dr, "F\,\mathrm{ev}^{\mathsf{C}}"] & \\
(FX_i)_{i\in I},G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[d, "(1)_I{,}\underline{G}"'] & (FX_i)_{i\in I},\underline{\mathsf{C}}((FX_i)_{i\in I};FY) \arrow[r, "\mathrm{ev}^{\mathsf{C}}"'] \arrow[d, "(1)_I{,}\underline{\mathsf{C}}(\triangleright;\lambda_Y)"'] \arrow[u, phantom, "=" near start, shift left=6ex] & FY \arrow[d, "\lambda_Y"] \\
(FX_i)_{i\in I},\underline{\mathsf{C}}((GX_i)_{i\in I};GY) \arrow[r, "(1)_I{,}\underline{\mathsf{C}}((\lambda_{X_i})_{i\in I};1)"] \arrow[d, "(\lambda_{X_i})_{i\in I}{,}1"'] & (FX_i)_{i\in I},\underline{\mathsf{C}}((FX_i)_{i\in I};GY) \arrow[dr, "\mathrm{ev}^{\mathsf{C}}"] \arrow[d, phantom, "="] & {} \arrow[u, phantom, "="] \\
(GX_i)_{i\in I},\underline{\mathsf{C}}((GX_i)_{i\in I};GY) \arrow[rr, "\mathrm{ev}^{\mathsf{C}}"] & & GY
\end{tikzcd}$$

Universality of evaluation implies the equation (see left–top hexagon)

$$\begin{tikzcd}
\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "u_F"] \arrow[d, "u_G"'] & F\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[rr, "\underline{F}"] \arrow[d, phantom, "="] & & \underline{\mathsf{C}}((FX_i)_{i\in I};FY) \arrow[d, "\underline{\mathsf{C}}(\triangleright;\lambda_Y)"] \\
G\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "\underline{G}"] & \underline{\mathsf{C}}((GX_i)_{i\in I};GY) \arrow[rr, "\underline{\mathsf{C}}((\lambda_{X_i})_{i\in I};1)"] & & \underline{\mathsf{C}}((FX_i)_{i\in I};GY)
\end{tikzcd}$$

In short form, we obtained the equation in C

$$
\begin{tikzcd}
\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "="] & \underline{\mathsf{C}}((FX_i)_{i\in I};FY) \arrow[d, "\underline{\mathsf{C}}(\triangleright;\lambda_Y)"] \\
\underline{\mathsf{C}}((GX_i)_{i\in I};GY) \arrow[r, "\underline{\mathsf{C}}((\lambda_{X_i})_{i\in I};1)"] & \underline{\mathsf{C}}((FX_i)_{i\in I};GY)
\end{tikzcd}
$$

Lemmata 4.16 and 4.17 allow to rewrite this equation as follows:

$$
\begin{tikzcd}
\underline{\mathsf{C}}((X_i)_{i\in I};Y) \arrow[r, "F',r_Y"] \arrow[d, "(r_{X_i})_{i\in I},G'"'] \arrow[dr, phantom, "="] & \underline{\mathsf{C}}((FX_i)_{i\in I};FY),\underline{\mathsf{C}}(FY;GY) \arrow[d, "\mu^{\mathsf{C}}_{\triangleright}"] \\
(\underline{\mathsf{C}}(FX_i;GX_i))_{i\in I},\underline{\mathsf{C}}((GX_i)_{i\in I};GY) \arrow[r, "\mu^{\mathsf{C}}_{\mathrm{id}_I}"] & \underline{\mathsf{C}}((FX_i)_{i\in I};GY)
\end{tikzcd}
\tag{4.32.2}
$$

This is precisely condition (4.4.1) of r being a multinatural transformation of C-multifunctors. □

Notice also that conversely, equation (4.32.2) implies commutativity of the exterior of diagram (4.32.1). Its particular case of $I = \varnothing$ implies condition (4.31.1).

Composition of augmented multifunctors $(F, u_F) : \mathsf{C} \to \mathsf{C}$ and $(G, u_G) : \mathsf{C} \to \mathsf{C}$ is defined as $(G \circ F, u_{G\circ F})$, where

$$
u_{G\circ F} = \bigl(\mathrm{Id} \xrightarrow{u_F} F \xrightarrow{u_G} G \circ F\bigr) = \bigl(\mathrm{Id} \xrightarrow{u_G} G \xrightarrow{Gu_F} G \circ F\bigr). \tag{4.32.3}
$$

Clearly, it is strictly associative.

4.33 Proposition. *Composition* (4.32.3) *of augmented (symmetric) multifunctors and usual left and right multiplication of multifunctors and multinatural transformations make the category of augmented multifunctors* $\mathrm{AugMltFun}(\mathsf{C})$ *into a strictly monoidal one. The correspondence* $\mathrm{AugMltFun}(\mathsf{C}) \to \mathsf{C}\text{-}\mathcal{MC}\mathrm{atm}(\underline{\mathsf{C}}, \underline{\mathsf{C}})$, $F \mapsto F'$, $\lambda \mapsto \dot{\lambda}$ *is a strictly monoidal functor with values in the category of (symmetric)* C*-multifunctors and their multinatural transformations.*

Proof. Let $\lambda : (F, u_F) \to (G, u_G) : \mathsf{C} \to \mathsf{C}$ be a multinatural transformation of augmented (symmetric) multifunctors, and let $(H, u_H) : \mathsf{C} \to \mathsf{C}$ be an augmented (symmetric) multifunctor. Then the multinatural transformation $H\lambda : H \circ F \to H \circ G : \mathsf{C} \to \mathsf{C}$ satisfies condition (4.31.1) due to

$$
\begin{aligned}
\bigl(\mathrm{Id} \xrightarrow{u_{H\circ F}} H \circ F \xrightarrow{H\lambda} H \circ G\bigr) &= \bigl(\mathrm{Id} \xrightarrow{u_H} H \xrightarrow{Hu_F} H \circ F \xrightarrow{H\lambda} H \circ G\bigr) \\
&= \bigl(\mathrm{Id} \xrightarrow{u_H} H \xrightarrow{Hu_G} H \circ G\bigr) = u_{H\circ G}.
\end{aligned}
$$

The multinatural transformation $\lambda = \lambda_{H-} : F \circ H \to G \circ H : \mathsf{C} \to \mathsf{C}$ satisfies condition (4.31.1) due to

$$\begin{aligned}\bigl(\mathrm{Id} \xrightarrow{u_{F\circ H}} F \circ H \xrightarrow{\lambda} G \circ H\bigr) &= \bigl(\mathrm{Id} \xrightarrow{u_H} H \xrightarrow{u_F} F \circ H \xrightarrow{\lambda} G \circ H\bigr) \\ &= \bigl(\mathrm{Id} \xrightarrow{u_H} H \xrightarrow{u_G} G \circ H\bigr) = u_{G\circ H}.\end{aligned}$$

Thus, these operations equip the category of augmented multifunctors with the strictly associative monoidal product, the composition. The unit object is $(\mathrm{Id}, \mathrm{id})$.

Let us verify compatibility of the map $F \mapsto F'$ with the composition of multifunctors:

$$\begin{aligned}F' \cdot G' = \bigl[&\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{u_F} F\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{F} \underline{\mathsf{C}}((FX_i)_{i\in I}; FY) \\ &\xrightarrow{u_G} G\underline{\mathsf{C}}((FX_i)_{i\in I}; FY) \xrightarrow{G} \underline{\mathsf{C}}((GFX_i)_{i\in I}; GFY)\bigr] \\ = \bigl[&\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{u_F} F\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{u_G} GF\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \\ &\xrightarrow{GF} G\underline{\mathsf{C}}((FX_i)_{i\in I}; FY) \xrightarrow{G} \underline{\mathsf{C}}((GFX_i)_{i\in I}; GFY)\bigr] \\ = \bigl[&\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \xrightarrow{u_{G\circ F}} GF\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \\ &\xrightarrow{G\circ F} \underline{\mathsf{C}}((GFX_i)_{i\in I}; GFY)\bigr] \\ = (&F \cdot G)'.\end{aligned}$$

One can prove compatibility with left and right multiplication of multifunctors and transformations and with composition of transformations. □

Not all natural transformations of interest are multinatural. That is why we consider also transformations subject to less conditions.

4.34 Definition. A *natural transformation of augmented multifunctors* $\lambda : (F, u_F) \to (G, u_G) : \mathsf{C} \to \mathsf{C}$ is a natural transformation (of ordinary functors) $\lambda : F \to G : \mathsf{C} \to \mathsf{C}$ such that condition (4.31.1) is satisfied and the following equation holds in C for an arbitrary morphism $e : X, W \to Y$ in C:

$$\begin{tikzcd} FX, W \arrow[r, "1{,}u_F"] \arrow[d, "\lambda_X{,}1"'] & FX, FW \arrow[r, "Fe"] \arrow[d, phantom, "="] & FY \arrow[d, "\lambda_Y"] \\ GX, W \arrow[r, "1{,}u_G"] & GX, GW \arrow[r, "Ge"] & GY \end{tikzcd} \tag{4.34.1}$$

Notice that if $\lambda : F \to G : \mathsf{C} \to \mathsf{C}$ is a multinatural transformation and satisfies (4.31.1), then it automatically satisfies (4.34.1). If $(F, u_F) \xrightarrow{\lambda} (G, u_G) \xrightarrow{\mu} (H, u_H) : \mathsf{C} \to \mathsf{C}$ are two natural transformations of augmented multifunctors, then their composition $\lambda \cdot \mu$ satisfies both conditions (4.31.1) and (4.34.1). Taking such natural transformations as morphisms we get wider category of augmented multifunctors $\mathsf{C} \to \mathsf{C}$.

4.35 Corollary. *Composition* (4.32.3) *of augmented multifunctors and usual left and right multiplication of ordinary functors and natural transformations make the category of augmented multifunctors into a strictly monoidal one. It is mapped by the strictly monoidal functor $F \mapsto F'$, $\lambda \mapsto \dot{\lambda}$ to the category of C-functors and their natural transformations.*

Proof. The reasoning of Propositions 4.30, 4.32 and 4.33 goes through for the particular case of $I = \mathbf{1}$. Notice that the following particular case of (4.34.1)

$$\begin{tikzcd}
FX, \underline{\mathsf{C}}(X;Y) \arrow[r, "1{,}u_F"] \arrow[d, "\lambda_X{,}1"'] & FX, F\underline{\mathsf{C}}(X;Y) \arrow[r, "F\,\mathrm{ev}^{\mathsf{C}}"] \arrow[d, phantom, "="] & FY \arrow[d, "\lambda_Y"] \\
GX, \underline{\mathsf{C}}(X;Y) \arrow[r, "1{,}u_G"] & GX, G\underline{\mathsf{C}}(X;Y) \arrow[r, "G\,\mathrm{ev}^{\mathsf{C}}"] & GY
\end{tikzcd} \tag{4.35.1}$$

is equivalent to equation

$$\begin{tikzcd}
\underline{\mathsf{C}}(X;Y) \arrow[r, "F'"] \arrow[d, "G'"'] \arrow[dr, phantom, "="] & \underline{\mathsf{C}}(FX;FY) \arrow[d, "\underline{\mathsf{C}}(1;\lambda_Y)"] \\
\underline{\mathsf{C}}(GX;GY) \arrow[r, "\underline{\mathsf{C}}(\lambda_X;1)"] & \underline{\mathsf{C}}(FX;GY)
\end{tikzcd} \tag{4.35.2}$$

required for natural transformations of C-functors. □

For any algebra (monad) $((F,u_F),m_F,\eta_F)$ in the monoidal category of augmented multifunctors the unit $\eta_F : (\mathrm{Id},\mathrm{id}) \to (F,u_F)$ coincides with u_F due to (4.31.1). We may describe all such monads as ordinary monads (F,m_F,u_F) in ordinary category C which satisfy additional properties:

– F is a multifunctor and $u_F : \mathrm{Id} \to F$ is multinatural;

– for any morphism $e : X,W \to Y$ in C the exterior of the following diagram commutes:

$$\begin{tikzcd}
F^2X, W \arrow[r, "1{,}u_F"] \arrow[d, "m_F{,}1"'] & F^2X, FW \arrow[r, "1{,}Fu_F"] \arrow[d, dashed, "m_F{,}1"] & F^2X, F^2W \arrow[r, "F^2e"] & F^2Y \arrow[d, "m_F"] \\
FX, W \arrow[r, "1{,}u_F"] & FX, FW \arrow[rr, "Fe"] & & FY
\end{tikzcd} \tag{4.35.3}$$

The pentagon containing the dashed arrow is not required to commute, however, it does commute in our main example of Proposition 10.20.

4.36 Proposition. *Let $E : \mathsf{C} \to \mathsf{D}$ be a (symmetric) multifunctor, and let $(F, u_F) : \mathsf{C} \to \mathsf{C}$, $(G, u_G) : \mathsf{D} \to \mathsf{D}$ be augmented (symmetric) multifunctors. Suppose that*

$$\big(\mathsf{C} \xrightarrow{F} \mathsf{C} \xrightarrow{E} \mathsf{D}\big) = \big(\mathsf{C} \xrightarrow{E} \mathsf{D} \xrightarrow{G} \mathsf{D}\big),$$

$$\big(\mathsf{C} \underset{F}{\overset{\mathrm{Id}}{\underset{u_F\Downarrow}{\rightrightarrows}}} \mathsf{C} \xrightarrow{E} \mathsf{D}\big) = \big(\mathsf{C} \xrightarrow{E} \mathsf{D} \underset{G}{\overset{\mathrm{Id}}{\underset{u_G\Downarrow}{\rightrightarrows}}} \mathsf{D}\big).$$

Then the following equation holds

$$\begin{tikzcd}
E\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \arrow[r, "EF'"] \arrow[d, "\underline{E}"'] \arrow[dr, phantom, "="] & E\underline{\mathsf{C}}((FX_i)_{i\in I}; FY) \arrow[d, "\underline{E}"] \\
\underline{\mathsf{D}}((EX_i)_{i\in I}; EY) \arrow[r, "G'"] & \underline{\mathsf{D}}((GEX_i)_{i\in I}; GEY)
\end{tikzcd}$$

Proof. This equation is the exterior of the following diagram

$$\begin{tikzcd}
E\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \arrow[r, "Eu_F", "\parallel"' description] \arrow[d, "\underline{E}"'] \arrow[dr, phantom, "\overset{u_G}{=}"] & EF\underline{\mathsf{C}}((X_i)_{i\in I}; Y) \arrow[r, "E\underline{F}"] \arrow[d, "G\underline{E}"] \arrow[dr, phantom, "="] & E\underline{\mathsf{C}}((FX_i)_{i\in I}; FY) \arrow[d, "\underline{E}"] \\
\underline{\mathsf{D}}((EX_i)_{i\in I}; EY) \arrow[r, "u_G"] & G\underline{\mathsf{D}}((EX_i)_{i\in I}; EY) \arrow[r, "\underline{G}"] & \underline{\mathsf{D}}((GEX_i)_{i\in I}; GEY)
\end{tikzcd}$$

The left square commutes by naturality of u_G, and the right square commutes by Lemma 4.25. □

Chapter 5

Kleisli multicategories

We consider the multicategory of T-coalgebras, where T is a multicomonad in a closed multicategory. It gives rise to a closed multicategory of free T-coalgebras, isomorphic to Kleisli multicategory of T. The particular case of lax representable multicategories is described in detail.

5.1 Coalgebras for a multicomonad. Let (T, Δ, ε) be a multicomonad on a closed symmetric multicategory C. This means that $T : \mathsf{C} \to \mathsf{C}$ is a multifunctor, $\Delta : T \to TT : \mathsf{C} \to \mathsf{C}$ and $\varepsilon : T \to \mathrm{Id} : \mathsf{C} \to \mathsf{C}$ are multinatural transformations, and the triple (T, Δ, ε) is a coalgebra in the strict monoidal category $\mathcal{MC}\mathrm{at}(\mathsf{C}, \mathsf{C})$. The following definition is analogous to the case of ordinary categories, treated by Beck [Bec03].

5.2 Definition. A *T-coalgebra* is a morphism $\delta : X \to TX$ in C such that the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\delta"] \arrow[d, "\delta"'] & TX \arrow[d, "T\delta"] \\ TX \arrow[r, "\Delta"] & TTX \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\delta"] \arrow[dr, equal] & TX \arrow[d, "\varepsilon"] \\ & X \end{tikzcd}$$

A morphism of T-coalgebras $(\delta_1 : X_1 \to TX_1)$, …, $(\delta_n : X_n \to TX_n)$ to a T-coalgebra $\beta : Y \to TY$ is a map $f \in \mathsf{C}(X_1, \dots, X_n; Y)$ such that $f \cdot \beta = (\delta_1, \dots, \delta_n) \cdot (Tf)$. T-coalgebras and their morphisms form a multicategory denoted C_T. The composition and the units of C_T are those of C.

5.3 Lemma. *The forgetful functor $\mathsf{C}_T \to \mathsf{C}$ has the right adjoint T: for any $Y \in \mathrm{Ob}\,\mathsf{C}$ the morphism $\Delta : TY \to TTY$ is a T-coalgebra; and for T-coalgebras $(\delta_1 : X_1 \to TX_1)$, …, $(\delta_n : X_n \to TX_n)$ there is a pair of mutually inverse natural isomorphisms*

$$\begin{aligned} \mathsf{C}(X_1, \dots, X_n; Y) &\to \mathsf{C}_T(X_1, \dots, X_n; TY), & f &\mapsto \hat{f} = (\delta_1, \dots, \delta_n) \cdot (Tf) \\ \mathsf{C}_T(X_1, \dots, X_n; TY) &\to \mathsf{C}(X_1, \dots, X_n; Y), & g &\mapsto \check{g} = g \cdot \varepsilon \end{aligned} \tag{5.3.1}$$

Proof. First of all, $\hat{f}$ is a morphism of T-coalgebras. Indeed, the following compositions

in C are equal:

$$\begin{aligned}\hat{f}\cdot\Delta &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{Tf} TY \xrightarrow{\Delta} TTY\bigr]\\ &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{(\Delta)_i} (TTX_i) \xrightarrow{TTf} TTY\bigr]\\ &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{(T\delta_i)} (TTX_i) \xrightarrow{TTf} TTY\bigr]\\ &= (\delta_1,\dots,\delta_n)\cdot(T\hat{f}).\end{aligned}$$

To show that the discussed maps are inverse to each other, we write

$$\begin{aligned}\check{\hat{f}} &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{Tf} TY \xrightarrow{\varepsilon} Y\bigr]\\ &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{(\varepsilon)} (X_i) \xrightarrow{f} Y\bigr] = f.\end{aligned}$$

We also have

$$\begin{aligned}\hat{\check{g}} &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{T\check{g}} TY\bigr]\\ &= \bigl[(X_i) \xrightarrow{(\delta_i)} (TX_i) \xrightarrow{Tg} TTY \xrightarrow{T\varepsilon} TY\bigr]\\ &= \bigl[(X_i) \xrightarrow{g} TY \xrightarrow{\Delta} TTY \xrightarrow{T\varepsilon} TY\bigr] = g,\end{aligned}$$

so the lemma is proven. □

For each T-coalgebra Z the equation $\delta\cdot T\delta = \delta\cdot\Delta : Z \to TTZ$ shows that $\delta : Z \to TZ$ is a morphism of T-coalgebras.

In the above assumptions consider an example of multinatural transformation (4.18.2). Denote by

$$\underline{T} : T\underline{\mathsf{C}}((X_i)_{i\in I};Y) \to \underline{\mathsf{C}}((TX_i)_{i\in I};TY) \tag{5.3.2}$$

the closing multinatural transformation of the multifunctor $T : \mathsf{C} \to \mathsf{C}$. It is determined by equation

$$T\,\mathrm{ev}_{(X_i);Y} = \bigl[(TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};Y) \xrightarrow{(1)_I,\underline{T}} (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I};TY) \xrightarrow{\mathrm{ev}_{(TX_i);TY}} TY\bigr].$$

Given T-coalgebras X_i, $i\in I$, and an object B of C denote by $\Theta_{(X_i);B} : T\underline{\mathsf{C}}((X_i)_{i\in I};B) \to \underline{\mathsf{C}}((X_i)_{i\in I};TB)$ the unique morphism that satisfies the equation

$$\begin{tikzcd} (X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[r, "(\delta)_I{,}1"] \arrow[d, "(1)_I{,}\Theta_{(X_i);B}"'] \arrow[dr, phantom, "="] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[d, "T\,\mathrm{ev}^{\mathsf{C}}_{(X_i);B}"] \\ (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};TB) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};TB}"] & TB \end{tikzcd} \tag{5.3.3}$$

Its existence and uniqueness follows from closedness of C.

5.4 Lemma. *So defined $\Theta_{(X_i);B}$ can be decomposed as follows:*

$$\Theta_{(X_i);B} = \bigl[T\underline{\mathsf{C}}((X_i)_{i\in I};B) \xrightarrow{\underline{T}} \underline{\mathsf{C}}((TX_i)_{i\in I};TB) \xrightarrow{\underline{\mathsf{C}}((\delta)_I;1)} \underline{\mathsf{C}}((X_i)_{i\in I};TB)\bigr].$$

Proof. There is a commutative diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[r, "(\delta)_I{,}1"] \arrow[d, "(1)_I{,}\underline{T}"'] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[d, "(1)_I{,}\underline{T}"'] \arrow[dd, bend left, "T\,\mathrm{ev}^{\mathsf{C}}_{(X_i);B}"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I};TB) \arrow[r, "(\delta)_I{,}1"] \arrow[d, "(1)_I{,}\underline{\mathsf{C}}((\delta)_I;1)"'] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I};TB) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(TX_i);TB}"'] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};TB) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};TB}"] & TB
\end{tikzcd}$$

Uniqueness implies the claimed equation. □

5.5 Definition. A diagram $X \xrightarrow{e} Y \overset{f}{\underset{g}{\rightrightarrows}} Z$ in a multicategory C is an *equalizer*, if for any family $(W_k)_{k\in K}$ of objects of C the induced diagram

$$\mathsf{C}((W_k)_{k\in K};X) \xrightarrow{\mathsf{C}(\triangleright;e)} \mathsf{C}((W_k)_{k\in K};Y) \overset{\mathsf{C}(\triangleright;f)}{\underset{\mathsf{C}(\triangleright;g)}{\rightrightarrows}} \mathsf{C}((W_k)_{k\in K};Z)$$

is an equalizer in $\mathcal{S}\mathrm{et}$. In particular, $ef = eg$. We say that a multicategory C *has equalizers* if each pair $f, g : Y \to Z$ of morphisms in C can be completed to an equalizer diagram.

A lax representable multicategory $\widehat{\mathcal{C}}$ has equalizers if and only if category $\mathcal{C}$ has.

5.6 Lemma. *For an arbitrary T-coalgebra (X,δ) in a multicategory C the diagram $X \xrightarrow{\delta} TX \overset{T\delta}{\underset{\Delta}{\rightrightarrows}} TTX$ is an equalizer in C.*

Proof. Let $(W_k)_{k\in K}$ be any family of objects of C. In the induced diagram

$$\mathsf{C}((W_k)_{k\in K};X) \xrightarrow{\mathsf{C}(\triangleright;\delta)} \mathsf{C}((W_k)_{k\in K};TX) \overset{\mathsf{C}(\triangleright;T\delta)}{\underset{\mathsf{C}(\triangleright;\Delta)}{\rightrightarrows}} \mathsf{C}((W_k)_{k\in K};TTX)$$

both paths are equal. Since $\delta : X \to TX$ is a split embedding with the splitting $\varepsilon : TX \to X$, the first mapping $\mathsf{C}(\triangleright;\delta)$ is a split embedding with the splitting $\mathsf{C}(\triangleright;\varepsilon)$.

Assume that both images of an element $g \in \mathsf{C}((W_k)_{k\in K};TX)$ coincide:

$$\bigl[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{T\delta} TTX\bigr] = \bigl[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{\Delta} TTX\bigr].$$

Then

$$\begin{aligned} g &= \big[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{\Delta} TTX \xrightarrow{\varepsilon} TX\big] \\ &= \big[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{T\delta} TTX \xrightarrow{\varepsilon} TX\big] \\ &= \big[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{\varepsilon} X \xrightarrow{\delta} TX\big] \qquad \text{(by naturality of } \varepsilon) \end{aligned}$$

comes from the element $f = \big[(W_k)_{k\in K} \xrightarrow{g} TX \xrightarrow{\varepsilon} X\big]$ of $\mathsf{C}((W_k)_{k\in K}; X)$. □

5.7 Lemma. *Let X_i be T-coalgebras, and let B be an object of C. Use the notation $\underline{\mathsf{C}'_T}((X_i)_{i\in I}; TB) = T\underline{\mathsf{C}}((X_i)_{i\in I}; B)$ and define*

$$\begin{aligned} \mathrm{ev}^{\mathsf{C}'_T}_{(X_i)_{i\in I};TB} &= \big[(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{(\delta)_I,1} (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\,\mathrm{ev}^{\mathsf{C}}} TB\big] \\ &= \big[(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{(1)_I,\Theta} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} TB\big]. \end{aligned}$$

Then $\varphi^{\mathsf{C}'_T}_{(Y_j)_{j\in J};(X_i)_{i\in I};TB}$, given by (4.7.1) with C_T resp. $\underline{\mathsf{C}'_T}$ in place of C resp. $\underline{\mathsf{C}}$, is a bijection.

Proof. Being a composition of T-coalgebra morphisms, $\mathrm{ev}^{\mathsf{C}'_T}_{(X_i)_{i\in I};TB}$ is a morphism of T-coalgebras itself.

Given a morphism of T-coalgebras $f : (X_i)_{i\in I}, (Y_j)_{j\in J} \to TB$, let us show that there exists a unique T-coalgebra morphism $g : (Y_j)_{j\in J} \to T\underline{\mathsf{C}}((X_i)_{i\in I}; B)$ such that

$$\begin{aligned} f &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1)_I,g} (X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\mathrm{ev}^{\mathsf{C}'_T}_{(X_i)_{i\in I};TB}} TB\big] \\ &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1)_I,g\cdot\Theta} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};TB}} TB\big] \\ &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(\delta)_I,g} (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\,\mathrm{ev}^{\mathsf{C}}} TB\big]. \end{aligned} \tag{5.7.1}$$

Assuming that such g exists we obtain:

$$\begin{aligned} \check{f} = f\cdot\varepsilon &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(\delta)_I,g} (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\,\mathrm{ev}^{\mathsf{C}}} TB \xrightarrow{\varepsilon} B\big] \qquad (5.7.2) \\ &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(\delta\cdot\varepsilon)_I,g\cdot\varepsilon} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} B\big] \\ &= \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1)_I,\check{g}} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} B\big]. \end{aligned}$$

By Lemma 5.3 T-coalgebra morphisms $g : (Y_j)_{j\in J} \to T\underline{\mathsf{C}}((X_i)_{i\in I}; B)$ are in bijective correspondence with morphisms $\check{g} = g\cdot\varepsilon : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; B)$. Therefore, the above equation determines g uniquely.

Let $p : (Y_j)_{j\in J} \to \underline{\mathsf{C}}((X_i)_{i\in I}; B)$ be a morphism that satisfies the following equation:

$$f\cdot\varepsilon = \big[(X_i)_{i\in I}, (Y_j)_{j\in J} \xrightarrow{(1),p} (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} B\big].$$

It exists by closedness of C. Put $g = \hat{p} = (\delta)_J \cdot Tp : (Y_j)_{j \in J} \to T\underline{\mathsf{C}}((X_i)_{i \in I}; B)$. So defined g is a T-coalgebra morphism and $\check{g} = p$ by Lemma 5.3. Thus for this g equation (5.7.2) holds. It implies equations (5.7.1) again by Lemma 5.3. The claim is proven. □

5.8 Definition. T-coalgebras of the form (TB, Δ) for some object B of C are called *free*. Multicategory C_T^f of free T-coalgebras has the same objects as C. Its morphisms, compositions and units are taken from C_T: $\mathsf{C}_T^f((A_i)_{i \in I}; B) \overset{\text{def}}{=} \mathsf{C}_T((TA_i)_{i \in I}; TB)$.

5.9 Corollary (to Lemma 5.7)**.** *If multicategory C is closed, then the multicategory C_T^f of free T-coalgebras is closed as well. The inner homomorphisms objects can be chosen as $\underline{\mathsf{C}_T^f}((A_i)_{i \in I}; B) = \underline{\mathsf{C}}((TA_i)_{i \in I}; B)$ and the evaluations as*

$$\begin{aligned} \mathrm{ev}^{\mathsf{C}_T^f}_{(A_i)_{i \in I}; B} &= \mathrm{ev}^{\mathsf{C}'_T}_{(TA_i)_{i \in I}; TB} \\ &= \big[(TA_i)_{i \in I}, T\underline{\mathsf{C}}((TA_i)_{i \in I}; B) \xrightarrow{(1)_I, \Theta} (TA_i)_{i \in I}, \underline{\mathsf{C}}((TA_i)_{i \in I}; TB) \xrightarrow{\mathrm{ev}^{\mathsf{C}}} TB\big]. \end{aligned}$$

5.10 Proposition. *The split embedding*

$$e = \big[\underline{\mathsf{C}'_T}((X_i)_{i \in I}; TB) = T\underline{\mathsf{C}}((X_i)_{i \in I}; B) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i \in I}; B) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i \in I}; TB)\big]$$

is the equalizer in C of a pair of T-coalgebra morphisms

$$\begin{tikzcd} \underline{\mathsf{C}'_T}((X_i)_{i \in I}; TB) \arrow[r, "\text{equalizer of the pair}", "e"'] & T\underline{\mathsf{C}}((X_i)_{i \in I}; TB) \arrow[r, "T\underline{\mathsf{C}}(\triangleright; \Delta)"] \arrow[dr, "\Delta"'] & T\underline{\mathsf{C}}((X_i)_{i \in I}; TTB), \\ & & TT\underline{\mathsf{C}}((X_i)_{i \in I}; TB) \arrow[u, "T\Theta"'] \end{tikzcd} \tag{5.10.1}$$

where X_i, $i \in I$, are T-coalgebras. Diagram (5.10.1) *is nothing else but the equalizer*

$$\underline{\mathsf{C}'_T}((X_i)_{i \in I}; TB) \xrightarrow{e} \underline{\mathsf{C}'_T}((X_i)_{i \in I}; TTB) \overset{\underline{\mathsf{C}'_T}(\triangleright; T\Delta)}{\underset{\underline{\mathsf{C}'_T}(\triangleright; \Delta)}{\rightrightarrows}} \underline{\mathsf{C}'_T}((X_i)_{i \in I}; TTTB).$$

The embedding e is split by the morphism $T\underline{\mathsf{C}}(\triangleright; \varepsilon) : T\underline{\mathsf{C}}((X_i)_{i \in I}; TB) \to T\underline{\mathsf{C}}((X_i)_{i \in I}; B)$.

Proof. We have

$$T\underline{\mathsf{C}}(\triangleright; \Delta) \cdot T\underline{\mathsf{C}}(\triangleright; \varepsilon) = T\underline{\mathsf{C}}(\triangleright; \Delta \cdot \varepsilon) = T\underline{\mathsf{C}}(\triangleright; \mathrm{id}) = \mathrm{id}$$

due to Lemma 4.13. Thus, the morphism $T\underline{\mathsf{C}}(\triangleright; \Delta)$ is an embedding, split by the morphism $T\underline{\mathsf{C}}(\triangleright; \varepsilon)$.

Let us prove that the composition

$$T\underline{\mathsf{C}}((X_i)_{i \in I}; Z) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i \in I}; Z) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i \in I}; TZ)$$

is an embedding, split by the morphism $T\underline{\mathsf{C}}(\triangleright;\varepsilon) : T\underline{\mathsf{C}}((X_i)_{i\in I};TZ) \to T\underline{\mathsf{C}}((X_i)_{i\in I};Z)$. This immediately follows from the equation

$$\big[T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \xrightarrow{\Theta} \underline{\mathsf{C}}((X_i)_{i\in I};TZ) \xrightarrow{\underline{\mathsf{C}}(\triangleright;\varepsilon)} \underline{\mathsf{C}}((X_i)_{i\in I};Z)\big] = \varepsilon, \tag{5.10.2}$$

which we are going to prove now. Indeed, we have a commutative diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "(\delta)_I{,}1"] \arrow[d, "(1)_I{,}\Theta_{(X_i);Z}"'] \arrow[rrd, bend left, "(1)_I{,}\varepsilon"] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[d, "T\,\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"'] \arrow[rd, "(\varepsilon)_I{,}\varepsilon"] & \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};TZ) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};TZ}"] \arrow[d, "(1)_I{,}\underline{\mathsf{C}}(\triangleright;\varepsilon)"'] & TZ \arrow[d, "\varepsilon"'] & (X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[ld, "\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i)_{i\in I};Z}"] & Z &
\end{tikzcd}$$

Uniqueness implies the required equation (5.10.2). Thus, $\Delta\cdot T\Theta : T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \to T\underline{\mathsf{C}}((X_i)_{i\in I};TZ)$ is an embedding, split by $T\underline{\mathsf{C}}(\triangleright;\varepsilon)$.

Let us prove that for an arbitrary morphism $g : B \to A$ in C we have $\underline{\mathsf{C}'_T}(\triangleright;Tg) = T\underline{\mathsf{C}}(\triangleright;g) : T\underline{\mathsf{C}}((X_i)_{i\in I};B) \to T\underline{\mathsf{C}}((X_i)_{i\in I};A)$. Indeed, there is only one top arrow which makes commutative exterior of the diagram

$$\begin{tikzcd}
(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[r, "(1)_I{,}\underline{\mathsf{C}'_T}(\triangleright;Tg)", "(1)_I{,}T\underline{\mathsf{C}}(\triangleright;g)"'] \arrow[d, "(\delta)_I{,}1"'] & (X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};A) \arrow[d, "(\delta)_I{,}1"] \\
(TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};B) \arrow[r, "(T1)_I{,}T\underline{\mathsf{C}}(\triangleright;g)"] \arrow[d, "T\,\mathrm{ev}^{\mathsf{C}}"'] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I};A) \arrow[d, "T\,\mathrm{ev}^{\mathsf{C}}"] \\
TB \arrow[r, "Tg"] & TA
\end{tikzcd}$$

In particular, $\underline{\mathsf{C}'_T}(\triangleright;T\Delta) = T\underline{\mathsf{C}}(\triangleright;\Delta) : T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \to T\underline{\mathsf{C}}((X_i)_{i\in I};TZ)$.

Let us prove now that $\underline{\mathsf{C}'_T}(\triangleright;\Delta) = \Delta\cdot T\Theta : T\underline{\mathsf{C}}((X_i)_{i\in I};Z) \to T\underline{\mathsf{C}}((X_i)_{i\in I};TZ)$ for an arbitrary object Z of C. In fact, substituting the right hand side in the defining diagram

of the left hand side we get the exterior of the following diagram:

$$\begin{tikzcd}
(X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(1)_I{,}\Delta"] \arrow[dd, "(\delta)_I{,}1"'] \arrow[dr, "(\delta)_I{,}\Delta"'] & (X_i)_{i\in I}, TT\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(1)_I{,}T\Theta"] & (X_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; TZ) \arrow[d, "(\delta)_I{,}1"] \\
& (TX_i)_{i\in I}, TT\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(1)_I{,}T\Theta"] \arrow[d, "(T\delta)_I{,}1"'] & (TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; TZ) \arrow[dd, "T\,\mathrm{ev}^{\mathsf{C}}"] \\
(TX_i)_{i\in I}, T\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[r, "(\Delta)_I{,}\Delta"] \arrow[d, "T\,\mathrm{ev}^{\mathsf{C}}"'] & (TTX_i)_{i\in I}, TT\underline{\mathsf{C}}((X_i)_{i\in I}; Z) \arrow[dr, "TT\,\mathrm{ev}^{\mathsf{C}}"] & \\
TZ \arrow[rr, "\Delta"] & & TTZ
\end{tikzcd}$$

Clearly, the diagram commutes, and the claimed equation is proven.

Let us verify that e gives an equalizer. We begin with proving that both left–to–right paths in the diagram of T-coalgebra morphisms

$$\begin{tikzcd}
T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \arrow[dr, "\Delta"'] & & T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \arrow[rr, "T\underline{\mathsf{C}}(\rhd;\Delta)"] \arrow[dr, "\Delta"'] & & T\underline{\mathsf{C}}((X_i)_{i\in I}; TTB) \\
& TT\underline{\mathsf{C}}((X_i)_{i\in I}; B) \arrow[ur, "T\Theta"'] & & TT\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \arrow[ur, "T\Theta"'] &
\end{tikzcd} \tag{5.10.3}$$

compose to the same morphism. Notice that this diagram is nothing else but

$$\underline{\mathsf{C}'_T}((X_i)_{i\in I}; TB) \xrightarrow{\underline{\mathsf{C}'_T}(\rhd;\Delta)} \underline{\mathsf{C}'_T}((X_i)_{i\in I}; TTB) \underset{\underline{\mathsf{C}'_T}(\rhd;\Delta)}{\overset{\underline{\mathsf{C}'_T}(\rhd;T\Delta)}{\rightrightarrows}} \underline{\mathsf{C}'_T}((X_i)_{i\in I}; TTTB).$$

Lemma 5.7 allows to conclude just as in Lemma 4.15 that these arrows compose to $\underline{\mathsf{C}'_T}(\rhd; \Delta\cdot T\Delta)$ and $\underline{\mathsf{C}'_T}(\rhd; \Delta\cdot\Delta)$, respectively. However, these two morphisms are equal. Hence the two compositions above coincide. One can prove this also directly using the following identity

$$\begin{aligned}
&\big[T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\Theta} \underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\underline{\mathsf{C}}(\rhd;\Delta)} \underline{\mathsf{C}}((X_i)_{i\in I}; TTB)\big] \\
&= \big[T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\Theta} \underline{\mathsf{C}}((X_i)_{i\in I}; TTB)\big],
\end{aligned}$$

whose proof we omit.

To prove that diagram (5.10.3) is an equalizer in C, we apply to it the functor $\mathsf{C}((W_k)_{k\in K}; -)$ for an arbitrary family $(W_k)_{k\in K}$ of objects of C. It remains to prove that an arbitrary morphism $g : (W_k)_{k\in K} \to T\underline{\mathsf{C}}((X_i)_{i\in I}; TB)$ in C such that

$$\begin{aligned}
&\big[(W_k)_{k\in K} \xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\underline{\mathsf{C}}(\rhd;\Delta)} T\underline{\mathsf{C}}((X_i)_{i\in I}; TTB)\big] \\
&\quad = \big[(W_k)_{k\in K} \xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i\in I}; TTB)\big]
\end{aligned} \tag{5.10.4}$$

comes from an element $f \in \mathsf{C}((W_k)_{k\in K}, T\underline{\mathsf{C}}((X_i)_{i\in I}; B))$.

Due to Lemma 4.15 we have

$$\bigl[\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\underline{\mathsf{C}}(\triangleright;\Delta)} \underline{\mathsf{C}}((X_i)_{i\in I}; TTB) \xrightarrow{\underline{\mathsf{C}}(\triangleright;T\varepsilon)} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] = \underline{\mathsf{C}}(\triangleright; \mathrm{id}) = \mathrm{id}\,.$$

Therefore, composing equation (5.10.4) with $T\underline{\mathsf{C}}(\triangleright; T\varepsilon)$ we get the presentation

$$\begin{aligned} g = \bigl[(W_k)_{k\in K} &\xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \\ &\xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i\in I}; TTB) \xrightarrow{T\underline{\mathsf{C}}(\triangleright;T\varepsilon)} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr]. \end{aligned} \tag{5.10.5}$$

Due to Lemma 5.4 the last two arrows can be transformed as follows:

$$\begin{aligned} &\bigl[T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\Theta} \underline{\mathsf{C}}((X_i)_{i\in I}; TTB) \xrightarrow{\underline{\mathsf{C}}(\triangleright;T\varepsilon)} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] \\ &= \bigl[T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T} \underline{\mathsf{C}}((TX_i)_{i\in I}; TTB) \xrightarrow{\underline{\mathsf{C}}((\delta)_I;1)} \underline{\mathsf{C}}((X_i)_{i\in I}; TTB) \xrightarrow{\underline{\mathsf{C}}(\triangleright;T\varepsilon)} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] \\ &= \bigl[T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T} \underline{\mathsf{C}}((TX_i)_{i\in I}; TTB) \xrightarrow{\underline{\mathsf{C}}(\triangleright;T\varepsilon)} \underline{\mathsf{C}}((TX_i)_{i\in I}; TB) \xrightarrow{\underline{\mathsf{C}}((\delta)_I;1)} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] \\ &= \bigl[T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\underline{\mathsf{C}}(\triangleright;\varepsilon)} T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T} \underline{\mathsf{C}}((TX_i)_{i\in I}; TB) \xrightarrow{\underline{\mathsf{C}}((\delta)_I;1)} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] \\ &= \bigl[T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\underline{\mathsf{C}}(\triangleright;\varepsilon)} T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{\Theta} \underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr]. \end{aligned}$$

We have used also Lemmata 4.13 and 4.22.

Thus, presentation (5.10.5) can be rewritten as follows:

$$\begin{aligned} g &= \bigl[(W_k)_{k\in K} \xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \\ &\qquad \xrightarrow{TT\underline{\mathsf{C}}(\triangleright;\varepsilon)} TT\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr] \\ &= \bigl[(W_k)_{k\in K} \xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\underline{\mathsf{C}}(\triangleright;\varepsilon)} T\underline{\mathsf{C}}((X_i)_{i\in I}; B) \\ &\qquad \xrightarrow{\Delta} TT\underline{\mathsf{C}}((X_i)_{i\in I}; B) \xrightarrow{T\Theta} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB)\bigr]. \end{aligned}$$

Therefore, g comes from the element

$$f = \bigl[(W_k)_{k\in K} \xrightarrow{g} T\underline{\mathsf{C}}((X_i)_{i\in I}; TB) \xrightarrow{T\underline{\mathsf{C}}(\triangleright;\varepsilon)} T\underline{\mathsf{C}}((X_i)_{i\in I}; B)\bigr].$$

Hence, diagram (5.10.3) is an equalizer. □

5.11 Example. Let $(T, \Delta, \varepsilon, \eta)$ be a multicomonad and an augmented comonad on a symmetric multicategory C. It means that (T, Δ, ε) is a multicomonad, and natural transformation of ordinary functors $\eta : \mathrm{Id} \to T : \mathsf{C} \to \mathsf{C}$ is a morphism of coalgebras in the strict monoidal category $\mathcal{C}at(\mathsf{C}, \mathsf{C})$. In other terms, equations

$$\bigl(X \xrightarrow{\eta} TX \xrightarrow{\varepsilon} X\bigr) = \mathrm{id},$$
$$\bigl(X \xrightarrow{\eta} TX \xrightarrow{\Delta} TTX\bigr) = \bigl(X \xrightarrow{\eta} TX \xrightarrow{\eta} TTX\bigr),$$

hold true for all objects X of $\mathcal{C}$. Notice that the latter composition is equal also to the composition $\bigl(X \xrightarrow{\eta} TX \xrightarrow{T\eta} TTX\bigr)$ due to naturality of η. Hence, $(X, \eta : X \to TX)$ is a T-coalgebra. Bijection (5.3.1) for such T-coalgebra (X, η) turns into bijection

$$\mathsf{C}(X;Y) \to \mathsf{C}_T((X,\eta);(TY,\Delta)), \quad f \mapsto \eta \cdot (Tf) = \bigl(X \xrightarrow{f} Y \xrightarrow{\eta} TY\bigr)$$

with an inverse map

$$\mathsf{C}_T((X,\eta);(TY,\Delta)) \to \mathsf{C}(X;Y), \qquad g \mapsto g \cdot \varepsilon.$$

5.12 Remark. Let $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric closed Monoidal category. Let $((T,\tau^I),\Delta,\varepsilon) : \mathcal{C} \to \mathcal{C}$ be a lax symmetric Monoidal comonad. Then $\mathsf{C} = \widehat{\mathcal{C}}$ is a symmetric multicategory and the symmetric multifunctor $T = \widehat{(T,\tau^I)}$ extends to a multicomonad $(T,\Delta,\varepsilon) : \mathsf{C} \to \mathsf{C}$. We abuse the notation by writing Δ, ε in place of $\widehat{\Delta}$, $\widehat{\varepsilon}$. We claim that T-coalgebras form a symmetric Monoidal category $(\mathcal{C}_T, \otimes^I, \lambda^f)$. Indeed, the tensor product of T-coalgebras $(\delta_i : X_i \to TX_i)_{i\in I}$ is given by the composition

$$\delta = \bigl[\otimes^{i\in I} X_i \xrightarrow{\otimes^{i\in I}\delta_i} \otimes^{i\in I} TX_i \xrightarrow{\tau^I} T \otimes^{i\in I} X_i\bigr]. \tag{5.12.1}$$

This comultiplication is counital, since

$$\begin{aligned}\bigl[\otimes^{i\in I} X_i \xrightarrow{\otimes^{i\in I}\delta_i} \otimes^{i\in I} TX_i \xrightarrow{\tau^I} T \otimes^{i\in I} X_i \xrightarrow{\varepsilon} \otimes^{i\in I} X_i\bigr] \\ = \bigl[\otimes^{i\in I} X_i \xrightarrow{\otimes^{i\in I}\delta_i} \otimes^{i\in I} TX_i \xrightarrow{\otimes^I \varepsilon} \otimes^{i\in I} X_i\bigr] = \mathrm{id},\end{aligned}$$

due to ε being a Monoidal transformation. The comultiplication δ is coassociative, since the diagram

$$\begin{tikzcd}
\otimes^{i\in I} X_i \arrow[r, "\otimes^{i\in I}\delta_i"] \arrow[d, "\otimes^{i\in I}\delta_i"'] \arrow[dr, phantom, "="] & \otimes^{i\in I} TX_i \arrow[r, "\tau^I"] \arrow[d, "\otimes^{i\in I} T\delta_i"] \arrow[dr, phantom, "="] & T \otimes^{i\in I} X_i \arrow[d, "T\otimes^{i\in I}\delta_i"] \\
\otimes^{i\in I} TX_i \arrow[r, "\otimes^I \Delta"] \arrow[d, "\tau^I"'] & \otimes^{i\in I} TTX_i \arrow[r, "\tau^I"] \arrow[d, phantom, "="] & T \otimes^{i\in I} TX_i \arrow[d, "T\tau^I"] \\
T \otimes^{i\in I} X_i \arrow[rr, "\Delta"] & & TT \otimes^{i\in I} X_i
\end{tikzcd}$$

commutes. Indeed, the left top square expresses coassociativity of δ_i, the right top square commutes by naturality of τ^I, the bottom pentagon commutes by Δ being a Monoidal transformation.

Let us verify that $\otimes^I : \mathcal{C}_T^I \to \mathcal{C}_T$ induced by $\otimes^I : \mathcal{C}^I \to \mathcal{C}$ is a functor. It takes a family $(f_i : X_i \to Y_i)_{i\in I}$ of T-coalgebra morphisms to the morphism $\otimes^{i\in I} f_i : \otimes^{i\in I} X_i \to \otimes^{i\in I} Y_i$ of

$\mathcal{C}$. Let us prove that this is a T-coalgebra morphism. Indeed, diagram

$$\begin{tikzcd}
\otimes^{i\in I}X_i \arrow[r, "\otimes^I\delta"] \arrow[d, "\otimes^{i\in I}f_i"'] \arrow[dr, phantom, "="] & \otimes^{i\in I}TX_i \arrow[r, "\tau"] \arrow[d, "\otimes^{i\in I}Tf_i"] \arrow[dr, phantom, "="] & T\otimes^{i\in I}X_i \arrow[d, "T\otimes^{i\in I}f_i"] \\
\otimes^{i\in I}Y_i \arrow[r, "\otimes^I\delta"] & \otimes^{i\in I}TY_i \arrow[r, "\tau"] & T\otimes^{i\in I}Y_i
\end{tikzcd}$$

commutes due to naturality of τ.

Given T-coalgebras X_i, the structure morphism $\lambda^\phi : \otimes^{i\in I}X_i \to \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i$ is a T-coalgebra morphism for any $\phi : I \to J$. Indeed, diagram

$$\begin{tikzcd}
\otimes^{i\in I}X_i \arrow[r, "\otimes^I\delta"] \arrow[d, "\lambda^\phi"'] \arrow[dr, phantom, "="] & \otimes^{i\in I}TX_i \arrow[r, "\tau"] \arrow[d, "\lambda^\phi"'] \arrow[dr, phantom, "="] & T\otimes^{i\in I}X_i \arrow[dd, "T\lambda^\phi"] \\
\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i \arrow[r, "\otimes^{j\in J}\otimes^{\phi^{-1}j}\delta"] & \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}TX_i \arrow[d, "\otimes^J\tau"'] & \\
 & \otimes^{j\in J}T\otimes^{i\in\phi^{-1}j}X_i \arrow[r, "\tau"] & T\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}X_i
\end{tikzcd}$$

commutes due to naturality of λ^ϕ and (T,τ) being a lax Monoidal functor. Hence, $(\mathcal{C}_T, \otimes^I, \lambda^f)$ is a symmetric Monoidal category.

5.13 Remark. Let $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ be a lax symmetric Monoidal category, and let $((T,\tau^I),\Delta,\varepsilon) : \mathcal{C} \to \mathcal{C}$ be a lax symmetric Monoidal comonad. It gives rise to a symmetric multicomonad $(\widehat{T},\widehat{\Delta},\widehat{\varepsilon}) : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}}$. The mapping

$$\widehat{T} : \widehat{\mathcal{C}}((X_i)_{i\in I};Y) = \mathcal{C}(\otimes^{i\in I}X_i, Y) \to \widehat{\mathcal{C}}((TX_i)_{i\in I};TY) = \mathcal{C}(\otimes^{i\in I}TX_i, TY)$$

is given by $\widehat{T}f = \tau^I \cdot Tf$.

Note that the categories $\widehat{\mathcal{C}_T}$ and $\widehat{\mathcal{C}}_{\widehat{T}}$ have the same set of objects. Namely, an object of both categories is a morphism $\delta : X \to TX = \widehat{T}X$ in $\mathcal{C}$ that satisfies equations $\delta\cdot\Delta = \delta\cdot T\delta = \delta\cdot\widehat{T}\delta$ and $\delta\cdot\varepsilon = \mathrm{id}_X$, that is, a T-coalgebra in $\mathcal{C}$. Let X_i, $i \in I$ and Y be T-coalgebras in $\mathcal{C}$. Then the set $\widehat{\mathcal{C}_T}((X_i)_{i\in I};Y) = \mathcal{C}_T(\otimes^{i\in I}X_i, Y)$ consists of morphisms $f : \otimes^{i\in I}X_i \to Y$ in $\mathcal{C}$ such that $f\cdot\delta = (\otimes^{i\in I}\delta_i)\cdot\tau^I\cdot Tf$. On the other hand, the set $\widehat{\mathcal{C}}_{\widehat{T}}((X_i)_{i\in I};Y)$ consists of morphisms $f : (X_i)_{i\in I} \to Y$ in $\widehat{\mathcal{C}}$ such that $f\cdot\delta = (\delta_i)\cdot\widehat{T}f$, equivalently, of morphisms $f : \otimes^{i\in I}X_i \to Y$ in $\mathcal{C}$ such that $f\cdot\delta = (\otimes^{i\in I}\delta_i)\cdot\tau^I\cdot Tf$. This implies that $\widehat{\mathcal{C}_T}((X_i)_{i\in I};Y) = \widehat{\mathcal{C}}_{\widehat{T}}((X_i)_{i\in I};Y)$. Since compositions in both multicategories are induced by the composition in $\mathcal{C}$, we obtain $\widehat{\mathcal{C}_T} = \widehat{\mathcal{C}}_{\widehat{T}}$.

5.14 Kleisli multicategories. Let (T, Δ, ε) be a symmetric multicomonad in a symmetric multicategory C, not necessarily closed. We are going to define Kleisli multicategories, which is a more familiar way to represent free T-coalgebras. This is a multicategory version of the well known notion of Kleisli category.

5.15 Definition. The *Kleisli* multicategory C^T is the symmetric multicategory with the same objects as in C, with the space of morphisms

$$\mathsf{C}^T(X_1, \dots, X_n; Y) = \mathsf{C}(TX_1, \dots, TX_n; Y)$$

(that is isomorphic to $\mathsf{C}_T(TX_1, \dots, TX_n; TY)$ via (5.3.1)); a composition $(f_1, \dots, f_n)g$ in C^T is the composition $(\hat{f}_1, \dots, \hat{f}_n)g$ in C; a unit 1_X in C^T is the map $\varepsilon : TX \to X$ in C.

5.16 Remark. Kleisli multicategory C^T is isomorphic to the multicategory C_T^f of free T-coalgebras. The isomorphism $\mathsf{C}^T \to \mathsf{C}_T^f$ is the identity map on objects. The bijective map on morphisms is given by

$$\begin{aligned} \hat{-} : \mathsf{C}^T((X_i)_{i\in I}; Y) = \mathsf{C}((TX_i)_{i\in I}; Y) \longrightarrow \mathsf{C}_T((TX_i)_{i\in I}; TY) = \mathsf{C}_T^f((X_i)_{i\in I}; Y), \\ f : (TX_i) \to Y \longmapsto \hat{f} = \big[(TX_i) \xrightarrow{(\Delta)_i} (TTX_i) \xrightarrow{Tf} TY\big]. \end{aligned}$$

It agrees with the composition μ_ϕ for a map $\phi : I \to J \in \mathcal{S}$, because the composition $\mu_\phi^{\mathsf{C}^T}((f_j)_{j\in J}, g) = \mu_\phi^{\mathsf{C}}((\hat{f}_j)_{j\in J}, g)$ of $f_j : (TX_i)_{i\in\phi^{-1}(j)} \to Y_j \in \mathsf{C}$ and $g : (TY_j)_{j\in J} \to Z \in \mathsf{C}$ is mapped by $\hat{-}$ to

$$\begin{aligned} &[\mu_\phi^{\mathsf{C}^T}((f_j)_{j\in J}, g)]\hat{} \\ &= \big[(TX_i)_{i\in I} \xrightarrow{(\Delta)_i} (TTX_i)_{i\in I} \xrightarrow{(T\Delta)_i} (TTTX_i)_{i\in I} \xrightarrow{(TTf_j)_j} (TTY_j)_{j\in J} \xrightarrow{Tg} TZ\big] \\ &= \big[(TX_i)_{i\in I} \xrightarrow{(\Delta)_i} (TTX_i)_{i\in I} \xrightarrow{(\Delta T)_i} (TTTX_i)_{i\in I} \xrightarrow{(TTf_j)_j} (TTY_j)_{j\in J} \xrightarrow{Tg} TZ\big] \\ &= \big[(TX_i)_{i\in I} \xrightarrow{(\Delta)_i} (TTX_i)_{i\in I} \xrightarrow{(Tf_j)_j} (TY_j)_{j\in J} \xrightarrow{(\Delta)_j} (TTY_j)_{j\in J} \xrightarrow{Tg} TZ\big] \\ &= \mu_\phi^{\mathsf{C}}((\hat{f}_j)_{j\in J}, \hat{g}). \end{aligned}$$

The map $\hat{-}$ agrees with the units, because the unit $\varepsilon : TX \to X \in \mathsf{C}$ is mapped to

$$\hat{\varepsilon} = \big[(TX) \xrightarrow{\Delta} TTX \xrightarrow{T\varepsilon} TX\big] = 1_{TX}.$$

We already know the following statement by Corollary 5.9.

5.17 Proposition. *For a symmetric closed multicategory C, Kleisli multicategory C^T is closed. The inner hom-objects are $\underline{\mathsf{C}^T}((X_i)_{i\in I}; Z) \overset{\text{def}}{=} \underline{\mathsf{C}}((TX_i)_{i\in I}; Z)$. The evaluation*

$$\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Z} \in \mathsf{C}^T\big((X_i)_{i\in I}, \underline{\mathsf{C}^T}((X_i)_{i\in I}; Z); Z\big) = \mathsf{C}\big((TX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I}; Z); Z\big)$$

is given by the following composition in C*:*

$$\begin{aligned}
&\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Z} \\
&\quad = \big[(TX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I};Z) \xrightarrow{(1_{TX_i}),\varepsilon} (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\big]. \qquad (5.17.1)
\end{aligned}$$

One can give a direct proof of this statement, which is left to the reader as an exercise.

5.18 Multiplication in a closed Kleisli multicategory. According to Proposition 4.10, for an arbitrary tree of height 2

$$t = (I \xrightarrow{\phi} J \to \mathbf{1} | (X_i)_{i\in I}, (Y_j)_{j\in J}, Z \in \mathrm{Ob}\,\mathsf{C}) \qquad (5.18.1)$$

there are morphisms

$$\begin{aligned}
\mu^{\mathsf{C}}_{\phi} &: (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \to \underline{\mathsf{C}}((X_i)_{i\in I};Z) \in \mathsf{C}, \\
\mu^{\mathsf{C}^T}_{\phi} &: (\underline{\mathsf{C}^T}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, \underline{\mathsf{C}^T}((Y_j)_{j\in J};Z) \to \underline{\mathsf{C}^T}((X_i)_{i\in I};Z) \in \mathsf{C}^T.
\end{aligned}$$

They are the only solutions of the following equations in C (resp. in C^T):

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}}_{(X_i);Y_j})_{j\in I},1"] \arrow[d, "1{,}\mu^{\mathsf{C}}_{\phi}"'] \arrow[dr, phantom, "="] & (Y_j)_{j\in J}, \underline{\mathsf{C}}((Y_j)_{j\in J};Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}}_{(Y_j);Z}"] \\
(X_i)_{i\in I}, \underline{\mathsf{C}}((X_i)_{i\in I};Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}}_{(X_i);Z}"] & Z
\end{tikzcd} \qquad (5.18.2)$$

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}^T}((X_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, \underline{\mathsf{C}^T}((Y_j)_{j\in J};Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Y_j})_{j\in I},1"] \arrow[d, "1{,}\mu^{\mathsf{C}^T}_{\phi}"'] \arrow[dr, phantom, "="] & (Y_j)_{j\in J}, \underline{\mathsf{C}^T}((Y_j)_{j\in J};Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}^T}_{(Y_j);Z}"'] \\
(X_i)_{i\in I}, \underline{\mathsf{C}^T}((X_i)_{i\in I};Z) \arrow[r, "\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Z}"] & Z
\end{tikzcd} \qquad (5.18.3)$$

Equation (5.18.3) is equivalent to the following equation in C:

$$\begin{aligned}
&\big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\qquad \xrightarrow{(\Delta)_I,(\Delta)_J,\Delta} (TTX_i)_{i\in I}, (TT\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, TT\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\qquad \xrightarrow{(1)_I,(T\varepsilon)_J,\varepsilon} (TTX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\xrightarrow{(T\,\mathrm{ev}^{\mathsf{C}}_{(TX_i);Y_j})_{j\in J},1} (TY_j)_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \xrightarrow{(1)_J,\varepsilon} (TY_j)_{j\in J}, \underline{\mathsf{C}}((TY_j)_{j\in J};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TY_j);Z}} Z\big] \\
&= \big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\qquad \xrightarrow{(\Delta)_I,(\Delta)_J,\Delta} (TTX_i)_{i\in I}, (TT\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J}, TT\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\qquad \xrightarrow{(\varepsilon)_I,T\mu^{\mathsf{C}^T}_{\phi}} (TX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I};Z) \xrightarrow{(1)_I,\varepsilon} (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\big].
\end{aligned}$$

Using multinaturality of ε and the identities $\Delta \cdot \varepsilon = 1$, $\Delta \cdot T\varepsilon = 1$, which are implied by the axioms of comonad, we reduce this equation to

$$
\begin{aligned}
&\bigl[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J}; Z) \\
&\xrightarrow{(\Delta)_I,(1)_J,1} (TTX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J}; Z) \\
&\xrightarrow{(T\operatorname{ev}^{\mathsf{C}}_{(TX_i);Y_j})_{j\in J},\varepsilon} (TY_j)_{j\in J}, \underline{\mathsf{C}}((TY_j)_{j\in J}; Z) \xrightarrow{\operatorname{ev}^{\mathsf{C}}_{(TY_j);Z}} Z\bigr] \\
&= \bigl[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J}; Z) \\
&\xrightarrow{(1)_I,\mu_\phi^{\underline{\mathsf{C}}^T}} (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I}; Z) \xrightarrow{\operatorname{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\bigr]. \qquad (5.18.4)
\end{aligned}
$$

5.19 Remark. The Kleisli multicategory C^T is isomorphic to the multicategory C_T^f of free T-coalgebras (see Remark 5.16). Both are closed, thus, one can choose the inner homomorphisms objects and evaluations in a coherent way. Namely, we choose $\underline{\mathsf{C}^T}((X_i)_{i\in I}; Z) = \underline{\mathsf{C}_T^f}((X_i)_{i\in I}; Z) = \underline{\mathsf{C}}((TX_i)_{i\in I}; Z)$ for arbitrary objects X_i, Z. Evaluation in C^T given by (5.17.1) induces a composition in C, which is a T-coalgebra morphism:

$$
\begin{aligned}
(\operatorname{ev}^{\mathsf{C}^T}_{(X_i);Z})\widehat{\ } &= \bigl[(TX_i)_{i\in I}, T\underline{\mathsf{C}^T}((X_i)_{i\in I}; Z) \xrightarrow{(\Delta)_{I\sqcup 1}} (TTX_i)_{i\in I}, TT\underline{\mathsf{C}^T}((X_i)_{i\in I}; Z) \xrightarrow{T\operatorname{ev}^{\mathsf{C}^T}_{(X_i);Z}} TZ\bigr] \\
&= \bigl[(TX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I}; Z) \xrightarrow{(\Delta)_I,1} (TTX_i)_{i\in I}, T\underline{\mathsf{C}}((TX_i)_{i\in I}; Z) \xrightarrow{T\operatorname{ev}^{\mathsf{C}}_{(TX_i);Z}} TZ\bigr] \\
&= \operatorname{ev}^{\mathsf{C}'_T}_{(TX_i);TZ} \qquad (5.19.1)
\end{aligned}
$$

by Lemma 5.7. The above morphism is also denoted $\operatorname{ev}^{\mathsf{C}_T^f}_{(X_i);Z}$. The isomorphism $\mathsf{C}^T \xrightarrow{\sim} \mathsf{C}_T^f$ of Remark 5.16 has identity closing transformation.

Denote by $T' : \mathsf{C}_T^f \to \mathsf{C}_T$ the multifunctor $X \mapsto TX$ given by identity map on morphisms. There are multifunctors $E = \bigl(\mathsf{C}_T^f \xrightarrow{T'} \mathsf{C}_T \xrightarrow{F} \mathsf{C}\bigr)$. The multifunctor F with $\operatorname{Ob} F = \mathrm{id}$ forgets the T-coalgebra structure and gives inclusion on morphisms. Given $A_k, B \in \operatorname{Ob}\mathsf{C}$, $k \in K$, we can write the closing transformation for E

$$
\theta_{(A_k);B} = \underline{E}_{(A_k);B} : T\underline{\mathsf{C}_T^f}((A_k)_{k\in K}; B) = T\underline{\mathsf{C}}((TA_k)_{k\in K}; B) \to \underline{\mathsf{C}}((TA_k)_{k\in K}; TB). \qquad (5.19.2)
$$

This is the unique morphism that satisfies the equation

$$
\begin{tikzcd}
(TA_k)_{k\in K}, T\underline{\mathsf{C}}((TA_k)_{k\in K}; B) \arrow[r, "(\Delta)_K,1"] \arrow[d, "(1)_K,\theta_{(A_k);B}"'] \arrow[dr, phantom, "="] & (TTA_k)_{k\in K}, T\underline{\mathsf{C}}((TA_k)_{k\in K}; B) \arrow[d, "T\operatorname{ev}^{\mathsf{C}}_{(TA_k);B}"] \\
(TA_k)_{k\in K}, \underline{\mathsf{C}}((TA_k)_{k\in K}; TB) \arrow[r, "\operatorname{ev}^{\mathsf{C}}_{(TA_k)_{k\in K};TB}"] & TB
\end{tikzcd}
$$

By (5.3.3) the morphism $\theta_{(A_k);B}$ coincides with $\Theta_{(TA_k);B}$.

5.20 Corollary (to Lemma 5.4)**.** *The morphism $\theta_{(A_k);B}$ can be decomposed as follows:*

$$\theta_{(A_k);B} = \big[T\underline{\mathsf{C}}((TA_k)_{k\in K};B) \xrightarrow{T} \underline{\mathsf{C}}((TTA_k)_{k\in K};TB) \xrightarrow{\underline{\mathsf{C}}((\Delta)_K;1)} \underline{\mathsf{C}}((TA_k)_{k\in K};TB)\big].$$

With this notation we can write the left hand side of (5.18.4) as follows:

$$\begin{aligned}
&\big[(TX_i)_{i\in I},(T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\quad\xrightarrow{(1)_I,(\theta_{(X_i);Y_j}),\varepsilon} (TX_i)_{i\in I},(\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};TY_j))_{j\in J},\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\quad\xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(TX_i);TY_j})_{j\in J},1} (TY_j)_{j\in J},\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TY_j);Z}} Z\big] \\
&= \big[(TX_i)_{i\in I},(T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\quad\xrightarrow{(1)_I,(\theta_{(X_i);Y_j})_{j\in J},\varepsilon} (TX_i)_{i\in I},(\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};TY_j))_{j\in J},\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \\
&\quad\xrightarrow{(1)_I,\mu^{\mathsf{C}}_\phi} (TX_i)_{i\in I},\underline{\mathsf{C}}((TX_i)_{i\in I};Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\big].
\end{aligned} \tag{5.20.1}$$

The last transformation is due to equation (5.18.2). Comparing (5.20.1) with the right hand side of (5.18.4) we conclude that

$$\begin{aligned}
\mu^{\mathsf{C}^T}_\phi = \big[(T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) &\xrightarrow{(\theta_{(X_i);Y_j})_{j\in J},\varepsilon} \\
(\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};TY_j))_{j\in J},\underline{\mathsf{C}}((TY_j)_{j\in J};Z) &\xrightarrow{\mu^{\mathsf{C}}_\phi} \underline{\mathsf{C}}((TX_i)_{i\in I};Z)\big].
\end{aligned} \tag{5.20.2}$$

5.21 Remark. We choose the closed structure data for the Kleisli multicategory C^T and the isomorphic to it multicategory C^f_T of free T-coalgebras coherent as in Remark 5.19. Therefore, for any map $\phi : I \to J$ in $\mathcal{S}$ the composition map $\mu^{\mathsf{C}^f_T}_\phi$ coincides with the T-coalgebra morphism $(\mu^{\mathsf{C}^T}_\phi)\hat{}$, because both are unique arrows which make commutative the diagram

$$\begin{tikzcd}
(TX_i)_{i\in I},(T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j};Y_j))_{j\in J},T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}^f_T})_J{,}1"', "\|" description, "((\mathrm{ev}^{\mathsf{C}^T})\hat{})_J{,}1" below] \arrow[d, "(1)_I{,}\mu^{\mathsf{C}^f_T}_\phi \,=\, (1)_I{,}(\mu^{\mathsf{C}^T}_\phi)\hat{}"] & (TY_j)_{j\in J},T\underline{\mathsf{C}}((TY_j)_{j\in J};Z) \arrow[d, "(\mathrm{ev}^{\mathsf{C}^T})\hat{} \,=\, \mathrm{ev}^{\mathsf{C}^f_T}"] \\
(TX_i)_{i\in I},T\underline{\mathsf{C}}((TX_i)_{i\in I};Z) \arrow[r, "(\mathrm{ev}^{\mathsf{C}^T})\hat{}", "\|" description, "\mathrm{ev}^{\mathsf{C}^f_T}" below] & TZ
\end{tikzcd} \tag{5.21.1}$$

This is a particular case of equation (4.10.1).

Furthermore, writing down diagram (4.12.2) for Kleisli multicategory C^T, and applying isomorphism of multicategories $\mathsf{C}^T \to \mathsf{C}^f_T$ of Remark 5.16 to it, we get the equation $\underline{\mathsf{C}^T}(\phi;g)\hat{} = \underline{\mathsf{C}^f_T}(\phi;\hat{g})$. Similarly, writing down diagram (4.12.3) for C^T, and applying isomorphism $\mathsf{C}^T \to \mathsf{C}^f_T$ to it, we get the equation $\underline{\mathsf{C}^T}(f;1)\hat{} = \underline{\mathsf{C}^f_T}(\hat{f};1)$.

5.22 Left and right multiplications in closed Kleisli multicategories. Let $f : (Y_j)_{j\in J} \to Z$ be a morphism in C^T, equivalently, a morphism $f : (TY_j)_{j\in J} \to Z$ in C. Let us compute the morphisms $\underline{\mathsf{C}^T}(f;1)$ and $\underline{\mathsf{C}^T}(\phi;f)$ in terms of the multicategory C.

The morphism $\underline{\mathsf{C}^T}(f;1) : \underline{\mathsf{C}^T}(Z;W) \to \underline{\mathsf{C}^T}((Y_j)_{j\in J};W)$ is uniquely determined via the diagram in C^T:

$$\begin{tikzcd}
(Y_j)_{j\in J}, \underline{\mathsf{C}^T}(Z;W) \arrow[r, "(1)_J{,}\underline{\mathsf{C}^T}(f;1)"] \arrow[d, "f{,}1"'] & (Y_j)_{j\in J}, \underline{\mathsf{C}^T}((Y_j)_{j\in J};W) \arrow[d, "\mathrm{ev}^{\mathsf{C}^T}_{(Y_j);W}"] \\
Z, \underline{\mathsf{C}^T}(Z;W) \arrow[r, "\mathrm{ev}^{\mathsf{C}^T}_{Z;W}"] & Z
\end{tikzcd}$$

Let $g = \underline{\mathsf{C}^T}(f;1)$. Commutativity of the above diagram is equivalent to the following equation in C:

$$\begin{aligned}
&\bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(\Delta)_J,\Delta} (TTY_j)_{j\in J}, TT\underline{\mathsf{C}}(TZ;W) \\
&\qquad \xrightarrow{(T\varepsilon),Tg} (TY_j)_{j\in J}, T\underline{\mathsf{C}}((TY_j)_{j\in J};W) \xrightarrow{(1)_I,\varepsilon} (TY_j)_{j\in J}, \underline{\mathsf{C}}((TY_j)_{j\in J};W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TY_j);W}} W\bigr] \\
&= \bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(\Delta)_J,\Delta} (TTY_j)_{j\in J}, TT\underline{\mathsf{C}}(TZ;W) \\
&\qquad \xrightarrow{Tf,\varepsilon} TZ, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{1,\varepsilon} TZ, \underline{\mathsf{C}}(TZ;W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{TZ;W}} W\bigr].
\end{aligned}$$

Using the identities $\Delta\cdot\varepsilon = 1$, $\Delta\cdot T\varepsilon = 1$ implied by the axioms of comonad, and naturality of ε, we can rewrite this equation as follows:

$$\begin{aligned}
&\bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(1)_J,g} (TY_j)_{j\in J}, \underline{\mathsf{C}}((TY_j)_{j\in J};W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TY_j);W}} W\bigr] \\
&= \bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(\Delta)_J,\varepsilon} (TTY_j)_{j\in J}, \underline{\mathsf{C}}(TZ;W) \\
&\qquad \xrightarrow{Tf,1} TZ, \underline{\mathsf{C}}(TZ;W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{TZ;W}} W\bigr]. \qquad (5.22.1)
\end{aligned}$$

Since $(\Delta)_J \cdot Tf = \hat{f}$, the right hand side of the above equation can be written as

$$\begin{aligned}
&\bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(1)_J,\varepsilon} (TY_j)_{j\in J}, \underline{\mathsf{C}}(TZ;W) \xrightarrow{\hat{f},1} TZ, \underline{\mathsf{C}}(TZ;W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{TZ;W}} W\bigr] \\
&= \bigl[(TY_j)_{j\in J}, T\underline{\mathsf{C}}(TZ;W) \xrightarrow{(1)_J,\varepsilon} (TY_j)_{j\in J}, \underline{\mathsf{C}}(TZ;W) \\
&\qquad \xrightarrow{(1)_J,\underline{\mathsf{C}}(\hat{f};1)} (TY_j)_{j\in J}, \underline{\mathsf{C}}((TY_j);W) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TY_j);W}} W\bigr].
\end{aligned}$$

The last transformation is due to equation (4.12.3). Comparing the obtained expression with the left hand side of (5.22.1) we conclude that

$$\underline{\mathsf{C}^T}(f;1) = \bigl[T\underline{\mathsf{C}}(TZ;W) \xrightarrow{\varepsilon} \underline{\mathsf{C}}(TZ;W) \xrightarrow{\underline{\mathsf{C}}(\hat{f};1)} \underline{\mathsf{C}}((TY_j)_{j\in J};W)\bigr]. \qquad (5.22.2)$$

Let $X_i \in \operatorname{Ob} \mathsf{C}$, $i \in I$ be a family of objects, $\phi : I \to J$ a map in $\operatorname{Mor} \mathcal{S}$. The morphism $\underline{\mathsf{C}}(\phi; f)$ is uniquely determined via the diagram in C^T:

$$\begin{tikzcd}
(X_i)_{i\in I}, (\underline{\mathsf{C}^T}((X_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \arrow[r, "(1)_I, \underline{\mathsf{C}^T}(\phi;f)"] \arrow[d, "(\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Y_j})_{j\in J}"'] & (X_i)_{i\in I}, \underline{\mathsf{C}^T}((X_i)_{i\in I}; Z) \arrow[d, "\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Z}"] \\
(Y_j)_{j\in J} \arrow[r, "f"] & Z
\end{tikzcd}$$

Let $h = \underline{\mathsf{C}^T}(\phi; f)$. Similarly to the above computations, the diagram can be expanded to the following equation in C:

$$\begin{aligned}
&\big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(1)_I, h} (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\big] \\
&\quad = \big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(\Delta)_I, (1)_J} \\
&\qquad (TTX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(T\,\mathrm{ev}^{\mathsf{C}}_{(TX_i);Y_j})} (TY_j)_{j\in J} \xrightarrow{f} Z\big]. \qquad (5.22.3)
\end{aligned}$$

Using equation (5.3.3) we can transform the right hand side of (5.22.1) into

$$\begin{aligned}
&\big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(1)_I, (\theta_{(X_i);Y_j})_{j\in J}} \\
&\qquad (TX_i)_{i\in I}, (\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; TY_j))_{j\in J} \xrightarrow{(\mathrm{ev}^{\mathsf{C}}_{(TX_i);Y_j})_{j\in J}} (TY_j)_{j\in J} \xrightarrow{f} Z\big] \\
&= \big[(TX_i)_{i\in I}, (T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(1)_I, (\theta_{(X_i);Y_j})_{j\in J}} \\
&\qquad (TX_i)_{i\in I}, (\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; TY_j))_{j\in J} \xrightarrow{(1)_I, \underline{\mathsf{C}}(\phi;f)} \\
&\qquad\qquad (TX_i)_{i\in I}, \underline{\mathsf{C}}((TX_i)_{i\in I}; Z) \xrightarrow{\mathrm{ev}^{\mathsf{C}}_{(TX_i);Z}} Z\big].
\end{aligned}$$

Comparing the obtained expression with the left hand side of (5.22.1) we conclude that

$$\begin{aligned}
\underline{\mathsf{C}^T}(\phi; f) = \big[&(T\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; Y_j))_{j\in J} \xrightarrow{(\theta_{(X_i)_{i\in\phi^{-1}j};Y_j})_{j\in J}} \\
&(\underline{\mathsf{C}}((TX_i)_{i\in\phi^{-1}j}; TY_j))_{j\in J} \xrightarrow{\underline{\mathsf{C}}(\phi;f)} \underline{\mathsf{C}}((TX_i)_{i\in I}; Z)\big]. \qquad (5.22.4)
\end{aligned}$$

5.23 Lax Monoidal case. Let $\mathcal{C} = (\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric closed Monoidal category. Let $((T, \tau^I), \Delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$ be a lax symmetric Monoidal comonad. Then $\mathsf{C} = \widehat{\mathcal{C}}$ is a symmetric multicategory and the symmetric multifunctor $T = \widehat{(T, \tau^I)}$ extends to a multicomonad $(T, \Delta, \varepsilon) : \mathsf{C} \to \mathsf{C}$. We abuse the notation by writing Δ, ε in place of $\widehat{\Delta}$, $\widehat{\varepsilon}$. Since $\mathcal{C}$ is closed, the symmetric multicategories C, C^T are closed as well. Inner homomorphism objects in C, C^T are

$$\underline{\mathsf{C}}((X_i)_{i\in I}; Y) = \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Y),$$
$$\underline{\mathsf{C}^T}((X_i)_{i\in I}; Y) = \underline{\mathsf{C}}((TX_i)_{i\in I}; Y) = \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y).$$

Evaluation morphism in C is

$$\mathrm{ev}^{\mathsf{C}}_{(X_i);Y} = \mathrm{ev}^{\mathcal{C}}_{\otimes^I X_i,Y} : (\otimes^{i\in I} X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Y) \to Y.$$

Evaluation morphism in C^T is

$$\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Y}$$
$$= \bigl[(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{1\otimes\varepsilon} (\otimes^{i\in I} TX_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{\mathrm{ev}^{\mathcal{C}}_{\otimes^I TX_i,Y}} Y\bigr].$$

The corresponding T-coalgebra morphism is given by (5.3.1):

$$\begin{aligned}
(\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Y})\widehat{\ } = \Bigl[&(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{(\otimes^I\Delta)\otimes\Delta} \\
&(\otimes^{i\in I} TTX_i) \otimes TT\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{(\tau^I\otimes 1)\cdot\tau^{\mathbf{2}}} \\
&T\bigl[(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y)\bigr] \xrightarrow{T\,\mathrm{ev}^{\mathsf{C}^T}_{(X_i);Y}} TY\Bigr] \\
= \Bigl[&(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{(\otimes^I\Delta)\otimes\Delta} (\otimes^{i\in I} TTX_i) \otimes TT\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \\
&\xrightarrow{1\otimes T\varepsilon} (\otimes^{i\in I} TTX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \\
&\xrightarrow{(\tau^I\otimes 1)\cdot\tau^{\mathbf{2}}} T\bigl[(\otimes^{i\in I} TX_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y)\bigr] \xrightarrow{T\,\mathrm{ev}^{\mathcal{C}}_{\otimes^I TX_i,Y}} TY\Bigr] \\
= \Bigl[&(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{(\otimes^I\Delta)\otimes 1} (\otimes^{i\in I} TTX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \\
&\xrightarrow{(\tau^I\otimes 1)\cdot\tau^{\mathbf{2}}} T\bigl[(\otimes^{i\in I} TX_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y)\bigr] \xrightarrow{T\,\mathrm{ev}^{\mathcal{C}}_{\otimes^I TX_i,Y}} TY\Bigr] \\
= \bigl[&(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \xrightarrow{1\otimes\theta_{(X_i);Y}} (\otimes^{i\in I} TX_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, TY) \\
&\xrightarrow{\mathrm{ev}^{\mathcal{C}}_{\otimes^I TX_i,TY}} TY\bigr].
\end{aligned} \tag{5.23.1}$$

The unique morphism

$$\theta_{(X_i);Y} : T\underline{\mathcal{C}}(\otimes^{i\in I} TX_i, Y) \to \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, TY)$$

which satisfies the above equation, exists by closedness of $\mathcal{C}$.

The closing multinatural transformation (4.18.2)

$$\underline{T}_{(X_i);Y} : T\underline{\mathcal{C}}(\otimes^{i\in I} X_i, Y) \to \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, TY),$$

for the multifunctor $T : \mathsf{C} \to \mathsf{C}$ is determined by equation

$$\begin{tikzcd}
(\otimes^{i\in I} TX_i) \otimes T\underline{\mathcal{C}}(\otimes^{i\in I} X_i, Y) \arrow[r, "1\otimes\underline{T}"] \arrow[d, "(\tau^I\otimes 1)\cdot\tau^{\mathbf{2}}"'] \arrow[dr, phantom, "="] & (\otimes^{i\in I} TX_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} TX_i, TY) \arrow[d, "\mathrm{ev}^{\mathcal{C}}"] \\
T\bigl[(\otimes^{i\in I} X_i) \otimes \underline{\mathcal{C}}(\otimes^{i\in I} X_i, Y)\bigr] \arrow[r, "T\,\mathrm{ev}^{\mathcal{C}}"] & TY
\end{tikzcd}$$

In particular, for $I = 1$ the equation reads:

$$
\begin{tikzcd}
TX \otimes T\underline{\mathcal{C}}(X,Y) \arrow[r, "1\otimes \underline{T}_{X,Y}"] \arrow[d, "\tau"'] \arrow[dr, phantom, "="] & TX \otimes \underline{\mathcal{C}}(TX,TY) \arrow[d, "\mathrm{ev}^{\mathcal{C}}"] \\
T\big[X \otimes \underline{\mathcal{C}}(X,Y)\big] \arrow[r, "T\,\mathrm{ev}^{\mathcal{C}}"] & TY
\end{tikzcd}
$$

5.24 Lemma. *We have*

$$\underline{T}_{(X_i);Y} = \big[T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \xrightarrow{\underline{T}_{\otimes^{i\in I}X_i,Y}} \underline{\mathcal{C}}(T\otimes^{i\in I}X_i,TY) \xrightarrow{\underline{\mathcal{C}}(\tau,TY)} \underline{\mathcal{C}}(\otimes^{i\in I}TX_i,TY)\big].$$

Proof. By closedness, the equation in question is equivalent to the commutativity of the exterior of the following diagram

$$
\begin{tikzcd}
& (\otimes^{i\in I}TX_i)\otimes T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \arrow[dl, "\tau^I\otimes 1"'] \arrow[dr, "1\otimes\underline{T}"] & \\
(T\otimes^{i\in I}X_i)\otimes T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \arrow[dr, "1\otimes\underline{T}"] \arrow[dd, "\tau"'] & & (\otimes^{i\in I}TX_i)\otimes\underline{\mathcal{C}}(T\otimes^{i\in I}X_i,TY) \arrow[dl, "\tau^I\otimes 1"'] \arrow[dd, "1\otimes\underline{\mathcal{C}}(\tau,1)"] \\
& (T\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(T\otimes^{i\in I}X_i,TY) \arrow[dd, "\mathrm{ev}^{\mathcal{C}}"] & \\
T\big[(\otimes^{i\in I}X_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y)\big] \arrow[dr, "T\,\mathrm{ev}^{\mathcal{C}}"'] & & (\otimes^{i\in I}TX_i)\otimes\underline{\mathcal{C}}(\otimes^{i\in I}TX_i,TY) \arrow[dl, "\mathrm{ev}^{\mathcal{C}}"] \\
& TY &
\end{tikzcd}
$$

Here the left parallelogram is the definition of $\underline{T}$, and the right parallelogram is the definition of $\underline{\mathcal{C}}(\tau,1)$. □

5.25 Corollary. *Morphisms* $\Theta_{(X_i);Y} : T\widehat{\underline{\mathcal{C}}}((X_i)_{i\in I};Y) \to \widehat{\underline{\mathcal{C}}}((X_i)_{i\in I};TY)$ *and* $\Theta_{\otimes^{i\in I}X_i,Y} : T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \to \underline{\mathcal{C}}(\otimes^{i\in I}X_i,TY)$ *coincide.*

Proof. Lemma 5.4 gives the decomposition which can be transformed via Lemma 5.24 to:

$$
\begin{aligned}
\Theta_{(X_i);Y} &= \big[T\widehat{\underline{\mathcal{C}}}((X_i)_{i\in I};Y) \xrightarrow{\underline{T}_{(X_i);Y}} \widehat{\underline{\mathcal{C}}}((TX_i)_{i\in I};TY) \xrightarrow{\widehat{\underline{\mathcal{C}}}((\delta_i)_{i\in I};1)} \widehat{\underline{\mathcal{C}}}((X_i)_{i\in I};TY)\big] \\
&= \big[T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \xrightarrow{\underline{T}_{\otimes^{i\in I}X_i,Y}} \underline{\mathcal{C}}(T\otimes^{i\in I}X_i,TY) \xrightarrow{\underline{\mathcal{C}}(\tau,1)} \underline{\mathcal{C}}(\otimes^{i\in I}TX_i,TY) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \xrightarrow{\underline{\mathcal{C}}(\otimes^{i\in I}\delta_i,1)} \underline{\mathcal{C}}(\otimes^{i\in I}X_i,TY)\big] \\
&= \big[T\underline{\mathcal{C}}(\otimes^{i\in I}X_i,Y) \xrightarrow{\underline{T}_{\otimes^{i\in I}X_i,Y}} \underline{\mathcal{C}}(T\otimes^{i\in I}X_i,TY) \xrightarrow{\underline{\mathcal{C}}(\delta,1)} \underline{\mathcal{C}}(\otimes^{i\in I}X_i,TY)\big] \\
&= \Theta_{\otimes^{i\in I}X_i,Y}
\end{aligned}
$$

due to formula (5.12.1). □

5.26 Monads for Kleisli multicategories. Let C be a $\mathcal{V}$-multicategory. Let $(T : \mathsf{C} \to \mathsf{C}, \Delta, \varepsilon)$ be a multicomonad in C, see Section 5.1. In particular, Δ and ε are multinatural transformations. Let $(M : \mathsf{C} \to \mathsf{C}, m : MM \to M, u : \mathrm{Id} \to M)$ be a monad in C. Namely, $M : \mathsf{C} \to \mathsf{C}$ is a multifunctor, $m : MM \to M : \mathsf{C} \to \mathsf{C}$ is a natural transformation and $u : \mathrm{Id} \to M : \mathsf{C} \to \mathsf{C}$ is a multinatural transformation. Let $\xi : MT \to TM : \mathsf{C} \to \mathsf{C}$ be a multinatural transformation. If it satisfies certain properties, then the monad M generates a monad (M^T, m^T, u^T) in the Kleisli multicategory C^T. These properties resemble distributivity laws between two monads, introduced by Beck [Bec69]. Let us describe the components of the new monad.

Let $M : \mathsf{C} \to \mathsf{C}$ be a (symmetric) $\mathcal{V}$-multifunctor, and let $\xi : MT \to TM : \mathsf{C} \to \mathsf{C}$ be a multinatural $\mathcal{V}$-transformation. Composition in C is denoted $\cdot = \cdot_{\mathsf{C}}$. Composition in C^T is denoted $\cdot_{\mathsf{C}^T}$. We are going to define a (symmetric) multifunctor $M^T : \mathsf{C}^T \to \mathsf{C}^T$ acting on objects in the same way as M, so $\operatorname{Ob} M^T = \operatorname{Ob} M$. It will act on morphisms via

$$\begin{aligned} M^T : \mathsf{C}^T((X_i)_{i\in I}; Y) = \mathsf{C}((X_iT)_{i\in I}; Y) &\xrightarrow{M} \mathsf{C}((X_iTM)_{i\in I}; YM) \\ &\xrightarrow{(\xi)_i \cdot_{\mathsf{C}} -} \mathsf{C}((X_iMT)_{i\in I}; YM) = \mathsf{C}^T((X_iM)_{i\in I}; YM). \end{aligned}$$

5.27 Proposition. *If the diagram*

$$\begin{tikzcd} MT \arrow[rr, "\xi"] \arrow[d, "\Delta"'] & & TM \arrow[d, "\Delta M"] \\ MTT \arrow[r, "\xi T"] & TMT \arrow[r, "\xi"] \arrow[u, phantom, "="] & TTM \end{tikzcd} \tag{5.27.1}$$

commutes and the equation

$$\bigl(MT \xrightarrow{\xi} TM \xrightarrow{\varepsilon M} M\bigr) = \varepsilon \tag{5.27.2}$$

holds true, then $M^T : \mathsf{C}^T \to \mathsf{C}^T$ defined above is a multifunctor.

Proof. Let $\phi : I \to J$ be a non-decreasing map of finite totally ordered sets (arbitrary map for symmetric multicategory case). The equation to prove is

$$\begin{tikzcd} \otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\mathsf{C}^T((X_i)_{i\in\phi^{-1}j}; Y_j)\bigr)_{j\in J}, \mathsf{C}^T((Y_j)_{j\in J}; Z)\bigr] \arrow[r, "\mu_\phi^{\mathsf{C}^T}"] \arrow[d, "\otimes^{J\sqcup\mathbf{1}} M^T"'] & \mathsf{C}^T((X_i)_{i\in I}; Z) \arrow[d, "M^T"] \\ \otimes^{J\sqcup\mathbf{1}}\bigl[\bigl(\mathsf{C}^T((X_iM^T)_{i\in\phi^{-1}j}; Y_jM^T)\bigr)_{j\in J}, \mathsf{C}^T((Y_jM^T)_{j\in J}; ZM^T)\bigr] \arrow[r, "\mu_\phi^{\mathsf{C}^T}"] & \mathsf{C}^T((X_iM^T)_{i\in I}; ZM^T) \end{tikzcd}$$

The top–right path composes to the morphism

$$\Big\langle \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iT)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_jT)_{j\in J};Z)\big] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[((\xi\cdot\Delta M)_{i\in\phi^{-1}j}\cdot(\_TM))_{j\in J},(\_M)]} \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iMT)_{i\in\phi^{-1}j};Y_jTM)\big)_{j\in J},\mathsf{C}((Y_jTM)_{j\in J};ZM)\big] \xrightarrow{\mu^{\mathsf{C}}_\phi} \mathsf{C}((X_iMT)_{i\in I};ZM)\Big\rangle$$

The left–bottom path gives

$$\Big\langle \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iT)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_jT)_{j\in J};Z)\big] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[((\Delta\cdot(\xi T))_{i\in\phi^{-1}j}\cdot(\_MT))_{j\in J},(\xi)_J\cdot(\_M)]} \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iMT)_{i\in\phi^{-1}j};Y_jMT)\big)_{j\in J},\mathsf{C}((Y_jMT)_{j\in J};ZM)\big] \xrightarrow{\mu^{\mathsf{C}}_\phi} \mathsf{C}((X_iMT)_{i\in I};ZM)\Big\rangle$$

$$= \Big\langle \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iT)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_jT)_{j\in J};Z)\big] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[((\Delta\cdot(\xi T))_{i\in\phi^{-1}j}\cdot(\_MT)\cdot\xi)_{j\in J},(\_M)]} \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iMT)_{i\in\phi^{-1}j};Y_jTM)\big)_{j\in J},\mathsf{C}((Y_jTM)_{j\in J};ZM)\big] \xrightarrow{\mu^{\mathsf{C}}_\phi} \mathsf{C}((X_iMT)_{i\in I};ZM)\Big\rangle$$

$$= \Big\langle \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iT)_{i\in\phi^{-1}j};Y_j)\big)_{j\in J},\mathsf{C}((Y_jT)_{j\in J};Z)\big] \xrightarrow{\otimes^{J\sqcup\mathbf{1}}[((\Delta\cdot(\xi T)\cdot\xi)_{i\in\phi^{-1}j}\cdot(\_MT))_{j\in J},(\_M)]} \otimes^{J\sqcup\mathbf{1}}\big[\big(\mathsf{C}((X_iMT)_{i\in\phi^{-1}j};Y_jTM)\big)_{j\in J},\mathsf{C}((Y_jTM)_{j\in J};ZM)\big] \xrightarrow{\mu^{\mathsf{C}}_\phi} \mathsf{C}((X_iMT)_{i\in I};ZM)\Big\rangle,$$

since the composition is associative, and due to multinaturality of ξ. Condition (5.27.1) implies that the two obtained compositions coincide.

M sends units to units if the composite

$$\mathsf{C}(XT;X) \xrightarrow{M} \mathsf{C}(XTM;XM) \xrightarrow{\xi\cdot_{\mathsf{C}}-} \mathsf{C}(XMT;XM)$$

sends ε to ε. This is precisely equation (5.27.2). □

5.28 Proposition. *Let $N : \mathsf{C} \to \mathsf{C}$ be another (symmetric) $\mathcal{V}$-multifunctor, and let $\psi : NT \to TN : \mathsf{C} \to \mathsf{C}$ be a multinatural $\mathcal{V}$-transformation. Assume that the pair (N,ψ) satisfies the same equation* (5.27.1) *as the pair (M,ξ) does. Then the composite multifunctor $MN : \mathsf{C} \to \mathsf{C}$ and the multinatural transformation $\chi = \big(MNT \xrightarrow{\psi} MTN \xrightarrow{\xi N} TMN\big)$ also satisfy equations* (5.27.1) *and* (5.27.2)*. Furthermore, the equation*

$$(M,\xi)^T\cdot(N,\psi)^T = (MN,\chi)^T : \mathsf{C}^T \to \mathsf{C}^T \tag{5.28.1}$$

holds true. Notice also that $(\mathrm{Id},\mathrm{id})^T = \mathrm{Id}$.

Proof. Equation (5.27.1) for (MN,χ) is the exterior of the following commutative diagram

$$\begin{tikzcd}
MNT \arrow[rrr, "\psi"] \arrow[d, "\Delta"'] & & & MTN \arrow[rrr, "\xi N"] \arrow[d, "\Delta N"'] & & & TMN \arrow[d, "\Delta MN"] \\
MNTT \arrow[r, "\psi T"] \arrow[drr, "\chi T"'] & MTNT \arrow[rr, "\psi"] \arrow[dr, "\xi NT"] & & MTTN \arrow[r, "\xi TN"] & TMTN \arrow[rr, "\xi N"] & & TTMN \\
& & TMNT \arrow[urr, "\psi"] \arrow[urrrr, "\chi"'] & & & &
\end{tikzcd}$$

Equation (5.28.1) follows from the computation

$$\begin{aligned}
M^T\cdot N^T = \bigl[&\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{M} \mathsf{C}((X_iTM)_{i\in I};YM) \xrightarrow{(\xi)_i\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMT)_{i\in I};YM) \\
&\xrightarrow{N} \mathsf{C}((X_iMTN)_{i\in I};YMN) \xrightarrow{(\psi)_i\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMNT)_{i\in I};YMN)\bigr] \\
= \bigl[&\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{M} \mathsf{C}((X_iTM)_{i\in I};YM) \xrightarrow{N} \mathsf{C}((X_iTMN)_{i\in I};YMN) \\
&\xrightarrow{(\xi N)_i\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMTN)_{i\in I};YMN) \xrightarrow{(\psi)_i\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMNT)_{i\in I};YMN)\bigr] = (MN)^T.
\end{aligned}$$

The composite of pairs (M,ξ) and (N,ψ) satisfies condition (5.27.2), as the following computation shows:

$$\bigl(MNT \xrightarrow{\psi} MTN \xrightarrow{\xi N} TMN \xrightarrow{\varepsilon MN} MN\bigr) = \bigl(MNT \xrightarrow{\psi} MTN \xrightarrow{\varepsilon N} MN\bigr) = \varepsilon.$$

The last statement is obvious. □

5.29 Proposition. *Let pairs* $(M,\xi : MT\to TM)$, $(N,\psi : NT\to TN)$ *satisfy conditions* (5.27.1) *and* (5.27.2). *Let* $p : M\to N : \mathsf{C}\to\mathsf{C}$ *be a (multi)natural transformation such that the equation*

$$\begin{tikzcd}
MT \arrow[r, "\xi"] \arrow[d, "pT"'] \arrow[dr, phantom, "="] & TM \arrow[d, "p"] \\
NT \arrow[r, "\psi"] & TN
\end{tikzcd} \tag{5.29.1}$$

holds. Then the formula $p^T = \varepsilon\cdot_{\mathsf{C}} p \in \mathsf{C}(XMT;XN) = \mathsf{C}^T(XM^T;XN^T)$, $X\in\operatorname{Ob}\mathsf{C}$, *gives a (multi)natural transformation* $p^T : M^T\to N^T : \mathsf{C}^T\to\mathsf{C}^T$.

Proof. The equation to prove reads as follows:

$$\begin{tikzcd}
\mathsf{C}^T((X_i)_{i\in I};Y) \arrow[r, "M^T"] \arrow[d, "N^T"'] \arrow[dr, phantom, "="] & \mathsf{C}^T((X_iM)_{i\in I};YM) \arrow[d, "-\cdot_{\mathsf{C}^T}p^T"] \\
\mathsf{C}^T((X_iN)_{i\in I};YN) \arrow[r, "(p^T)_I\cdot_{\mathsf{C}^T}-"] & \mathsf{C}^T((X_iM)_{i\in I};YN)
\end{tikzcd}$$

where $I = \mathbf{1}$ if p is a natural transformation. Expanding out the top-right path yields

$$\big[\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{M} \mathsf{C}((X_iTM)_{i\in I};YM) \xrightarrow{(\xi)_I\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMT)_{i\in I};YM)$$
$$\xrightarrow{(\Delta)_I\cdot_{\mathsf{C}}(\_T)} \mathsf{C}((X_iMT)_{i\in I};YMT) \xrightarrow{-\cdot_{\mathsf{C}}\varepsilon\cdot_{\mathsf{C}}p} \mathsf{C}((X_iMT)_{i\in I};YN)\big]$$
$$= \big[\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{M} \mathsf{C}((X_iTM)_{i\in I};YM) \xrightarrow{(\xi)_I\cdot_{\mathsf{C}}-\cdot_{\mathsf{C}}p} \mathsf{C}((X_iMT)_{i\in I};YN)\big],$$

due to the identity

$$(\Delta)_I \cdot_{\mathsf{C}} (\_T) \cdot_{\mathsf{C}} \varepsilon = (\Delta)_I \cdot_{\mathsf{C}} (\varepsilon)_I \cdot_{\mathsf{C}} - = \mathrm{id} : \mathsf{C}((X_iMT)_{i\in I};YM) \to \mathsf{C}((X_iMT)_{i\in I};YM)$$

implied by the naturality of ε and axioms of a comonad. The left-bottom path is equal to

$$\big[\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{N} \mathsf{C}((X_iTN)_{i\in I};YN) \xrightarrow{(\psi)_I\cdot_{\mathsf{C}}-}$$
$$\mathsf{C}((X_iNT)_{i\in I};YN) \xrightarrow{(\widehat{p^T})_I\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMT)_{i\in I};YN)\big]$$
$$= \big[\mathsf{C}((X_iT)_{i\in I};Y) \xrightarrow{N} \mathsf{C}((X_iTN)_{i\in I};YN) \xrightarrow{(p)_I\cdot_{\mathsf{C}}-}$$
$$\mathsf{C}((X_iTM)_{i\in I};YN) \xrightarrow{(\xi)_I\cdot_{\mathsf{C}}-} \mathsf{C}((X_iMT)_{i\in I};YN)\big],$$

due to the equation

$$\widehat{p^T} = \big[XMT \xrightarrow{\Delta} XMTT \xrightarrow{\varepsilon T} XMT \xrightarrow{pT} XNT\big] = pT$$

and assumption (5.29.1). The obtained expressions coincide by the (multi)naturality of p. □

5.30 Proposition. *Let $(M,\xi : MT \to TM)$, $(N,\psi : NT \to TN)$, $(K,\chi : KT \to TK)$ be pairs satisfying conditions (5.27.1) and (5.27.2). Suppose $M \xrightarrow{p} N \xrightarrow{q} K : \mathsf{C} \to \mathsf{C}$ are (multi)natural transformations that satisfy condition (5.29.1). Then the following hold.*

(a) The composite $p \cdot q$ satisfies condition (5.29.1) and $p^T \cdot q^T = (p\cdot q)^T : M^T \to K^T : \mathsf{C} \to \mathsf{C}$.

(b) The transformation $M \cdot q : MN \to MK : \mathsf{C} \to \mathsf{C}$ satisfies condition (5.29.1) and

$$\left(\begin{tikzcd} \mathsf{C} \arrow[r, "(M{,}\xi)"] & \mathsf{C} \arrow[r, bend left, "(N{,}\psi)"] \arrow[r, bend right, "(K{,}\chi)"'] \arrow[r, phantom, "\Downarrow q"] & \mathsf{C} \end{tikzcd}\right)^T = \left(\begin{tikzcd} \mathsf{C}^T \arrow[r, "M^T"] & \mathsf{C}^T \arrow[r, bend left, "N^T"] \arrow[r, bend right, "K^T"'] \arrow[r, phantom, "\Downarrow q^T"] & \mathsf{C}^T \end{tikzcd}\right).$$

(c) The transformation $p \cdot K : MK \to NK : \mathsf{C} \to \mathsf{C}$ satisfies condition (5.29.1) and

$$\left(\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "(M{,}\xi)"] \arrow[r, bend right, "(N{,}\psi)"'] \arrow[r, phantom, "\Downarrow p"] & \mathsf{C} \arrow[r, "(K{,}\chi)"] & \mathsf{C} \end{tikzcd}\right)^T = \left(\begin{tikzcd} \mathsf{C}^T \arrow[r, bend left, "M^T"] \arrow[r, bend right, "N^T"'] \arrow[r, phantom, "\Downarrow p^T"] & \mathsf{C}^T \arrow[r, "K^T"] & \mathsf{C}^T \end{tikzcd}\right).$$

Proof. For the proof of (a), substitute the definitions of p^T and q^T into the equation in question. We obtain

$$\begin{aligned} p^T \cdot q^T &= \big[XMT \xrightarrow{\Delta} XMTT \xrightarrow{\varepsilon T} XMT \xrightarrow{pT} XNT \xrightarrow{\varepsilon} XN \xrightarrow{q} XK\big] \\ &= \big[XMT \xrightarrow{pT} XNT \xrightarrow{\varepsilon} XN \xrightarrow{q} XK\big] \\ &= \big[XMT \xrightarrow{\varepsilon} XM \xrightarrow{p} XN \xrightarrow{q} XK\big] = (p \cdot q)^T, \end{aligned}$$

by axioms of a comonad and by the naturality of ε.

The first claim in part (b) is clear. The equation holds by definition, as

$$(M^T \cdot q^T)_X = (q^T)_{XM^T} = (q^T)_{XM} = \varepsilon_{XM} \cdot q_{XM} = (M \cdot q)^T_X,$$

for each $X \in \operatorname{Ob} \mathsf{C}$.

The first claim in part (c) is clear. To prove the equation, note that

$$\begin{aligned} p^T \cdot K^T &= \big[XMKT \xrightarrow{\chi} XMTK \xrightarrow{\varepsilon K} XMK \xrightarrow{pK} XNK\big] \\ &= \big[XMKT \xrightarrow{\varepsilon} XMK \xrightarrow{pK} XNK\big] = (p \cdot K)^T, \end{aligned}$$

by condition (5.27.2). The proposition is proven. □

5.31 Corollary. *There is a strict monoidal category whose objects are pairs (M, ξ) consisting of a (symmetric) multifunctor $M : \mathsf{C} \to \mathsf{C}$ and a multinatural transformation $\xi : MT \to TM$ satisfying conditions* (5.27.1) *and* (5.27.2)*. A morphism $(M, \xi) \to (N, \psi)$ is a (multi)natural transformation $p : M \to N : \mathsf{C} \to \mathsf{C}$ satisfying condition* (5.29.1)*. The tensor product is composition, see Proposition 5.28. The correspondence $(M, \xi) \mapsto M^T$ is a strict monoidal functor from this category to $(\mathcal{S})\mathcal{MC}\mathrm{at}(\mathsf{C}, \mathsf{C})$.*

Proof. The proof follows from Propositions 5.28–5.30. □

5.32 Theorem. *Assume that equations* (5.27.1), (5.27.2), *and the equations*

$$\begin{tikzcd} MMT \arrow[r, "\xi"] \arrow[d, "mT"'] & MTM \arrow[r, "\xi M"] & TMM \arrow[d, "m"] \\ MT \arrow[rr, "\xi"] & = & TM \end{tikzcd} \tag{5.32.1}$$

$$\big(T \xrightarrow{uT} MT \xrightarrow{\xi} TM\big) = u \tag{5.32.2}$$

hold. Then $u^T = \varepsilon \cdot_{\mathsf{C}} u \in \mathsf{C}(XT; XM) = \mathsf{C}^T(X; XM^T)$ is a multinatural transformation $u^T : \mathrm{Id} \to M^T : \mathsf{C}^T \to \mathsf{C}^T$, and $m^T = \varepsilon \cdot m \in \mathsf{C}(XMMT; XM)$ is a natural transformation $m^T : M^T M^T \to M^T : \mathsf{C}^T \to \mathsf{C}^T$. The triple (M^T, m^T, u^T) is a monad in C^T. Furthermore, if m is multinatural, then m^T is multinatural as well, and (M^T, m^T, u^T) is a multimonad.

Proof. Conditions (5.32.1) and (5.32.2) are precisely condition (5.29.1) for m and u respectively. The strict monoidal functor $-^T$ sends an algebra in one strict monoidal category to an algebra in the other one. □

Part II

A_∞-categories

Chapter 6

The tensor comonad on quivers

Let $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda^f)$ be a symmetric closed Monoidal abelian $\mathscr{U}$-category. We assume in addition that arbitrary $\mathscr{U}$-small limits and colimits[1] exist in $\mathcal{V}$, that $\mathscr{U}$-small filtering colimits in $\mathcal{V}$ commute with finite projective limits. Closedness of $\mathcal{V}$ implies that the tensor product commutes with arbitrary $\mathscr{U}$-small colimits. The endomorphism ring $\Bbbk = \mathcal{V}(\mathbb{1}_\mathcal{V}, \mathbb{1}_\mathcal{V})$ of the unit object $\mathbb{1}_\mathcal{V}$ is $\mathscr{U}$-small and commutative. Notice that the category $\mathcal{V}$ is $\Bbbk$-linear.

Actually, we are interested only in two examples: $\mathcal{V} = \mathbf{gr} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$ is the category of *all* $\mathscr{U}$-small graded $\Bbbk$-modules and $\mathcal{V} = \mathbf{dg} = \mathbf{dg}(\Bbbk\text{-}\mathbf{Mod})$ is the category of *all* $\mathscr{U}$-small differential graded $\Bbbk$-modules (complexes). Commutation of filtering colimits with finite limits follows from [GV73, Corollaire 2.9]. The symmetric Monoidal structure comes from Example 3.26. The reader may assume that $\mathcal{V}$ means one of these two categories. By abuse of notation we shall denote the unit object of $\mathcal{V}$ also by $\Bbbk$. This is convenient in both examples.

We construct two symmetric Monoidal structures in the category of $\mathcal{V}$-quivers and a lax symmetric Monoidal comonad $T^{\geqslant 1}$ in this category. This is the tensor module without the 0-th term. Coalgebras over this comonad are described in this chapter. We discuss also their relationship with augmented counital coassociative coalgebras in the category of $\mathcal{V}$-quivers.

From now on we suppose that in all Monoidal categories encountered in the sequel $\otimes^L = \mathrm{Id}$ and $\rho^L : \otimes^L \to \mathrm{Id}$ is the identity morphism, for each 1-element set L. One can manage to fulfill this condition for all categories below.

6.1 The symmetric Monoidal category of quivers. The category of $\mathcal{V}$-quivers defined in Section 3.2 is denoted ${}^{\mathcal{V}}\mathscr{Q}$. When $\mathcal{V} = \mathbf{gr}$, the category of graded $\Bbbk$-linear quivers ${}^{\mathbf{gr}}\mathscr{Q}$ is denoted simply $\mathscr{Q}$. When $\mathcal{V} = \mathbf{dg}$, the category of differential graded $\Bbbk$-linear quivers ${}^{\mathbf{dg}}\mathscr{Q}$ is denoted also ${}^{d}\mathscr{Q}$. The category ${}^{\mathcal{V}}\mathscr{Q}$ has a natural symmetric Monoidal structure ${}^{\mathcal{V}}\mathscr{Q}_p = ({}^{\mathcal{V}}\mathscr{Q}, \boxtimes^I, \lambda^f)$. For given $\mathcal{V}$-quivers $\mathcal{Q}_i$ the quiver $\boxtimes^{i\in I}\mathcal{Q}_i$ has the set of objects $\prod_{i\in I} \operatorname{Ob}\mathcal{Q}_i$ and the objects of morphisms $(\boxtimes^{i\in I}\mathcal{Q}_i)((X_i)_{i\in I}, (Y_i)_{i\in I}) = \otimes^{i\in I}\mathcal{Q}_i(X_i, Y_i)$. Isomorphisms λ^f are those of $\mathcal{V}$. When $\mathcal{V} = \mathbf{gr}$ or $\mathcal{V} = \mathbf{dg}$ we use the notation $\mathscr{Q}_p = {}^{\mathbf{gr}}\mathscr{Q}_p$ and ${}^{d}\mathscr{Q}_p = {}^{\mathbf{dg}}\mathscr{Q}_p$. The unit object $\mathbb{1}_p = \boxtimes^{\varnothing}()$ of $\mathscr{Q}_p$ and ${}^{d}\mathscr{Q}_p$ is the (differential) graded $\Bbbk$-quiver with a unique object $*$ and the module of homomorphisms $\mathbb{1}_p(*, *) = \Bbbk$.

An arbitrary $\mathscr{U}$-small set S generates a $\mathcal{V}$-quiver $\Bbbk S$, whose set of objects is S, and

[1] limits and colimits of functors $\mathcal{D} \to \mathcal{V}$, where the category $\mathcal{D}$ is $\mathscr{U}$-small

the object of morphisms is $\Bbbk S(X,Y) = \Bbbk$ if $X = Y \in S$ and $\Bbbk S(X,Y) = 0$ if $X \neq Y$. A map $f : R \to S$ induces the $\Bbbk$-quiver morphism $\Bbbk f : \Bbbk R \to \Bbbk S$, $\operatorname{Ob}\Bbbk f = f$, $\Bbbk f = \mathrm{id}_\Bbbk : \Bbbk R(X,X) \to \Bbbk S(Xf, Xf)$ for all $X \in R$, and $\Bbbk f = 0 : \Bbbk R(X,Y) \to \Bbbk S(Xf, Yf)$ if $X \neq Y \in R$. Given a quiver $\mathcal{C}$, we abbreviate the quiver $\Bbbk \operatorname{Ob}\mathcal{C}$ to $\Bbbk\mathcal{C}$. For a quiver morphism $f : \mathcal{A} \to \mathcal{B}$ we denote by $\Bbbk f$ the quiver map $\Bbbk \operatorname{Ob} f : \Bbbk\mathcal{A} \to \Bbbk\mathcal{B}$.

Let S be a $\mathscr{U}$-small set. Let ${}^{\mathcal{V}}\mathscr{Q}/S$ be the category of $\mathcal{V}$-quivers $\mathcal{C}$ with the set of objects S, whose morphisms are morphisms of $\mathcal{V}$-quivers $f : \mathcal{A} \to \mathcal{B}$ such that $\operatorname{Ob} f = \mathrm{id}_S$. The $\Bbbk$-linear abelian category ${}^{\mathcal{V}}\mathscr{Q}/S$ admits the following structure of a Monoidal category $({}^{\mathcal{V}}\mathscr{Q}/S, \otimes^I, \lambda^f)$:

$$(\otimes^{i\in\mathbf{n}}\mathcal{Q}_i)(X,Z) = \bigoplus_{Y_i\in S,\, 0\leqslant i\leqslant n}^{Y_0=X,\, Y_n=Z} \otimes^{i\in\mathbf{n}}\mathcal{Q}_i(Y_{i-1}, Y_i).$$

In particular, $\otimes^{\mathbf{0}}() = \Bbbk S$ is the unit object. Isomorphisms λ^f extend totally those of $\mathcal{V}$.

Given a family of quivers $\mathcal{A}_i \in {}^{\mathcal{V}}\mathscr{Q}/S$, $i \in I$, we can form their direct sum $\oplus_{i\in I}\mathcal{A}_i \in {}^{\mathcal{V}}\mathscr{Q}/S$. The same conclusion applies when we have bijections $\operatorname{Ob}\mathcal{A}_i \simeq S$, along which the quiver structure can be transported to the set S. Such bijections will be obvious and implicit in our applications.

Recall that a $\mathcal{V}$-quiver is a particular case of a $\mathcal{V}$-span. Let $\mathcal{A}$, $\mathcal{B}$ be $\mathcal{V}$-quivers. A *$\mathcal{V}$-span morphism* $r : \mathcal{A} \to \mathcal{B}$ is by Section 3.5 a pair of maps $f = \operatorname{Ob}_s r$, $g = \operatorname{Ob}_t r : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ and a collection of morphisms $r : \mathcal{A}(X,Y) \to \mathcal{B}(Xf, Yg)$ in $\mathcal{V}$, given for each pair X, Y of objects of $\mathcal{A}$. In particular, if $f^i : \mathcal{A} \to \mathcal{B}$ is a finite family of quiver morphisms such that $\operatorname{Ob} f^i = \phi$ does not depend on i, then there is their sum, the quiver morphism $\sum_i f^i : \mathcal{A} \to \mathcal{B}$.

The Monoidal categories ${}^{\mathcal{V}}\mathscr{Q}/S$ can be included into a bicategory of $\mathcal{V}$-spans, however, we shall use only part of this picture. We shall need tensor products of $\mathcal{V}$-span morphisms (not in maximal generality), so we start to define them now.

Let $\mathcal{A}_i$, $\mathcal{B}_i$, $i \in \mathbf{n}$, be $\mathcal{V}$-quivers with $\operatorname{Ob}\mathcal{A}_i = S$, $\operatorname{Ob}\mathcal{B}_i = R$ for all $i \in \mathbf{n}$. Let $f_i : \mathcal{A}_i \to \mathcal{B}_i$, $i \in \mathbf{n}$, be $\mathcal{V}$-span morphisms such that $\operatorname{Ob}_t f_i = \operatorname{Ob}_s f_{i+1} : S \to R$ for all $1 \leqslant i < n$. (In the case $n = 0$ a map $\operatorname{Ob} : S \to R$ has to be given.) Define a $\mathcal{V}$-span morphism $f = \otimes^{i\in\mathbf{n}} f_i : \otimes_S^{i\in\mathbf{n}}\mathcal{A}_i \to \otimes_R^{i\in\mathbf{n}}\mathcal{B}_i$ with object maps $\operatorname{Ob}_s f = \operatorname{Ob}_s f_1 : S \to R$, $\operatorname{Ob}_t f = \operatorname{Ob}_t f_n : S \to R$ by the mappings

$$\begin{aligned}
(\otimes_S^{i\in\mathbf{n}}\mathcal{A}_i)(X,Z) &= \bigoplus_{Y_i\in S,\, 0\leqslant i\leqslant n}^{Y_0=X,\, Y_n=Z} \otimes^{i\in\mathbf{n}}\mathcal{A}_i(Y_{i-1}, Y_i) \\
&\xrightarrow{\oplus\otimes^{i\in\mathbf{n}} f_i} \bigoplus_{Y_i\in S,\, 0\leqslant i\leqslant n}^{Y_0=X,\, Y_n=Z} \otimes^{i\in\mathbf{n}}\mathcal{B}_i(Y_{i-1}.(\operatorname{Ob}_s f_i), Y_i.(\operatorname{Ob}_t f_i)) \longrightarrow \\
&\bigoplus_{W_i\in R,\, 0\leqslant i\leqslant n}^{W_0=X.(\operatorname{Ob}_s f_1),\, W_n=Z.(\operatorname{Ob}_t f_n)} \otimes^{i\in\mathbf{n}}\mathcal{B}_i(W_{i-1}, W_i) = (\otimes_R^{i\in\mathbf{n}}\mathcal{B}_i)(X.(\operatorname{Ob}_s f), Z.(\operatorname{Ob}_t f)),
\end{aligned}$$

where the last mapping is induced by identity mappings of direct summands indexed by $(Y_0,\dots,Y_n)$ and by $(Y_0.(\mathrm{Ob}_s f_1),\dots,Y_{n-1}.(\mathrm{Ob}_s f_n),Y_n.(\mathrm{Ob}_t f_n))$ (by (Y_0) and by $(Y_0.\mathrm{Ob})$ if $n=0$ and $X=Y$).

When all $f_i:\mathcal{A}_i\to\mathcal{B}_i$ are quiver morphisms with $\mathrm{Ob}\, f_i=\phi:S\to R$, the resulting $\otimes^{i\in\mathbf{n}}f_i$ is a quiver morphism again. In particular, if $f:\mathcal{A}\to\mathcal{B}$ is a $\mathcal{V}$-quiver morphism, then the $\mathcal{V}$-quiver morphism

$$T^nf=f^{\otimes n}=\otimes^{i\in\mathbf{n}}f:T^n\mathcal{A}=\mathcal{A}^{\otimes n}=\otimes^{i\in\mathbf{n}}_{\mathrm{Ob}\,\mathcal{A}}\mathcal{A}\to T^n\mathcal{B}=\mathcal{B}^{\otimes n}=\otimes^{i\in\mathbf{n}}_{\mathrm{Ob}\,\mathcal{B}}\mathcal{B}$$

is defined for $n\geqslant 0$. For example, $T^1f=f:T^1\mathcal{A}=\mathcal{A}\to T^1\mathcal{B}=\mathcal{B}$ and $T^0f=\Bbbk\,\mathrm{Ob}\,f:T^0\mathcal{A}=\Bbbk\,\mathrm{Ob}\,\mathcal{A}\to T^0\mathcal{B}=\Bbbk\,\mathrm{Ob}\,\mathcal{B}$. So the functors $T^n:{}^{\mathcal{V}}\mathcal{Q}\to{}^{\mathcal{V}}\mathcal{Q}$ are defined for $n\geqslant 0$.

6.1.1 The tensor monad in ${}^{\mathcal{V}}\mathcal{Q}/S$. For a $\mathcal{V}$-quiver $\mathcal{A}$ consider the $\mathcal{V}$-quivers $T^n\mathcal{A}=\mathcal{A}^{\otimes n}=\otimes^{\mathbf{n}}\mathcal{A}$, $n\geqslant 0$, the tensor powers of $\mathcal{A}$ in ${}^{\mathcal{V}}\mathcal{Q}/\mathrm{Ob}\,\mathcal{A}$. Thus,

$$T^n\mathcal{A}(A,B)=\bigoplus_{\substack{A_0,\dots,A_n\in\mathrm{Ob}\,\mathcal{A}\\ A_0=A,\,A_n=B}}\mathcal{A}(A_0,A_1)\otimes\dots\otimes\mathcal{A}(A_{n-1},A_n)$$

for $n>0$. We define also the functors $T,T^{\geqslant 1},T^{\leqslant 1}:{}^{\mathcal{V}}\mathcal{Q}\to{}^{\mathcal{V}}\mathcal{Q}$ as direct sums of T^n:

$$T\mathcal{A}(A,B)=\bigoplus_{n=0}^{\infty}T^n\mathcal{A}(A,B),\quad T^{\geqslant 1}\mathcal{A}(A,B)=\bigoplus_{n=1}^{\infty}T^n\mathcal{A}(A,B),\quad T^{\leqslant 1}\mathcal{A}=\Bbbk\mathcal{A}\oplus\mathcal{A}$$

on objects. On quiver morphisms $f:\mathcal{A}\to\mathcal{B}$ they are defined by

$$Tf=\bigoplus_{n=0}^{\infty}T^nf:T\mathcal{A}\to T\mathcal{B},\quad T^{\geqslant 1}f=\bigoplus_{n=1}^{\infty}T^nf:T^{\geqslant 1}\mathcal{A}\to T^{\geqslant 1}\mathcal{B},\quad T^{\leqslant 1}f=\Bbbk f\oplus f.$$

Notice that T can be chosen equal to $T^{\leqslant 1}\circ T^{\geqslant 1}$, which we assume from now on.

Thus, the functor $T:{}^{\mathcal{V}}\mathcal{Q}\to{}^{\mathcal{V}}\mathcal{Q}$ is given by the formula

$$T\mathcal{C}(X,Y)=\bigoplus_{\substack{X_0,\dots,X_m\in\mathrm{Ob}\,\mathcal{C}\\ X_0=X,\,X_m=Y}}^{m\geqslant 0}\otimes^{j\in\mathbf{m}}\mathcal{C}(X_{j-1},X_j).$$

Therefore, its square is

$$TT\mathcal{C}(X,Y)=\bigoplus_{\substack{X_0,\dots,X_m\in\mathrm{Ob}\,\mathcal{C}\\ X_0=X,\,X_m=Y}}^{g:\mathbf{m}\to\mathbf{n}\in\mathcal{O}}\otimes^{p\in\mathbf{n}}\otimes^{j\in g^{-1}p}\,\mathcal{C}(X_{j-1},X_j),$$

where the summation extends over all isotonic maps $g : \mathbf{m} \to \mathbf{n}$. In general,

$$(T)^k\mathcal{C}(X,Y) = \bigoplus_{\substack{\mathbf{m_1} \xrightarrow{g_1} \mathbf{m_2} \xrightarrow{g_2} \dots \xrightarrow{g_{k-1}} \mathbf{m_k} \\ X_0,\dots,X_{m_1} \in \operatorname{Ob}\mathcal{C} \\ X_0 = X,\, X_{m_1} = Y}} \otimes^{j_k \in \mathbf{m_k}} \otimes^{j_{k-1} \in g_{k-1}^{-1} j_k} \dots \otimes^{j_1 \in g_1^{-1} j_2} \mathcal{C}(X_{j_1 - 1}, X_{j_1}).$$

The summation extends over composable sequences of isotonic maps $g_p : \mathbf{m_p} \to \mathbf{m_{p+1}}$, $1 \leqslant p < k$.

The endofunctor $T : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ has a structure of a monad. The multiplication is given by the natural transformation $\mu : TT \to T$ which is the sum of morphisms

$$(\lambda^g)^{-1} : \otimes^{p \in \mathbf{n}} \otimes^{j \in g^{-1}p} \mathcal{C}(X_{j-1}, X_j) \to \otimes^{j \in \mathbf{m}} \mathcal{C}(X_{j-1}, X_j).$$

Associativity of μ on the summand of $TTT\mathcal{C}$ labeled by $\mathbf{k} \xrightarrow{f} \mathbf{m} \xrightarrow{g} \mathbf{n}$ is guaranteed by equation (2.10.3). The unit $\eta : \mathrm{Id} \to T$ is given by $\mathrm{in}_1 : \mathcal{C} = T^1\mathcal{C} \hookrightarrow T\mathcal{C}$. In the left hand side of

$$\big(T \xrightarrow{\eta \circ T} TT \xrightarrow{\mu} T\big) = \mathrm{id}$$

the image of $\eta \circ T$ is labeled by maps $g : \mathbf{m} \to \mathbf{1}$. Such a map is unique and $\lambda^{\mathbf{m} \to \mathbf{1}} = \mathrm{id}$. This proves the above equation. In the left hand side of

$$\big(T \xrightarrow{T \circ \eta} TT \xrightarrow{\mu} T\big) = \mathrm{id}$$

the image of $T \circ \eta$ is labeled by $\mathrm{id} : \mathbf{m} \to \mathbf{m}$, and $\lambda^{\mathrm{id}} = \mathrm{id}$. So the above equation is proven. Thus, $(T, \mu, \eta) : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ is a monad.

Recall that a *T-algebra* is an object $\mathcal{C}$ of ${}^{\mathcal{V}}\mathcal{Q}$, equipped with a morphism $\alpha : T\mathcal{C} \to \mathcal{C}$ (the action) such that

$$\begin{tikzcd} TT\mathcal{C} \arrow[r, "T(\alpha)"] \arrow[d, "\mu"'] \arrow[dr, phantom, "="] & T\mathcal{C} \arrow[d, "\alpha"] \\ T\mathcal{C} \arrow[r, "\alpha"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "\eta"] \arrow[dr, equal, "="] & T\mathcal{C} \arrow[d, "\alpha"] \\ & \mathcal{C} \end{tikzcd}$$

For the monad T described above T-algebras are precisely $\mathcal{V}$-categories.

Suppose that $\mathcal{A}_i^j$ are graded quivers, $i \in I$, $j \in \mathbf{m}$. Assume that $\operatorname{Ob}\mathcal{A}_i^j = S_i$ does not depend on j. Define $S = \prod_{i\in I} S_i$. Denote by $\otimes_{S_i}$ the tensor product in ${}^{\mathcal{V}}\mathcal{Q}/S_i$. There is an isomorphism of graded quivers

$$\overline{\varkappa} : \otimes_S^{j \in \mathbf{m}} \boxtimes^{i \in I} \mathcal{A}_i^j \to \boxtimes^{i \in I} \otimes_{S_i}^{j \in \mathbf{m}} \mathcal{A}_i^j, \tag{6.1.1}$$

identity on objects, which is a direct sum of permutation isomorphisms

$$\sigma_{(12)} : \otimes^{j \in \mathbf{m}} \otimes^{i \in I} \mathcal{A}_i^j(X_i^{j-1}, X_i^j) \to \otimes^{i \in I} \otimes^{j \in \mathbf{m}} \mathcal{A}_i^j(X_i^{j-1}, X_i^j),$$

where $X_i^j \in S_i$, $0 \leqslant j \leqslant m$. In the particular case $\mathcal{A}_i^j = \mathcal{A}_i$ we get the isomorphisms

$$\overline{\varkappa} : T^m \boxtimes^{i\in I} \mathcal{A}_i \to \boxtimes^{i\in I} T^m \mathcal{A}_i.$$

They are the only non-trivial matrix elements of the embedding of functors

$$\varkappa^I : T \boxtimes^{i\in I} \mathcal{A}_i \to \boxtimes^{i\in I} T\mathcal{A}_i.$$

6.1.2 Biunital $\mathcal{V}$-quivers. A *biunital quiver* is a $\mathcal{V}$-quiver $\mathcal{C}$ together with a pair of morphisms $\Bbbk\mathcal{C} \xrightarrow{\eta} \mathcal{C} \xrightarrow{\varepsilon} \Bbbk\mathcal{C}$ in ${}^{\mathcal{V}}\mathcal{Q}/\operatorname{Ob}\mathcal{C}$, whose composition is $\eta\varepsilon = \mathrm{id}_{\Bbbk\mathcal{C}}$. Biunital **gr**-quivers are called *augmented quivers* by Keller [Kel06a], who studied them independently of the present work, while analyzing [Lyu03].

Biunital quivers with arbitrary $\mathscr{U}$-small sets of objects form a category denoted ${}^{\mathcal{V}}\mathcal{Q}_{bu}$. A *morphism* from a biunital quiver $\varepsilon : \mathcal{A} \rightleftarrows \Bbbk\mathcal{A} : \eta$ to a biunital quiver $\varepsilon : \mathcal{B} \rightleftarrows \Bbbk\mathcal{B} : \eta$ is a $\mathcal{V}$-quiver morphism $f : \mathcal{A} \to \mathcal{B}$ such that $f\varepsilon = \varepsilon(\Bbbk f)$, $\eta f = (\Bbbk f)\eta$. The category ${}^{\mathcal{V}}\mathcal{Q}_{bu}$ is equivalent to the category of $\mathcal{V}$-quivers ${}^{\mathcal{V}}\mathcal{Q}$ via functors ${}^{\mathcal{V}}\mathcal{Q}_{bu} \to {}^{\mathcal{V}}\mathcal{Q}$, $(\varepsilon : \mathcal{C} \rightleftarrows \Bbbk\mathcal{C} : \eta) \mapsto \overline{\mathcal{C}} = \operatorname{Im}(1 - \varepsilon\eta) = \operatorname{Ker}\varepsilon$, and ${}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$, $\mathcal{A} \mapsto T^{\leqslant 1}\mathcal{A} = (\mathrm{pr}_1 : \Bbbk\mathcal{A} \oplus \mathcal{A} \rightleftarrows \Bbbk\mathcal{A} : \mathrm{in}_1)$, quasi-inverse to each other. Nevertheless, the category ${}^{\mathcal{V}}\mathcal{Q}_{bu}$ is technically useful.

There is a faithful (forgetful) functor $F : {}^{\mathcal{V}}\mathcal{Q}_{bu} \to {}^{\mathcal{V}}\mathcal{Q}$, $(\varepsilon : \mathcal{C} \rightleftarrows \Bbbk\mathcal{C} : \eta) \mapsto \mathcal{C}$. The category ${}^{\mathcal{V}}\mathcal{Q}_{bu}$ inherits a symmetric Monoidal structure from ${}^{\mathcal{V}}\mathcal{Q}$ via F, namely,

$$\boxtimes_{bu}^{i\in I}(\mathcal{A}_i \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \Bbbk\mathcal{A}_i) \overset{\text{def}}{=} \bigl(\boxtimes^{i\in I}\mathcal{A}_i \underset{\boxtimes^I\eta}{\overset{\boxtimes^I\varepsilon}{\rightleftarrows}} \boxtimes^{i\in I} \Bbbk\mathcal{A}_i \underset{\sim}{\overset{\lambda_{\mathcal{V}}^{\varnothing\to I}}{\longleftarrow}} \Bbbk(\boxtimes^{i\in I}\mathcal{A}_i)\bigr).$$

The unit object is the one-object quiver $\varepsilon : \Bbbk = \Bbbk : \eta$, and the tensor product of morphisms f_i of biunital quivers is $\boxtimes^{i\in I} f_i$. The structure isomorphism λ_{bu}^f is determined by the condition: the pair

$$(F, \mathrm{id}) : {}^{\mathcal{V}}\mathcal{Q}_{bu} = ({}^{\mathcal{V}}\mathcal{Q}_{bu}, \boxtimes_{bu}^I, \lambda_{bu}^f) \to ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I, \lambda^f) = {}^{\mathcal{V}}\mathcal{Q}_p$$

is a symmetric Monoidal functor. Thus, we must have

$$\lambda_{bu}^f = \lambda^f : \boxtimes_{bu}^{i\in I}(\mathcal{A}_i \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \Bbbk\mathcal{A}_i) \to \boxtimes_{bu}^{j\in J} \boxtimes_{bu}^{i\in f^{-1}j} (\mathcal{A}_i \underset{\eta}{\overset{\varepsilon}{\rightleftarrows}} \Bbbk\mathcal{A}_i).$$

One can easily show that this is a morphism of biunital quivers. Clearly, equation (2.5.4) is satisfied, and ${}^{\mathcal{V}}\mathcal{Q}_{bu}$ is, indeed, a symmetric Monoidal category, equipped with a symmetric Monoidal functor $(F, \mathrm{id}) : {}^{\mathcal{V}}\mathcal{Q}_{bu} \to {}^{\mathcal{V}}\mathcal{Q}_p$.

This symmetric Monoidal structure $({}^{\mathcal{V}}\mathcal{Q}_{bu}, \boxtimes_{bu}^I, \lambda_{bu}^f)$ translates via the equivalence $T^{\leqslant 1} : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$ to a new symmetric Monoidal structure ${}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes_u^I, \lambda_u^f)$ on ${}^{\mathcal{V}}\mathcal{Q}$. Explicitly it is given by

$$\boxtimes_u^{i\in I}\mathcal{A}_i = \bigoplus_{j_i\in\{0,1\},\, i\in I}^{\sum_i j_i > 0} \boxtimes^{i\in I} T^{j_i}\mathcal{A}_i = \bigoplus_{\varnothing\neq S\subset I} \boxtimes^{i\in I} T^{\chi(i\in S)}\mathcal{A}_i,$$

where $\chi(i \in S) = 1$ if $i \in S$, and $\chi(i \in S) = 0$ if $i \notin S$. In particular, $\operatorname{Ob} \boxtimes_u^{i\in I} \mathcal{A}_i = \operatorname{Ob} \boxtimes^{i\in I} \mathcal{A}_i = \prod_{i\in I} \operatorname{Ob} \mathcal{A}_i$. The following canonical isomorphism is implied by additivity of $\boxtimes$,

$$\vartheta^I = \bigl[\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{A}_i) \xrightarrow{\sim} \bigoplus_{j_i\in\{0,1\},\, i\in I} \boxtimes^{i\in I} T^{j_i}\mathcal{A}_i \xrightarrow{\sim} T^0(\boxtimes^{i\in I}\mathcal{A}_i) \oplus \boxtimes_u^{i\in I}\mathcal{A}_i = T^{\leqslant 1}(\boxtimes_u^{i\in I}\mathcal{A}_i)\bigr],$$

and by the isomorphism $\boxtimes^{i\in I} T^0\mathcal{A}_i \xrightarrow{\sim} T^0(\boxtimes^{i\in I}\mathcal{A}_i)$, consisting of identifications $(\lambda_{\mathcal{V}}^{\varnothing\to I})^{-1} : \otimes^I \Bbbk \xrightarrow{\sim} \Bbbk$. Actually, $\vartheta^I : \boxtimes_{bu}^{i\in I} T^{\leqslant 1}\mathcal{A}_i \to T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i$ is an isomorphism of biunital quivers. Therefore, for an arbitrary map $f : I \to J$ the following composition

$$\begin{aligned} T^{\leqslant 1}(\boxtimes_u^{i\in I}\mathcal{A}_i) &\xrightarrow{(\vartheta^I)^{-1}} \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{A}_i) \xrightarrow{\lambda^f} \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}(T^{\leqslant 1}\mathcal{A}_i) \\ &\xrightarrow{\boxtimes^{j\in J}\vartheta^{f^{-1}j}} \boxtimes^{j\in J} T^{\leqslant 1}(\boxtimes_u^{i\in f^{-1}j}\mathcal{A}_i) \xrightarrow{\vartheta^J} T^{\leqslant 1}(\boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}\mathcal{A}_i) \end{aligned} \tag{6.1.2}$$

is an isomorphism of biunital quivers. Hence, it equals $T^{\leqslant 1}(\lambda_u^f)$ for some uniquely determined isomorphism

$$\lambda_u^f : \boxtimes_u^{i\in I}\mathcal{A}_i \to \boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}\mathcal{A}_i.$$

Uniqueness implies that it satisfies (2.10.3). Therefore, ${}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes_u^I, \lambda_u^f)$ is a symmetric Monoidal category. The above equation has the form (2.17.2), so it implies that

$$(T^{\leqslant 1}, \vartheta^I) : {}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes_u^I, \lambda_u^f) \to ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I, \lambda^f) = {}^{\mathcal{V}}\mathcal{Q}_p \tag{6.1.3}$$

is a symmetric Monoidal functor.

In particular, the unit object $\mathbb{1}_u = \boxtimes_u^{\varnothing}()$ of ${}^{\mathcal{V}}\mathcal{Q}_u$ is the $\mathcal{V}$-quiver with a unique object $*$ and zero object of homomorphisms, and

$$\begin{aligned} &(\mathcal{A}\boxtimes_u \mathcal{B})\bigl((A,B),(A',B')\bigr) \\ &\simeq \begin{cases} \mathcal{A}(A,A')\otimes\mathcal{B}(B,B'), & A\neq A',\ B\neq B', \\ \mathcal{A}(A,A')\otimes\mathcal{B}(B,B')\oplus\Bbbk\otimes\mathcal{B}(B,B'), & A=A',\ B\neq B', \\ \mathcal{A}(A,A')\otimes\mathcal{B}(B,B')\oplus\mathcal{A}(A,A')\otimes\Bbbk, & A\neq A',\ B=B', \\ \mathcal{A}(A,A')\otimes\mathcal{B}(B,B')\oplus\mathcal{A}(A,A')\otimes\Bbbk\oplus\Bbbk\otimes\mathcal{B}(B,B'), & A=A',\ B=B'. \end{cases} \end{aligned} \tag{6.1.4}$$

The equivalence $T^{\leqslant 1} : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$ is, by construction, a symmetric Monoidal equivalence. Its second component is the same isomorphism ϑ^I in ${}^{\mathcal{V}}\mathcal{Q}$, so we denote it by the same symbol, abusing notation:

$$(T^{\leqslant 1}, \vartheta^I) : {}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes_u^I, \lambda_u^f) \to ({}^{\mathcal{V}}\mathcal{Q}_{bu}, \boxtimes_{bu}^I, \lambda_{bu}^f) = {}^{\mathcal{V}}\mathcal{Q}_{bu}.$$

Functor (6.1.3) decomposes as

$$(T^{\leqslant 1}, \vartheta^I) = \bigl({}^{\mathcal{V}}\mathcal{Q}_u \xrightarrow{(T^{\leqslant 1},\vartheta^I)} {}^{\mathcal{V}}\mathcal{Q}_{bu} \xrightarrow{(F,\mathrm{id})} {}^{\mathcal{V}}\mathcal{Q}_p\bigr).$$

6.2 Lemma. *The functor $T : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ admits a lax symmetric Monoidal structure*

$$(T, \widetilde{\tau}^I) : {}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes_u^I, \lambda_u^f) \to ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I, \lambda^f) = {}^{\mathcal{V}}\mathcal{Q}_p.$$

Proof. We are going to define morphisms $\widetilde{\tau}^I : \boxtimes^{i\in I}(T\mathcal{A}_i) \to T(\boxtimes_u^{i\in I}\mathcal{A}_i)$. Given quivers $\mathcal{A}_i$ we find that

$$\boxtimes^{i\in I}(T\mathcal{A}_i) = \bigoplus_{(m_i)\in\mathbb{Z}_{\geqslant 0}^I} \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i$$

is a direct sum over $(m_i) \in \mathbb{Z}_{\geqslant 0}^I$. On the other hand,

$$T(\boxtimes_u^{i\in I}\mathcal{A}_i) = \bigoplus_{m=0}^{\infty} T^m(\boxtimes_u^{i\in I}\mathcal{A}_i) = \bigoplus_{m=0}^{\infty} \bigoplus_{S\subset I\times\mathbf{m}}^{\mathrm{pr}_2 S=\mathbf{m}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i$$

decomposes into direct sum over pairs (m, S), where $m \in \mathbb{Z}_{\geqslant 0}$, and subset $S \subset I \times \mathbf{m}$ satisfies the condition $\mathrm{pr}_2 S = \mathbf{m}$.

Define $\widetilde{\tau}^I : \boxtimes^{i\in I}(T\mathcal{A}_i) \to T(\boxtimes_u^{i\in I}\mathcal{A}_i)$ to be the identity map on objects $(X_i)_{i\in I}$. Define the only non-trivial matrix coefficients of $\widetilde{\tau}^I$ to be the isomorphisms

$$\widetilde{\tau}^I : \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i, \qquad (6.2.1)$$

where $S_i = \{p \in \mathbf{m} \mid (i,p) \in S\}$ satisfy the condition $|S_i| = m_i$ for all $i \in I$. Here $\overline{\varkappa}$ is given by (6.1.1). If $|S_i| \neq m_i$ for some i, the corresponding matrix coefficient of $\widetilde{\tau}^I$ vanishes.

Let $f : I \to J$ be a map of finite sets. We want to verify equation (2.17.2) for $(T, \widetilde{\tau}^I)$. For a given family $(\mathcal{A}_i)_{i\in I}$ of quivers it reads:

$$\begin{aligned}
\big[\boxtimes^{i\in I}(T\mathcal{A}_i) \xrightarrow[\sim]{\lambda^f} \boxtimes^{j\in J} \boxtimes^{i\in f^{-1}j} T\mathcal{A}_i \xrightarrow{\boxtimes^{j\in J}\widetilde{\tau}^{f^{-1}j}} \boxtimes^{j\in J} T(\boxtimes_u^{i\in f^{-1}j}\mathcal{A}_i) \\
\xrightarrow{\widetilde{\tau}^J} T(\boxtimes_u^{j\in J} \boxtimes_u^{i\in f^{-1}j} \mathcal{A}_i)\big] \\
= \big[\boxtimes^{i\in I}(T\mathcal{A}_i) \xrightarrow{\widetilde{\tau}^I} T \boxtimes_u^{i\in I} \mathcal{A}_i \xrightarrow[\sim]{T\lambda_u^f} T(\boxtimes_u^{j\in J} \boxtimes_u^{i\in f^{-1}j} \mathcal{A}_i)\big].
\end{aligned}$$

Given non-negative integers $(m_i)_{i\in I}$ and m, we restrict the above equation to the corresponding direct summands:

$$\begin{aligned}
\big[\boxtimes^{i\in I}(T^{m_i}\mathcal{A}_i) \xrightarrow{\lambda^f} \boxtimes^{j\in J} \boxtimes^{i\in f^{-1}j} T^{m_i}\mathcal{A}_i \\
\xrightarrow{\boxtimes^{j\in J}\widetilde{\tau}^{f^{-1}j}} \bigoplus_{(p_j)\in\mathbb{Z}_{\geqslant 0}^J} \boxtimes^{j\in J} T^{p_j}(\boxtimes_u^{i\in f^{-1}j}\mathcal{A}_i) \xrightarrow{\widetilde{\tau}^J} T^m(\boxtimes_u^{j\in J} \boxtimes_u^{i\in f^{-1}j} \mathcal{A}_i)\big] \\
= \big[\boxtimes^{i\in I}(T^{m_i}\mathcal{A}_i) \xrightarrow{\widetilde{\tau}^I} T^m \boxtimes_u^{i\in I} \mathcal{A}_i \xrightarrow{T^m\lambda_u^f} T^m(\boxtimes_u^{j\in J} \boxtimes_u^{i\in f^{-1}j} \mathcal{A}_i)\big]. \qquad (6.2.2)
\end{aligned}$$

The right hand side projects via permutation isomorphisms to summands of the middle term, corresponding to subsets $S \subset I \times \mathbf{m}$ such that $\operatorname{pr}_2 S = \mathbf{m}$ and $m_i = |S \cap (\{i\} \times \mathbf{m})|$. The left hand side uses permutation isomorphisms to summands indexed by $(p_j)_j \in \mathbb{Z}^J_{\geqslant 0}$, subsets $S_j \subset f^{-1}j \times \mathbf{p_j}$ such that $\operatorname{pr}_2 S_j = \mathbf{p_j}$ and $m_i = |S_j \cap (\{i\} \times \mathbf{p_j})|$ for all $i \in f^{-1}j$, and by a subset $R \subset J \times \mathbf{m}$ such that $\operatorname{pr}_2 R = \mathbf{m}$ and $p_j = |R \cap (\{j\} \times \mathbf{m})|$. Define subsets $P_j = \operatorname{pr}_2[R \cap (\{j\} \times \mathbf{m})] \subset \mathbf{m}$, then $|P_j| = p_j$. Using the unique isotonic bijection $P_j \simeq \mathbf{p_j}$ we can replace $S_j \subset f^{-1}j \times \mathbf{p_j}$ with subsets $S'_j \subset f^{-1}j \times P_j$ such that $\operatorname{pr}_2 S'_j = P_j$ and $m_i = |S'_j \cap (\{i\} \times P_j)|$. Notice that families $(R, S'_j)_{j \in J}$ and subsets $S \subset I \times \mathbf{m}$ with the required properties are in bijection. Indeed, given S we define

$$\begin{gathered}
R = \{(j,k) \in J \times \mathbf{m} \mid \exists i \in f^{-1}j : (i,k) \in S\}, \\
S'_j = S \cap (f^{-1}j \times \mathbf{m}) \subset f^{-1}j \times P_j = f^{-1}j \times \operatorname{pr}_2[R \cap (\{j\} \times \mathbf{m})].
\end{gathered}$$

Vice versa, given $(R, S'_j)_{j \in J}$ we define

$$S = \{(i,k) \in I \times \mathbf{m} \mid (fi,k) \in R, \quad (i,k) \in S'_{fi}\}.$$

Let $i \in I$ and $j = fi$. Then

$$|S'_j \cap (\{i\} \times P_j)| = |S'_j \cap (\{i\} \times \mathbf{m})| = |S \cap (f^{-1}j \times \mathbf{m}) \cap (\{i\} \times \mathbf{m})| = |S \cap (\{i\} \times \mathbf{m})|,$$

so these numbers are not equal or equal to m_i simultaneously. Correspondingly, the matrix elements of both sides of (6.2.2) simultaneously vanish or give the following natural transformations:

$$\begin{aligned}
\Big[& \boxtimes^{i \in I} \otimes^{\mathbf{m_i}} \mathcal{A}_i \xrightarrow{\lambda^f} \boxtimes^{j \in J} \boxtimes^{i \in f^{-1}j} \otimes^{\mathbf{m_i}} \mathcal{A}_i \xrightarrow{\boxtimes^{j \in J} \boxtimes^{i \in f^{-1}j} \lambda^{S_j \cap (\{i\} \times \mathbf{p_j}) \hookrightarrow \mathbf{p_j}}} \\
& \boxtimes^{j \in J} \boxtimes^{i \in f^{-1}j} \otimes^{q_j \in \mathbf{p_j}} T^{\chi((i,q_j) \in S_j)} \mathcal{A}_i \xrightarrow{\boxtimes^{j \in J} \overline{\varkappa}^{-1}} \boxtimes^{j \in J} \otimes^{q_j \in \mathbf{p_j}} \boxtimes^{i \in f^{-1}j} T^{\chi((i,q_j) \in S_j)} \mathcal{A}_i \\
& \xrightarrow{\boxtimes^{j \in J} \lambda^{P_j \hookrightarrow \mathbf{m}}} \boxtimes^{j \in J} \otimes^{k \in \mathbf{m}} T^{\chi((j,k) \in R)} \boxtimes^{i \in f^{-1}j} T^{\chi((i,k) \in S'_j)} \mathcal{A}_i \\
& \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{k \in \mathbf{m}} \boxtimes^{j \in J} T^{\chi((j,k) \in R)} \boxtimes^{i \in f^{-1}j} T^{\chi((i,k) \in S'_j)} \mathcal{A}_i \Big] \\
= \Big[& \boxtimes^{i \in I} \otimes^{\mathbf{m_i}} \mathcal{A}_i \xrightarrow{\boxtimes^{i \in I} \lambda^{S \cap (\{i\} \times \mathbf{m}) \hookrightarrow \mathbf{m}}} \boxtimes^{i \in I} \otimes^{k \in \mathbf{m}} T^{\chi((i,k) \in S)} \mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} \\
& \otimes^{k \in \mathbf{m}} \boxtimes^{i \in I} T^{\chi((i,k) \in S)} \mathcal{A}_i \xrightarrow{\otimes^{\mathbf{m}} \lambda^f} \otimes^{k \in \mathbf{m}} \boxtimes^{j \in J} \boxtimes^{i \in f^{-1}j} T^{\chi((fi,k) \in R)} T^{\chi((i,k) \in S'_j)} \mathcal{A}_i \\
& = \otimes^{k \in \mathbf{m}} \boxtimes^{j \in J} T^{\chi((j,k) \in R)} \boxtimes^{i \in f^{-1}j} T^{\chi((i,k) \in S'_j)} \mathcal{A}_i \Big].
\end{aligned}$$

The above maps can be specified on objects $X^n_i \in \operatorname{Ob} \mathcal{A}_i$ for $i \in I$, $0 \leqslant n \leqslant m_i$. Using objects $A^n_i = \mathcal{A}_i(X^{n-1}_i, X^n_i) \in \operatorname{Ob} \mathcal{V}$, $1 \leqslant n \leqslant m_i$, one can rewrite the above equation in terms of symmetric Monoidal category $(\mathcal{V}, \otimes, \lambda^f)$ only. By Lemma 2.33 and Remark 2.34 the obtained equation between natural transformations holds automatically. Therefore, $(T, \widetilde{\tau}^I)$ is a lax symmetric Monoidal functor. $\square$

6.3 Proposition. *The endofunctor* $T^{\geqslant 1} : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ *admits a unique lax symmetric Monoidal structure* $(T^{\geqslant 1}, \tau^I) : {}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I_u, \lambda^f_u) \to ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I_u, \lambda^f_u) = {}^{\mathcal{V}}\mathcal{Q}_u$ *such that*

$$(T, \widetilde{\tau}^I) = (T^{\leqslant 1}, \vartheta^I) \circ (T^{\geqslant 1}, \tau^I) : ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I_u, \lambda^f_u) \to ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I, \lambda^f). \tag{6.3.1}$$

Proof. The lax symmetric Monoidal functor $(T, \widetilde{\tau}^I) : {}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_p$ of Lemma 6.2 admits a lifting $(T, \widetilde{\tau}^I) : {}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$ such that

$$(T, \widetilde{\tau}^I) = \big({}^{\mathcal{V}}\mathcal{Q}_u \xrightarrow{(T,\widetilde{\tau}^I)} {}^{\mathcal{V}}\mathcal{Q}_{bu} \xrightarrow{(F,\mathrm{id})} {}^{\mathcal{V}}\mathcal{Q}_p\big).$$

We abuse the notation because the already defined transformation $\widetilde{\tau}^I : \boxtimes^{i\in I}(T\mathcal{A}_i) \to T(\boxtimes^{i\in I}_u \mathcal{A}_i)$ is a morphism of biunital quivers. Hence, it is a natural transformation between functors ${}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$. Equations (2.6.1) for new and old transformations coincide. Thus, $(T, \widetilde{\tau}^I) : {}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$ is a lax symmetric Monoidal functor.

Since $(T^{\leqslant 1}, \vartheta^I) : {}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_{bu}$ is a symmetric Monoidal equivalence, it has a quasi-inverse $(\operatorname{Ker}\varepsilon, \gamma) : {}^{\mathcal{V}}\mathcal{Q}_{bu} \to {}^{\mathcal{V}}\mathcal{Q}_u$, $(\varepsilon : \mathcal{C} \rightleftarrows \Bbbk\mathcal{C} : \eta) \mapsto \operatorname{Ker}\varepsilon$, which gives $(T^{\geqslant 1}, \tau^I)$ up to an isomorphism. Actually, since $T = T^{\leqslant 1} \circ T^{\geqslant 1}$ and ϑ^I is invertible, there is a unique natural transformation $\tau^I : \boxtimes^{i\in I}_u (T^{\geqslant 1}\mathcal{A}_i) \to T^{\geqslant 1}(\boxtimes^{i\in I}_u \mathcal{A}_i)$ such that equation (6.3.1) holds. In particular, the composition

$$T^{\leqslant 1} \boxtimes^{i\in I}_u (T^{\geqslant 1}\mathcal{A}_i) \xrightarrow{(\vartheta^I)^{-1}} \boxtimes^{i\in I} (T\mathcal{A}_i) \xrightarrow{\widetilde{\tau}^I} T(\boxtimes^{i\in I}_u \mathcal{A}_i) = T^{\leqslant 1}T^{\geqslant 1}(\boxtimes^{i\in I}_u \mathcal{A}_i) \tag{6.3.2}$$

equals $T^{\leqslant 1}(\tau^I)$. Clearly, $(T^{\geqslant 1}, \tau^I)$ is a lax symmetric Monoidal functor. □

6.4 Remark. The presentation

$$(T^{\geqslant 1}, \tau^I) = \big({}^{\mathcal{V}}\mathcal{Q}_u \xrightarrow{(T,\widetilde{\tau}^I)} {}^{\mathcal{V}}\mathcal{Q}_{bu} \xrightarrow{(\operatorname{Ker}\varepsilon,\mathrm{id})} {}^{\mathcal{V}}\mathcal{Q}_u\big)$$

implies that

$$\tau = \Big(\boxtimes^{i\in I}_u T^{\geqslant 1}\mathcal{A}_i = \bigoplus_{0\neq(m_i)\in\mathbb{Z}^I_{\geqslant 0}} \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i \xrightarrow{\sum\widetilde{\tau}} T^{\geqslant 1} \boxtimes^{i\in I}_u \mathcal{A}_i\Big). \tag{6.4.1}$$

6.5 The tensor comonad $T^{\geqslant 1}$.

The functor $T^{\geqslant 1} : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ is given by the formula

$$T^{\geqslant 1}\mathcal{C}(X,Y) = \bigoplus^{m>0}_{\substack{X_0,\dots,X_m\in\operatorname{Ob}\mathcal{C}\\ X_0=X,\, X_m=Y}} \otimes^{j\in\mathbf{m}} \mathcal{C}(X_{j-1}, X_j). \tag{6.5.1}$$

Therefore, its square is

$$T^{\geqslant 1}T^{\geqslant 1}\mathcal{C}(X,Y) = \bigoplus^{\substack{n>0\\ g:\mathbf{m}\twoheadrightarrow\mathbf{n}\in\mathcal{O}}}_{\substack{X_0,\dots,X_m\in\operatorname{Ob}\mathcal{C}\\ X_0=X,\, X_m=Y}} \otimes^{p\in\mathbf{n}} \otimes^{j\in g^{-1}p} \mathcal{C}(X_{j-1}, X_j), \tag{6.5.2}$$

where the summation extends over all isotonic surjections $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$ with non-empty $\mathbf{n}$. In general,

$$(T^{\geqslant 1})^k \mathcal{C}(X,Y) = \bigoplus_{\substack{\mathbf{m_1} \xrightarrow{g_1}\!\!\!\!\twoheadrightarrow \mathbf{m_2} \xrightarrow{g_2}\!\!\!\!\twoheadrightarrow \dots \xrightarrow{g_{k-1}}\!\!\!\!\twoheadrightarrow \mathbf{m_k} \\ m_k>0}}^{\substack{X_0,\dots,X_{m_1} \in \operatorname{Ob}\mathcal{C} \\ X_0=X,\, X_{m_1}=Y}} \otimes^{j_k \in \mathbf{m_k}} \otimes^{j_{k-1} \in g_{k-1}^{-1} j_k} \dots \otimes^{j_1 \in g_1^{-1} j_2} \mathcal{C}(X_{j_1-1}, X_{j_1}).$$

The summation extends over composable sequences of isotonic surjections $g_p : \mathbf{m_p} \twoheadrightarrow \mathbf{m_{p+1}}$, $1 \leqslant p < k$, $m_k > 0$, that is, over non-empty plane staged trees with k stages.

The endofunctor $T^{\geqslant 1} : {}^{\mathcal{V}}\mathcal{Q} \to {}^{\mathcal{V}}\mathcal{Q}$ has a structure of an augmented comonad $(T^{\geqslant 1}, \Delta, \varepsilon, \eta)$. The comultiplication is given by the natural transformation $\Delta : T^{\geqslant 1} \to T^{\geqslant 1} T^{\geqslant 1}$ which is a sum of morphisms

$$\lambda^g : \otimes^{j \in \mathbf{m}} \mathcal{C}(X_{j-1}, X_j) \to \otimes^{p \in \mathbf{n}} \otimes^{j \in g^{-1} p} \mathcal{C}(X_{j-1}, X_j). \tag{6.5.3}$$

That is, for each summand of (6.5.2) labeled by an isotonic surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$ there exists a unique summand of (6.5.1) which is mapped to it by λ^g, namely, the summand labeled by the source m of g. Coassociativity of Δ

$$\begin{aligned}
&\Bigl[\bigoplus_{k>0} \otimes^{j \in \mathbf{k}} (\mathcal{C})_j \xrightarrow{\Delta} \bigoplus_{h:\mathbf{k} \twoheadrightarrow \mathbf{n} \neq \mathbf{0}} \otimes^{p \in \mathbf{n}} \otimes^{j \in h^{-1} p} (\mathcal{C})_j \xrightarrow{T^{\geqslant 1}(\Delta)} \bigoplus_{\mathbf{k} \xrightarrow{f}\!\!\!\!\twoheadrightarrow \mathbf{m} \xrightarrow{g}\!\!\!\!\twoheadrightarrow \mathbf{n} \neq \mathbf{0}} \otimes^{p \in \mathbf{n}} \otimes^{q \in g^{-1} p} \otimes^{j \in f^{-1} q} (\mathcal{C})_j\Bigr] \\
&= \Bigl[\bigoplus_{k>0} \otimes^{j \in \mathbf{k}} (\mathcal{C})_j \xrightarrow{\Delta} \bigoplus_{f:\mathbf{k} \twoheadrightarrow \mathbf{m} \neq \mathbf{0}} \otimes^{q \in \mathbf{m}} \otimes^{j \in f^{-1} q} (\mathcal{C})_j \xrightarrow{\Delta} \bigoplus_{\mathbf{k} \xrightarrow{f}\!\!\!\!\twoheadrightarrow \mathbf{m} \xrightarrow{g}\!\!\!\!\twoheadrightarrow \mathbf{n} \neq \mathbf{0}} \otimes^{p \in \mathbf{n}} \otimes^{q \in g^{-1} p} \otimes^{j \in f^{-1} q} (\mathcal{C})_j\Bigr]
\end{aligned} \tag{6.5.4}$$

is proven as follows. Only summand labeled by $h = g \circ f : \mathbf{k} \twoheadrightarrow \mathbf{n}$ and no other is mapped to the summand labeled by $\mathbf{k} \overset{f}{\twoheadrightarrow} \mathbf{m} \overset{g}{\twoheadrightarrow} \mathbf{n}$ in the left hand side. Therefore the required equation reduces to

$$\begin{aligned}
&\bigl[\otimes^{j \in \mathbf{k}} (\mathcal{C})_j \xrightarrow{\lambda^{g \circ f}} \otimes^{p \in \mathbf{n}} \otimes^{j \in f^{-1} g^{-1} p} (\mathcal{C})_j \xrightarrow{\otimes^{p \in \mathbf{n}} \lambda^{f : f^{-1} g^{-1} p \to g^{-1} p}} \otimes^{p \in \mathbf{n}} \otimes^{q \in g^{-1} p} \otimes^{j \in f^{-1} q} (\mathcal{C})_j\bigr] \\
&\qquad = \bigl[\otimes^{j \in \mathbf{k}} (\mathcal{C})_j \xrightarrow{\lambda^f} \otimes^{q \in \mathbf{m}} \otimes^{j \in f^{-1} q} (\mathcal{C})_j \xrightarrow{\lambda^g} \otimes^{p \in \mathbf{n}} \otimes^{q \in g^{-1} p} \otimes^{j \in f^{-1} q} (\mathcal{C})_j\bigr].
\end{aligned}$$

This is precisely property (2.10.3) for λ.

The counit is given by the transformation $\varepsilon = \mathrm{pr}_1 : T^{\geqslant 1} \to \mathrm{Id}$, $\mathrm{pr}_1 : T^{\geqslant 1}\mathcal{A} \to \mathcal{A} \in {}^{\mathcal{V}}\mathcal{Q}/\operatorname{Ob}\mathcal{A}$. We have

$$\bigl(T^{\geqslant 1} \xrightarrow{\Delta} T^{\geqslant 1} T^{\geqslant 1} \xrightarrow{\varepsilon} T^{\geqslant 1}\bigr) = \mathrm{id},$$

because for each $m \in \mathbb{Z}_{>0}$ there exists only one isotonic surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{1}$. Furthermore, $\lambda^{\mathbf{m} \to \mathbf{1}} = \mathrm{id}$. Also

$$\bigl(T^{\geqslant 1} \xrightarrow{\Delta} T^{\geqslant 1} T^{\geqslant 1} \xrightarrow{T^{\geqslant 1}(\varepsilon)} T^{\geqslant 1}\bigr) = \mathrm{id}, \tag{6.5.5}$$

because for each $m \in \mathbb{Z}_{>0}$ there exists only one isotonic surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$ such that $|g^{-1}p| = 1$ for each $p \in \mathbf{n}$, namely, $\mathrm{id} : \mathbf{m} \to \mathbf{m}$. Furthermore, $\lambda^{\mathrm{id}_\mathbf{m}} = \mathrm{id}$. Therefore, $(T^{\geqslant 1}, \Delta, \varepsilon)$ is a comonad.

The morphism of comonads $\eta : \mathrm{Id} \to T^{\geqslant 1} : {}^{\mathcal{V}}\mathscr{Q} \to {}^{\mathcal{V}}\mathscr{Q}$ is given by the transformation $\eta = \mathrm{in}_1 : \mathcal{A} \to T^{\geqslant 1}\mathcal{A} \in {}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{A}$. Clearly, $\eta \cdot \varepsilon = \mathrm{in}_1 \cdot \mathrm{pr}_1 = \mathrm{id}_\mathcal{A}$ and

$$\bigl(\mathcal{A} \xrightarrow{\eta} T^{\geqslant 1}\mathcal{A} \xrightarrow{\Delta} T^{\geqslant 1}T^{\geqslant 1}\mathcal{A}\bigr) = \bigl(\mathcal{A} \xrightarrow{\eta} T^{\geqslant 1}\mathcal{A} \xrightarrow{\eta} T^{\geqslant 1}T^{\geqslant 1}\mathcal{A}\bigr),$$

because there is only one surjection $g : \mathbf{1} \twoheadrightarrow \mathbf{n}$, namely, $\mathrm{id}_\mathbf{1}$. Thus, $(T^{\geqslant 1}, \Delta, \varepsilon, \eta)$ is an augmented comonad.

6.6 The $T^{\geqslant 1}$-comodule T.

The functor $T : {}^{\mathcal{V}}\mathscr{Q} \to {}^{\mathcal{V}}\mathscr{Q}$, given by the formula

$$T\mathcal{C} = \oplus_{m \geqslant 0} \otimes^{j \in \mathbf{m}} \mathcal{C},$$

has a $T^{\geqslant 1}$-comodule structure (coaction)

$$\widetilde{\Delta} = T^{\leqslant 1}(\Delta) : T \to TT^{\geqslant 1} = T \circ T^{\geqslant 1} = T^{\geqslant 1} \cdot T.$$

Indeed, applying $T^{\leqslant 1}$ to (6.5.4) we get

$$\bigl[T\mathcal{C} \xrightarrow{\widetilde{\Delta}} TT^{\geqslant 1}\mathcal{C} \xrightarrow{T(\Delta)} TT^{\geqslant 1}T^{\geqslant 1}\mathcal{C}\bigr] = \bigl[T\mathcal{C} \xrightarrow{\widetilde{\Delta}} TT^{\geqslant 1}\mathcal{C} \xrightarrow{\widetilde{\Delta}(T^{\geqslant 1})} TT^{\geqslant 1}T^{\geqslant 1}\mathcal{C}\bigr],$$

so coaction is coassociative. Applying $T^{\leqslant 1}$ to (6.5.5) we get

$$\bigl(T\mathcal{C} \xrightarrow{\widetilde{\Delta}} TT^{\geqslant 1}\mathcal{C} \xrightarrow{T(\varepsilon)} T\mathcal{C}\bigr) = \mathrm{id}_\mathcal{C},$$

so coaction is counital.

Notice that

$$TT^{\geqslant 1}\mathcal{C} = \oplus_{g:\mathbf{m} \twoheadrightarrow \mathbf{n}} \otimes^{p \in \mathbf{n}} \otimes^{j \in g^{-1}p} \mathcal{C},$$

where the summation extends over all isotonic surjections $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$. The components of $\widetilde{\Delta}$ are again given by (6.5.3).

In general,

$$T(T^{\geqslant 1})^{k-1}\mathcal{C} = \bigoplus_{\mathbf{m_1} \overset{g_1}{\twoheadrightarrow} \mathbf{m_2} \overset{g_2}{\twoheadrightarrow} \dots \overset{g_{k-1}}{\twoheadrightarrow} \mathbf{m_k}} \otimes^{j_k \in \mathbf{m_k}} \otimes^{j_{k-1} \in g_{k-1}^{-1} j_k} \dots \otimes^{j_1 \in g_1^{-1} j_2} \mathcal{C}.$$

The summation extends over composable sequences of isotonic surjections $g_p : \mathbf{m_p} \twoheadrightarrow \mathbf{m_{p+1}}$, $1 \leqslant p < k$, that is, over all plane staged trees with k stages.

6.7 Coalgebras over the comonad $T^{\geqslant 1}$. Let us describe coalgebras $\delta : \mathcal{C} \to T^{\geqslant 1}\mathcal{C}$ over $T^{\geqslant 1} : {}^{\mathcal{V}}\mathscr{Q} \to {}^{\mathcal{V}}\mathscr{Q}$ (see Definition 5.2). Since in equation $\bigl(\mathcal{C} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{C} \xrightarrow{\varepsilon} \mathcal{C}\bigr) = \mathrm{id}_{\mathcal{C}}$ the morphism ε is in ${}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$, we find that $\operatorname{Ob}\delta = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$, that is, $\delta \in {}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Therefore, the class of all $T^{\geqslant 1}$-coalgebras is the union over all $\mathscr{U}$-small sets S of classes of coalgebras over the comonad $T^{\geqslant 1} : {}^{\mathcal{V}}\mathscr{Q}/S \to {}^{\mathcal{V}}\mathscr{Q}/S$.

A $T^{\geqslant 1}$-coalgebra structure $\delta : \mathcal{C} \to T^{\geqslant 1}\mathcal{C}$ of an object $\mathcal{C}$ of ${}^{\mathcal{V}}\mathscr{Q}$ gives a sequence of morphisms $\delta\,\mathrm{pr}_1 = \overline{\Delta}^{(1)} = \mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $\delta\,\mathrm{pr}_2 = \overline{\Delta}^{(2)} = \overline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, $\delta\,\mathrm{pr}_3 = \overline{\Delta}^{(3)} = \overline{\Delta}(1\otimes\overline{\Delta}) = \overline{\Delta}(\overline{\Delta}\otimes 1) : \mathcal{C} \to \mathcal{C}^{\otimes 3}$, etc. In the other words, a $T^{\geqslant 1}$-coalgebra is a coassociative coalgebra $(\mathcal{C}, \overline{\Delta})$ in ${}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ together with a morphism $\delta : \mathcal{C} \to T^{\geqslant 1}\mathcal{C}$ such that

$$\delta\,\mathrm{pr}_k = \overline{\Delta}^{(k)} : \mathcal{C} \to \mathcal{C}^{\otimes k} \tag{6.7.1}$$

for all $k \geqslant 1$, where $\overline{\Delta}^{(k)}$ is an iteration of $\overline{\Delta}$.

The countable direct sums $\oplus_{i=1}^{\infty}\mathcal{A}_i$ in ${}^{\mathcal{V}}\mathscr{Q}/S$ are given by

$$(\oplus_{i=1}^{\infty}\mathcal{A}_i)(X,Y) = \bigoplus_{i=1}^{\infty}\bigl(\mathcal{A}_i(X,Y)\bigr), \qquad X, Y \in S.$$

For a given coassociative coalgebra $(\mathcal{C}, \overline{\Delta})$ there exists a morphism $\delta : \mathcal{C} \to T^{\geqslant 1}\mathcal{C}$ which satisfies (6.7.1) if and only if for all objects X, Y of $\mathcal{C}$ the map

$$\overline{\Delta}^{(\bullet)} = (\overline{\Delta}^{(1)}, \overline{\Delta}^{(2)}, \dots, \overline{\Delta}^{(k)}, \dots) : \mathcal{C}(X,Y) \to \prod_{k=1}^{\infty}\mathcal{C}^{\otimes k}(X,Y)$$

factors through $\beta : \oplus_{k=1}^{\infty}\mathcal{C}^{\otimes k}(X,Y) \hookrightarrow \prod_{k=1}^{\infty}\mathcal{C}^{\otimes k}(X,Y)$. In the cases $\mathcal{V} = \mathbf{gr}$ or $\mathcal{V} = \mathbf{dg}$ this condition means that for each $z \in \mathcal{C}(X,Y)^d$ there exists a positive integer k such that $z\overline{\Delta}^{(k)} = 0$ (thus, $z\overline{\Delta}^{(m)} = 0$ for all $m \geqslant k$). In these cases the following statement becomes obvious.

6.8 Proposition. *A $T^{\geqslant 1}$-coalgebra $\mathcal{C}$ in ${}^{\mathcal{V}}\mathscr{Q}$ is a coassociative coalgebra $(\mathcal{C}, \overline{\Delta} : \mathcal{C} \to \mathcal{C}\otimes\mathcal{C})$ in the monoidal category ${}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ such that for each pair X, Y of objects of $\mathcal{C}$*

$$\mathcal{C}(X,Y) = \operatorname*{colim}_{k\to\infty}\operatorname{Ker}\bigl(\overline{\Delta}^{(k)} : \mathcal{C}(X,Y) \to \mathcal{C}^{\otimes k}(X,Y)\bigr). \tag{6.8.1}$$

The class of coalgebras in $\mathscr{Q}$ which satisfy (6.8.1) is studied by Keller [Kel06a] under the name of *cocomplete cocategories.*

Proof. Let $(\mathcal{C}, \overline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C})$ be a coassociative coalgebra in the monoidal category ${}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$. Denote by $\mathrm{pr}_k : \prod_{m=1}^{\infty}\mathcal{C}^{\otimes m}(X,Y) \to \mathcal{C}^{\otimes k}(X,Y)$ and by $\mathrm{in}_k : \mathcal{C}^{\otimes k}(X,Y) \to$

$\prod_{m=1}^{\infty} \mathcal{C}^{\otimes m}(X,Y)$ the natural projection and injection. Consider the idempotent endomorphism $\phi_n = 1 - \sum_{k=1}^{n} \mathrm{pr}_k \cdot \mathrm{in}_k : \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y) \to \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)$. Its kernel is $\operatorname{Ker} \phi_n = \oplus_{k=1}^{n} \mathcal{C}^{\otimes k}(X,Y)$. We have

$$\coprod_{k\geqslant 1} \mathcal{C}^{\otimes k}(X,Y) \simeq \oplus_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y) = \operatorname*{colim}_{n\to\infty} \oplus_{k=1}^{n} \mathcal{C}^{\otimes k}(X,Y) = \operatorname*{colim}_{n\to\infty} \operatorname{Ker} \phi_n.$$

By definition of kernel there is a pull-back square

$$\begin{tikzcd}
\operatorname{Ker}(\overline{\Delta}^{(\bullet)} \cdot \phi_n) \arrow[r, "\overline{\Delta}^{(\bullet)}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Ker} \phi_n \arrow[d, hook] & \\
\mathcal{C}(X,Y) \arrow[r, "\overline{\Delta}^{(\bullet)}"] & \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y) \arrow[r, "\phi_n"] & \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y).
\end{tikzcd}$$

Since countable filtering colimits in $\mathcal{V}$ commute with finite limits it induces the pull-back square

$$\begin{tikzcd}
\operatorname*{colim}_{n\to\infty} \operatorname{Ker}(\overline{\Delta}^{(\bullet)} \cdot \phi_n) \arrow[r] \arrow[d, hook, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & \oplus_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y) \arrow[d, hook, "\beta"] \\
\mathcal{C}(X,Y) \arrow[r, "\overline{\Delta}^{(\bullet)}"'] \arrow[ur, dotted, "\delta"] & \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)
\end{tikzcd}$$

We claim that α is an isomorphism if and only if there exists $\delta : \mathcal{C}(X,Y) \to \oplus_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)$ such that $\overline{\Delta}^{(\bullet)} = \delta \cdot \beta$. Clearly, the first implies the second. Conversely, the second implies that

$$\begin{tikzcd}
\mathcal{C}(X,Y) \arrow[r, "\delta"] \arrow[d, equal] \arrow[dr, phantom, "\lrcorner", very near start] & \oplus_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y) \arrow[d, hook, "\beta"] \\
\mathcal{C}(X,Y) \arrow[r, "\overline{\Delta}^{(\bullet)}"] & \prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)
\end{tikzcd}$$

is another pull-back square, since β is a monomorphism. Thus, α is an isomorphism.

Therefore, $\mathcal{C}$ is a $T^{\geqslant 1}$-coalgebra if and only if α is an isomorphism, that is,

$$\begin{aligned}
\mathcal{C}(X,Y) &= \operatorname*{colim}_{n\to\infty} \operatorname{Ker}\Big((0,\dots,0,\overline{\Delta}^{(n+1)},\overline{\Delta}^{(n+2)},\dots) : \mathcal{C}(X,Y) \to \textstyle\prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)\Big) \\
&= \operatorname*{colim}_{n\to\infty} \operatorname{Ker}\Big((0,\dots,0,\overline{\Delta}^{(n+1)},0,0,\dots) : \mathcal{C}(X,Y) \to \textstyle\prod_{k=1}^{\infty} \mathcal{C}^{\otimes k}(X,Y)\Big).
\end{aligned}$$

This implies the proposition. $\square$

Definition 5.2 tells that morphisms of $T^{\geqslant 1}$-coalgebras are $\mathcal{V}$-quiver morphisms $f : \mathcal{C} \to \mathcal{D}$ such that

$$\big(\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{D}\big) = \big(\mathcal{C} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{C} \xrightarrow{T^{\geqslant 1}f} T^{\geqslant 1}\mathcal{D}\big).$$

Equivalently,

$$\big(\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{\overline{\Delta}} \mathcal{D} \otimes \mathcal{D}\big) = \big(\mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f\otimes f} \mathcal{D} \otimes \mathcal{D}\big). \tag{6.8.2}$$

This describes the category ${}^{\mathcal{V}}\mathcal{Q}_{T^{\geqslant 1}}$ of $T^{\geqslant 1}$-coalgebras.

6.9 Example. Any quiver $\mathcal{C} \in \operatorname{Ob}{}^{\mathcal{V}}\mathcal{Q}$ can be turned into a coassociative coalgebra with zero comultiplication $(\mathcal{C}, \overline{\Delta} = 0 : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C})$. The corresponding $T^{\geqslant 1}$-coalgebra $(\mathcal{C}, \mathrm{in}_1 : \mathcal{C} \to T^{\geqslant 1}\mathcal{C})$ is that of Example 5.11. It uses the augmentation homomorphism $\mathrm{in}_1 : \mathrm{Id} \to T^{\geqslant 1}$.

6.10 Remark. Let $\mathcal{C}$, $\mathcal{D}$ be $T^{\geqslant 1}$-coalgebras, and let $\overline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, $\overline{\Delta} : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ be the corresponding coassociative comultiplications. Let $f, g : \mathcal{C} \to \mathcal{D}$ be $T^{\geqslant 1}$-coalgebras morphisms such that $\operatorname{Ob} f = \operatorname{Ob} g$. Then the morphism of quivers $f + g : \mathcal{C} \to \mathcal{D}$ is well defined. It is a $T^{\geqslant 1}$-coalgebra morphism if and only if

$$(\mathcal{C} \xrightarrow{f+g} \mathcal{D} \xrightarrow{\overline{\Delta}} \mathcal{D} \otimes \mathcal{D}) = (\mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{(f+g)\otimes(f+g)} \mathcal{D} \otimes \mathcal{D}).$$

This equation is equivalent to

$$(\mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f\otimes g+g\otimes f} \mathcal{D} \otimes \mathcal{D}) = 0.$$

In particular, if the structure of a $T^{\geqslant 1}$-coalgebra on $\mathcal{C}$ is given by the augmentation morphism as in Example 6.9, then $\overline{\Delta} = 0 : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and the above equation holds independently of f, g. In this case, the sum of any two $T^{\geqslant 1}$-coalgebra morphisms is a $T^{\geqslant 1}$-morphism as well.

The following result is a corollary of Lemma 5.3. It is independently obtained for graded $\Bbbk$-linear quivers by Keller [Kel06a, Lemma 5.2], who omits the proof.

6.11 Corollary (cf. Keller [Kel06a])**.** *Let $\mathcal{C}$ be a $T^{\geqslant 1}$-coalgebra, and let $\mathcal{A}$ be a $\mathcal{V}$-quiver. Then there is a natural bijection*

$${}^{\mathcal{V}}\mathcal{Q}_{T^{\geqslant 1}}(\mathcal{C}, T^{\geqslant 1}\mathcal{A}) \to {}^{\mathcal{V}}\mathcal{Q}(\mathcal{C}, \mathcal{A}), \qquad (f : \mathcal{C} \to T^{\geqslant 1}\mathcal{A}) \mapsto \big(\mathcal{C} \xrightarrow{f} T^{\geqslant 1}\mathcal{A} \xrightarrow{\mathrm{pr}_1} \mathcal{A}\big).$$

Given a coassociative coalgebra $(\mathcal{C}, \overline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C})$ in the monoidal category ${}^{\mathcal{V}}\mathcal{Q}/\operatorname{Ob}\mathcal{C}$ we construct a coassociative coalgebra structure Δ' on the object $T^{\leqslant 1}\mathcal{C} = \Bbbk\mathcal{C} \oplus \mathcal{C}$ of the category ${}^{\mathcal{V}}\mathcal{Q}/\operatorname{Ob}\mathcal{C}$, namely,

$$\Delta' = \big(T^{\leqslant 1}\mathcal{C} \xrightarrow{\mathrm{pr}_1} \mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mathrm{in}_1 \otimes \mathrm{in}_1} T^{\leqslant 1}\mathcal{C} \otimes T^{\leqslant 1}\mathcal{C}\big). \tag{6.11.1}$$

The above Δ' satisfies conditions

$$\eta\Delta' = 0, \qquad \Delta'(1 \otimes \varepsilon) = 0, \qquad \Delta'(\varepsilon \otimes 1) = 0, \tag{6.11.2}$$

where $\eta = \mathrm{in}_0 : \Bbbk\mathcal{C} \to T^{\leqslant 1}\mathcal{C}$, $\varepsilon = \mathrm{pr}_0 : T^{\leqslant 1}\mathcal{C} \to \Bbbk\mathcal{C}$.

On the other hand, if $\varepsilon : \mathcal{A} \rightleftarrows \Bbbk\mathcal{A} : \eta$ satisfies $\eta\varepsilon = \mathrm{id}_{\Bbbk\mathcal{A}}$ in ${}^{\mathcal{V}}\!\mathcal{Q}/\operatorname{Ob}\mathcal{A}$, then $\mathcal{A}$ decomposes as $\Bbbk\mathcal{C} \oplus \mathcal{C}$. Any coassociative comultiplication $\Delta' : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ in ${}^{\mathcal{V}}\!\mathcal{Q}/\operatorname{Ob}\mathcal{A}$ which satisfies equations (6.11.2) has a unique presentation of the form

$$\Delta' = \bigl(\mathcal{A} \xrightarrow{\mathrm{pr}_2} \mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{\mathrm{in}_2 \otimes \mathrm{in}_2} \mathcal{A} \otimes \mathcal{A}\bigr)$$

with

$$\overline{\Delta} = \Delta'\big|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}. \tag{6.11.3}$$

Furthermore, such Δ' gives rise to the coassociative comultiplication

$$\Delta_0 = \Delta' + \mathrm{id}_{\mathcal{A}} \otimes \eta + \eta \otimes \mathrm{id}_{\mathcal{A}} - \varepsilon\eta \otimes \eta, \tag{6.11.4}$$

so that $(\mathcal{A}, \Delta_0, \varepsilon)$ is a counital coassociative coalgebra in ${}^{\mathcal{V}}\!\mathcal{Q}/\operatorname{Ob}\mathcal{A}$, and $\eta : \Bbbk\mathcal{A} \to \mathcal{A}$ is a homomorphism of counital coalgebras. In other words, the object $(\mathcal{A}, \varepsilon, \eta)$ is a counital coassociative coalgebra in ${}^{\mathcal{V}}\!\mathcal{Q}_{bu}/\operatorname{Ob}\mathcal{A}$. All these statements are straightforward.

Vice versa, a counital coassociative coalgebra in ${}^{\mathcal{V}}\!\mathcal{Q}_{bu}/\operatorname{Ob}\mathcal{A}$ can be interpreted as a homomorphism of counital coalgebras $\eta : \Bbbk\mathcal{A} \to (\mathcal{A}, \Delta_0, \varepsilon)$ in ${}^{\mathcal{V}}\!\mathcal{Q}/\operatorname{Ob}\mathcal{A}$, that is, an augmented counital coassociative coalgebra $(\mathcal{A}, \Delta_0, \varepsilon, \eta)$. It gives rise to the coassociative comultiplication

$$\Delta' = \Delta_0 - \mathrm{id}_{\mathcal{A}} \otimes \eta - \eta \otimes \mathrm{id}_{\mathcal{A}} + \varepsilon\eta \otimes \eta \tag{6.11.5}$$

which satisfies equations (6.11.2). It corresponds in turn to the coassociative comultiplication $\overline{\Delta} = \Delta_0\big|_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ in ${}^{\mathcal{V}}\!\mathcal{Q}/\operatorname{Ob}\mathcal{C}$.

Morphisms $f : \mathcal{A} \to \mathcal{B}$ of considered coalgebras are required to preserve comultiplication:

$$\bigl(\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{\Delta_0} \mathcal{B} \otimes \mathcal{B}\bigr) = \bigl(\mathcal{A} \xrightarrow{\Delta_0} \mathcal{A} \otimes \mathcal{A} \xrightarrow{f \otimes f} \mathcal{B} \otimes \mathcal{B}\bigr)$$

(and similarly for Δ'), and to preserve ε, η:

$$\bigl(\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{\varepsilon} \Bbbk\mathcal{B}\bigr) = \bigl(\mathcal{A} \xrightarrow{\varepsilon} \Bbbk\mathcal{A} \xrightarrow{\Bbbk f} \Bbbk\mathcal{B}\bigr),$$
$$\bigl(\Bbbk\mathcal{A} \xrightarrow{\eta} \mathcal{A} \xrightarrow{f} \mathcal{B}\bigr) = \bigl(\Bbbk\mathcal{A} \xrightarrow{\Bbbk f} \Bbbk\mathcal{B} \xrightarrow{\eta} \mathcal{B}\bigr).$$

This gives categories of coalgebras of various kinds. In particular, we have the category $\mathrm{ac}{}^{\mathcal{V}}\!\mathcal{Q}$ of augmented counital coassociative coalgebras $(\mathcal{A}, \Delta_0, \varepsilon, \eta)$ in ${}^{\mathcal{V}}\!\mathcal{Q}$, whose morphisms are required to satisfy all three identities.

We summarize the above remarks in the following lemma.

6.12 Lemma. *Formulas (6.11.1), (6.11.3)–(6.11.5) define equivalences* $f \mapsto T^{\leqslant 1}f$, $\mathcal{C} \mapsto (\mathrm{pr}_0 : T^{\leqslant 1}\mathcal{C} \rightleftarrows T^0\mathcal{C} : \mathrm{in}_0)$ *of the category* $\mathrm{c}^{\mathcal{V}}\mathcal{Q}$ *of coassociative coalgebras* $(\mathcal{C}, \overline{\Delta})$ *in* ${}^{\mathcal{V}}\mathcal{Q}$ *with the category* $\mathrm{ac}^{\mathcal{V}}\mathcal{Q}$ *and the following two categories*

$$\{(\mathcal{A}, \Delta', \varepsilon, \eta) \in {}^{\mathcal{V}}\mathcal{Q} \mid \eta\varepsilon = 1,\ \eta\Delta' = 0,\ \Delta'(1 \otimes \varepsilon) = 0,\ \Delta'(\varepsilon \otimes 1) = 0,\ \Delta' - \textit{coassociative}\},$$

$$\mathrm{c}^{\mathcal{V}}\mathcal{Q}_{bu} = \{\textit{counital coassociative coalgebras } \mathcal{A} \textit{ in } {}^{\mathcal{V}}\mathcal{Q}_{bu}\}.$$

In particular, categories of coalgebras with fixed set of objects of the four above types are equivalent.

This statement is dual to the usual procedure of adjoining a unit to an associative algebra.

6.13 Example. Let $\mathcal{B}$ be a $\mathcal{V}$-quiver. Then $\mathcal{A} = T\mathcal{B} = T^{\leqslant 1}T^{\geqslant 1}\mathcal{B}$ has comultiplication $\Delta_0 : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, whose iterations

$$\Delta_0^{(0)} = \mathrm{pr}_0 : T\mathcal{B} \to \Bbbk\mathcal{B}, \qquad \Delta_0^{(1)} = \mathrm{id}_{\mathcal{A}}, \qquad \Delta_0^{(2)} = \Delta_0,$$
$$\Delta_0^{(r)} = \Delta_0(\Delta_0^{(r-1)} \otimes 1) : \mathcal{A} \to T^r\mathcal{A} \quad \text{for } r \geqslant 2$$

are described as follows. Their matrix coefficients are given by the isomorphisms $(\Delta_0^{(r)})_n^{m_1 \dots m_r} = \lambda^f : T^n\mathcal{B} \to \otimes^{i \in \mathbf{r}} T^{m_i}\mathcal{B}$, where $(f : \mathbf{n} \to \mathbf{r}) \in \mathcal{O}$ has $|f^{-1}i| = m_i$ for all $i \in \mathbf{r}$. The $\mathcal{V}$-quiver $\mathcal{A} = T\mathcal{B}$ also has the non-counital comultiplication $\Delta' : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (see (6.11.1)) whose iterations can be defined as $\Delta'^{(0)} = \mathrm{pr}_0 : T\mathcal{B} \to \Bbbk\mathcal{B}$, $\Delta'^{(1)} = \mathrm{id}_{\mathcal{A}} - \mathrm{pr}_0 \cdot \mathrm{in}_0 : T\mathcal{B} \to T\mathcal{B}$, $\Delta'^{(2)} = \Delta'$, $\Delta'^{(r)} = \Delta'(\Delta'^{(r-1)} \otimes 1) : \mathcal{A} \to T^r\mathcal{A}$ for $r \geqslant 2$. The reason for such a definition of $\Delta'^{(0)}$, $\Delta'^{(1)}$ is that matrix coefficients of all $\Delta'^{(r)}$ are given by the isomorphisms $(\Delta'^{(r)})_n^{m_1 \dots m_r} = \lambda^f : T^n\mathcal{B} \to \otimes^{i \in \mathbf{r}} T^{m_i}\mathcal{B}$, if f is surjective, where $(f : \mathbf{n} \to \mathbf{r}) \in \mathcal{O}$ has $|f^{-1}i| = m_i$ for all $i \in \mathbf{r}$. If f is not surjective, the coefficient $(\Delta'^{(r)})_n^{m_1 \dots m_r}$ vanishes.

6.14 Proposition. *The category* $\mathrm{ac}^{\mathcal{V}}\mathcal{Q}$ *of augmented counital coassociative coalgebras in* ${}^{\mathcal{V}}\mathcal{Q}$ *has a symmetric Monoidal structure. The tensor product of a family* $((\mathcal{A}_i, \Delta_0, \varepsilon, \eta))_{i \in I}$ *is* $\boxtimes^{i \in I}\mathcal{A}_i$ *equipped with the operations*

$$\Delta_0 = \bigl[\boxtimes^{i \in I}\mathcal{A}_i \xrightarrow{\boxtimes^{i \in I}\Delta_0} \boxtimes^{i \in I}(\mathcal{A}_i \otimes \mathcal{A}_i) \xrightarrow{\overline{\varkappa}^{-1}} (\boxtimes^{i \in I}\mathcal{A}_i) \otimes (\boxtimes^{i \in I}\mathcal{A}_i)\bigr],$$
$$\varepsilon = \bigl[\boxtimes^{i \in I}\mathcal{A}_i \xrightarrow{\boxtimes^{i \in I}\varepsilon} \boxtimes^{i \in I} T^0\mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} T^0 \boxtimes^{i \in I} \mathcal{A}_i\bigr],$$
$$\eta = \bigl[T^0 \boxtimes^{i \in I} \mathcal{A}_i \xrightarrow{\overline{\varkappa}} \boxtimes^{i \in I} T^0\mathcal{A}_i \xrightarrow{\boxtimes^{i \in I}\eta} \boxtimes^{i \in I} \mathcal{A}_i\bigr].$$

The isomorphisms $\lambda^f : \boxtimes^{i \in I}\mathcal{A}_i \to \boxtimes^{j \in J} \boxtimes^{i \in f^{-1}j} \mathcal{A}_i$ *are those of* ${}^{\mathcal{V}}\mathcal{Q}_p$. *The symmetric Monoidal category* $(\mathrm{ac}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I, \lambda^f)$ *is denoted* $\mathrm{ac}^{\mathcal{V}}\mathcal{Q}_p$.

Proof. The proof is by direct computations. It is left as an exercise to the reader. □

6.15 Lax symmetric Monoidal comonad $T^{\geqslant 1}$. Now we combine the lax symmetric Monoidal structure of the functor $T^{\geqslant 1}$ with its comonad structure.

6.16 Proposition. *The data $((T^{\geqslant 1},\tau^I),\Delta,\varepsilon) : ({}^{\mathcal V}\mathcal{Q},\boxtimes^I_u,\lambda^f_u) \to ({}^{\mathcal V}\mathcal{Q},\boxtimes^I_u,\lambda^f_u)$ constitute a lax symmetric Monoidal comonad.*

Proof. Let us prove that $\varepsilon = \mathrm{pr}_1 : T^{\geqslant 1} \to \mathrm{Id} : {}^{\mathcal V}\mathcal{Q}_u \to {}^{\mathcal V}\mathcal{Q}_u$ is a Monoidal transformation. Equation (2.20.1) reads:

$$\boxtimes^I_u\varepsilon = \bigl[\boxtimes^I_u(T^{\geqslant 1}\mathcal{A}_i) \xrightarrow{\tau^I} T^{\geqslant 1}(\boxtimes^I_u\mathcal{A}_i) \xrightarrow{\varepsilon} \boxtimes^I_u \mathcal{A}_i\bigr]$$

for all quivers $\mathcal{A}_i$. Applying $T^{\leqslant 1}$ we rewrite this equation in equivalent form

$$\boxtimes^{i\in I}\mathrm{pr}_{0,1} = \bigl[\boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\widetilde{\tau}^I} T(\boxtimes^{i\in I}_u\mathcal{A}_i) \xrightarrow{\mathrm{pr}_{0,1}} T^{\leqslant 1}\boxtimes^{i\in I}_u \mathcal{A}_i \xrightarrow{(\vartheta^I)^{-1}} \boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i\bigr],$$

where $\mathrm{pr}_{0,1} = T^{\leqslant 1}\varepsilon : T\mathcal{A} \to T^{\leqslant 1}\mathcal{A}$ is the natural projection. That is, for all $(m_i) \in \mathbb{Z}^I_{\geqslant 0}$ and $m \in \{0,1\}$

$$\widetilde{\tau}^I\cdot\mathrm{pr}_m = \bigl(\boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \xrightarrow{\boxtimes^I\mathrm{pr}_{0,1}} \boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i \xrightarrow{\vartheta^I} T^{\leqslant 1}\boxtimes^{i\in I}_u \mathcal{A}_i \xrightarrow{\mathrm{pr}_m} T^m \boxtimes^{i\in I}_u \mathcal{A}_i\bigr).$$

The target decomposes over subsets $S \subset I\times\mathbf{m}$ such that $\mathrm{pr}_2 S = \mathbf{m}$. The non-trivial components of $\widetilde{\tau}^I$ satisfy the equation $m_i = |S\cap(\{i\}\times\mathbf{m})|$. In this case the inequalities $m_i \leqslant m \leqslant |S| \leqslant \sum_{i\in I} m_i$ imply the following. If some of m_i are greater than 1, then both sides vanish. If $m_i \leqslant 1$ for all $i\in I$, the equation can be written in short form

$$\widetilde{\tau}^I = \vartheta^I : \boxtimes^{i\in I}T^{m_i}\mathcal{A}_i \to T^m \boxtimes^{i\in I}_u \mathcal{A}_i.$$

If all $m_i = 0$, then the only non-vanishing component of $\widetilde{\tau}^I$ corresponds to $m=0$, $S=\varnothing$, and both sides are identity maps. If all $m_i\in\{0,1\}$ and $m_j = 1$ for some $j\in I$, then the only non-vanishing component from (6.2.1) corresponds to $m=1$, and $S\subset I\times\mathbf{1}$ such that $i\times 1\in S$ if and only if $m_i = 1$. Such component of $\widetilde{\tau}^I$ is an embedding, that coincides with ϑ^I. Therefore, ε is a Monoidal transformation.

Let us prove that comultiplication $\Delta : T^{\geqslant 1} \to T^{\geqslant 1}T^{\geqslant 1} : {}^{\mathcal V}\mathcal{Q}_u \to {}^{\mathcal V}\mathcal{Q}_u$ is a Monoidal transformation. Equivalently, coaction $\widetilde{\Delta} \overset{\text{def}}{=} T^{\leqslant 1}\Delta : T \to T\circ T^{\geqslant 1} : {}^{\mathcal V}\mathcal{Q}_u \to {}^{\mathcal V}\mathcal{Q}_p$ is a Monoidal transformation. Equation (2.20.1) takes the form

$$\begin{tikzcd}
{}^{\mathcal V}\mathcal{Q}^I \arrow[r, bend left, "T^I"] \arrow[r, bend right, "(T^{\geqslant 1})^I\cdot T^I"'] \arrow[r, phantom, "\widetilde{\Delta}^I\Downarrow"] \arrow[d, "\boxtimes^I_u"'] & {}^{\mathcal V}\mathcal{Q}^I \arrow[d, "\boxtimes^I"] \arrow[dl, Rightarrow, "\psi^I"'] \\
{}^{\mathcal V}\mathcal{Q} \arrow[r, "T^{\geqslant 1}\cdot T"'] & {}^{\mathcal V}\mathcal{Q}
\end{tikzcd}
=
\begin{tikzcd}
{}^{\mathcal V}\mathcal{Q}^I \arrow[r, "T^I"] \arrow[d, "\boxtimes^I_u"'] & {}^{\mathcal V}\mathcal{Q}^I \arrow[d, "\boxtimes^I"] \arrow[dl, Rightarrow, "\widetilde{\tau}^I"] \\
{}^{\mathcal V}\mathcal{Q} \arrow[r, bend left, "T"] \arrow[r, bend right, "T^{\geqslant 1}\cdot T"'] \arrow[r, phantom, "\widetilde{\Delta}\Downarrow"] & {}^{\mathcal V}\mathcal{Q}
\end{tikzcd}
\tag{6.16.1}$$

for each $I \in \operatorname{Ob} \mathcal{S}$, where the transformation

$$\psi^I = \begin{tikzcd} {}^{\mathcal{V}}\mathcal{Q}^I \arrow[r, "(T^{\geqslant 1})^I"] \arrow[d, "\boxtimes^I_u"'] & {}^{\mathcal{V}}\mathcal{Q}^I \arrow[r, "T^I"] \arrow[d, "\boxtimes^I_u"'] \arrow[dl, Rightarrow, "\tau^I"'] & {}^{\mathcal{V}}\mathcal{Q}^I \arrow[d, "\boxtimes^I"] \arrow[dl, Rightarrow, "\tilde{\tau}^I"'] \\ {}^{\mathcal{V}}\mathcal{Q} \arrow[r, "T^{\geqslant 1}"'] & {}^{\mathcal{V}}\mathcal{Q} \arrow[r, "T"'] & {}^{\mathcal{V}}\mathcal{Q} \end{tikzcd}$$

turns $(T^{\geqslant 1} \cdot T, \psi^I)$ into a lax symmetric Monoidal functor.

6.17 Lemma. *The natural transformation*

$$\psi^I = \big[\boxtimes^I (TT^{\geqslant 1}\mathcal{A}_i) \xrightarrow{\tilde{\tau}^I} T(\boxtimes^I_u (T^{\geqslant 1}\mathcal{A}_i)) \xrightarrow{T(\tau^I)} TT^{\geqslant 1}(\boxtimes^I_u \mathcal{A}_i)\big]$$

is the map

$$\boxtimes^{i\in I} \bigoplus_{g_i:\mathbf{m_i}\twoheadrightarrow\mathbf{n_i}} \otimes^{t\in\mathbf{n_i}} T^{g_i^{-1}t}\mathcal{A}_i \xrightarrow{\tilde{\tau}^I} \bigoplus_{n\geqslant 0} \otimes^{q\in\mathbf{n}} \bigoplus_{r_i(q)\geqslant 0}^{\forall q\, \exists i\, r_i(q)>0} \boxtimes^{i\in I} T^{r_i(q)}\mathcal{A}_i$$
$$\xrightarrow{T(\tau^I)} \bigoplus_{g:\mathbf{m}\twoheadrightarrow\mathbf{n}} \otimes^{q\in\mathbf{n}} \otimes^{p\in g^{-1}q} \bigoplus_{S\subset I\times\mathbf{m}}^{\mathrm{pr}_2 S=\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i.$$

For each summand of the target there is only one summand of the source mapped into it non-trivially. Namely, for each isotonic surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$ and subset $S \subset I \times \mathbf{m}$ such that $\mathrm{pr}_2 S = \mathbf{m}$, there is a unique family of isotonic surjections $(g_i : \mathbf{m_i} \twoheadrightarrow \mathbf{n_i})_{i\in I}$ and the permutation isomorphism

$$\Big[\boxtimes^{i\in I} \otimes^{t\in\mathbf{n_i}} T^{g_i^{-1}t}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\lambda^{[(1\times g)S]\cap(\{i\}\times\mathbf{n})\hookrightarrow\mathbf{n}}}$$
$$\boxtimes^{i\in I} \otimes^{q\in\mathbf{n}} T^{\chi((i,q)\in(1\times g)S)} \otimes^{S\cap(\{i\}\times g^{-1}q)} \mathcal{A}_i \xrightarrow{\varkappa^{-1}} \otimes^{q\in\mathbf{n}} \boxtimes^{i\in I} \otimes^{S\cap(\{i\}\times g^{-1}q)} \mathcal{A}_i$$
$$\xrightarrow{\otimes^{q\in\mathbf{n}}\boxtimes^{i\in I}\lambda^{S\cap(\{i\}\times g^{-1}q)\hookrightarrow g^{-1}q}} \otimes^{q\in\mathbf{n}} \boxtimes^{i\in I} \otimes^{p\in g^{-1}q} T^{\chi((i,p)\in S)}\mathcal{A}_i$$
$$\xrightarrow{\otimes^{q\in\mathbf{n}}\varkappa^{-1}} \otimes^{q\in\mathbf{n}} \otimes^{p\in g^{-1}q} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i\Big] \tag{6.17.1}$$

giving all non-trivial matrix elements of ψ^I.

Proof. For each summand of the third term, labeled by $(g : \mathbf{m} \twoheadrightarrow \mathbf{n}, S)$, there is only one summand of the second term non-trivially mapped into it, namely, the one labeled by $(n, (r_i(q))_{q\in\mathbf{n}})$, where $r_i(q) = |S \cap (\{i\} \times g^{-1}q)|$. For each summand of the second term, labeled by $(n, (r_i(q))_{q\in\mathbf{n}})$, there is only one summand of the first term non-trivially mapped into it, namely, the one labeled by

$$\Big(m_i = \sum_{q\in\mathbf{n}} r_i(q), n_i = |\{q \in \mathbf{n} \mid r_i(q) > 0\}|, g_i\Big)_{i\in I},$$

where

$$g_i : \mathbf{m_i} \simeq S \cap (\{i\} \times \mathbf{m}) \xrightarrow{1\times g} \big[(1 \times g)S\big] \cap (\{i\} \times \mathbf{n}) \simeq \mathbf{n_i}$$

corresponds to the partition of $m_i = \sum_{q\in\mathbf{n}} r_i(q)$ into n_i non-vanishing summands $r_i(q)$. The discussed matrix element is given by (6.17.1). □

Equation (6.16.1) that we are proving can be rewritten as follows:

$$\begin{tikzcd}
\boxtimes^I(T\mathcal{A}_i) \arrow[r, "\widetilde{\tau}^I"] \arrow[d, "\boxtimes^I(\widetilde{\Delta})_i"'] \arrow[dr, phantom, "="] & T(\boxtimes^I_u \mathcal{A}_i) \arrow[d, "\widetilde{\Delta}"] \\
\boxtimes^I(TT^{\geqslant 1}\mathcal{A}_i) \arrow[r, "\psi^I"] & TT^{\geqslant 1}(\boxtimes^I_u \mathcal{A}_i)
\end{tikzcd}$$

where $\widetilde{\Delta} = T^{\leqslant 1}(\Delta)$. For each summand of the target labeled by $\big(g : \mathbf{m} \twoheadrightarrow \mathbf{n}, S \subset I \times \mathbf{m} \mid \mathrm{pr}_2 S = \mathbf{m}\big)$, there is a unique summand of the source mapped non-trivially to it by $\psi^I \circ (\boxtimes^I(\widetilde{\Delta})_i)$, namely, the one labeled by $(m_i)_{i\in I}$ in

$$\boxtimes^I(T\mathcal{A}_i) = \bigoplus_{(m_i)\in\mathbb{Z}^I_{\geqslant 0}} \boxtimes^{i\in I} T^{m_i}\mathcal{A}_i.$$

The corresponding matrix element is a functorial isomorphism. Its explicit form can be obtained by Lemma 6.17 and by formula (6.5.3) for matrix elements of $\widetilde{\Delta}$.

For the same summand of the target there is a unique summand of

$$T(\boxtimes^I_u \mathcal{A}_i) = \bigoplus_{m=0}^{\infty} \bigoplus_{S\subset I\times\mathbf{m}}^{\mathrm{pr}_2 S=\mathbf{m}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I}\, T^{\chi((i,p)\in S)}\mathcal{A}_i,$$

namely, the one labeled by (m, S) which is mapped to it by $\widetilde{\Delta}$ non-trivially. The corresponding matrix element is a functorial isomorphism. Furthermore, there is a unique summand of the source mapped non-trivially to the previous one by $\widetilde{\tau}^I$, namely, the one labeled by $(m'_i)_{i\in I}$, where

$$m'_i = |S \cap (\{i\} \times \mathbf{m})| = \sum_{q\in\mathbf{n}} |S \cap (\{i\} \times g^{-1}q)| = \sum_{q\in\mathbf{n}} r_i(q) = m_i.$$

The non-trivial matrix elements of both sides of equation $\psi^I \circ (\boxtimes^I(\widetilde{\Delta})_i) = \widetilde{\Delta} \circ \widetilde{\tau}^I$ are

functorial isomorphisms presented below:

$$
\begin{aligned}
\big[\boxtimes^{i\in I} \otimes^{\mathbf{m_i}} \mathcal{A}_i &\xrightarrow{\boxtimes^{i\in I}\lambda^{g_i}} \boxtimes^{i\in I} \otimes^{t\in\mathbf{n_i}} T^{g_i^{-1}t}\mathcal{A}_i \xrightarrow{\boxtimes^{i\in I}\lambda^{[(1\times g)S]\cap(\{i\}\times\mathbf{n})\hookrightarrow\mathbf{n}}} \\
&\boxtimes^{i\in I} \otimes^{q\in\mathbf{n}} T^{\chi((i,q)\in(1\times g)S)} \otimes^{S\cap(\{i\}\times g^{-1}q)} \mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{q\in\mathbf{n}} \boxtimes^{i\in I} \otimes^{S\cap(\{i\}\times g^{-1}q)} \mathcal{A}_i \\
&\xrightarrow{\otimes^{q\in\mathbf{n}}\boxtimes^{i\in I}\lambda^{S\cap(\{i\}\times g^{-1}q)\hookrightarrow g^{-1}q}} \otimes^{q\in\mathbf{n}} \boxtimes^{i\in I} \otimes^{p\in g^{-1}q} T^{\chi((i,p)\in S)}\mathcal{A}_i \\
&\xrightarrow{\otimes^{q\in\mathbf{n}}\overline{\varkappa}^{-1}} \otimes^{q\in\mathbf{n}} \otimes^{p\in g^{-1}q} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i\big] \\
= \big[\boxtimes^{i\in I} \otimes^{\mathbf{m_i}} \mathcal{A}_i &\xrightarrow{\boxtimes^{i\in I}\lambda^{S\cap(\{i\}\times\mathbf{m})\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} \\
&\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i \xrightarrow{\lambda^g} \otimes^{q\in\mathbf{n}} \otimes^{p\in g^{-1}q} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i\big].
\end{aligned}
$$

The above maps can be specified on objects $X_i^n \in \operatorname{Ob}\mathcal{A}_i$ for $i \in I$, $0 \leqslant n \leqslant m_i$. Using objects $A_i^n = \mathcal{A}_i(X_i^{n-1}, X_i^n)$ of $\mathcal{V}$, $1 \leqslant n \leqslant m_i$, one can rewrite the above equation in terms of symmetric Monoidal category $(\mathcal{V}, \otimes, \lambda^f)$ only. By Lemma 2.33 and Remark 2.34 the obtained equation between natural transformations holds automatically. Therefore, all matrix elements of $\psi^I \circ (\boxtimes^I(\widetilde{\Delta})_i)$ and $\widetilde{\Delta}\circ\widetilde{\tau}^I$ coincide, and the proposition is proven. □

6.18 Corollary. *The category ${}^{\mathcal{V}}\mathscr{Q}_{uT^{\geqslant 1}} = ({}^{\mathcal{V}}\mathscr{Q}_{T^{\geqslant 1}}, \boxtimes_u^I, \lambda_u^f)$ of $T^{\geqslant 1}$-coalgebras in ${}^{\mathcal{V}}\mathscr{Q}$ is symmetric Monoidal.*

We shall often use $T^{\geqslant 1}$-coalgebras in the sequel. Let us describe in more familiar terms the symmetric Monoidal structure of the category ${}^{\mathcal{V}}\mathscr{Q}_{uT^{\geqslant 1}}$ obtained in Remark 5.12. As noticed in Section 6.7 a $T^{\geqslant 1}$-coalgebra $\delta : \mathcal{C} \to T^{\geqslant 1}\mathcal{C}$ in ${}^{\mathcal{V}}\mathscr{Q}$ determines a coassociative coalgebra $\overline{\Delta} = \delta \cdot \mathrm{pr}_2 : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$. From $\overline{\Delta}$ we obtain via Lemma 6.12 a counital coassociative comultiplication $\Delta_0 : T^{\leqslant 1}\mathcal{C} \to T^{\leqslant 1}\mathcal{C} \otimes T^{\leqslant 1}\mathcal{C}$ on $T^{\leqslant 1}\mathcal{C} = (\varepsilon = \mathrm{pr}_0 : \Bbbk\mathcal{C} \oplus \mathcal{C} \rightleftarrows \Bbbk\mathcal{C} : \mathrm{in}_0 = \eta)$ with the counit ε. In particular, a $T^{\geqslant 1}$-coalgebra $\mathcal{C}$ in ${}^{\mathcal{V}}\mathscr{Q}$ gives rise to a counital coassociative coalgebra $T^{\leqslant 1}\mathcal{C}$ in ${}^{\mathcal{V}}\mathscr{Q}/\operatorname{Ob}\mathcal{C}$ with the augmentation coalgebra homomorphism $\eta : \Bbbk\mathcal{C} \to T^{\leqslant 1}\mathcal{C}$.

Using decompositions $T^{\leqslant 1}\mathcal{C} = \Bbbk\mathcal{C} \oplus \mathcal{C}$ and

$$T^{\leqslant 1}\mathcal{C} \otimes T^{\leqslant 1}\mathcal{C} = \Bbbk\mathcal{C} \otimes \Bbbk\mathcal{C} \oplus \Bbbk\mathcal{C} \otimes \mathcal{C} \oplus \mathcal{C} \otimes \Bbbk\mathcal{C} \oplus \mathcal{C} \otimes \mathcal{C}$$

we can represent the comultiplication Δ_0 in $T^{\leqslant 1}\mathcal{C}$ by the matrix

$$\Delta_0 = \begin{pmatrix} \lambda^{\varnothing\to\mathbf{2}} & 0 & 0 & 0 \\ 0 & \lambda^{\cdot\,\mathsf{I}} & \lambda^{\mathsf{I}\,\cdot} & \overline{\Delta} \end{pmatrix}.$$

6.19 Proposition. *The full and faithful functor $T^{\leqslant 1} : {}^{\mathcal{V}}\mathscr{Q}_{T^{\geqslant 1}} \to \mathrm{ac}{}^{\mathcal{V}}\mathscr{Q}$ gives rise to a symmetric Monoidal functor $(T^{\leqslant 1}, \vartheta^I) : {}^{\mathcal{V}}\mathscr{Q}_{uT^{\geqslant 1}} \to \mathrm{ac}{}^{\mathcal{V}}\mathscr{Q}_p$.*

Proof. The considered functor is a composition of the full and faithful embedding $\iota : {}^{\mathcal{V}}\mathscr{Q}_{T^{\geqslant 1}} \hookrightarrow \mathrm{c}{}^{\mathcal{V}}\mathscr{Q}$ and the equivalence $T^{\leqslant 1} : \mathrm{c}{}^{\mathcal{V}}\mathscr{Q} \to \mathrm{ac}{}^{\mathcal{V}}\mathscr{Q}$ due to Lemma 6.12. Thus, it is full and faithful itself.

Let $\mathcal{C}_i$ be $T^{\geqslant 1}$-coalgebras for $i \in I$. The quiver $\boxtimes_u^{i\in I}\mathcal{C}_i$ has a structure of a $T^{\geqslant 1}$-coalgebra, described in Remark 5.12. The augmented counital coalgebra $(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i, \Delta_0, \varepsilon, \eta)$ is associated with it. Let us prove that $\vartheta^I : \boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \to T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i$ is an isomorphism in $\mathrm{ac}^{\mathcal{V}}\mathscr{Q}$, where the first augmented counital coalgebra structure $\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i, \Delta_0, \varepsilon, \eta)$ is obtained via Proposition 6.14.

We have $\varepsilon = \bigl(T^{\leqslant 1}\mathcal{C}_i \xrightarrow{\mathrm{pr}_0} T^0\mathcal{C}_i = T^0T^{\leqslant 1}\mathcal{C}_i\bigr)$ and $\eta = \bigl(T^0T^{\leqslant 1}\mathcal{C}_i = T^0\mathcal{C}_i \xrightarrow{\mathrm{in}_0} T^{\leqslant 1}\mathcal{C}_i\bigr)$. First of all ϑ^I preserves the counit ε:

$$\begin{tikzcd}
\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \arrow[rr, "\vartheta^I"] \arrow[d, "\boxtimes^I \mathrm{pr}_0"'] \arrow[drr, "\varepsilon"] & & T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, "\mathrm{pr}_0"] \arrow[dr, "\varepsilon"] \arrow[d, phantom, "="] & T^0\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[d, equal] \\
\boxtimes^{i\in I}T^0\mathcal{C}_i \arrow[r, equal] & \boxtimes^{i\in I}T^0T^{\leqslant 1}\mathcal{C}_i \arrow[r, "\overline{\varkappa}^{-1}"'] & T^0\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \arrow[r, equal, "T^0\vartheta^I"'] & T^0T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i
\end{tikzcd}$$

Indeed, both paths give identity map on $\boxtimes^{i\in I}T^0\mathcal{C}_i$, and vanish on other direct summands of the source. Inverting arrows and replacing ε with η, pr_0 with in_0, we conclude that ϑ^I preserves the augmentation η.

Let us prove that ϑ^I agrees with the comultiplication Δ_0. This is expressed by the left pentagon of the diagram:

$$\begin{tikzcd}
\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \arrow[r, "\vartheta"] \arrow[d, "\boxtimes^I\Delta_0"'] & T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[d, "\mathrm{pr}_1"] \arrow[dddl, "\Delta_0"'] \\
\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i \otimes T^{\leqslant 1}\mathcal{C}_i) \arrow[d, "\overline{\varkappa}^{-1}"'] & \boxtimes_u^{i\in I}\mathcal{C}_i \arrow[d, "\delta"] \\
(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i) \otimes (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i) \arrow[d, "\vartheta\otimes\vartheta"'] & T^{\geqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[d, "\mathrm{pr}_2"] \\
(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \otimes (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \arrow[r, "\mathrm{pr}_1\otimes\mathrm{pr}_1"] & (\boxtimes_u^{i\in I}\mathcal{C}_i) \otimes (\boxtimes_u^{i\in I}\mathcal{C}_i)
\end{tikzcd} \tag{6.19.1}$$

Composing the left pentagon with the quiver maps

$$(\varepsilon \otimes 1)(\lambda^{\cdot\,|})^{-1},\ (1 \otimes \varepsilon)(\lambda^{|\,\cdot})^{-1} : (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \otimes (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \to T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i$$

we get the map ϑ in both sides of the obtained equation. Thus, it remains to compose the left pentagon with the projection

$$\mathrm{pr}_1 \otimes \mathrm{pr}_1 : (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \otimes (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \to (\boxtimes_u^{i\in I}\mathcal{C}_i) \otimes (\boxtimes_u^{i\in I}\mathcal{C}_i).$$

Precisely that is done in the above diagram. We have only to show commutativity of its

exterior. The top–right path can be computed via the following commutative diagram:

$$\begin{tikzcd}
\boxtimes^{i\in I}T^{\leqslant1}\mathcal{C}_i \arrow[r, "\vartheta"] \arrow[d, "\boxtimes^I T^{\leqslant1}\delta"'] & T^{\leqslant1}\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, "\mathrm{pr}_1"] \arrow[d, "T^{\leqslant1}\boxtimes_u^I\delta"'] & \boxtimes_u^{i\in I}\mathcal{C}_i \arrow[d, "\boxtimes_u^I\delta"'] \arrow[dd, bend left=60, "\delta\cdot\mathrm{pr}_2"] \\
\boxtimes^{i\in I}T^{\leqslant1}T^{\geqslant1}\mathcal{C}_i \arrow[r, "\vartheta"] \arrow[d, equal] & T^{\leqslant1}\boxtimes_u^{i\in I}T^{\geqslant1}\mathcal{C}_i \arrow[r, "\mathrm{pr}_1"] \arrow[d, "T^{\leqslant1}\tau"'] & \boxtimes_u^{i\in I}T^{\geqslant1}\mathcal{C}_i \arrow[d, "\tau\cdot\mathrm{pr}_2"'] \\
\boxtimes^{i\in I}T\mathcal{C}_i \arrow[r, "\widetilde{\tau}"] & T\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, "\mathrm{pr}_2"] & T^2\boxtimes_u^{i\in I}\mathcal{C}_i
\end{tikzcd}$$

It equals the above left–down path. We have to prove that the latter equals the left–down path of diagram (6.19.1). We shall prove the equation between matrix elements. Fix a direct summand $\boxtimes^{i\in I}T^{n_i}\mathcal{C}_i$ of the source, $n_i\in\{0,1\}$ for $i\in I$. Fix a direct summand $\otimes^{p\in\mathbf{2}}\boxtimes^{i\in I}T^{\chi((i,p)\in S)}\mathcal{C}_i$ of the target. Here the subset $S\subset I\times\mathbf{2}$ satisfies the condition $\mathrm{pr}_2 S=\mathbf{2}$. Denote $S_i=\{p\in\mathbf{2}\mid(i,p)\in S\}$ and $m_i=|S_i|\leqslant 2$. The required equation between matrix elements takes the form

$$\begin{tikzcd}
\boxtimes^{i\in I}T^{n_i}\mathcal{C}_i \arrow[r, "\boxtimes^{i\in I}(T^{\leqslant1}\delta)_{n_i,m_i}"] \arrow[d, "\boxtimes^{i\in I}(\Delta_0)_{n_i,S_i}"'] & \boxtimes^{i\in I}T^{m_i}\mathcal{C}_i \arrow[d, "\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{2}}"] \\
\boxtimes^{i\in I}\otimes^{p\in\mathbf{2}}T^{\chi(p\in S_i)}\mathcal{C}_i \arrow[r, equal] \arrow[d, "\varkappa^{-1}"'] \arrow[dr, phantom, "="] & \boxtimes^{i\in I}\otimes^{p\in\mathbf{2}}T^{\chi((i,p)\in S)}\mathcal{C}_i \arrow[d, "\varkappa^{-1}"] \\
\otimes^{p\in\mathbf{2}}\boxtimes^{i\in I}T^{\chi(p\in S_i)}\mathcal{C}_i \arrow[r, equal] & \otimes^{p\in\mathbf{2}}\boxtimes^{i\in I}T^{\chi((i,p)\in S)}\mathcal{C}_i
\end{tikzcd}$$

due to (6.2.1). The top square is implied by the following equation, which holds true for each $i\in I$

$$\begin{tikzcd}[column sep=huge]
\bigoplus_{n_i=0}^{1}\boxtimes^{i\in I}T^{n_i}\mathcal{C}_i \arrow[r, "\begin{pmatrix}1&0&\frac{0}{\Delta}\\0&1&\end{pmatrix}", "T^{\leqslant1}\delta"'] \arrow[d, "\begin{pmatrix}\lambda^{\varnothing\to\mathbf{2}}&0&0&\frac{0}{\Delta}\\0&\lambda^{\cdot\,\shortmid}&\lambda^{\shortmid\,\cdot}&\end{pmatrix}"', "\Delta_0"] & \bigoplus_{m_i=0}^{2}\boxtimes^{i\in I}T^{m_i}\mathcal{C}_i \arrow[d, "\lambda^{S_i\hookrightarrow\mathbf{2}}"', "\begin{pmatrix}\lambda^{\varnothing\to\mathbf{2}}&0&0&0\\0&\lambda^{\cdot\,\shortmid}&\lambda^{\shortmid\,\cdot}&0\\0&0&0&1\end{pmatrix}"] \\
\bigoplus_{S_i\subset\mathbf{2}}\otimes^{p\in\mathbf{2}}T^{\chi(p\in S_i)}\mathcal{C}_i \arrow[r, equal] & \bigoplus_{S_i\subset\mathbf{2}}\otimes^{p\in\mathbf{2}}T^{\chi(p\in S_i)}\mathcal{C}_i
\end{tikzcd}$$

It is obvious once written in matrix form.

The constructed natural isomorphisms $\vartheta^I:\boxtimes^I\circ T^{\leqslant1}\to T^{\leqslant1}\circ\boxtimes_u^I$ satisfy necessary equations (2.6.1) because $(T^{\leqslant1},\vartheta^I):{}^{\mathcal{V}}\mathcal{Q}_u\to{}^{\mathcal{V}}\mathcal{Q}_p$ is a symmetric Monoidal functor due to considerations preceding (6.1.3). $\square$

Chapter 7

Closedness of the Kleisli multicategory of quivers

Let $\mathcal{V} = (\mathcal{V}, \otimes^I, \lambda^f)$ be a symmetric *closed* Monoidal abelian $\mathscr{U}$-category. As in Chapter 6 we assume that arbitrary $\mathscr{U}$-small limits and colimits exist in $\mathcal{V}$, that $\mathscr{U}$-small filtering colimits in $\mathcal{V}$ commute with finite projective limits. Closedness of $\mathcal{V}$ implies that the tensor product commutes with arbitrary $\mathscr{U}$-small colimits. The reader may assume that $\mathcal{V}$ means $\mathbf{gr}$ or $\mathbf{dg}$.

In this chapter we prove that closedness of $\mathcal{V}$ implies closedness of symmetric Monoidal categories of quivers ${}^{\mathcal{V}}\mathscr{Q}_p$ and ${}^{\mathcal{V}}\mathscr{Q}_u$. Moreover, left and right actions of quiver morphisms in $\underline{{}^{\mathcal{V}}\mathscr{Q}_p}$ and $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}$ are determined by such actions for morphisms of $\mathcal{V}$ in $\underline{\mathcal{V}}$. Compositions in $\underline{{}^{\mathcal{V}}\mathscr{Q}_p}$ and $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}$ are related with the composition in $\underline{\mathcal{V}}$ by simple formulas. The lax symmetric Monoidal comonad $T^{\geqslant 1}$ provides the closed symmetric multicategory $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$. An isomorphic multicategory Q is obtained from it via shift of the objects by [1]. This is a precursor without differentials of the multicategory of A_∞-categories considered in the next chapter. The composition in $\underline{\mathsf{Q}}$ is expressed via composition in $\underline{\mathscr{Q}_u}$. We introduce the notion of $T^{\geqslant 1}$-coderivations and show that these are ordinary coalgebra coderivations.

7.1 Closedness of symmetric Monoidal category ${}^{\mathcal{V}}\mathscr{Q}_p$. Let us prove that symmetric Monoidal category of $\mathcal{V}$-quivers ${}^{\mathcal{V}}\mathscr{Q}_p$ is closed. Let $\mathcal{A}$, $\mathcal{C}$ be $\mathcal{V}$-quivers. Define the $\mathcal{V}$-quiver $\mathcal{V}\text{-span}(\mathcal{A},\mathcal{C})$ as follows. Its set of objects is $\operatorname{Ob}(\mathcal{V}\text{-span}(\mathcal{A},\mathcal{C})) = \mathcal{S}\mathrm{et}(\operatorname{Ob}\mathcal{A}, \operatorname{Ob}\mathcal{C})$, the set of mappings $\operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{C}$. The objects of morphisms are

$$\mathcal{V}\text{-span}(\mathcal{A},\mathcal{C})(f,g) = \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathcal{V}}\big(\mathcal{A}(X,Y), \mathcal{C}(Xf,Yg)\big) \in \operatorname{Ob}\mathcal{V}.$$

For example, when $\mathcal{V} = \mathbf{gr}$, objects of $\mathbf{gr}\text{-span}(\mathcal{A},\mathcal{B})$ are maps $f : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, and the graded component $\mathbf{gr}\text{-span}(\mathcal{A},\mathcal{B})(f,g)^d$ of the $\Bbbk$-module of morphisms consists of $\Bbbk$-span morphisms $r : \mathcal{A} \to \mathcal{B}$ with $\operatorname{Ob}_s r = f$, $\operatorname{Ob}_t r = g$ of degree $\deg r = d$, that is,

$$\mathbf{gr}\text{-span}(\mathcal{A},\mathcal{B})(f,g) = \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathbf{gr}}\big(\mathcal{A}(X,Y), \mathcal{B}(Xf,Yg)\big) \in \operatorname{Ob}\mathbf{gr}.$$

7.2 Proposition. *The symmetric Monoidal category ${}^{\mathcal{V}}\mathscr{Q}_p = ({}^{\mathcal{V}}\mathscr{Q}, \boxtimes^I, \lambda^f)$ is closed with*

the inner homomorphisms objects $\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C}) = \mathcal{V}\text{-span}(\mathcal{A},\mathcal{C})$ *and evaluations*

$$\mathrm{ev}^{{}^{\mathcal{V}}\mathcal{Q}_p} : \mathcal{A} \boxtimes \mathcal{V}\text{-span}(\mathcal{A},\mathcal{C}) \to \mathcal{C}, \qquad (X,f) \mapsto Xf,$$

$$\mathrm{ev}^{{}^{\mathcal{V}}\mathcal{Q}_p} = \bigl[\mathcal{A}(X,Y) \otimes \mathcal{V}\text{-span}(\mathcal{A},\mathcal{C})(f,g) \xrightarrow{1\otimes \mathrm{pr}_{(X,Y)}} \mathcal{A}(X,Y) \otimes \underline{\mathcal{V}}\bigl(\mathcal{A}(X,Y), \mathcal{C}(Xf,Yg)\bigr) \xrightarrow{\mathrm{ev}^{\mathcal{V}}} \mathcal{C}(Xf,Yg)\bigr], \qquad a\otimes r \mapsto a.r_{X,Y}.$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be $\mathcal{V}$-quivers. Consider the morphism

$$\varphi_{{}^{\mathcal{V}}\mathcal{Q}_p} : {}^{\mathcal{V}}\mathcal{Q}(\mathcal{B}, \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})) \longrightarrow {}^{\mathcal{V}}\mathcal{Q}(\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}),$$

$$\bigl[f : \mathcal{B} \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})\bigr] \longmapsto \bigl[\mathcal{A}\boxtimes\mathcal{B} \xrightarrow{1\boxtimes f} \mathcal{A}\boxtimes \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C}) \xrightarrow{\mathrm{ev}^{{}^{\mathcal{V}}\mathcal{Q}_p}} \mathcal{C}\bigr].$$

We have to prove that it is invertible. The inverse morphism is constructed as follows.
Let $g : \mathcal{A}\boxtimes\mathcal{B} \to \mathcal{C}$ be a $\mathcal{V}$-quiver morphism. Define the map

$$\mathrm{Ob}\, f : \mathrm{Ob}\,\mathcal{B} \to \mathrm{Ob}\,\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C}) = \mathcal{S}et(\mathrm{Ob}\,\mathcal{A}, \mathrm{Ob}\,\mathcal{C}), \qquad U \longmapsto (Uf : X \mapsto (X,U)g).$$

Thus $X.(Uf) = (X,U)g$. The morphisms in $\mathcal{V}$

$$g_{(X,U),(Y,V)} : \mathcal{A}(X,Y) \otimes \mathcal{B}(U,V) \to \mathcal{C}((X,U)g,(Y,V)g)$$

can be presented in the form

$$\varphi_{\mathcal{V}}^{-1} g_{(X,U),(Y,V)} : \mathcal{B}(U,V) \to \underline{\mathcal{V}}\bigl(\mathcal{A}(X,Y), \mathcal{C}((X,U)g,(Y,V)g)\bigr).$$

Thus for any pair U, V of objects of $\mathcal{B}$ there is a morphism in $\mathcal{V}$

$$f_{U,V} = \bigl[\mathcal{B}(U,V) \xrightarrow{(\varphi_{\mathcal{V}}^{-1} g_{(X,U),(Y,V)})_{X,Y}} \prod_{X,Y\in\mathrm{Ob}\,\mathcal{A}} \underline{\mathcal{V}}\bigl(\mathcal{A}(X,Y), \mathcal{C}(X.(Uf), Y.(Vf))\bigr) = \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})(Uf,Vf)\bigr].$$

The collection of such morphisms gives a $\mathcal{V}$-quiver morphism $\varphi_{{}^{\mathcal{V}}\mathcal{Q}_p}^{-} g = f : \mathcal{B} \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})$. This defines a morphism $\varphi_{{}^{\mathcal{V}}\mathcal{Q}_p}^{-} : {}^{\mathcal{V}}\mathcal{Q}(\mathcal{A}\boxtimes\mathcal{B},\mathcal{C}) \to {}^{\mathcal{V}}\mathcal{Q}(\mathcal{B}, \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C}))$.

Closedness of $\mathcal{S}et$ implies that $\mathcal{S}et(\mathrm{Ob}\,\mathcal{B}, \mathrm{Ob}\,\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C}))$ and $\mathcal{S}et(\mathrm{Ob}(\mathcal{A}\boxtimes\mathcal{B}), \mathrm{Ob}\,\mathcal{C})$ are in bijection. Closedness of $\mathcal{V}$ extends this bijection to objects of morphisms. One can verify directly that the morphisms $\varphi_{{}^{\mathcal{V}}\mathcal{Q}_p}$ and $\varphi_{{}^{\mathcal{V}}\mathcal{Q}_p}^{-}$ are inverse to each other. □

Let $h : \mathcal{A} \to \mathcal{B}$ be a morphism in ${}^{\mathcal{V}}\mathcal{Q}_p$. It gives rise to the morphism $\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(h,1) : \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{B},\mathcal{C}) \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})$ in ${}^{\mathcal{V}}\mathcal{Q}_p$. It is described by the following equation

$$\begin{tikzcd} \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{B},\mathcal{C})(f,g) \arrow[r, "\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(h{,}1)"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "="] & \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A},\mathcal{C})(hf,hg) \arrow[d, "\mathrm{pr}"] \\ \underline{\mathcal{V}}(\mathcal{B}(Xh,Yh), \mathcal{C}(Xhf,Yhg)) \arrow[r, "\underline{\mathcal{V}}(h{,}1)"] & \underline{\mathcal{V}}(\mathcal{A}(X,Y), \mathcal{C}(Xhf,Yhg)) \end{tikzcd}$$

Let $h : \mathcal{B} \to \mathcal{C}$ be a morphism in ${}^{\mathcal{V}}\mathscr{Q}_p$. It gives rise to the morphism $\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(1,h) : \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{B}) \to \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{C})$ uniquely determined by the following equation

$$\begin{tikzcd}
\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{B})(f,g) \arrow[r, "\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(1,h)"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "="] & \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{C})(fh,gh) \arrow[d, "\mathrm{pr}"] \\
\underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{B}(Xf,Yg)) \arrow[r, "\underline{\mathcal{V}}(1,h)"] & \underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{C}(Xfh,Ygh))
\end{tikzcd}$$

Being a closed Monoidal category ${}^{\mathcal{V}}\mathscr{Q}_p$ has associative multiplication, the quiver morphism

$$\mu^{\underline{{}^{\mathcal{V}}\mathscr{Q}_p}} : \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{B}) \boxtimes \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{B},\mathcal{C}) \to \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{C})$$

given for arbitrary $\mathcal{V}$-quivers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. It takes a pair of maps (f,h) to their composition $fh : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{C}$. On morphisms it is determined by the following equation

$$\begin{tikzcd}
\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{B})(f,g) \otimes \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{B},\mathcal{C})(h,k) \arrow[d, "\mathrm{pr}\otimes\mathrm{pr}"'] \arrow[r, "\mu^{\underline{{}^{\mathcal{V}}\mathscr{Q}_p}}"] & \underline{{}^{\mathcal{V}}\mathscr{Q}_p}(\mathcal{A},\mathcal{C})(fh,gk) \arrow[d, "\mathrm{pr}"] \\
\underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{B}(Xf,Yg)) \otimes \underline{\mathcal{V}}(\mathcal{B}(Xf,Yg),\mathcal{C}(Xfh,Ygk)) \arrow[r, "\mu^{\underline{\mathcal{V}}}"'] & \underline{\mathcal{V}}(\mathcal{A}(X,Y),\mathcal{C}(Xfh,Ygk))
\end{tikzcd}$$

Proof of these facts is left to the reader. More complicated reasoning of Sections 7.6.1–7.6.3 can serve as a model.

7.3 Example. Let us describe the category of differential graded quivers ${}^{d}\mathscr{Q}$ in terms of symmetric closed Monoidal category $\mathscr{Q}_p$ of graded $\Bbbk$-linear quivers. An object $\mathcal{C}$ of ${}^{d}\mathscr{Q}$ is a graded quiver $\mathcal{C}$ equipped with degree 1 maps $d : \mathcal{C}(X,Y) \to \mathcal{C}(X,Y)$, $d^2 = 0$ for all pairs X, Y of objects of $\mathcal{C}$. Equivalently, an element $d \in \underline{\mathscr{Q}_p}(\mathcal{C},\mathcal{C})(\mathrm{id}_{\operatorname{Ob}\mathcal{C}},\mathrm{id}_{\operatorname{Ob}\mathcal{C}})^1$ is given such that $\mu^{\underline{\mathscr{Q}_p}}(d \otimes d) = 0$. A morphism $f : \mathcal{A} \to \mathcal{B}$ of differential graded quivers is a morphism of graded quivers that commutes with the differential, $\bigl(\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{d} \mathcal{B}\bigr) = \bigl(\mathcal{A} \xrightarrow{d} \mathcal{A} \xrightarrow{f} \mathcal{B}\bigr)$. Equivalently, the following equation holds:

$$\bigl[\mathbb{1}_p[-1] \xrightarrow{d} \underline{\mathscr{Q}_p}(\mathcal{B},\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(f,1)} \underline{\mathscr{Q}_p}(\mathcal{A},\mathcal{B})\bigr] = \bigl[\mathbb{1}_p[-1] \xrightarrow{d} \underline{\mathscr{Q}_p}(\mathcal{A},\mathcal{A}) \xrightarrow{\underline{\mathscr{Q}_p}(1,f)} \underline{\mathscr{Q}_p}(\mathcal{A},\mathcal{B})\bigr].$$

Here $\mathbb{1}_p[-1]$ is the graded quiver with the unique object $*$ and the module of morphisms $\mathbb{1}_p[-1](*,*) = \Bbbk[-1]$ concentrated in degree 1.

7.4 Closedness of symmetric Monoidal category ${}^{\mathcal{V}}\mathcal{Q}_u$. Let us prove that symmetric Monoidal category of quivers ${}^{\mathcal{V}}\mathcal{Q}_u$ with modified tensor product is also closed.

7.5 Proposition. *The symmetric Monoidal category ${}^{\mathcal{V}}\mathcal{Q}_u = ({}^{\mathcal{V}}\mathcal{Q}, \boxtimes^I_u, \lambda^f_u)$ of $\mathcal{V}$-quivers is closed with $\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})$ being a $\mathcal{V}$-quiver, whose objects are morphisms of $\mathcal{V}$-quivers $\mathcal{A} \to \mathcal{C}$ and the object of morphisms between $f, g : \mathcal{A} \to \mathcal{C}$ is given by*

$$\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})(f, g) = \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, \mathcal{C})(\operatorname{Ob} f, \operatorname{Ob} g) \in \operatorname{Ob} \mathcal{V}.$$

Proof. To establish a bijection

$$\psi : {}^{\mathcal{V}}\mathcal{Q}(\mathcal{B}, \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})) \xrightarrow{\sim} {}^{\mathcal{V}}\mathcal{Q}(\mathcal{A} \boxtimes_u \mathcal{B}, \mathcal{C}) \tag{7.5.1}$$

note that due to (6.1.4) the second functor assigns to a triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $\mathcal{V}$-quivers the following data: a function $\operatorname{Ob} \mathcal{A} \times \operatorname{Ob} \mathcal{B} \ni (A, B) \mapsto (A, B)f \in \operatorname{Ob} \mathcal{C}$, and morphisms

$$f_{A,A';B,B'} : \mathcal{A}(A, A') \otimes \mathcal{B}(B, B') \to \mathcal{C}((A, B)f, (A', B')f), \tag{7.5.2}$$
$$f_{A,A';B} : \mathcal{A}(A, A') \otimes \Bbbk \to \mathcal{C}((A, B)f, (A', B)f), \tag{7.5.3}$$
$$f_{A;B,B'} : \Bbbk \otimes \mathcal{B}(B, B') \to \mathcal{C}((A, B)f, (A, B')f) \tag{7.5.4}$$

in $\mathcal{V}$ for all $A, A' \in \operatorname{Ob} \mathcal{A}$, $B, B' \in \operatorname{Ob} \mathcal{B}$.

On the other hand, a morphism $\mathcal{B} \to \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})$ is given by the following data: a map on objects $\operatorname{Ob} \mathcal{B} \to \operatorname{Ob} \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})$, $B \mapsto \bigl(A \mapsto (A, B)f\bigr)$ together with (7.5.3) and a map on morphisms

$$\mathcal{B}(B, B') \to \prod_{A,A' \in \operatorname{Ob} \mathcal{A}} \underline{\mathcal{V}}\bigl(T^{\leqslant 1}\mathcal{A}(A, A'), \mathcal{C}((A, B)f, (A', B')f)\bigr),$$

which is equivalent to a pair consisting of collection of maps (7.5.4) and of the map

$$\mathcal{B}(B, B') \to \prod_{A,A' \in \operatorname{Ob} \mathcal{A}} \underline{\mathcal{V}}\bigl(\mathcal{A}(A, A'), \mathcal{C}((A, B)f, (A', B')f)\bigr).$$

The latter map is the same as collection of maps (7.5.2). Therefore, bijection (7.5.1) is constructed. Let us describe the corresponding evaluations.

Denote by $\Upsilon : \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C}) \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, \mathcal{C})$ the quiver morphism which gives the map $f \mapsto \operatorname{Ob} f$ on objects and identity map on morphisms. Take $\mathcal{B} = \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})$ and consider the evaluation morphism $\operatorname{ev}^{{}^{\mathcal{V}}\mathcal{Q}_u}_{\mathcal{A},\mathcal{C}} : \mathcal{A} \boxtimes_u \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C}) \to \mathcal{C}$, corresponding to $\mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})$. The identity morphism $\mathrm{id}_{\mathcal{B}}$ is described by the assignment on objects $(g : \mathcal{A} \to \mathcal{C}) \mapsto (X \mapsto Xg)$ together with $g_{X,Y} : \mathcal{A}(X, Y) \to \mathcal{C}(Xg, Yg)$, and on morphisms it is given by the identity morphism

$$\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}, \mathcal{C})(f, g) = \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, \mathcal{C})(\operatorname{Ob} f, \operatorname{Ob} g) = \prod_{X,Y \in \operatorname{Ob} \mathcal{A}} \underline{\mathcal{V}}\bigl(T^{\leqslant 1}\mathcal{A}(X, Y), \mathcal{C}(Xf, Yg)\bigr)$$

in $\mathcal{V}$. Therefore, on objects evaluation acts as $\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_u}_{\mathcal{A},\mathcal{C}}(X,f) = Xf$, and on morphisms via

$$\mathrm{ev}'_{\mathcal{A},\mathcal{C}} = \big[T^{\leqslant 1}\mathcal{A} \boxtimes \underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C}) \xrightarrow{1\boxtimes\Upsilon} T^{\leqslant 1}\mathcal{A} \boxtimes \underline{{}^{\mathcal{V}}\!\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C}) \xrightarrow{\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_p}} \mathcal{C}\big], \tag{7.5.5}$$

$$\mathrm{ev}''_{\mathcal{A},\mathcal{C}} : \mathcal{A} \boxtimes T^0\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C}) \to \mathcal{C},$$

$$\mathrm{ev}''_{\mathcal{A},\mathcal{C}} = \big[\mathcal{A}(X,Y) \otimes T^0\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C})(f,f) \xrightarrow[\sim]{(\lambda^{\mathsf{I}\cdot})^{-1}} \mathcal{A}(X,Y) \xrightarrow{f_{X,Y}} \mathcal{C}(Xf,Yf)\big]. \tag{7.5.6}$$

The map $\varphi_{\mathcal{B};\mathcal{A};\mathcal{C}}$ given by (4.7.1) can be written as

$$\begin{array}{ccccc} {}^{\mathcal{V}}\!\mathcal{Q}(\mathcal{B},\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C})) & \xrightarrow{\mathcal{A}\boxtimes_u -} & {}^{\mathcal{V}}\!\mathcal{Q}(\mathcal{A}\boxtimes_u\mathcal{B},\mathcal{A}\boxtimes_u\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C})) & \xrightarrow{-\cdot\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_u}_{\mathcal{A},\mathcal{C}}} & {}^{\mathcal{V}}\!\mathcal{Q}(\mathcal{A}\boxtimes_u\mathcal{B},\mathcal{C}), \\ f & \longmapsto & \mathrm{id}_{\mathcal{A}}\boxtimes_u f & \longmapsto & (\mathrm{id}_{\mathcal{A}}\boxtimes_u f)\cdot\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_u}_{\mathcal{A},\mathcal{C}}. \end{array}$$

Explicit computation shows that it coincides with the bijection ψ given by (7.5.1), as it has to be for an arbitrary closed monoidal category. Indeed, $\mathrm{Ob}\,\varphi(f) = \mathrm{Ob}\,\psi(f)$. Restricting $\varphi(f),\psi(f) : \mathcal{A}\boxtimes_u\mathcal{B} = (T^{\leqslant 1}\mathcal{A}\boxtimes\mathcal{B}) \oplus (\mathcal{A}\boxtimes T^0\mathcal{B}) \to \mathcal{C}$ to $T^{\leqslant 1}\mathcal{A}\boxtimes\mathcal{B}$ we get

$$\begin{aligned} \varphi(f) &= (\mathrm{id}_{T^{\leqslant 1}\mathcal{A}}\boxtimes f)\cdot\mathrm{ev}'_{\mathcal{A},\mathcal{C}} \\ &= \big[T^{\leqslant 1}\mathcal{A}\boxtimes\mathcal{B} \xrightarrow{1\boxtimes f} T^{\leqslant 1}\mathcal{A}\boxtimes\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C}) \xrightarrow{1\boxtimes\Upsilon} T^{\leqslant 1}\mathcal{A}\boxtimes\underline{{}^{\mathcal{V}}\!\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C}) \xrightarrow{\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_p}} \mathcal{C}\big]. \end{aligned}$$

Restricted further to $\mathcal{A}\boxtimes\mathcal{B}$ this gives (7.5.2), and restricted to $T^0\mathcal{A}\boxtimes\mathcal{B}$ this gives (7.5.4), which coincides with $\psi(f)$. Restriction to $\mathcal{A}\boxtimes T^0\mathcal{B}$ gives

$$\begin{aligned} \varphi(f) &= (\mathrm{id}_{\mathcal{A}}\boxtimes T^0 f)\cdot\mathrm{ev}''_{\mathcal{A},\mathcal{C}} \\ &= \big[\mathcal{A}(A,A')\otimes T^0\mathcal{B}(B,B) \xrightarrow{1\otimes T^0 f} \mathcal{A}(A,A')\otimes T^0\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C})((\_,B)f,(\_,B)f) \\ &\qquad \xrightarrow[\sim]{(\lambda^{\mathsf{I}\cdot})^{-1}} \mathcal{A}(A,A') \xrightarrow{f_{A,A';B}} \mathcal{C}((A,B)f,(A',B)f)\big]. \end{aligned}$$

Being just (7.5.3) this coincides with $\psi(f)$. Therefore, $\varphi_{\mathcal{B};\mathcal{A};\mathcal{C}}$ is bijective and the Monoidal category ${}^{\mathcal{V}}\!\mathcal{Q}_u$ is closed. □

7.6 Multiplications in ${}^{\mathcal{V}}\!\mathcal{Q}_u$. Let us describe various multiplications in the closed Monoidal category ${}^{\mathcal{V}}\!\mathcal{Q}_u$. They are obtained as a result of computations directly from the definitions. The details are left to the reader.

7.6.1 Left multiplication in ${}^{\mathcal{V}}\!\mathcal{Q}_u$. Let $h : \mathcal{A} \to \mathcal{B}$ be a morphism in ${}^{\mathcal{V}}\!\mathcal{Q}_u$. It gives rise to a morphism $\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(h,1) : \underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{B},\mathcal{C}) \to \underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C})$ in ${}^{\mathcal{V}}\!\mathcal{Q}_u$ found from the following diagram

$$\begin{tikzcd} \mathcal{A}\boxtimes_u\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{B},\mathcal{C}) \arrow[r, "1\boxtimes_u\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(h,1)"] \arrow[d, "h\boxtimes_u 1"'] & \mathcal{A}\boxtimes_u\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{A},\mathcal{C}) \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_u}"] \\ \mathcal{B}\boxtimes_u\underline{{}^{\mathcal{V}}\!\mathcal{Q}_u}(\mathcal{B},\mathcal{C}) \arrow[r, "\mathrm{ev}^{{}^{\mathcal{V}}\!\mathcal{Q}_u}"] & \mathcal{C} \end{tikzcd}$$

An object f of the graded quiver ${}^{\mathcal{V}}\mathscr{Q}_u(\mathcal{B},\mathcal{C})$, that is, a quiver morphism $f:\mathcal{B}\to\mathcal{C}$, is mapped by $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1)$ to the composite hf. For a pair of quiver morphisms $f,g:\mathcal{B}\to\mathcal{C}$, the $\Bbbk$-linear map $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1):\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{B},\mathcal{C})(f,g)\to\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C})(hf,hg)$ fits into the commutative diagram

$$\begin{tikzcd} \underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{B},\mathcal{C})(f,g) \arrow[r, "\mathrm{pr}"] \arrow[d, "\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1)"'] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{B}(Xh,Yh),\mathcal{C}(Xhf,Yhg)) \arrow[d, "\underline{\mathcal{V}}(T^{\leqslant 1}h,1)"] \\ \underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C})(hf,hg) \arrow[r, "\mathrm{pr}"] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{C}(Xhf,Yhg)) \end{tikzcd} \tag{7.6.1}$$

for each pair of objects $X,Y\in\operatorname{Ob}\mathcal{B}$. In other notation,

$$\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1)=\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}h,1):\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{B},\mathcal{C})(\operatorname{Ob}f,\operatorname{Ob}g)\to\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})(\operatorname{Ob}hf,\operatorname{Ob}hg).$$

Equivalently, $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1)$ satisfies the equation

$$\begin{aligned}\big[\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{B},\mathcal{C})\xrightarrow{\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(h,1)}\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C})\xrightarrow{\Upsilon}\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})\big]&\\ =\big[\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{B},\mathcal{C})\xrightarrow{\Upsilon}\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{B},\mathcal{C})\xrightarrow{\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}h,1)}\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})\big].&\end{aligned} \tag{7.6.2}$$

7.6.2 Right multiplication in ${}^{\mathcal{V}}\mathscr{Q}_u$. Let $h:\mathcal{B}\to\mathcal{C}$ be a morphism in ${}^{\mathcal{V}}\mathscr{Q}_u$. It gives rise to a morphism $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(1,h):\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{B})\to\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C})$ uniquely determined by the following diagram

$$\begin{tikzcd} \mathcal{A}\boxtimes_u\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \arrow[r, "1\boxtimes_u\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(1,h)"] \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\mathscr{Q}_u}"'] & \mathcal{A}\boxtimes_u\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C}) \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\mathscr{Q}_u}"] \\ \mathcal{B} \arrow[r, "h"] & \mathcal{C} \end{tikzcd}$$

A quiver morphism $f:\mathcal{A}\to\mathcal{B}$ is mapped by $\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(1,h)$ to the composite fh. For a pair of quiver morphisms $f,g:\mathcal{A}\to\mathcal{B}$, the diagram

$$\begin{tikzcd} \underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{B})(f,g) \arrow[r, "\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(1,h)"] \arrow[d, "\mathrm{pr}"'] & \underline{{}^{\mathcal{V}}\mathscr{Q}_u}(\mathcal{A},\mathcal{C})(fh,gh) \arrow[d, "\mathrm{pr}"] \\ \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{B}(Xf,Yg)) \arrow[r, "\underline{\mathcal{V}}(1,h)"] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{C}(Xfh,Ygh)) \end{tikzcd} \tag{7.6.3}$$

commutes, for each pair of objects $X,Y\in\operatorname{Ob}\mathcal{A}$, thus determining the top $\Bbbk$-linear map unambiguously. In other notation,

$$\underline{{}^{\mathcal{V}}\mathscr{Q}_u}(1,h)=\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(1,h):\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{B})(\operatorname{Ob}f,\operatorname{Ob}g)\to\underline{{}^{\mathcal{V}}\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})(\operatorname{Ob}fh,\operatorname{Ob}gh). \tag{7.6.4}$$

Equivalently, $\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(1,h)$ satisfies the equation

$$
\begin{aligned}
&\big[\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \xrightarrow{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(1,h)} \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{C}) \xrightarrow{\Upsilon} \underline{{}^{\mathcal{V}}\!\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})\big] \\
&\qquad = \big[\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \xrightarrow{\Upsilon} \underline{{}^{\mathcal{V}}\!\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{B}) \xrightarrow{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_p}(1,h)} \underline{{}^{\mathcal{V}}\!\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{C})\big]. \qquad (7.6.5)
\end{aligned}
$$

7.6.3 Composition in the closed Monoidal category ${}^{\mathcal{V}}\!\mathscr{Q}_u$. For arbitrary quivers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ there is a quiver morphism

$$\mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}} : \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C}) \to \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{C})$$

uniquely determined by the following equation in ${}^{\mathcal{V}}\!\mathscr{Q}_u$:

$$
\begin{tikzcd}
\mathcal{A} \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C}) \arrow[r, "\lambda_u^{\mathsf{IV}}"] \arrow[d, "\lambda_u^{\mathsf{VI}}"'] & \mathcal{A} \boxtimes_u \big(\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C})\big) \arrow[d, "1 \boxtimes_u \mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}}"] \\
\big(\mathcal{A} \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B})\big) \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C}) \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\!\mathscr{Q}_u} \boxtimes_u 1"'] \arrow[r, phantom, "="] & \mathcal{A} \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{C}) \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\!\mathscr{Q}_u}"] \\
\mathcal{B} \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C}) \arrow[r, "\mathrm{ev}^{{}^{\mathcal{V}}\!\mathscr{Q}_u}"] & \mathcal{C}
\end{tikzcd}
\qquad (7.6.6)
$$

It is not difficult to conclude that $\mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}}$ maps a pair of quiver morphisms $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{B} \to \mathcal{C}$ to their composite $fg : \mathcal{A} \to \mathcal{C}$. The $\Bbbk$-linear maps

$$\mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}} : \big(\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B}) \boxtimes_u \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C})\big)((f,h),(g,k)) \to \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B})(fh,gk)$$

are found unambiguously from the following diagrams:

$$
\begin{tikzcd}
\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B})(f,g) \otimes \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C})(h,k) \arrow[r, "\mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}}"] \arrow[d, "\mathrm{pr}\otimes\mathrm{pr}"'] \arrow[dr, phantom, "="] & \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{C})(fh,gk) \arrow[d, "\mathrm{pr}"'] \\
\begin{array}{l} \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{B}(Xf,Yg)) \otimes \\ \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{B}(Xf,Yg),\mathcal{C}(Xfh,Ygk)) \end{array} \arrow[r, "-\cdot\mathrm{in}_1\cdot-"] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{C}(Xfh,Ygk))
\end{tikzcd}
\qquad (7.6.7)
$$

for arbitrary objects $X, Y \in \operatorname{Ob}\mathcal{A}$ and quiver maps $f, g : \mathcal{A} \to \mathcal{B}$, $h, k : \mathcal{B} \to \mathcal{C}$;

$$
\begin{tikzcd}
T^0\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{B})(f,f) \otimes \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{B},\mathcal{C})(h,k) \arrow[r, "\mu_{\underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}}"] \arrow[d, "\mathrm{pr}"'] & \underline{{}^{\mathcal{V}}\!\mathscr{Q}_u}(\mathcal{A},\mathcal{C})(fh,fk) \arrow[d, "\mathrm{pr}"] \\
\underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{B}(Xf,Yf),\mathcal{C}(Xfh,Yfk)) \arrow[r, "\underline{\mathcal{V}}(T^{\leqslant 1}f,1)"] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y),\mathcal{C}(Xfh,Xfk))
\end{tikzcd}
\qquad (7.6.8)
$$

for arbitrary objects $X, Y \in \operatorname{Ob}\mathcal{A}$ and quiver maps $f : \mathcal{A} \to \mathcal{B}$, $h, k : \mathcal{B} \to \mathcal{C}$;

$$\begin{tikzcd}
\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})(f,g) \otimes T^0\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{B},\mathcal{C})(h,h) \arrow[r, "\mu^{{}^{\mathcal{V}}\mathcal{Q}_u}"] \arrow[d, "\mathrm{pr}"'] & \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{C})(fh,gh) \arrow[d, "\mathrm{pr}"] \\
\underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y), \mathcal{B}(Xf,Yg)) \arrow[r, "\underline{\mathcal{V}}(1,h_{Xf,Yg})"] & \underline{\mathcal{V}}(T^{\leqslant 1}\mathcal{A}(X,Y), \mathcal{C}(Xfh,Ygh))
\end{tikzcd} \tag{7.6.9}$$

for arbitrary objects $X, Y \in \operatorname{Ob}\mathcal{A}$ and quiver maps $f, g : \mathcal{A} \to \mathcal{B}$, $h : \mathcal{B} \to \mathcal{C}$.

7.7 Example. The closing transformation $\underline{T^{\leqslant 1}} : T^{\leqslant 1}\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B}) \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, T^{\leqslant 1}\mathcal{B})$ for the symmetric Monoidal functor $(T^{\leqslant 1}, \vartheta) : {}^{\mathcal{V}}\mathcal{Q}_u \to {}^{\mathcal{V}}\mathcal{Q}_p$ can be found from the commutative diagram

$$\begin{tikzcd}
T^{\leqslant 1}\mathcal{A} \boxtimes T^{\leqslant 1}\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B}) \arrow[r, "1\boxtimes\underline{T^{\leqslant 1}}"] \arrow[d, "\vartheta^{\mathbf{2}}"'] & T^{\leqslant 1}\mathcal{A} \boxtimes \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, T^{\leqslant 1}\mathcal{B}) \arrow[d, "\mathrm{ev}^{{}^{\mathcal{V}}\mathcal{Q}_p}"] \\
T^{\leqslant 1}[\mathcal{A} \boxtimes_u \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})] \arrow[r, "T^{\leqslant 1}\mathrm{ev}^{{}^{\mathcal{V}}\mathcal{Q}_u}"] & T^{\leqslant 1}\mathcal{B}
\end{tikzcd} \tag{7.7.1}$$

see Section 4.18. It takes $f \in {}^{\mathcal{V}}\mathcal{Q}(\mathcal{A},\mathcal{B})$ to $\operatorname{Ob} f = \operatorname{Ob} T^{\leqslant 1} f : \operatorname{Ob} T^{\leqslant 1}\mathcal{A} \to \operatorname{Ob} T^{\leqslant 1}\mathcal{B}$ by Corollary 4.20. Its restriction to $T^1\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})$ is

$$\underline{T^{\leqslant 1}} = \bigl[\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})(f,g) = \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A},\mathcal{B})(\operatorname{Ob} f, \operatorname{Ob} g) \\ \xrightarrow{\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(1,\mathrm{in}_1)} \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, T^{\leqslant 1}\mathcal{B})(\operatorname{Ob} T^{\leqslant 1} f, \operatorname{Ob} T^{\leqslant 1} g)\bigr]. \tag{7.7.2}$$

Its restriction to $T^0\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})$ is

$$\underline{T^{\leqslant 1}} : T^0\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A},\mathcal{B})(f,f) \longrightarrow \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(T^{\leqslant 1}\mathcal{A}, T^{\leqslant 1}\mathcal{B})(\operatorname{Ob} T^{\leqslant 1} f, \operatorname{Ob} T^{\leqslant 1} g), \\ 1 \longmapsto \bigl(T^{\leqslant 1} f : (T^{\leqslant 1}\mathcal{A})(X,Y) \to (T^{\leqslant 1}\mathcal{B})(Xf,Yf)\bigr)_{X,Y\in\operatorname{Ob}\mathcal{A}}. \tag{7.7.3}$$

7.8 Closing transformation for the functor $\boxtimes^I_u$. The tensor multiplication functors $\boxtimes^I : {}^{\mathcal{V}}\mathcal{Q}^I_p \to {}^{\mathcal{V}}\mathcal{Q}_p$, $\boxtimes^I_u : {}^{\mathcal{V}}\mathcal{Q}^I_u \to {}^{\mathcal{V}}\mathcal{Q}_u$ determine the closing transformations

$$\underline{\boxtimes^I} : \boxtimes^{i\in I}\underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\mathcal{A}_i,\mathcal{B}_i) \to \underline{{}^{\mathcal{V}}\mathcal{Q}_p}(\boxtimes^{i\in I}\mathcal{A}_i, \boxtimes^{i\in I}\mathcal{B}_i), \\ \underline{\boxtimes^I_u} : \boxtimes^{i\in I}_u\underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \to \underline{{}^{\mathcal{V}}\mathcal{Q}_u}(\boxtimes^{i\in I}_u\mathcal{A}_i, \boxtimes^{i\in I}_u\mathcal{B}_i),$$

as for arbitrary closed symmetric Monoidal category, see Example 4.26. In the case of ${}^{\mathcal{V}}\mathcal{Q}_p$ the closing transformation $\underline{\boxtimes^I}$ coincides with the closing transformation $\underline{\otimes^I}$ for $\mathcal{V}$ on objects of morphisms, see the end of Example 4.26.

The transformation $\underline{\boxtimes^I_u}$ satisfies the commutative diagram

$$\begin{tikzcd}
(\boxtimes_u^{i\in I}\mathcal{A}_i)\boxtimes_u(\boxtimes_u^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i)) \arrow[r, "\sigma_{(12)}"] \arrow[d, "1\boxtimes_u\underline{\boxtimes^I_u}"'] & \boxtimes_u^{i\in I}(\mathcal{A}_i\boxtimes_u {}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i)) \arrow[d, "\boxtimes^I_u \mathrm{ev}^{{}^{\mathcal{V}}\mathscr{Q}_u}"] \\
(\boxtimes_u^{i\in I}\mathcal{A}_i)\boxtimes_u {}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\boxtimes_u^{i\in I}\mathcal{B}_i) \arrow[r, "\mathrm{ev}^{{}^{\mathcal{V}}\mathscr{Q}_u}"] & \boxtimes_u^{i\in I}\mathcal{B}_i
\end{tikzcd}$$

By Corollary 4.20 $\operatorname{Ob}\underline{\boxtimes^I_u} = \boxtimes^I_u$. This map takes a family $(f_i : \mathcal{A}_i \to \mathcal{B}_i)_{i\in I}$ of quiver morphisms to the quiver morphism $\boxtimes_u^{i\in I} f_i : \boxtimes_u^{i\in I}\mathcal{A}_i \to \boxtimes_u^{i\in I}\mathcal{B}_i$.

The Monoidal isomorphism of symmetric Monoidal functors $\vartheta : \boxtimes^{i\in I}(T^{\leqslant 1})^I \to T^{\leqslant 1}\boxtimes_u^{i\in I} : {}^{\mathcal{V}}\mathscr{Q}_u^I \to {}^{\mathcal{V}}\mathscr{Q}_p$ (see Example 4.26) implies by Lemmata 4.24 and 4.25 that the following equation holds:

$$\begin{tikzcd}
T^{\leqslant 1}\boxtimes_u^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \arrow[r, "T^{\leqslant 1}\underline{\boxtimes^I_u}"] & T^{\leqslant 1}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\boxtimes_u^{i\in I}\mathcal{B}_i) \arrow[r, "\underline{T^{\leqslant 1}}"] \arrow[d, phantom, "="] & {}^{\mathcal{V}}\underline{\mathscr{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i, T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{B}_i) \\
\boxtimes^{i\in I}T^{\leqslant 1}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \arrow[u, "\vartheta", "\wr"'] \arrow[r, "\boxtimes^I\underline{T^{\leqslant 1}}"] & \boxtimes^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A}_i,T^{\leqslant 1}\mathcal{B}_i) \arrow[r, "\underline{\boxtimes^I}"] & {}^{\mathcal{V}}\underline{\mathscr{Q}_p}(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{A}_i,\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{B}_i) \arrow[u, "{}^{\mathcal{V}}\underline{\mathscr{Q}_p}(\vartheta^{-1},\vartheta)"', "\wr"]
\end{tikzcd} \tag{7.8.1}$$

A solution $\underline{\boxtimes^I_u}$ of this equation exists by construction. Its uniqueness is guaranteed by injectivity of $\underline{T^{\leqslant 1}}$. By (7.7.2) the solution $\underline{\boxtimes^I_u}$ is determined by the equation

$$\begin{tikzcd}
\boxtimes_u^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \arrow[r, "\underline{\boxtimes^I_u}"] \arrow[d, hook, "\mathrm{in}_1"'] & {}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\boxtimes_u^{i\in I}\mathcal{B}_i) \arrow[d, "\Upsilon"] \\
T^{\leqslant 1}\boxtimes_u^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \arrow[d, "\vartheta^{-1}"', "\wr"] & {}^{\mathcal{V}}\underline{\mathscr{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i,\boxtimes_u^{i\in I}\mathcal{B}_i) \arrow[d, hook, "{}^{\mathcal{V}}\underline{\mathscr{Q}_p}(1,\mathrm{in}_1)"] \\
\boxtimes^{i\in I}T^{\leqslant 1}{}^{\mathcal{V}}\underline{\mathscr{Q}_u}(\mathcal{A}_i,\mathcal{B}_i) \arrow[d, "\boxtimes^I\underline{T^{\leqslant 1}}"'] \arrow[r, phantom, "="] & {}^{\mathcal{V}}\underline{\mathscr{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i, T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{B}_i) \\
\boxtimes^{i\in I}{}^{\mathcal{V}}\underline{\mathscr{Q}_p}(T^{\leqslant 1}\mathcal{A}_i,T^{\leqslant 1}\mathcal{B}_i) \arrow[r, "\underline{\boxtimes^I}"] & {}^{\mathcal{V}}\underline{\mathscr{Q}_p}(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{A}_i,\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{B}_i) \arrow[u, "\wr", "{}^{\mathcal{V}}\underline{\mathscr{Q}_p}(\vartheta^{-1},\vartheta)"']
\end{tikzcd}$$

Notice that the natural morphism of quivers Υ is faithful, moreover, gives identity maps of objects of morphisms. It takes a quiver morphism f to $\operatorname{Ob} f$.

7.9 Various multicategories of quivers. From now on we consider only $\mathscr{Q} = {}^{\mathbf{gr}}\mathscr{Q}$ and later also ${}^{d}\mathscr{Q} = {}^{\mathbf{dg}}\mathscr{Q}$. Multimaps $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ in $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ are morphisms of quivers $f : \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}$, or equivalently, morphisms of quivers $\overline{f} : \boxtimes^{i\in I}T\mathcal{A}_i \to \mathcal{B}$ such that

$\overline{f}\big|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = 0$. A bijection between the two presentations is established as follows. Given $f : \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}$, we construct

$$\overline{f} = \big[\boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow[\sim]{\vartheta^I} T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{T^{\leqslant 1}f} T^{\leqslant 1}\mathcal{B} \xrightarrow{\mathrm{pr}_1} \mathcal{B}\big] = \widehat{T^{\leqslant 1}}f \cdot \mathrm{pr}_1 . \tag{7.9.1}$$

Given $\overline{f} : \boxtimes^{i\in I}T\mathcal{A}_i \to \mathcal{B}$, we construct

$$f = \big[\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xhookrightarrow{\mathrm{in}_1} T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow[\sim]{(\vartheta^I)^{-1}} \boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\overline{f}} \mathcal{B}\big]. \tag{7.9.2}$$

Let $f : \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}$ be a morphism of $\mathscr{Q}_u$. The corresponding $T^{\geqslant 1}$-coalgebra morphism $\hat{f} : \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \to T^{\geqslant 1}\mathcal{B}$ defined via (5.3.1) is given by the composition in the upper row in the following commutative diagram

$$\begin{tikzcd}
\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \arrow[r, "\boxtimes_u^I\Delta"] \arrow[d, hook] & \boxtimes_u^{i\in I}T^{\geqslant 1}T^{\geqslant 1}\mathcal{A}_i \arrow[r, "\tau^I"] \arrow[d, hook] & T^{\geqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \arrow[r, "T^{\geqslant 1}f"] \arrow[d, hook] & T^{\geqslant 1}\mathcal{B} \arrow[d, hook] \\
\boxtimes^{i\in I}T\mathcal{A}_i \arrow[r, "\boxtimes^I\tilde{\Delta}"] & \boxtimes^{i\in I}TT^{\geqslant 1}\mathcal{A}_i \arrow[r, "\tilde{\tau}^I"] & T\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \arrow[r, "Tf"] & T\mathcal{B}
\end{tikzcd} \tag{7.9.3}$$

The composition in the lower row is denoted $\tilde{f}$. It restricts to 0-th component as

$$\big[\boxtimes^{i\in I}T^0\mathcal{A}_i = \boxtimes^{i\in I}T^0T^{\geqslant 1}\mathcal{A}_i = T^0\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{T^0f} T^0\mathcal{B}\big] = T^0f,$$

which gives $\operatorname{Ob} f$ on objects and identity map on morphisms. Together with diagram (7.9.3) this proves that

$$\tilde{f} = \big(\boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\vartheta} T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{T^{\leqslant 1}\hat{f}} T\mathcal{B}\big) = \widehat{T^{\leqslant 1}}\hat{f}. \tag{7.9.4}$$

7.10 Remark. According to Proposition 6.19 the first factor ϑ is an isomorphism of augmented coalgebras in $\mathscr{Q}_p$. The second factor $T^{\leqslant 1}\hat{f}$ is a morphism of augmented coalgebras by Lemma 6.12. Therefore, $\tilde{f} : \boxtimes^{i\in I}T\mathcal{A}_i \to T\mathcal{B}$ is an augmented counital coalgebra homomorphism in $\mathscr{Q}_p$ as well (with respect to Δ_0, $\varepsilon = \mathrm{pr}_0$, $\eta = \mathrm{in}_0$).

Notice also that

$$\begin{aligned}
\tilde{f}\cdot\mathrm{pr}_1 &= \big(\boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\vartheta} T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\mathrm{pr}_1} \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\hat{f}} T^{\geqslant 1}\mathcal{B} \xrightarrow{\mathrm{pr}_1} \mathcal{B}\big) \\
&= \big(\boxtimes^{i\in I}T\mathcal{A}_i \xrightarrow{\vartheta} T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\mathrm{pr}_1} \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{f} \mathcal{B}\big) = \overline{f}
\end{aligned} \tag{7.10.1}$$

due to Lemma 5.3.

7.11 Definition. *Components* of a morphism $\overline{f} : \boxtimes^{i\in I}T\mathcal{A}_i \to \mathcal{B}$ in $\mathscr{A} \overset{\text{def}}{=} \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ are its restrictions $f_j : \boxtimes^{i\in I}T^{j^i}\mathcal{A}_i \to \mathcal{B}$ to direct summands of the source, where $j = (j^i)_{i\in I} \in (\mathbb{Z}_{\geqslant 0})^I \smallsetminus \{0\}$. Such a morphism f is *strict* if its arbitrary component f_j vanishes unless $|j| = 1$.

If components of $\overline{f} : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ are $f_j : \boxtimes^{i\in I} T^{j^i}\mathcal{A}_i \to \mathcal{B}$, then matrix coefficients of $\tilde{f}$ are given by

$$\sum_{\substack{j_1,\dots,j_m\in(\mathbb{Z}_{\geqslant 0})^I\smallsetminus\{0\}\\ j_1+\dots+j_m=(\ell^i)_{i\in I}}} \bigl(\boxtimes^{i\in I} T^{\ell^i}\mathcal{A}_i \xrightarrow{\boxtimes^I \lambda^{g_i}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{j^i_p}\mathcal{A}_i \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{j^i_p}\mathcal{A}_i \xrightarrow{f_{j_1}\otimes\dots\otimes f_{j_m}} T^m\mathcal{B}\bigr), \quad (7.11.1)$$

where the isotonic maps $g_i : \boldsymbol{\ell}^{\mathbf{i}} \to \mathbf{m}$ correspond to partition of ℓ^i into summands $j^i_p = |g_i^{-1}(p)|$, $p \in \mathbf{m}$.

If $I = \varnothing$, then $\boxtimes_u^{i\in\varnothing} T^{\geqslant 1}\mathcal{A}_i = \mathbb{1}_u$ is the quiver with one object and zero module of morphisms. Therefore, morphisms $f : \boxtimes_u^{i\in\varnothing} T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}$ of $\mathcal{Q}_u$ and the corresponding $T^{\geqslant 1}$-coalgebra morphisms $\hat{f} : \boxtimes_u^{i\in\varnothing} T^{\geqslant 1}\mathcal{A}_i \to T^{\geqslant 1}\mathcal{B}$ are identified with objects of $\mathcal{B}$.

The structure of lax Monoidal comonad $\bigl((T^{\geqslant 1}, \tau), \Delta, \varepsilon\bigr)$ on the symmetric closed Monoidal category $\mathcal{Q}_u$ of graded $\Bbbk$-quivers implies by Proposition 5.17 that $\mathscr{A}$ is a symmetric closed multicategory with the inner homomorphisms objects

$$\underline{\widehat{\mathcal{Q}_u}^{T^{\geqslant 1}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) = \underline{\mathcal{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}) = \underline{\mathcal{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}) \xrightarrow[\sim]{\underline{\mathcal{Q}_p}(\vartheta^I,1)} \underline{\mathcal{Q}_p}(\boxtimes^{i\in I} T\mathcal{A}_i, \mathcal{B}).$$

Composition in $\mathscr{A}$ is described as follows. Let $\phi : I \to J$ be a map in $\mathcal{S}$. The composition of given multimaps $f_j : \boxtimes_u^{i\in\phi^{-1}j} T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}_j$, $h : \boxtimes_u^{j\in J} T^{\geqslant 1}\mathcal{B}_j \to \mathcal{C}$ is found as

$$\begin{split}(f_j)\cdot h = \bigl[\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\boxtimes_u^I \Delta} \boxtimes_u^{i\in I} T^{\geqslant 1}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\lambda_u^\phi} \boxtimes_u^{j\in J} \boxtimes_u^{i\in\phi^{-1}j} T^{\geqslant 1}T^{\geqslant 1}\mathcal{A}_i \\ \xrightarrow{\boxtimes_u^{j\in J}\tau^{\phi^{-1}j}} \boxtimes_u^{j\in J} T^{\geqslant 1} \boxtimes_u^{i\in\phi^{-1}j} T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\boxtimes_u^{j\in J} T^{\geqslant 1} f_j} \boxtimes_u^{j\in J} T^{\geqslant 1}\mathcal{B}_j \xrightarrow{h} \mathcal{C}\bigr]. \quad (7.11.2)\end{split}$$

Using (6.1.2) and (6.3.2) we obtain the second presentation of composition:

$$\begin{split}\overline{(f_j)\cdot h} = \bigl[\boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\boxtimes^I \tilde{\Delta}} \boxtimes^{i\in I} TT^{\geqslant 1}\mathcal{A}_i \xrightarrow{\lambda^\phi} \boxtimes^{j\in J} \boxtimes^{i\in\phi^{-1}j} TT^{\geqslant 1}\mathcal{A}_i \\ \xrightarrow{\boxtimes^{j\in J}\tilde{\tau}^{\phi^{-1}j}} \boxtimes^{j\in J} T \boxtimes_u^{i\in\phi^{-1}j} T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\boxtimes^{j\in J} T f_j} \boxtimes^{j\in J} T\mathcal{B}_j \xrightarrow{\overline{h}} \mathcal{C}\bigr]. \quad (7.11.3)\end{split}$$

If $I = \varnothing$, then $T^{\leqslant 1} \boxtimes_u^{i\in\varnothing} T^{\geqslant 1}\mathcal{A}_i = \mathbb{1}_p$ and $\boxtimes^{i\in\varnothing} T\mathcal{A}_i = \mathbb{1}_p$ are quivers with one object and the module $\Bbbk$ of morphisms. Therefore, $\underline{\mathscr{A}}(\,;\mathcal{B}) = \underline{\mathcal{Q}_p}(\mathbb{1}_p, \mathcal{B}) \simeq \mathcal{B}$ as it has to be in a closed multicategory.

Evaluation $\mathrm{ev}^{\mathscr{A}}_{(\mathcal{A}_i);\mathcal{B}}$ in $\mathscr{A}$ as a morphism in $\mathcal{Q}$ is the following composition

$$\begin{split}\boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}\underline{\mathcal{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B})] \xrightarrow{\lambda_u^{I\sqcup\mathbf{1}\to\mathbf{2}}} (\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathcal{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}) \\ \xrightarrow{1\boxtimes_u \mathrm{pr}_1} (\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i) \boxtimes_u \underline{\mathcal{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}) \xrightarrow{\mathrm{ev}^{\mathcal{Q}_u}_{\boxtimes_u^I T^{\geqslant 1}\mathcal{A}_i, \mathcal{B}}} \mathcal{B}.\end{split}$$

The notation $sM = M[1]$ (for shifted grading) is extended from $\Bbbk$-modules M to graded $\Bbbk$-linear quivers $\mathcal{A}$. Thus, $\operatorname{Ob} s\mathcal{A} = \operatorname{Ob}\mathcal{A}[1] = \operatorname{Ob}\mathcal{A}$ and $s\mathcal{A}(X,Y) = \mathcal{A}[1](X,Y) = (\mathcal{A}(X,Y))[1]$. Denote $\mathsf{Q} = {}^{[1]}\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ the symmetric multicategory, whose objects are graded $\Bbbk$-linear quivers, and multimaps are

$$\mathsf{Q}((\mathcal{A}_i)_{i\in I};\mathcal{B}) = \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}((\mathcal{A}_i[1])_{i\in I};\mathcal{B}[1]) = \mathscr{Q}(\boxtimes_u^{i\in I} T^{\geqslant 1}(\mathcal{A}_i[1]),\mathcal{B}[1])$$
$$\simeq \{f \in \mathscr{Q}(\boxtimes^{i\in I} T(\mathcal{A}_i[1]),\mathcal{B}[1]) \mid f\big|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = 0\}.$$

The multiquiver Q is isomorphic to $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ via the shift map $\mathcal{A} \mapsto s\mathcal{A} = \mathcal{A}[1]$. We equip Q with compositions coming from $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$.

The above implies that Q is a symmetric closed multicategory with

$$\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) = s^{-1}\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}}((s\mathcal{A}_i)_{i\in I};s\mathcal{B}) = \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i, s\mathcal{B})[-1]$$
$$= \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i, s\mathcal{B})[-1] \xrightarrow[\sim]{s\underline{\mathscr{Q}_p}(\vartheta^I,1)s^{-1}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in I} T s\mathcal{A}_i, s\mathcal{B})[-1]. \quad (7.11.4)$$

Its evaluation is the following composite morphism in $\mathscr{Q}$:

$$\operatorname{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}} = \big[(\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i, s\mathcal{B})$$
$$\xrightarrow{1\boxtimes_u \operatorname{pr}_1} (\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i, s\mathcal{B}) \xrightarrow{\operatorname{ev}^{\mathscr{Q}_u}_{\boxtimes_u^I T^{\geqslant 1} s\mathcal{A}_i, s\mathcal{B}}} s\mathcal{B}\big].$$

We shall identify an arrow $r : f \to g$ in $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})[1]$ with the tuple $(r_k)_{k\in(\mathbb{Z}_{\geqslant 0})^I}$, where $r_k : \operatorname{Ob} f \to \operatorname{Ob} g$ is an arrow in $\underline{\mathscr{Q}_p}(\boxtimes^{i\in I} T^{k_i} s\mathcal{A}_i, s\mathcal{B})$, $k = (k_i)_{i\in I}$. In particular, r_0 is an element of

$$\underline{\mathscr{Q}_p}(\boxtimes^{i\in I} T^0 s\mathcal{A}_i, s\mathcal{B})(f,g) \cong \prod_{A_i\in\operatorname{Ob}\mathcal{A}_i}^{i\in I} s\mathcal{B}(f(A_i)_{i\in I}, g(A_i)_{i\in I}).$$

The elements r_k are *components* of r.

7.12 Proposition. *The maps $\mathcal{A} \mapsto Ts\mathcal{A}$, $f \mapsto \tilde{f}$ define a fully faithful multifunctor $\mathsf{Q} = {}^{[1]}\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \to \widehat{\mathrm{ac}\mathscr{Q}_p}$.*

Proof. We obtain the described multifunctor as the composition of four multifunctors:

$$\mathsf{Q} \xrightarrow{\sim} \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \xrightarrow{\sim} \widehat{\mathscr{Q}}^{f}_{uT^{\geqslant 1}} \xrightarrow{\widehat{T^{\geqslant 1}}'} \widehat{\mathscr{Q}}_{uT^{\geqslant 1}} = \widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \xrightarrow{\widehat{T^{\leqslant 1}}} \widehat{\mathrm{ac}\mathscr{Q}_p}.$$

The first multifunctor is given by $\mathcal{A} \mapsto s\mathcal{A} = \mathcal{A}[1]$, $f \mapsto f$, it is an isomorphism of multicategories by definition of Q. The second multifunctor $s\mathcal{A} \mapsto s\mathcal{A}$, $f \mapsto \hat{f}$ is described

in Remark 5.16, it is an isomorphism of multicategories. The third multifunctor $s\mathcal{A} \mapsto T^{\geqslant 1}s\mathcal{A}$ is identity on morphisms, see Remark 5.19. Finally, the fourth multifunctor comes from the symmetric Monoidal functor $(T^{\leqslant 1}, \vartheta^I) : \mathscr{Q}_{uT^{\geqslant 1}} \to \mathrm{ac}\mathscr{Q}_p$ of Proposition 6.19, it is fully faithful by the same proposition. Their composition is $\mathcal{A} \mapsto T^{\leqslant 1}T^{\geqslant 1}s\mathcal{A} = Ts\mathcal{A}$, $f \mapsto \widehat{T^{\leqslant 1}\hat{f}} = \tilde{f}$ by (7.9.4). □

7.13 Closing transformation for the multifunctor $T^{\geqslant 1}$. We shall work with the lax symmetric Monoidal endofunctor $(T^{\geqslant 1}, \tau) : \mathscr{Q}_u \to \mathscr{Q}_u$ of the symmetric closed Monoidal category $\mathscr{Q}_u$. According to Section 4.18 there exists closing multinatural transformation (5.3.2) for the multifunctor $\widehat{(T^{\geqslant 1}, \tau)}$

$$\underline{T^{\geqslant 1}} : T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B}) \to \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i, T^{\geqslant 1}\mathcal{B}).$$

It is determined by equation

$$\begin{tikzcd}[column sep=small]
(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B}) \arrow[d, "\tau"'] \arrow[dr, "1\boxtimes_u \underline{T^{\geqslant 1}}"] & \\
& (\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i, T^{\geqslant 1}\mathcal{B}) \arrow[d, "\mathrm{ev}^{\mathscr{Q}_u}"] \\
T^{\geqslant 1}\big[(\boxtimes_u^{i\in I}\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B})\big] \arrow[r, phantom, "="] \arrow[r, "T^{\geqslant 1}\mathrm{ev}^{\mathscr{Q}_u}"'] & T^{\geqslant 1}\mathcal{B}
\end{tikzcd} \tag{7.13.1}$$

An object of $T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B})$, which is a morphism of quivers $f : \boxtimes_u^{i\in I}\mathcal{A}_i \to \mathcal{B}$, is mapped to the morphism of quivers $\underline{T^{\geqslant 1}}f : \boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \to T^{\geqslant 1}\mathcal{B}$, an object of $\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i, T^{\geqslant 1}\mathcal{B})$, computed via recipe of Proposition 7.5. On objects $X_i \in \operatorname{Ob}\mathcal{A}_i$ we have $\underline{T^{\geqslant 1}}f : (X_i)_{i\in I} \mapsto f(X_i)_{i\in I}$, hence, $\operatorname{Ob}\underline{T^{\geqslant 1}}f = \operatorname{Ob} f$. To find $\underline{T^{\geqslant 1}}f$ on morphisms let us write restriction of the left bottom path of diagram (7.13.1) to the following direct summand

$$\begin{aligned}
(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i) \boxtimes T^0T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B}) &\xrightarrow{\tau\boxtimes 1} \oplus_{m>0}\, (T^m \boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes T^0\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B}) \\
&\xrightarrow[\sim]{\oplus 1\boxtimes\lambda^{\varnothing\to\mathbf{m}}} \oplus_{m>0}\, (T^m \boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes T^mT^0\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B}) \\
&\xrightarrow[\sim]{\oplus\overline{\varkappa}^{-1}} \oplus_{m>0}\, T^m\big[(\boxtimes_u^{i\in I}\mathcal{A}_i) \boxtimes T^0\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B})\big] \xrightarrow{\oplus T^m\,\mathrm{ev}''} \oplus_{m>0}\, T^m\mathcal{B}.
\end{aligned}$$

Clearly, its restriction to f gives

$$\underline{T^{\geqslant 1}}f = \big[\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i \xrightarrow{\tau} T^{\geqslant 1} \boxtimes_u^{i\in I} \mathcal{A}_i \xrightarrow{T^{\geqslant 1}f} T^{\geqslant 1}\mathcal{B}\big].$$

To find $\underline{T^{\geqslant 1}}$ on morphisms let us write restriction of equation (7.13.1) to the direct summand $(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i) \boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i, \mathcal{B})$. Replacing it with an isomorphic quiver

we come to the following equation:

$$\begin{aligned}
&\bigl[(\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B}) \xrightarrow{\widetilde{\tau}^I\boxtimes 1} (T\boxtimes_u^{i\in I}\mathcal{A}_i)\boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B})\\
&\qquad\xrightarrow{\widetilde{\tau}^{\mathbf{2}}} T^{\geqslant 1}\bigl[(\boxtimes_u^{i\in I}\mathcal{A}_i)\boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B})\bigr] \xrightarrow{T^{\geqslant 1}\operatorname{ev}^{\mathscr{Q}_u}} T^{\geqslant 1}\mathcal{B}\bigr]\\
&= \bigl[(\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B}) \xrightarrow{\vartheta^I\boxtimes\underline{T^{\geqslant 1}}}\\
&\qquad (T^{\leqslant 1}\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i)\boxtimes\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i,T^{\geqslant 1}\mathcal{B}) \xrightarrow{\operatorname{ev}'} T^{\geqslant 1}\mathcal{B}\bigr].
\end{aligned}\tag{7.13.2}$$

For arbitrary quivers $\mathcal{C}$, $\mathcal{D}$ we expand the transformation

$$\widetilde{\tau}^{\mathbf{2}} : T\mathcal{C}\boxtimes T\mathcal{D}\to T(\mathcal{C}\boxtimes_u\mathcal{D})$$

which occurs in the above equation, into

$$\begin{aligned}
\widetilde{\tau}^{\mathbf{2}} = \Bigl[T^k\mathcal{C}\boxtimes T^j\mathcal{D} &\xrightarrow{(\lambda^{g:\mathbf{k}\hookrightarrow\mathbf{m}}\boxtimes\lambda^{h:\mathbf{j}\hookrightarrow\mathbf{m}})}\\
&\bigoplus_{\substack{g:\mathbf{k}\hookrightarrow\mathbf{m}\\ h:\mathbf{j}\hookrightarrow\mathbf{m}}}^{\operatorname{Im}g\cup\operatorname{Im}h=\mathbf{m}} (\otimes^{p\in\mathbf{m}}T^{\chi(p\in\operatorname{Im}g)}\mathcal{C})\boxtimes(\otimes^{p\in\mathbf{m}}T^{\chi(p\in\operatorname{Im}h)}\mathcal{D})\\
&\xrightarrow[\sim]{\oplus\varkappa^{-1}} \bigoplus_{\substack{g:\mathbf{k}\hookrightarrow\mathbf{m}\\ h:\mathbf{j}\hookrightarrow\mathbf{m}}}^{\operatorname{Im}g\cup\operatorname{Im}h=\mathbf{m}} \otimes^{p\in\mathbf{m}}(T^{\chi(p\in\operatorname{Im}g)}\mathcal{C}\boxtimes T^{\chi(p\in\operatorname{Im}h)}\mathcal{D}) = T^m(\mathcal{C}\boxtimes_u\mathcal{D})\Bigr].
\end{aligned}$$

For an arbitrary quiver $\mathcal{C}\in\operatorname{Ob}\mathscr{Q}$ and a positive integer M introduce a morphism of quivers

$$\nu^{\leqslant M} : T\mathcal{C}\to TT^{\leqslant 1}\mathcal{C},$$
$$\nu^{\leqslant M}_{kn} : T^k\mathcal{C}\to\oplus_{S\subset\mathbf{n}}\otimes^{p\in\mathbf{n}}T^{\chi(p\in S)}\mathcal{C} = T^nT^{\leqslant 1}\mathcal{C},$$

as follows. By definition, the matrix element $\nu^{\leqslant M}_{kn}$ vanishes, if $n>k+M$. Its summand corresponding to subset $S\subset\mathbf{n}$ vanishes unless $|S|=k$. If $n\leqslant k+M$ and $|S|=k$, the summand is defined as $\lambda^{g:\mathbf{k}\hookrightarrow\mathbf{n}} : T^k\mathcal{C}\xrightarrow{\sim}\otimes^{p\in\mathbf{n}}T^{\chi(p\in S)}\mathcal{C}$, where the image of isotonic embedding $g:\mathbf{k}\hookrightarrow\mathbf{n}$ is S. Thus, $\nu^{\leqslant M} : T\to TT^{\leqslant 1}$ is a natural transformation. We shall use it in such formulas, where dependence on M is irrelevant, provided the truncation parameter M is large enough.

The only solution of (7.13.2) is the following. Let $f^p:\boxtimes_u^{i\in I}\mathcal{A}_i\to\mathcal{B}$, $0\leqslant p\leqslant n$, $n>0$, be morphisms of quivers. Homogeneous elements

$$r^p\in\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B})(f^{p-1},f^p) = \underline{\mathscr{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i,\mathcal{B})(\overline{f^{p-1}},\overline{f^p}),$$

where $\overline{f^p}=\operatorname{Ob}\overline{f^p}$ refer to quiver morphisms

$$\overline{f^p} = \bigl[T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i\xrightarrow{\operatorname{pr}_1}\boxtimes_u^{i\in I}\mathcal{A}_i\xrightarrow{f^p}\mathcal{B}\bigr],$$

form a sequence

$$r = \big(f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{n-1} \xrightarrow{r^n} f^n\big) : \boxtimes_u^{i\in I} \mathcal{A}_i \to \mathcal{B}.$$

Up to an isomorphism ϑ, we may write $\overline{f^p}$ as the quiver morphism $\overline{f^p} : \boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i \to \mathcal{B}$. It is determined by its components $f^p_j = f^p_{(j_i)}$, $f^p_0 = 0$, $j = (j_i)_{i\in I}$, $j_i \in \{0,1\}$, $i \in I$. Each r^p can be viewed as an element of $\underline{\mathscr{Q}_p}(\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i, \mathcal{B})(\overline{f^{p-1}}, \overline{f^p})$, so it is determined by its components $r^p_j = r^p_{(j_i)}$, $j = (j_i)_{i\in I}$, $j_i \in \{0,1\}$, $i \in I$. The sequence r is mapped by $\underline{T^{\geqslant 1}}$ to

$$\underline{T^{\geqslant 1}}(r^1 \otimes \dots \otimes r^n) = \Big[\boxtimes^{i\in I} T\mathcal{A}_i \xrightarrow{\widetilde{\tau}^I} T \boxtimes_u^{i\in I} \mathcal{A}_i \xrightarrow{\nu^{\leqslant M}} TT^{\leqslant 1} \boxtimes_u^{i\in I} \mathcal{A}_i \xrightarrow{\sum \overline{f^0}^{\otimes p_0} \otimes r^1 \otimes \overline{f^1}^{\otimes p_1} \otimes r^2 \otimes \dots \otimes \overline{f^{n-1}}^{\otimes p_{n-1}} \otimes r^n \otimes \overline{f^n}^{\otimes p_n}} T^{\geqslant 1}\mathcal{B}\Big], \quad (7.13.3)$$

where $M \geqslant n$ is an arbitrary integer. The last map can be specified as

$$\sum_{p_0+\dots+p_n+n=m} \overline{f^0}^{\otimes p_0} \otimes r^1 \otimes \overline{f^1}^{\otimes p_1} \otimes r^2 \otimes \dots \otimes \overline{f^{n-1}}^{\otimes p_{n-1}} \otimes r^n \otimes \overline{f^n}^{\otimes p_n} : T^m T^{\leqslant 1} \boxtimes_u^{i\in I} \mathcal{A}_i \to T^m \mathcal{B}.$$

7.14 $T^{\geqslant 1}$-coderivations. Let $\mathcal{C}$, $\mathcal{D}$ be $T^{\geqslant 1}$-coalgebras. Complete the pair of morphisms on the right to an equalizer diagram in $\mathscr{Q}$:

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\text{equalizer of the pair}", "e"'] & T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[rr, "T^{\geqslant 1}\underline{\mathscr{Q}_u}(1,\delta)"] \arrow[dr, "\Delta"'] & & T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C}, T^{\geqslant 1}\mathcal{D}) \\
& & T^{\geqslant 1}T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[ur, "T^{\geqslant 1}\Theta"'] &
\end{tikzcd}$$

A quiver map $f : \mathcal{C} \to \mathcal{D}$ is an object of the kernel $\mathcal{E}$ if and only if equation

$$f_\bullet T^{\geqslant 1}\underline{\mathscr{Q}_u}(1,\delta) = f_\bullet(\Delta \cdot T^{\geqslant 1}\Theta) \qquad (7.14.1)$$

holds. By Lemma 5.4

$$\Theta = \big[T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \xrightarrow{\underline{T^{\geqslant 1}}} \underline{\mathscr{Q}_u}(T^{\geqslant 1}\mathcal{C}, T^{\geqslant 1}\mathcal{D}) \xrightarrow{\underline{\mathscr{Q}_u}(\delta,1)} \underline{\mathscr{Q}_u}(\mathcal{C}, T^{\geqslant 1}\mathcal{D})\big].$$

Using results of Sections 7.6.1, 7.6.2, and 7.13 we find that (7.14.1) is equivalent to

$$f \cdot \delta = \delta \cdot T^{\geqslant 1} f,$$

that is, f is an object of $\mathcal{E}$ if and only if f is a morphism of $T^{\geqslant 1}$-coalgebras. Equivalently, $f : (\mathcal{C}, \overline{\Delta}) \to (\mathcal{D}, \overline{\Delta})$ is a morphism of coassociative coalgebras.

We describe now part of the quiver $\mathcal{E}$. Define the quiver of coderivations $\operatorname{Coder}(\mathcal{C},\mathcal{D})$ via a pull-back square in the left part of the commutative diagram:

$$
\begin{tikzcd}
\operatorname{Coder}(\mathcal{C},\mathcal{D}) \arrow[r, hook, "e'"] \arrow[dd] \arrow[ddr, phantom, very near start, "\lrcorner"] & \underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[rr, "\underline{\mathscr{Q}_u}(1{,}\delta)"] \arrow[dr, hook, "\mathrm{in}_1"] \arrow[dd, hook, "\mathrm{in}_1"'] & & \underline{\mathscr{Q}_u}(\mathcal{C},T^{\geqslant1}\mathcal{D}) \arrow[dd, hook, "\mathrm{in}_1"] \\
& & T^{\geqslant1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[ur, "\Theta"'] \arrow[dd, hook, "\mathrm{in}_1"] & \\
\mathcal{E} \arrow[r, hook, "e"] & T^{\geqslant1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[rr, "T^{\geqslant1}\underline{\mathscr{Q}_u}(1{,}\delta)"] \arrow[dr, "\Delta"'] & & T^{\geqslant1}\underline{\mathscr{Q}_u}(\mathcal{C},T^{\geqslant1}\mathcal{D}) \\
& & T^{\geqslant1}T^{\geqslant1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D}) \arrow[ur, "T^{\geqslant1}\Theta"'] &
\end{tikzcd}
$$

Thus, $\operatorname{Coder}(\mathcal{C},\mathcal{D})$ is the intersection of subobjects $\mathcal{E}$ and $\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D})$ of $T^{\geqslant1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D})$. Objects of $\operatorname{Coder}(\mathcal{C},\mathcal{D})$ are again $T^{\geqslant1}$-coalgebra morphisms $\mathcal{C}\to\mathcal{D}$. One can verify that e' is the equalizer of the upper pair of maps $\underline{\mathscr{Q}_u}(1,\delta)$ and $\eta\cdot\Theta$. Let us compute the graded modules of morphisms of $\operatorname{Coder}(\mathcal{C},\mathcal{D})$.

Let $f,g:\mathcal{C}\to\mathcal{D}$ be $T^{\geqslant1}$-coalgebra morphisms. An element $r\in\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{D})(f,g)=\underline{\mathscr{Q}_p}(T^{\leqslant1}\mathcal{C},\mathcal{D})(f,g)$ belongs to $\operatorname{Coder}(\mathcal{C},\mathcal{D})(f,g)$ if and only if equation

$$r\centerdot\underline{\mathscr{Q}_u}(1,\delta)=r\centerdot\Theta \tag{7.14.2}$$

holds. The left hand side of (7.14.2) equals $r\cdot\delta$ by (7.6.3). The right hand side is $r\centerdot(\underline{T^{\geqslant1}}\cdot\underline{\mathscr{Q}_u}(\delta,1))$. By (7.13.3) and (7.6.1) the latter is equal to

$$\bigl[T^{\leqslant1}\mathcal{C}\xrightarrow{T^{\leqslant1}\delta}T^{\leqslant1}T^{\geqslant1}\mathcal{C}=T\mathcal{C}\xrightarrow{\nu^{\leqslant1}}TT^{\leqslant1}\mathcal{C}\xrightarrow{\sum\overline{f}^{\otimes p}\otimes r\otimes\overline{g}^{\otimes q}}T^{\geqslant1}\mathcal{D}\bigr],$$

where

$$\overline{f}=\bigl[T^{\leqslant1}\mathcal{C}\xrightarrow{\mathrm{pr}_1}\mathcal{C}\xrightarrow{f}\mathcal{D}\bigr],\qquad\overline{g}=\bigl[T^{\leqslant1}\mathcal{C}\xrightarrow{\mathrm{pr}_1}\mathcal{C}\xrightarrow{g}\mathcal{D}\bigr].$$

We conclude that equation (7.14.2) is equivalent to

$$\begin{aligned}&\bigl[T^{\leqslant1}\mathcal{C}\xrightarrow{r}\mathcal{D}\xrightarrow{\delta}T^{\geqslant1}\mathcal{D}\bigr]\\&\qquad=\bigl[T^{\leqslant1}\mathcal{C}\xrightarrow{T^{\leqslant1}\delta}T^{\leqslant1}T^{\geqslant1}\mathcal{C}=T\mathcal{C}\xrightarrow{\nu^{\leqslant1}}TT^{\leqslant1}\mathcal{C}\xrightarrow{\sum\overline{f}^{\otimes p}\otimes r\otimes\overline{g}^{\otimes q}}T^{\geqslant1}\mathcal{D}\bigr].\end{aligned}\tag{7.14.3}$$

Let us find all solutions r of this equation. We call them $T^{\geqslant1}$-coderivations.

A $\Bbbk$-span morphism $r:T^{\leqslant1}\mathcal{C}\to\mathcal{D}$ can be presented by a pair of $\Bbbk$-span morphisms

$$r_0=\bigl[T^0\mathcal{C}\xhookrightarrow{\mathrm{in}_0}T^{\leqslant1}\mathcal{C}\xrightarrow{r}\mathcal{D}\bigr],\qquad r_+=\bigl[\mathcal{C}\xhookrightarrow{\mathrm{in}_1}T^{\leqslant1}\mathcal{C}\xrightarrow{r}\mathcal{D}\bigr].$$

Restricting equation (7.14.3) to $T^0\mathcal{C}$ we get

$$\bigl[T^0\mathcal{C}\xrightarrow{r_0}\mathcal{D}\xrightarrow{\delta}T^{\geqslant1}\mathcal{D}\bigr]=\bigl[T^0\mathcal{C}\xrightarrow{r_0}\mathcal{D}\xhookrightarrow{\mathrm{in}_1}T^{\geqslant1}\mathcal{D}\bigr].$$

This property is satisfied if and only if

$$\big[T^0\mathcal{C} \xrightarrow{r_0} \mathcal{D} \xrightarrow{\overline{\Delta}} \mathcal{D} \otimes \mathcal{D}\big] = 0. \tag{7.14.4}$$

Restricting equation (7.14.3) to $T^1\mathcal{C}$ we get

$$\begin{aligned}
&\big[\mathcal{C} \xrightarrow{r_+} \mathcal{D} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{D}\big] \\
&= \Big[\mathcal{C} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{C} = \bigoplus_{m=1}^{\infty} T^m\mathcal{C} \xrightarrow{\sum_{p+1+q=m} f^{\otimes p}\otimes r_+ \otimes g^{\otimes q}} \bigoplus_{m=1}^{\infty} T^m\mathcal{D}\Big] \\
&\quad + \Big[\mathcal{C} \xrightarrow{\delta} T^{\geqslant 1}\mathcal{C} = \bigoplus_{m=1}^{\infty} T^m\mathcal{C} \xrightarrow{\oplus \lambda^{\mathbf{m}\hookrightarrow\mathbf{m+1}}} \\
&\qquad \bigoplus_{m=1}^{\infty} \bigoplus_{p+q=m} T^p\mathcal{C} \otimes T^0\mathcal{C} \otimes T^q\mathcal{C} \xrightarrow{\sum_{p+q=m} f^{\otimes p}\otimes r_0 \otimes g^{\otimes q}} \bigoplus_{m=1}^{\infty} T^{m+1}\mathcal{D}\Big].
\end{aligned} \tag{7.14.5}$$

Composing this equation with the projection $\mathrm{pr}_1 : T^{\geqslant 1}\mathcal{D} \to \mathcal{D}$ we get the identity $r_+ = r_+$, which imposes no conditions on r. When we compose the equation with the projection $\mathrm{pr}_2 : T^{\geqslant 1}\mathcal{D} \to T^2\mathcal{D}$ we obtain a non-trivial condition

$$\begin{aligned}
\big[\mathcal{C} \xrightarrow{r_+} \mathcal{D} \xrightarrow{\overline{\Delta}} \mathcal{D} \otimes \mathcal{D}\big] &= \big[\mathcal{C} \xrightarrow{\overline{\Delta}} \mathcal{C} \otimes \mathcal{C} \xrightarrow{f\otimes r_+ + r_+ \otimes g} \mathcal{D} \otimes \mathcal{D}\big] \\
&+ \big[\mathcal{C} \xrightarrow{\sim} \mathcal{C} \otimes T^0\mathcal{C} \xrightarrow{f\otimes r_0} \mathcal{D} \otimes \mathcal{D}\big] + \big[\mathcal{C} \xrightarrow{\sim} T^0\mathcal{C} \otimes \mathcal{C} \xrightarrow{r_0\otimes g} \mathcal{D} \otimes \mathcal{D}\big].
\end{aligned} \tag{7.14.6}$$

We claim that compositions of (7.14.5) with projections $\mathrm{pr}_k : T^{\geqslant 1}\mathcal{D} \to T^k\mathcal{D}$ for $k > 2$ give equations which follow from already obtained conditions (7.14.4) and (7.14.6). Let us prove this by induction on $k \geqslant 2$. Assume that equation (7.14.5) composed with pr_k is satisfied:

$$r_+ \cdot \overline{\Delta}^{(k)} = \overline{\Delta}^{(k)} \cdot \sum_{p+1+q=k} f^{\otimes p} \otimes r_+ \otimes g^{\otimes q} + \overline{\Delta}^{(k-1)} \cdot \sum_{p+q=k-1} f^{\otimes p} \otimes r_0 \otimes g^{\otimes q}.$$

Together with (7.14.4) and (7.14.6) it implies that composition of (7.14.5) with pr_{k+1} is satisfied:

$$\begin{aligned}
r_+ \cdot \overline{\Delta}^{(k+1)} &= r_+ \cdot \overline{\Delta}^{(k)} \cdot (\overline{\Delta} \otimes 1^{\otimes(k-1)}) \\
&= \overline{\Delta}^{(k)} \cdot \sum_{p+1+q=k}^{p>0} f\overline{\Delta} \otimes f^{\otimes(p-1)} \otimes r_+ \otimes g^{\otimes q} + \overline{\Delta}^{(k)} \cdot (r_+\overline{\Delta} \otimes g^{\otimes k}) \\
&\quad + \overline{\Delta}^{(k-1)} \cdot \sum_{p+q=k-1}^{p>0} f\overline{\Delta} \otimes f^{\otimes(p-1)} \otimes r_0 \otimes g^{\otimes q} \\
&= \overline{\Delta}^{(k)} \cdot (\overline{\Delta} \otimes 1) \cdot \sum_{t+q=k} f^{\otimes t} \otimes r_+ \otimes g^{\otimes q} + \overline{\Delta}^{(k)} \cdot \sum_{t+q=k} f^{\otimes t} \otimes r_0 \otimes g^{\otimes q}.
\end{aligned}$$

Therefore, equation (7.14.3) for $T^{\geqslant 1}$-coderivations is equivalent to the pair of equations (7.14.4) and (7.14.6). It admits another equivalent presentation. Namely, $T^{\geqslant 1}$-coderivations are in bijection with $\Bbbk$-span morphisms $r' : T^{\leqslant 1}\mathcal{C} \to T^{\leqslant 1}\mathcal{D}$ with $\mathrm{Ob}_s r' = \mathrm{Ob} f$, $\mathrm{Ob}_t r' = \mathrm{Ob}\, g$ which satisfy the equation

$$\begin{aligned}&\bigl[T^{\leqslant 1}\mathcal{C} \xrightarrow{r'} T^{\leqslant 1}\mathcal{D} \xrightarrow{\Delta_0} T^{\leqslant 1}\mathcal{D} \otimes T^{\leqslant 1}\mathcal{D}\bigr] \\ &\qquad = \bigl[T^{\leqslant 1}\mathcal{C} \xrightarrow{\Delta_0} T^{\leqslant 1}\mathcal{C} \otimes T^{\leqslant}\mathcal{C} \xrightarrow{T^{\leqslant 1}f\otimes r' + r'\otimes T^{\leqslant 1}g} T^{\leqslant 1}\mathcal{D} \otimes T^{\leqslant 1}\mathcal{D}\bigr]. \qquad (7.14.7)\end{aligned}$$

Such r' are called $(T^{\leqslant 1}f, T^{\leqslant 1}g)$-coderivations, or, for the sake of brevity, (f,g)-coderivations, as in [Lyu03, Section 2]. Notice that

$$T^{\leqslant 1}f, T^{\leqslant 1}g : (T^{\leqslant 1}\mathcal{C}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0) \to (T^{\leqslant 1}\mathcal{D}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0)$$

are homomorphisms of augmented coassociative counital coalgebras.

Let us prove equivalence of (7.14.7) with the pair of (7.14.4) and (7.14.6). Equation (7.14.7) implies immediately that

$$r' = \bigl[T^{\leqslant 1}\mathcal{C} \xrightarrow{r} \mathcal{D} \xhookrightarrow{\mathrm{in}_1} T^{\leqslant 1}\mathcal{D}\bigr]$$

for some $\Bbbk$-span morphism r. Indeed, composing (7.14.7) with $\varepsilon \otimes \varepsilon : T^{\leqslant 1}\mathcal{D} \otimes T^{\leqslant 1}\mathcal{D} \to T^0\mathcal{D} \otimes T^0\mathcal{D}$ we get $r' \cdot \varepsilon = r' \cdot \varepsilon + r' \cdot \varepsilon$, hence $r' \cdot \varepsilon = 0$. This $r = (r_0, r_+)$ satisfies equations (7.14.4) and (7.14.6). Indeed, restriction of (7.14.7) to $T^0\mathcal{C}$ gives one equation

$$\bigl[T^0\mathcal{C} \xrightarrow{r_0} \mathcal{D} \xrightarrow{\bar{\Delta}} \mathcal{D} \otimes \mathcal{D}\bigr] = 0,$$

which is (7.14.4), and two identities

$$\begin{aligned}\bigl[T^0\mathcal{C} \xrightarrow{r_0} \mathcal{D} \xrightarrow{\sim} T^0\mathcal{D} \otimes \mathcal{D}\bigr] &= \bigl[T^0\mathcal{C} \xrightarrow{\sim} T^0\mathcal{C} \otimes T^0\mathcal{C} \xrightarrow{T^0 f\otimes r_0} T^0\mathcal{D} \otimes \mathcal{D}\bigr], \\ \bigl[T^0\mathcal{C} \xrightarrow{r_0} \mathcal{D} \xrightarrow{\sim} \mathcal{D} \otimes T^0\mathcal{D}\bigr] &= \bigl[T^0\mathcal{C} \xrightarrow{\sim} T^0\mathcal{C} \otimes T^0\mathcal{C} \xrightarrow{r_0\otimes T^0 g} \mathcal{D} \otimes T^0\mathcal{D}\bigr].\end{aligned}$$

Restriction of (7.14.7) to $T^1\mathcal{C}$ gives equation (7.14.6) and two more identities.

When $\mathcal{D} = T^{\geqslant 1}\mathcal{B}$ we know by Proposition 5.10 that the split embedding

$$e = \bigl[T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \xrightarrow{\Delta} T^{\geqslant 1}T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \xrightarrow{T^{\geqslant 1}\Theta} T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C}, T^{\geqslant 1}\mathcal{B})\bigr]$$

is the equalizer of the pair $T^{\geqslant 1}\underline{\mathscr{Q}_u}(1,\Delta)$ and $\Delta \cdot T^{\geqslant 1}\Theta$. The following diagram is commutative:

$$\begin{tikzcd}
\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, hook, "\mathrm{in}_1"] \arrow[d, hook, "\mathrm{in}_1"'] & T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, "\Theta"] \arrow[d, hook, "\mathrm{in}_1"'] & \underline{\mathscr{Q}_u}(\mathcal{C}, T^{\geqslant 1}\mathcal{B}) \arrow[d, hook, "\mathrm{in}_1"] \\
T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, "\Delta"] & T^{\geqslant 1}T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, "T^{\geqslant 1}\Theta"] & T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C}, T^{\geqslant 1}\mathcal{B})
\end{tikzcd} \qquad (7.14.8)$$

Due to (5.10.2) the composite upper row is an embedding split by the morphism $\underline{\mathscr{Q}_u}(1,\varepsilon)$.

7.15 Proposition. *Let $\mathcal{C}$ be a $T^{\geqslant 1}$-coalgebra, $\mathcal{B}$ a graded $\Bbbk$-quiver. The exterior of diagram* (7.14.8)

$$\begin{tikzcd} \underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, hook, "\mathrm{in}_1\cdot\Theta"] \arrow[d, hook, "\mathrm{in}_1"'] & \underline{\mathscr{Q}_u}(\mathcal{C},T^{\geqslant 1}\mathcal{B}) \arrow[d, hook, "\mathrm{in}_1"] \\ T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \arrow[r, hook, "\Delta\cdot T^{\geqslant 1}\Theta"] & T^{\geqslant 1}\underline{\mathscr{Q}_u}(\mathcal{C},T^{\geqslant 1}\mathcal{B}) \end{tikzcd}$$

is a pull-back square. Thus the subquiver of coderivations $\mathrm{Coder}(\mathcal{C},T^{\geqslant 1}\mathcal{B})$ *identifies with the image* $\Theta(\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}))$ *of the embedding* $\mathrm{in}_1\cdot\Theta$.

Proof. The image of $\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B})$ is contained in $\mathrm{Coder}(\mathcal{C},T^{\geqslant 1}\mathcal{B})$ by definition of the latter. The action of Θ on objects $\mathrm{Ob}\,\Theta : \mathrm{Ob}\,\underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \to \mathrm{Ob}\,\mathrm{Coder}(\mathcal{C},T^{\geqslant 1}\mathcal{B})$ is bijective by Lemma 5.3 (see also Corollary 6.11). The inverse map is given by $(f : \mathcal{C} \to T^{\geqslant 1}\mathcal{B}) \mapsto (\check{f} = f\cdot\varepsilon = f\cdot\mathrm{pr}_1 : \mathcal{C}\to\mathcal{B})$.

Let us prove that $\mathrm{in}_1\cdot\Theta : \underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \to \mathrm{Coder}(\mathcal{C},T^{\geqslant 1}\mathcal{B})$ is an isomorphism of quivers. Proposition 5.10 implies that this is a split embedding with the splitting $\underline{\mathscr{Q}_u}(1,\varepsilon)$. We have to show that an (f,g)-coderivation $r : T^{\leqslant 1}\mathcal{C} \to T^{\geqslant 1}\mathcal{B}$ is uniquely determined by $\check{r} = r\varepsilon = \bigl(T^{\leqslant 1}\mathcal{C} \xrightarrow{r} T^{\geqslant 1}\mathcal{B} \xrightarrow{\mathrm{pr}_1} \mathcal{B}\bigr)$. Using equation (7.14.7) we get the commutative diagram,

$$\begin{tikzcd} & T^{\geqslant 1}\mathcal{B} \arrow[d, "\Delta_0^{(l)}"'] \arrow[dr, "\mathrm{pr}_l"] & \\ T^{\leqslant 1}\mathcal{C} \arrow[ur, "r"] \arrow[dr, "\Delta_0^{(l)}"'] & T^lT^{\geqslant 1}\mathcal{B} \arrow[r, "T^l\,\mathrm{pr}_1"] & T^l\mathcal{B} \\ & T^lT^{\leqslant 1}\mathcal{C} \arrow[u, "\sum f^{\otimes q}\otimes r\otimes g^{\otimes t}"] \arrow[ur, "\sum_{q+1+t=l}\check{f}^{\otimes q}\otimes\check{r}\otimes\check{g}^{\otimes t}"'] & \end{tikzcd} \tag{7.15.1}$$

which implies that r is recovered from $r\cdot\mathrm{pr}_1$ via

$$r = \sum_{q+1+t=l>0} \bigl(T^{\leqslant 1}\mathcal{C} \xrightarrow{\Delta_0^{(l)}} T^lT^{\leqslant 1}\mathcal{C} \xrightarrow{\check{f}^{\otimes q}\otimes\check{r}\otimes\check{g}^{\otimes t}} T^l\mathcal{B} \xrightarrow{\mathrm{in}_l} T^{\geqslant 1}\mathcal{B}\bigr).$$

The sum is finite on any element of $(T^{\leqslant 1}\mathcal{C})(X,Y)$ due to $\mathcal{C}$ being a $T^{\geqslant 1}$-coalgebra and due to formulas (6.11.1) and (6.11.4) relating $\overline{\Delta}$ and Δ_0. Hence, $\Theta : \underline{\mathscr{Q}_u}(\mathcal{C},\mathcal{B}) \to \mathrm{Coder}(\mathcal{C},T^{\geqslant 1}\mathcal{B})$ is bijective on morphisms. □

Let us consider now the case of several arguments. Let $(\mathcal{C}_i)_{i\in I}$, $\mathcal{D}$ be $T^{\geqslant 1}$-coalgebras. Then the equalizer diagram in $\underline{\widehat{\mathscr{Q}}_u}$

$$\mathcal{E} \xrightarrow{e} T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}((\mathcal{C}_i)_{i\in I};\mathcal{D}) \underset{\Delta\cdot T^{\geqslant 1}\Theta_{(\mathcal{C}_i);\mathcal{D}}}{\overset{T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}(\triangleright;\delta)}{\rightrightarrows}} T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}((\mathcal{C}_i)_{i\in I};T^{\geqslant 1}\mathcal{D})$$

coincides with the equalizer diagram in $\mathscr{Q}$

$$\mathcal{E} \xrightarrow{e} T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{C}_i, \mathcal{D}) \underset{\Delta\cdot T^{\geqslant 1}\Theta_{\boxtimes_u^{i\in I}\mathcal{C}_i,\mathcal{D}}}{\overset{T^{\geqslant 1}\underline{\mathscr{Q}_u}(1,\delta)}{\rightrightarrows}} T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}\mathcal{C}_i, T^{\geqslant 1}\mathcal{D}).$$

Indeed, $\widehat{\underline{\mathscr{Q}_u}}(\triangleright;\delta) = \underline{\mathscr{Q}_u}(1,\delta)$ by Lemma 4.27 and $\Theta_{(\mathcal{C}_i);\mathcal{D}} = \Theta_{\boxtimes_u^{i\in I}\mathcal{C}_i,\mathcal{D}}$ by Corollary 5.25. Therefore, the coderivation quiver $\mathrm{Coder}((\mathcal{C}_i)_{i\in I};\mathcal{D})$ introduced similarly to $\mathrm{Coder}(\mathcal{C},\mathcal{D})$ coincides with $\mathrm{Coder}(\boxtimes_u^{i\in I}\mathcal{C}_i,\mathcal{D})$. Its objects are $T^{\geqslant 1}$-coalgebra morphisms $f : (\mathcal{C}_i)_{i\in I} \to \mathcal{D}$, or equivalently, $f : \boxtimes_u^{i\in I}\mathcal{C}_i \to \mathcal{D}$. Let us describe the elements of $\mathrm{Coder}(\boxtimes_u^{i\in I}\mathcal{C}_i,\mathcal{D})(f,g)$, the (f,g)-coderivations, in equivalent form. The latter are $\Bbbk$-span morphisms $r' : T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \to T^{\leqslant 1}\mathcal{D}$ with $\mathrm{Ob}_s r' = \mathrm{Ob} f$, $\mathrm{Ob}_t r' = \mathrm{Ob}\, g$ which satisfy the equation

$$\begin{aligned}&\big[T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \xrightarrow{r'} T^{\leqslant 1}\mathcal{D} \xrightarrow{\Delta_0} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big]\\ &\quad= \big[T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \xrightarrow{\Delta_0} (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i)\otimes(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \xrightarrow{T^{\leqslant 1}f\otimes r' + r'\otimes T^{\leqslant 1}g} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big].\end{aligned} \tag{7.15.2}$$

They are in bijection with the $\Bbbk$-span morphisms

$$r'' = \big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \underset{\sim}{\xrightarrow{\vartheta^I}} T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \xrightarrow{r'} T^{\leqslant 1}\mathcal{D}\big].$$

Composing equation (7.15.2) with $\vartheta^I : \boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \to T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i$, which is an isomorphism of augmented counital coassociative algebras by Proposition 6.19, we get

$$\begin{aligned}&\big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{r''} T^{\leqslant 1}\mathcal{D} \xrightarrow{\Delta_0} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big]\\ &= \big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{\vartheta^I} T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \xrightarrow{\Delta_0} (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i)\otimes(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i)\\ &\qquad\qquad \xrightarrow{T^{\leqslant 1}f\otimes r' + r'\otimes T^{\leqslant 1}g} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big]\\ &= \big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\Delta_0} \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i\otimes T^{\leqslant 1}\mathcal{C}_i) \xrightarrow{\varkappa^{-1}} (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\otimes(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\\ &\qquad\qquad \xrightarrow{\vartheta^I\otimes\vartheta^I} (T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i)\otimes(T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i) \xrightarrow{T^{\leqslant 1}f\otimes r' + r'\otimes T^{\leqslant 1}g} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big]\\ &= \big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\Delta_0} \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i\otimes T^{\leqslant 1}\mathcal{C}_i) \xrightarrow{\varkappa^{-1}} (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\otimes(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\\ &\qquad\qquad \xrightarrow{\widehat{T^{\leqslant 1}f}\otimes r'' + r''\otimes\widehat{T^{\leqslant 1}g}} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big].\end{aligned}$$

Therefore, (f,g)-coderivations are $\Bbbk$-span morphisms $r'' : \boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \to T^{\leqslant 1}\mathcal{D}$ with $\mathrm{Ob}_s r'' = \mathrm{Ob} f$, $\mathrm{Ob}_t t'' = \mathrm{Ob}\, g$ which satisfy the equation

$$\begin{aligned}&\big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{r''} T^{\leqslant 1}\mathcal{D} \xrightarrow{\Delta_0} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big]\\ &= \big[\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in I}\Delta_0} \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i\otimes T^{\leqslant 1}\mathcal{C}_i) \xrightarrow{\varkappa^{-1}} (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\otimes(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\\ &\qquad\qquad \xrightarrow{\widehat{T^{\leqslant 1}f}\otimes r'' + r''\otimes\widehat{T^{\leqslant 1}g}} T^{\leqslant 1}\mathcal{D}\otimes T^{\leqslant 1}\mathcal{D}\big].\end{aligned}$$

7.16 Transformation θ. Formula (5.23.1) describes the $T^{\geqslant 1}$-coalgebra morphism

$$
\begin{aligned}
(\mathrm{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}})\widehat{\ } = \Big[(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \\
\xrightarrow{(\boxtimes^I_u\Delta)\boxtimes_u 1} (\boxtimes^{i\in I}_u T^{\geqslant 1}T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \xrightarrow{\tau^{I\sqcup\mathbf{1}}} \\
T^{\geqslant 1}\big[(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big] \xrightarrow{T^{\geqslant 1}\mathrm{ev}^{\mathscr{Q}_u}_{\boxtimes^I_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}}} T^{\geqslant 1}s\mathcal{B}\Big] \\
= \Big[(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \xrightarrow{1\boxtimes_u\theta_{(\mathcal{A}_i);\mathcal{B}}} \\
(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_u}_{\boxtimes^I_u T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}}} T^{\geqslant 1}s\mathcal{B}\Big]. \quad (7.16.1)
\end{aligned}
$$

According to bijection (7.5.1) the quiver map

$$\theta_{(\mathcal{A}_i);\mathcal{B}} : T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \to \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B})$$

assigns to an object (a morphism of quivers) $f : \boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i \to s\mathcal{B}$, the object (a morphism of quivers)

$$\big[\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\boxtimes^I_u\Delta} \boxtimes^{i\in I}_u T^{\geqslant 1}T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\tau^I} T^{\geqslant 1}\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{T^{\geqslant 1}f} T^{\geqslant 1}s\mathcal{B}\big] = \hat{f}.$$

Indeed, denote $\mathcal{C} = \boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i$, $\mathcal{D} = \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})$ and compute (7.5.3). The matrix element

$$\widetilde{\tau}^{\mathbf{2}} = \big[T^k\mathcal{C} \boxtimes T^0T^{\geqslant 1}\mathcal{D} \hookrightarrow T^k\mathcal{C} \boxtimes_u T^{\geqslant 1}\mathcal{D} \xrightarrow{\tau^{\mathbf{2}}} T^m(\mathcal{C} \boxtimes_u \mathcal{D})\big]$$

given by (6.2.1) produces the identity map $(X, f) \mapsto (X, f)$ on objects, and it does not vanish only if $k = m$, being the isomorphism

$$T^m\mathcal{C} \boxtimes T^0\mathcal{D} \xrightarrow[\sim]{1\boxtimes\lambda^{\varnothing\to\mathbf{m}}} T^m\mathcal{C} \boxtimes T^mT^0\mathcal{D} \xrightarrow[\sim]{\overline{\varkappa}^{-1}} T^m(\mathcal{C} \boxtimes T^0\mathcal{D}),$$

$$[\otimes^{p\in\mathbf{m}}\mathcal{C}(X_{p-1}, X_p)] \otimes T^0\mathcal{D}(f, f) \xrightarrow{\sim} \otimes^{p\in\mathbf{m}}[\mathcal{C}(X_{p-1}, X_p) \otimes T^0\mathcal{D}(f, f)]$$

on morphisms.

To compute the quiver map $\theta_{(\mathcal{A}_i);\mathcal{B}}$ on morphisms via (7.5.2), (7.5.4), we find the restriction of the left hand side of (7.16.1) in the form:

$$
\begin{aligned}
\Big[T^{\leqslant 1}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \simeq (\boxtimes^{i\in I}Ts\mathcal{A}_i) \boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \\
\xrightarrow{(\boxtimes^I\widetilde{\Delta})\boxtimes 1} (\boxtimes^{i\in I}TT^{\geqslant 1}s\mathcal{A}_i) \boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \\
\xrightarrow{\widetilde{\tau}^I\boxtimes 1} (T\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \\
\xrightarrow{\widetilde{\tau}^{\mathbf{2}}} T^{\geqslant 1}\big[(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big] \xrightarrow{T^{\geqslant 1}\mathrm{ev}^{\mathscr{Q}_u}_{\boxtimes^I_u T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}}} T^{\geqslant 1}s\mathcal{B}\Big].
\end{aligned}
$$

Starting from elements $r^t \in \underline{\mathscr{Q}_p}(\boxtimes^{i\in I}Ts\mathcal{A}_i, s\mathcal{B})(f^{t-1}, f^t)$, $1 \leqslant t \leqslant m$, which can be collected into a diagram

$$r = \bigl(f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{m-1} \xrightarrow{r^m} f^m\bigr) : \boxtimes^{i\in I}Ts\mathcal{A}_i \to s\mathcal{B},$$

we can compute the element $\hat{r} = (r^1 \otimes \dots \otimes r^m)\theta_{(\mathcal{A}_i);\mathcal{B}} \in \underline{\mathscr{Q}_p}(\boxtimes^{i\in I}Ts\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B})$ as follows:

$$\hat{r} = \Bigl[\boxtimes^{i\in I}Ts\mathcal{A}_i \xrightarrow{\boxtimes^I\tilde{\Delta}} \boxtimes^{i\in I} TT^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\tilde{\tau}^I} T\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\nu^{\leqslant M}} TT^{\leqslant 1}\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i$$
$$\xrightarrow[\sim]{T(\vartheta^I)^{-1}} T\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow{\sum \overline{f^0}^{\otimes p_0}\otimes r^1\otimes\overline{f^1}^{\otimes p_1}\otimes r^2\otimes\dots\otimes\overline{f^{m-1}}^{\otimes p_{m-1}}\otimes r^m\otimes\overline{f^m}^{\otimes p_m}} T^{\geqslant 1}s\mathcal{B}\Bigr], \quad (7.16.2)$$

where the objects are omitted. Respectively, linear maps

$$r^t : \boxtimes^{i\in I}Ts\mathcal{A}_i((X_i),(Y_i)) \to s\mathcal{B}((X_i)f^{t-1},(Y_i)f^t)$$

or quiver maps

$$\overline{f^t} = \bigl[\boxtimes^{i\in I}Ts\mathcal{A}_i \xrightarrow[\sim]{\vartheta^I} T^{\leqslant 1}\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\mathrm{pr}_1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{f^t} s\mathcal{B}\bigr]$$

are used. Here $M \geqslant m$ is an arbitrary integer. Composition (7.16.2) does not depend on $M \in [m,\infty[$, since all $\overline{f^t}$ vanish on $T^0\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \simeq \boxtimes^{i\in I}T^0s\mathcal{A}_i$.

The same element can be written as

$$\hat{r} = \Bigl[\boxtimes^{i\in I}Ts\mathcal{A}_i \xrightarrow{\boxtimes^I\tilde{\Delta}} \boxtimes^{i\in I} TT^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\tilde{\tau}^I} T\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i$$
$$\xrightarrow{\nu^{\leqslant M}} TT^{\leqslant 1}\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow[\sim]{T(\vartheta^I)^{-1}} T^n\boxtimes^{i\in I} Ts\mathcal{A}_i$$
$$\xrightarrow{\sum_{k_0+\dots+k_m+m=n} \hat{f^0}_{k_0p_0}\otimes r^1\otimes\hat{f^1}_{k_1p_1}\otimes\dots\otimes r^m\otimes\hat{f^m}_{k_mp_m}} T^{p_0+\dots+p_m+m}s\mathcal{B}\Bigr],$$

where $n = k_0+\dots+k_m+m$, and $\hat{f^q}_{k_qp_q} : T^{k_q}\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i \to T^{p_q}s\mathcal{B}$ is the matrix coefficient of $\hat{f^q}$. Another presentation of this formula is

$$\hat{r} = (r^1 \otimes \dots \otimes r^m)\theta_{(\mathcal{A}_i);\mathcal{B}} = \text{signed permutation and insertion of units} \times$$
$$\times \sum_{\substack{p_s\geqslant 0 \\ i_t^s, j_s\in(\mathbb{Z}_{\geqslant 0})^n;\, i_t^s\neq 0}} f^0_{i_1^0}\otimes\dots\otimes f^0_{i_{p_0}^0}\otimes r^1_{j_1}\otimes f^1_{i_1^1}\otimes\dots\otimes f^1_{i_{p_1}^1}\otimes\dots\otimes r^m_{j_m}\otimes f^m_{i_1^m}\otimes\dots\otimes f^m_{i_{p_m}^m}. \quad (7.16.3)$$

Important special case is $m = 1$; the element $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is taken by

$\theta_{(\mathcal{A}_i);\mathcal{B}}$ to the element

$$\hat{r} = (r)\theta_{(\mathcal{A}_i);\mathcal{B}} = \Big[\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow{\boxtimes^I\widetilde{\Delta}} \boxtimes^{i\in I} TT^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\widetilde{\tau}^I} T\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i$$
$$\xrightarrow{\nu^{\leqslant M}} TT^{\leqslant 1}\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow[\sim]{T(\vartheta^I)^{-1}} T\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow{\sum \overline{f}^{\otimes p}\otimes r\otimes\overline{g}^{\otimes q}} T^{p+1+q}s\mathcal{B}\Big]$$
$$= \begin{matrix}\text{signed permutation}\\ \text{\& insertion of units}\end{matrix} \times \sum_{\substack{p,q\geqslant 0\\ i_s,j,k_t\in(\mathbb{Z}_{\geqslant 0})^n;\, i_s,k_t\neq 0}} f_{i_1}\otimes\cdots\otimes f_{i_p}\otimes r_j\otimes g_{k_1}\otimes\cdots\otimes g_{k_q}. \quad (7.16.4)$$

7.17 Multiplication in the closed multicategory Q.

By Proposition 4.10, for an arbitrary tree

$$t = (I \xrightarrow{\phi} J \to \mathbf{1}|(\mathcal{A}_i)_{i\in I}, (\mathcal{B}_j)_{j\in J}, \mathcal{C} \in \operatorname{Ob}\mathsf{Q})$$

of height 2 there is a morphism in $\mathscr{Q}_u$

$$\mu_\phi^{\mathsf{Q}} : \boxtimes^{J\sqcup\mathbf{1}}\big[(T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\phi^{-1}j}T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}_j))_{j\in J}, T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{j\in J}T^{\geqslant 1}s\mathcal{B}_j, s\mathcal{C})\big]$$
$$\to \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{C}).$$

Using (5.20.2) we conclude that

$$\mu_\phi^{\mathsf{Q}} = \big[\boxtimes_u^{J\sqcup\mathbf{1}}\big[(T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\phi^{-1}j}T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}_j))_{j\in J}, T^{\geqslant 1}\underline{\mathscr{Q}_u}(\boxtimes_u^{j\in J}T^{\geqslant 1}s\mathcal{B}_j, s\mathcal{C})\big]$$
$$\xrightarrow{\boxtimes_u^{J\sqcup\mathbf{1}}[(\theta_{(\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j})_{j\in J},\mathrm{pr}_1]} \boxtimes_u^{J\sqcup\mathbf{1}}\big[(\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\phi^{-1}j}T^{\geqslant 1}\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}_j))_{j\in J}, \underline{\mathscr{Q}_u}(\boxtimes_u^{j\in J}T^{\geqslant 1}s\mathcal{B}_j, s\mathcal{C})\big]$$
$$\xrightarrow{\mu_\phi^{\underline{\mathscr{Q}_u}}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{C})\big]. \quad (7.17.1)$$

Chapter 8

A_∞-categories

A_∞-categories can be characterized as free $T^{\geqslant 1}$-coalgebras with a differential. The symmetric multicategory of graded quivers Q was constructed in Chapter 7. Adding differentials to the picture we obtain the symmetric multicategory A_∞ of A_∞-categories and A_∞-functors. Besides a definition of multicategory type these admit a description in terms of coalgebras. Namely, A_∞-categories are augmented counital coassociative coalgebras in the category ${}^d\mathscr{Q}_p$ of differential graded quivers which come from free $T^{\geqslant 1}$-coalgebras in $\mathscr{Q}$. Respectively, A_∞-functors are morphisms of such augmented differential coalgebras. One of the main features of A_∞-categories is that the multicategory A_∞ is closed. Closedness of A_∞ has somewhat unexpected consequences, for instance, homotopy Gerstenhaber algebra structure on the cohomological Hochschild complex of a differential graded category.

8.1 Differentials. According to Proposition 7.15 there is a quiver map

$$\theta\big|_{T^1} = \big[\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\sim} \operatorname{Coder}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \hookrightarrow \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\big].$$

It takes an arbitrary element $b \in \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})(\mathrm{pr}_1, \mathrm{pr}_1)$ to the $T^{\geqslant 1}$-coderivation $b\theta \in \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})(\mathrm{id}_{T^{\geqslant 1}s\mathcal{A}}, \mathrm{id}_{T^{\geqslant 1}s\mathcal{A}})$.

8.2 Definition. An object of A_∞ (an A_∞-category) is a graded $\Bbbk$-quiver $\mathcal{A}$ together with a differential, i.e., an arrow $b : \mathrm{id}_\mathcal{A} \to \mathrm{id}_\mathcal{A}$ in the graded $\Bbbk$-quiver $s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) = \underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}}(s\mathcal{A}; s\mathcal{A}) = \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})$ of degree 1 such that $(b\theta) \cdot \mathrm{in}_1 \cdot b = 0$ in $\underline{\mathscr{Q}_p}$, and with additional condition $b_0 = 0$.

Equivalently, b is given by a $T^{\geqslant 1}$-coalgebra morphism

$$\hat{b} = \big[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{in}_1} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\big],$$

see Example 5.11. Here $\mathbb{1}_p[-1]$ is equipped with the $T^{\geqslant 1}$-coalgebra structure $\eta = \mathrm{in}_1 : \mathbb{1}_p[-1] \to T^{\geqslant 1}\mathbb{1}_p[-1]$, which is the augmentation morphism of $T^{\geqslant 1}$.

Condition $(\hat{b}\theta) \cdot \mathrm{in}_1 \cdot b = 0$ in $\underline{\mathscr{Q}_p}$, or explicitly

$$\big[\mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta \boxtimes b} \underline{\mathscr{Q}_p}(Ts\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \xrightarrow{-\cdot \mathrm{in}_1 \cdot -} \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A})\big] = 0,$$

is equivalent to the condition

$$\big[\mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \hookrightarrow \mathbb{1}_p[-1] \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta \boxtimes_u b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\big] = 0 \quad (8.2.1)$$

due to (7.6.7). This can be written also as

$$\big[\mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \hookrightarrow \mathbb{1}_p[-1] \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\hat{b} \boxtimes_u \hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mu^{\underline{\mathsf{Q}}}} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\big] = 0 \quad (8.2.2)$$

due to presentation (5.20.2) of $\mu^{\underline{\mathsf{Q}}}$.

Formula (7.16.4) with $M = 0$ gives the composition in $\underline{\mathscr{Q}_p}$

$$\hat{b}\theta = \big[Ts\mathcal{A} \xrightarrow{\tilde{\Delta}} TT^{\geqslant 1}s\mathcal{A} \xrightarrow{\sum \mathrm{pr}_1^{\otimes i} \otimes b \otimes \mathrm{pr}_1^{\otimes k}} T^{\geqslant 1}s\mathcal{A}\big].$$

More explicit formula follows. Any integers $i, k \geqslant 0$, $j > 0$ define a summand of the matrix element between $T^m s\mathcal{A}$ and $T^n s\mathcal{A}$, where $m = i+j+k$ and $n = i+1+k$. Namely, they define the non-decreasing surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$, $g(l) = l$ if $l \leqslant i$, $g(l) = i+1$ if $i < l \leqslant i+j$, and $g(l) = l-j+1$ if $l > i+j$. We have the composition in $\underline{\mathscr{Q}_p}$

$$(\hat{b}\theta)_{mn} = \sum_{i+j+k=m}^{i+1+k=n} \big[\otimes^{\mathbf{m}} s\mathcal{A} \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} s\mathcal{A} \xrightarrow{\otimes^{\mathbf{n}}[(\mathrm{id})_{\mathbf{i}}, b_j, (\mathrm{id})_{\mathbf{k}}]} \otimes^{p\in\mathbf{n}} s\mathcal{A}\big].$$

An easier way to write it is

$$\hat{b}\theta = \sum_{i+j+k=n} 1^{\otimes i} \otimes b_j \otimes 1^{\otimes k} : T^n s\mathcal{A} \to T^{\geqslant 1}s\mathcal{A}.$$

Equation $(\hat{b}\theta) \cdot \mathrm{in}_1 \cdot b = 0$ in $\underline{\mathscr{Q}_p}$ has the explicit form:

$$\sum_{i+j+k=n} (1^{\otimes i} \otimes b_j \otimes 1^{\otimes k}) b_{i+1+k} = 0 : T^n s\mathcal{A} \to s\mathcal{A}, \qquad n > 0.$$

We also view the differential b as an element of $\underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A})(\mathrm{id}_{\mathrm{Ob}\,\mathcal{A}}, \mathrm{id}_{\mathrm{Ob}\,\mathcal{A}})$. Consider the element $\tilde{b} = (\hat{b}\theta)\,\mathrm{in}_1 \underline{T^{\leqslant 1}} \in \underline{\mathscr{Q}_p}(Ts\mathcal{A}, Ts\mathcal{A})(\mathrm{id}_{\mathrm{Ob}\,\mathcal{A}}, \mathrm{id}_{\mathrm{Ob}\,\mathcal{A}})$. This is a (1,1)-coderivation $\tilde{b} = \hat{b}\theta \cdot \mathrm{in}_1 : (Ts\mathcal{A}, \Delta_0) \to (Ts\mathcal{A}, \Delta_0)$. It is determined by the equations

$$\mathrm{in}_n \cdot \tilde{b} = \sum_{i+j+k=n} 1^{\otimes i} \otimes b_j \otimes 1^{\otimes k} : T^n s\mathcal{A} \to Ts\mathcal{A}.$$

We are going to prove that $\tilde{b}^2 = 0$ in the sense of $\underline{\mathscr{Q}_p}$.

Denote by $\angle$ the quiver embedding[1] $\mathbb{1}_p[-2] \xrightarrow{\sim} \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] \hookrightarrow \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1]$.

8.3 Lemma. *Condition* (8.2.2) *is equivalent to the condition*

$$\begin{split}\bigl[\mathbb{1}_p[-2] \xhookrightarrow{\angle} \mathbb{1}_p[-1] \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\hat{b}\boxtimes_u\hat{b}} \\ T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{(\mu^{\mathsf{Q}})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\bigr] = 0. \quad (8.3.1)\end{split}$$

For any quiver morphism $b : \mathbb{1}_p[-1] \to s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$ the composite left hand side is a $T^{\geqslant 1}$-coalgebra homomorphism, where the source is equipped with the $T^{\geqslant 1}$-coalgebra structure $(\mathbb{1}_p[-2], \mathrm{in}_1)$.

Proof. If equation (8.3.1) holds, then composing it with $\mathrm{pr}_1 : T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \to s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$ we obtain equation (8.2.2).

Assume now that condition (8.2.1) holds. Due to (5.20.2) the left hand side of (8.3.1) expands to the following composition of quiver morphisms:

$$\begin{split}\bigl[\mathbb{1}_p[-2] \xrightarrow{\sim} \mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \xrightarrow{b\boxtimes b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \\ \xrightarrow{\mathrm{in}_1\boxtimes\mathrm{in}_1} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\Delta\boxtimes\Delta} T^{\geqslant 1}T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes T^{\geqslant 1}T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \\ \hookrightarrow T^{\geqslant 1}T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\tau} T^{\geqslant 1}\bigl(T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\bigr) \\ \xrightarrow{T^{\geqslant 1}(\theta\boxtimes_u\mathrm{pr}_1)} T^{\geqslant 1}\bigl(\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr) \xrightarrow{T^{\geqslant 1}\mu^{\mathscr{Q}_u}} T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr].\end{split}$$

Since in_1 is an augmentation of the comonad $T^{\geqslant 1}$, we may replace $\Delta\boxtimes\Delta$ with $\mathrm{in}_1\boxtimes\mathrm{in}_1$. This allows to rewrite the above composition as

$$\begin{split}\bigl[\mathbb{1}_p[-2] \xrightarrow{\sim} \mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta\boxtimes b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \\ \hookrightarrow T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \\ \xrightarrow{\tau} T^{\geqslant 1}\bigl(\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr) \xrightarrow{T^{\geqslant 1}\mu^{\mathscr{Q}_u}} T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr]. \quad (8.3.2)\end{split}$$

Applying formula (6.4.1) for $I = \mathbf{2}$, $m_1 = m_2 = 1$ we find that the composition is a sum of three quiver morphisms. The first is indexed by $S = \mathbf{2}\times\mathbf{1}$, its target is $T^1\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})$, see (6.2.1). The second and the third are indexed by $S = \{(1,1),(2,2)\} \subset \mathbf{2}\times\mathbf{2}$ and by

[1] Warning: it is not a $T^{\geqslant 1}$-coalgebra morphism.

$S = \{(1,2),(2,1)\} \subset \mathbf{2}\times\mathbf{2}$, their target is $T^2\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})$. The second is

$$\begin{aligned}
\bigl[\mathbb{1}_p[-2] &\xrightarrow{\sim} \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta\boxtimes b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\lambda^{\mathsf{I}}\cdot\boxtimes\lambda\cdot^{\mathsf{I}}} \\
&\bigl(\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\otimes T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\bigr)\boxtimes\bigl(T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\otimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr) \\
&\xrightarrow{\varkappa} \bigl(\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr)\otimes\bigl(T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr) \\
&\qquad\qquad\xrightarrow{\mu^{\mathscr{Q}_u}\otimes\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\otimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr] \\
&= \bigl[\mathbb{1}_p[-2] \xrightarrow[\sim]{\lambda^{\varnothing\to\mathbf{2}}} \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] \xrightarrow{b\otimes b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\otimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr]
\end{aligned}$$

due to (7.6.8) and (7.6.9), since $\hat{b}\cdot\theta\cdot\mathrm{pr}_1 = b$. Similar computation for the third summand gives the expression $-\lambda^{\varnothing\to\mathbf{2}}\cdot(b\otimes b)$. Thus, the second and the third summands cancel each other. Therefore, (8.3.2) is equal to the first summand

$$\begin{aligned}
\bigl[\mathbb{1}_p[-2] &\xrightarrow{\sim} \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta\boxtimes b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \hookrightarrow \\
&\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes_u\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{in}_1} T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A})\bigr],
\end{aligned}$$

which vanishes by (8.2.1). Without this vanishing assumptions we see that this composition ends up with in_1, hence, its image is contained in $\operatorname{Ker}\overline{\Delta}$. By (6.8.2) the whole above expression is always a $T^{\geqslant 1}$-coalgebra homomorphism. □

8.4 Remark. According to Remark 5.21 equation (8.3.1) is equivalent to the following equation:

$$\begin{aligned}
\bigl[\mathbb{1}_p[-2] &\xrightarrow{\angle} \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1] \xrightarrow{\hat{b}\boxtimes_u\hat{b}} T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A})\boxtimes_u T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A}) \\
&\qquad\xrightarrow{\mu^{\widehat{\mathscr{Q}_u}{}^f_{T^{\geqslant 1}}}} T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A})\bigr] = 0.
\end{aligned}$$

Composing it with $\theta : T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A}) \to \widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; T^{\geqslant 1}s\mathcal{A})$ and taking into account equations (5.19.2) and (4.21.1) we obtain

$$\begin{aligned}
\bigl[\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] &\hookrightarrow \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1] \xrightarrow{\hat{b}\boxtimes_u\hat{b}} T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A})\boxtimes_u T^{\geqslant 1}\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; s\mathcal{A}) \\
&\xrightarrow{\theta\boxtimes_u\theta} \widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; T^{\geqslant 1}s\mathcal{A})\boxtimes_u\widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; T^{\geqslant 1}s\mathcal{A}) \xrightarrow{\mu^{\widehat{\mathscr{Q}_u}}} \widehat{\underline{\mathscr{Q}_u}}(T^{\geqslant 1}s\mathcal{A}; T^{\geqslant 1}s\mathcal{A})\bigr] = 0,
\end{aligned}$$

or equivalently:

$$\bigl[\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta\boxtimes\hat{b}\theta} \underline{\mathscr{Q}_p}(Ts\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \xrightarrow{-\cdot\mathrm{in}_1\cdot-} \underline{\mathscr{Q}_p}(Ts\mathcal{A}, T^{\geqslant 1}s\mathcal{A})\bigr] = 0,$$

that is, $(\hat{b}\theta)\cdot\mathrm{in}_1\cdot(\hat{b}\theta) = 0$ in $\underline{\mathscr{Q}_p}$. This equivalent to $\tilde{b}^2 = (\hat{b}\theta)\cdot\mathrm{in}_1\cdot(\hat{b}\theta)\cdot\mathrm{in}_1 = 0$ in $\underline{\mathscr{Q}_p}$. We have deduced this equation from the equation $(\hat{b}\theta)\cdot\mathrm{in}_1\cdot b = 0$ in $\underline{\mathscr{Q}_p}$. Actually, they are equivalent, since the latter can be obtained from the former by composing with pr_1.

Since $\hat{b} : \mathbb{1}_p[-1] \to T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) = T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$ is a $T^{\geqslant 1}$-coalgebra morphism, morphism φ for $\widehat{\mathscr{Q}}'_{uT^{\geqslant 1}}$ gives another $T^{\geqslant 1}$-coalgebra morphism

$$\begin{aligned}
\varphi(\hat{b}) &= \bigl[T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{1\boxtimes_u \hat{b}} T^{\geqslant 1}s\mathcal{A} \boxtimes_u T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{A}\bigr] \\
&= \bigl[T^{\geqslant 1}s\mathcal{A}, \mathbb{1}_p[-1] \xrightarrow{\Delta,\hat{b}} T^{\geqslant 1}T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{T^{\geqslant 1}\mathrm{ev}^{\mathscr{Q}_u}} T^{\geqslant 1}s\mathcal{A}\bigr] \\
&= \bigl[T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{1\boxtimes_u \hat{b}\theta} T^{\geqslant 1}s\mathcal{A} \boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_u}} T^{\geqslant 1}s\mathcal{A}\bigr].
\end{aligned}$$

The formulas are due to Lemma 5.7. The last one shows that the restriction of $\varphi(\hat{b})$ to $T^{\geqslant 1}s\mathcal{A} \boxtimes T^0\mathbb{1}_p[-1]$ coincides with $(\lambda^{\mathsf{I}\cdot})^{-1}$ by (7.5.6).

8.5 Lemma. *Condition* (8.3.1) *on b is equivalent to the following one:*

$$\bigl[Ts\mathcal{A} \boxtimes \mathbb{1}_p[-2] \xrightarrow{\sim} Ts\mathcal{A} \boxtimes \mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{b})\boxtimes 1} T^{\geqslant 1}s\mathcal{A} \boxtimes \mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{b})} T^{\geqslant 1}s\mathcal{A}\bigr] = 0. \tag{8.5.1}$$

Proof. According to equation (5.19.1) the morphism $(\mathrm{ev}^{\mathsf{Q}})\hat{}$ identifies with the evaluation $\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}$. The left hand side of equation (8.3.1) is a $T^{\geqslant 1}$-coalgebra morphism by Lemma 8.3. Due to Lemma 5.7 this equation can be written in equivalent form as

$$\begin{aligned}
&\bigl[T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-2] \xrightarrow{1\boxtimes_u \angle} T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-1] \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{1\boxtimes_u \hat{b}\boxtimes_u \hat{b}} \\
&T^{\geqslant 1}s\mathcal{A} \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{1\boxtimes_u (\mu^{\mathsf{Q}})\hat{}} T^{\geqslant 1}s\mathcal{A} \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{A}\bigr] \\
&\qquad = \bigl[T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-2] \xrightarrow{1\boxtimes_u 0} T^{\geqslant 1}s\mathcal{A} \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{A}\bigr],
\end{aligned} \tag{8.5.2}$$

where zero morphism $0 : \mathbb{1}_p[-2] \to T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$ takes the object $*$ to $\mathrm{id}^{\mathsf{Q}}_{\mathcal{A}} = \mathrm{pr}_1 : T^{\geqslant 1}s\mathcal{A} \to s\mathcal{A}$. By Lemma 5.7 the right hand side equals

$$\bigl[T^{\geqslant 1}s\mathcal{A} \boxtimes_u \mathbb{1}_p[-2] \xrightarrow{1\boxtimes_u 0} T^{\geqslant 1}s\mathcal{A} \boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_u}} T^{\geqslant 1}s\mathcal{A}\bigr],$$

where $0 : * \mapsto \mathrm{id}_{T^{\geqslant 1}s\mathcal{A}}$. This composition equals $(\lambda^{\mathsf{I}\cdot})^{-1}$ on $T^{\geqslant 1}s\mathcal{A} \boxtimes T^0\mathbb{1}_p[-2]$ by (7.5.6). The left hand side of (8.5.2) also equals $(\lambda^{\mathsf{I}\cdot})^{-1}$ on $T^{\geqslant 1}s\mathcal{A} \boxtimes T^0\mathbb{1}_p[-2]$. Therefore, it suffices to consider restriction of equation (8.5.2) to $Ts\mathcal{A} \boxtimes \mathbb{1}_p[-2]$, on which the right hand side vanishes.

Using diagram (5.21.1) the left hand side of (8.5.2) can be transformed to

$$\begin{aligned}
&\bigl[T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-2] \xrightarrow{1\boxtimes_u\angle} T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{1\boxtimes_u\hat{b}\boxtimes_u\hat{b}} \\
&T^{\geqslant 1}s\mathcal{A}\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}{}^{f}_{T^{\geqslant 1}}}\boxtimes_u 1} T^{\geqslant 1}s\mathcal{A}\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}{}^{f}_{T^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{A}\bigr] \\
&= \bigl[T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-2] \xrightarrow{1\boxtimes_u\angle} T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1] \\
&\qquad \xrightarrow{1\boxtimes_u\hat{b}\boxtimes_u 1} T^{\geqslant 1}s\mathcal{A}\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}{}^{f}_{T^{\geqslant 1}}}\boxtimes_u 1} T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-1] \\
&\qquad\qquad \xrightarrow{1\boxtimes_u\hat{b}} T^{\geqslant 1}s\mathcal{A}\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}{}^{f}_{T^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{A}\bigr].
\end{aligned}$$

Its restriction to $Ts\mathcal{A}\boxtimes\mathbb{1}_p[-2]$ equals the left hand side of (8.5.1) and the lemma is proven. □

8.6 Lemma. *Given* $\varphi(\hat{b}) = \varphi^{\widehat{\mathscr{Q}_u}{}'_{T^{\geqslant 1}}}(\hat{b})$*, one can restore* $b : \mathbb{1}_p[-1] \to \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) = \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A})$ *from the following equations in* $\widehat{\mathscr{Q}_u}$ *and* $\mathscr{Q}$*:*

$$\bigl[T^{\geqslant 1}s\mathcal{A}, \mathbb{1}_p[-1] \xrightarrow{\varphi^{\widehat{\mathscr{Q}_u}{}'_{T^{\geqslant 1}}}(\hat{b})} T^{\geqslant 1}s\mathcal{A} \xrightarrow{\mathrm{pr}_1} s\mathcal{A}\bigr] \tag{8.6.1}$$
$$= \bigl[T^{\geqslant 1}s\mathcal{A}, \mathbb{1}_p[-1] \xrightarrow{1,b} T^{\geqslant 1}s\mathcal{A}, \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}}} s\mathcal{A}\bigr] = \varphi^{\widehat{\mathscr{Q}_u}}(b), \tag{8.6.2}$$
$$\bigl[Ts\mathcal{A}\boxtimes\mathbb{1}_p[-1] \hookrightarrow T^{\geqslant 1}s\mathcal{A}\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\varphi^{\widehat{\mathscr{Q}_u}{}'_{T^{\geqslant 1}}}(\hat{b})} T^{\geqslant 1}s\mathcal{A} \xrightarrow{\mathrm{pr}_1} s\mathcal{A}\bigr] \tag{8.6.3}$$
$$= \bigl[Ts\mathcal{A}\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes b} Ts\mathcal{A}\boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\bigr] = \varphi^{\mathscr{Q}_p}(b). \tag{8.6.4}$$

Proof. Expression (8.6.1) expands to the composition in $\widehat{\mathscr{Q}_u}$:

$$\begin{aligned}
&\bigl[T^{\geqslant 1}s\mathcal{A}, \mathbb{1}_p[-1] \xrightarrow{\Delta,\hat{b}} T^{\geqslant 1}T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{T^{\geqslant 1}\mathrm{ev}^{\widehat{\mathscr{Q}_u}}} T^{\geqslant 1}s\mathcal{A} \xrightarrow{\mathrm{pr}_1} s\mathcal{A}\bigr] \\
&= \bigl[T^{\geqslant 1}s\mathcal{A}, \mathbb{1}_p[-1] \xrightarrow{\Delta,\hat{b}} T^{\geqslant 1}T^{\geqslant 1}s\mathcal{A}, T^{\geqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \\
&\qquad\qquad \xrightarrow{\mathrm{pr}_1,\mathrm{pr}_1} T^{\geqslant 1}s\mathcal{A}, \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}_u}}} s\mathcal{A}\bigr]
\end{aligned}$$

by multinaturality of $\varepsilon = \mathrm{pr}_1$. Clearly, this is equal to (8.6.2).

By the above considerations expression (8.6.3) expands to a composition in $\mathscr{Q}$:

$$\bigl[Ts\mathcal{A}\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes b} T^{\leqslant 1}T^{\geqslant 1}s\mathcal{A}\boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \hookrightarrow T^{\geqslant 1}s\mathcal{A}\boxtimes_u \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_u}} s\mathcal{A}\bigr].$$

By (7.5.5) this composition is equal to (8.6.4). □

8.7 A_∞-functors. In agreement with notation introduced just above Lemma 4.16 an arbitrary morphism of quivers $f : \mathcal{A} \to \mathcal{B}$ comes from a morphism of quivers $\dot{f} = \varphi^{-1}(f) : \mathbb{1}_p \to \underline{\mathscr{Q}_p}(\mathcal{A},\mathcal{B})$, namely, $\operatorname{Ob}\dot{f} : * \mapsto \operatorname{Ob} f$ and $\dot{f} : 1 \mapsto f \in \underline{\mathscr{Q}_p}(\mathcal{A},\mathcal{B})(\operatorname{Ob} f, \operatorname{Ob} f)^0$. By abuse of notation $\dot{f}$ can be also denoted f.

Recall that an element $f \in \mathsf{Q}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$, that is, a quiver morphism $f : \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1} s\mathcal{A}_i \to s\mathcal{B}$, has associated quiver morphisms $\overline{f} = \vartheta^I \cdot T^{\leqslant 1} f \cdot \mathrm{pr}_1 : \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \to s\mathcal{B}$, see equation (7.9.1), $\hat{f} : \boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i \to T^{\geqslant 1} s\mathcal{B}$ and $\tilde{f} : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B}$, see diagram (7.9.3).

8.8 Definition. A morphism in $\mathsf{A}_\infty((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$, called *$A_\infty$-functor*, is an element $f \in \mathsf{Q}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$ commuting with the differentials. This means that the following compositions in $\underline{\mathscr{Q}_p}$ are equal

$$\bigl(\boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{\dot{\tilde{f}}} Ts\mathcal{B} \xrightarrow{b} s\mathcal{B}\bigr) = \bigl(\boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{\sum_{i=1}^n 1^{\boxtimes(i-1)}\boxtimes\tilde{b}\boxtimes 1^{\boxtimes(n-i)}} \boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{\dot{\overline{f}}} s\mathcal{B}\bigr). \quad (8.8.1)$$

Due to (7.11.1) the above equation can be written in components as

$$\sum_{\substack{i_1,\dots,i_m\in(\mathbb{Z}_{\geqslant 0})^n\smallsetminus\{0\}\\ i_1+\dots+i_m=(\ell^1,\dots,\ell^n)}} \boxtimes^I \lambda \cdot \overline{\varkappa}^{-1} \cdot (f_{i_1}\otimes\dots\otimes f_{i_m}) b_m$$
$$= \sum_{q=1}^n \sum_{\substack{r+k+t=\ell^q\\ r+1+t=p}} \bigl(1^{\boxtimes(q-1)}\boxtimes(1^{\otimes r}\otimes b_k\otimes 1^{\otimes t})\boxtimes 1^{\boxtimes(n-q)}\bigr) f_{(\ell^1,\dots,\ell^{q-1},p,\ell^{q+1},\dots,\ell^n)} : \boxtimes^{i\in\mathbf{n}} T^{\ell^i} s\mathcal{A}_i \to s\mathcal{B}$$

for all $(\ell^1,\dots,\ell^n) \in (\mathbb{Z}_{\geqslant 0})^n \smallsetminus \{0\}$. If $n = 0$ the commutation with differentials holds true automatically.

8.9 Definition. An A_∞-functor f is *strict* if it is strict as a morphism in Q, that is, if its arbitrary component f_j vanishes for $j \in \mathbb{Z}^n_{\geqslant 0}$ unless $|j| = 1$.

Let $f : X \to Y$ be a morphism in an arbitrary closed Monoidal category $\mathcal{C}$. It defines the morphism $\dot{f} = \varphi^{-1}(f) : \mathbb{1} \to \underline{\mathcal{C}}(X,Y)$. For an arbitrary object Z of $\mathcal{C}$ the equations

$$\underline{\mathcal{C}}(1,f) = \bigl[\underline{\mathcal{C}}(Z,X) \xrightarrow[\sim]{\lambda^{\mathsf{I}\cdot}} \underline{\mathcal{C}}(Z,X)\otimes\mathbb{1} \xrightarrow{1\otimes\dot{f}} \underline{\mathcal{C}}(Z,X)\otimes\underline{\mathcal{C}}(X,Y) \xrightarrow{\mu^{\underline{\mathcal{C}}}} \underline{\mathcal{C}}(Z,Y)\bigr],$$
$$\underline{\mathcal{C}}(f,1) = \bigl[\underline{\mathcal{C}}(Y,Z) \xrightarrow[\sim]{\lambda^{\cdot\mathsf{I}}} \mathbb{1}\otimes\underline{\mathcal{C}}(Y,Z) \xrightarrow{\dot{f}\otimes 1} \underline{\mathcal{C}}(X,Y)\otimes\underline{\mathcal{C}}(Y,Z) \xrightarrow{\mu^{\underline{\mathcal{C}}}} \underline{\mathcal{C}}(X,Z)\bigr].$$

hold true (cf. Lemma 4.16 and Lemma 4.17). We shall use these identities to transform equation (8.8.1) in $\underline{\mathscr{Q}_p}$ to an equation between quiver morphisms. The differential b can be presented as the quiver morphism $b : \mathbb{1}_p[-1] \to \underline{\mathscr{Q}_p}(Ts\mathcal{B}, s\mathcal{B})$, $* \mapsto \mathrm{id}_{\operatorname{Ob}\mathcal{B}}$, $1 \mapsto (b : Ts\mathcal{B}(X,Y) \to s\mathcal{B}(X,Y))_{X,Y\in\operatorname{Ob}\mathcal{B}}$ of degree 0. Similarly, $\tilde{b}$ can be viewed as the quiver

morphism $\tilde{b} : \mathbb{1}_p[-1] \to \underline{\mathscr{Q}_p}(TsA_i, TsA_i)$, $* \mapsto \mathrm{id}_{\mathrm{Ob}\,A_i}$. Commutation condition (8.8.1) can be presented as

$$\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{b} \underline{\mathscr{Q}_p}(Ts\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(\tilde{f},1)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, s\mathcal{B})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^{n} \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{{}^t(\dot{\mathrm{id}}^{\boxtimes(i-1)}\boxtimes\tilde{b}\boxtimes\dot{\mathrm{id}}^{\boxtimes(n-i)})_i} \boxtimes^{i\in\mathbf{n}} \underline{\mathscr{Q}_p}(TsA_i, TsA_i) \\
&\qquad \xrightarrow{\boxtimes^{\mathbf{n}}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, \boxtimes^{i\in\mathbf{n}} TsA_i) \xrightarrow{\underline{\mathscr{Q}_p}(1,\bar{f})} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, s\mathcal{B})\big]. \qquad (8.9.1)
\end{aligned}$$

Here summation over $i \in \mathbf{n}$ is implied by inner product of a row and a column. Let us transform the left hand side of this equation.

Due to (7.9.4) the left hand side equals

$$\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\Upsilon} \underline{\mathscr{Q}_p}(Ts\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(\vartheta\cdot T^{\leqslant 1}\hat{f},1)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, s\mathcal{B})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_u}(\hat{f},1)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, s\mathcal{B}) \\
&\qquad \xrightarrow{\Upsilon} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(\vartheta,1)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, s\mathcal{B})\big],
\end{aligned}$$

where we have used (7.6.2).

Notation. Let X be an object of a graded $\Bbbk$-linear quiver $\mathcal{C}$. Then $\dot{X}$ denotes the only quiver morphism $\mathbb{1}_u \to \mathcal{C}$, $* \mapsto X$. This generalizes notation introduced above Lemma 4.16.

The morphism $\dot{X}$ induces the morphism $T^0\dot{X} : \mathbb{1}_p = T^0\mathbb{1}_u \to T^0\mathcal{C}$, $* \mapsto X$, which gives identity map $\mathrm{id}_\Bbbk$ on morphisms. In the particular case of an object $f : A \to \mathcal{B}$ of $\mathcal{C} = \underline{\mathscr{Q}_u}(A, \mathcal{B})$ there is $\dot{f} : \mathbb{1}_u \to \underline{\mathscr{Q}_u}(A, \mathcal{B})$, $* \mapsto f$. Using these notations we notice that

$$\begin{aligned}
\dot{\mathrm{id}}_{TsA_i} = \big[\mathbb{1}_p &\xrightarrow{T^0\dot{\mathrm{id}}_{T^{\geqslant 1}sA_i}} T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}sA_i, T^{\geqslant 1}sA_i) \xhookrightarrow{\mathrm{in}_0} T^{\leqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}sA_i, T^{\geqslant 1}sA_i) \\
&\xrightarrow{\underline{T^{\leqslant 1}}} \underline{\mathscr{Q}_p}(TsA_i, TsA_i)\big]
\end{aligned}$$

as formula (7.7.3) shows. This allows to rewrite commutation condition (8.9.1) as follows:

$$\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{b} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_u}(\hat{f},1)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, s\mathcal{B}) \xrightarrow{\Upsilon} \underline{\mathscr{Q}_p}(T^{\leqslant 1}\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, s\mathcal{B})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^{n} \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{{}^t(\boxtimes^{\mathbf{n}}((T^0\dot{\mathrm{id}}_{T^{\geqslant 1}sA_j})_{j<i}, b\theta\cdot\mathrm{in}_1, (T^0\dot{\mathrm{id}}_{T^{\geqslant 1}sA_j})_{j>i}))_i} \\
&\boxtimes^{i\in\mathbf{n}} T^{\leqslant 1}\underline{\mathscr{Q}_u}(T^{\geqslant 1}sA_i, T^{\geqslant 1}sA_i) \xrightarrow{\boxtimes^{\mathbf{n}}\underline{T^{\leqslant 1}}} \boxtimes^{i\in\mathbf{n}} \underline{\mathscr{Q}_p}(TsA_i, TsA_i) \xrightarrow{\boxtimes^{\mathbf{n}}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} TsA_i, \boxtimes^{i\in\mathbf{n}} TsA_i) \\
&\qquad \xrightarrow{\underline{\mathscr{Q}_p}(\vartheta^{-1},1)} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, \boxtimes^{i\in\mathbf{n}} TsA_i) \xrightarrow{\underline{\mathscr{Q}_p}(1,\bar{f})} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant 1}sA_i, s\mathcal{B})\big].
\end{aligned}$$

Using (7.8.1) we rewrite the right hand side of the above equation as follows:

$$\Big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{\oplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 \dot{\mathrm{id}}_{T^{\geqslant1}s\mathcal{A}_j})_{j<i}, b\theta, (T^0 \dot{\mathrm{id}}_{T^{\geqslant1}s\mathcal{A}_j})_{j>i})}$$

$$\bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, T^{\geqslant1}s\mathcal{A}_j))_{j<i}, \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i), (T^0 \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, T^{\geqslant1}s\mathcal{A}_j))_{j>i})$$

$$\hookrightarrow \boxtimes^{i\in\mathbf{n}} T^{\leqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\vartheta^{\mathbf{n}}} T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{T^{\leqslant1}\underline{\boxtimes_u^{\mathbf{n}}}}$$

$$T^{\leqslant1} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\underline{T^{\leqslant1}}} \underline{\mathscr{Q}_p}(T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i)$$

$$\xrightarrow{\underline{\mathscr{Q}_p}(1,\vartheta^{-1}\cdot\overline{f})} \underline{\mathscr{Q}_p}(T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\Big].$$

Explicit formula (7.7.2) for $\underline{T^{\leqslant1}}$ allows to simplify this expression. We use also the identity

$$\dot{\mathrm{id}}_{T^{\geqslant1}s\mathcal{A}_i} = \big(\mathbb{1}_u \xrightarrow{\dot{\mathrm{pr}}_1} T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, s\mathcal{A}_j) \xrightarrow{\theta} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, T^{\geqslant1}s\mathcal{A}_j)\big).$$

The resulting composition is

$$\Big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{\oplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 \dot{\mathrm{pr}}_1)_{j<i}, \hat{b}, (T^0 \dot{\mathrm{pr}}_1)_{j>i})}$$

$$\bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, s\mathcal{A}_j))_{j<i}, T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, s\mathcal{A}_i), (T^0 T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, s\mathcal{A}_j))_{j>i})$$

$$\xrightarrow{\oplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0\theta)_{j<i}, \theta, (T^0\theta)_{j>i})}$$

$$\bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, T^{\geqslant1}s\mathcal{A}_j))_{j<i}, \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i), (T^0 \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, T^{\geqslant1}s\mathcal{A}_j))_{j>i})$$

$$\hookrightarrow \boxtimes_u^{i\in\mathbf{n}} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\underline{\boxtimes_u^{\mathbf{n}}}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i)$$

$$\xrightarrow{\Upsilon} \underline{\mathscr{Q}_p}(T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\underline{\mathscr{Q}_p}(1,\mathrm{in}_1\cdot\vartheta^{-1}\cdot\overline{f})} \underline{\mathscr{Q}_p}(T^{\leqslant1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\Big].$$

Notice that $\mathrm{in}_1 \cdot \vartheta^{-1} \cdot \overline{f} = f$ by (7.9.2). Using equation (7.6.5) we transform commutation condition (8.9.1) to the form:

$$\Big[\mathbb{1}_p[-1] \xrightarrow{b} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_u}(\hat{f},1)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\Big]$$

$$= \Big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{\oplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 \dot{\mathrm{pr}}_1)_{j<i}, \hat{b}, (T^0 \dot{\mathrm{pr}}_1)_{j>i})}$$

$$\bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0 T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, s\mathcal{A}_j))_{j<i}, T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, s\mathcal{A}_i), (T^0 T^{\geqslant1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j, s\mathcal{A}_j))_{j>i})$$

$$\hookrightarrow \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, s\mathcal{A}_i) \xrightarrow{\boxtimes_u^{\mathbf{n}}\theta} \boxtimes_u^{i\in\mathbf{n}} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i)$$
$$\xrightarrow{\boxtimes_u^{\mathbf{n}}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i, \boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\underline{\mathscr{Q}_u}(1,f)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\big]. \quad (8.9.2)$$

Recall that the morphism $\underline{\mathsf{Q}}(f;1) \in \mathsf{Q}\big(\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B});\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\big)$ coincides with the morphism in $\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}$:

$$\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(f;1) : s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) = \underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(s\mathcal{B};s\mathcal{B}) \to \underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}((s\mathcal{A}_i)_{i\in\mathbf{n}};s\mathcal{B}) = s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}).$$

Due to (5.22.2) the latter is given by the composition of quiver morphisms

$$\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(f;1) = \big[T^{\geqslant1}\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\mathrm{pr}_1} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_u}(\hat{f},1)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\big].$$

Recall also that the morphism $\underline{\mathsf{Q}}(1;f) \in \mathsf{Q}((\underline{\mathsf{Q}}(\mathcal{A}_i,\mathcal{A}_i))_{i\in\mathbf{n}};\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}))$ coincides with the morphism in $\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}$:

$$(s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in\mathbf{n}} = (\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(s\mathcal{A}_i;s\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(1;f)} \underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}((s\mathcal{A}_i)_{i\in\mathbf{n}};s\mathcal{B}) = s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}).$$

Due to (5.22.4) the latter is given by the composition in $\widehat{\mathscr{Q}_u}$

$$\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(1;f) = \big[(T^{\geqslant1}\underline{\widehat{\mathscr{Q}_u}}(T^{\geqslant1}s\mathcal{A}_i, s\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{(\theta)_{\mathbf{n}}} (\underline{\widehat{\mathscr{Q}_u}}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i))_{i\in\mathbf{n}}$$
$$\xrightarrow{\underline{\widehat{\mathscr{Q}_u}}(1;f)} \underline{\widehat{\mathscr{Q}_u}}((T^{\geqslant1}s\mathcal{A}_i)_{i\in\mathbf{n}}, s\mathcal{B})\big].$$

Due to Lemma 4.27 this coincides with the composition of quiver morphisms

$$\underline{\widehat{\mathscr{Q}_u}^{T^{\geqslant1}}}(1;f) = \big[\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, s\mathcal{A}_i) \xrightarrow{\boxtimes_u^{\mathbf{n}}\theta} \boxtimes_u^{i\in\mathbf{n}} \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i, T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\boxtimes_u^{\mathbf{n}}}$$
$$\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i, \boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i) \xrightarrow{\underline{\mathscr{Q}_u}(1,f)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i, s\mathcal{B})\big].$$

This allows to rewrite commutation condition (8.9.2) as the composition of quiver morphisms

$$\big[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\underline{\mathsf{Q}}(f;1)} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\big]$$
$$= \big[\mathbb{1}_p[-1] \xrightarrow{(\lambda^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((\mathbb{1}_p)_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)_{j>i}) \xrightarrow{\oplus_{i=1}^n\boxtimes^{\mathbf{n}}((T^0\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j})_{j<i}, b, (T^0\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j})_{j>i})}$$
$$\bigoplus_{i=1}^n \boxtimes^{\mathbf{n}}((T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j<i}, s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i), (T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j>i})$$
$$\hookrightarrow \boxtimes_u^{i\in\mathbf{n}} s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\boxtimes_u^{i\in\mathbf{n}}\mathrm{in}_1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\underline{\mathsf{Q}}(1;f)} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\big]. \quad (8.9.3)$$

Compose this equation with $\mathrm{in}_1 : s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$. Notice that

$$\begin{aligned}&\bigl[s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\underline{\mathsf{Q}}(f;1)} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr]\\ &\quad= \bigl[s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\Delta} T^{\geqslant1}T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{T^{\geqslant1}\underline{\mathsf{Q}}(f;1)} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr],\end{aligned}$$

since in_1 is an augmentation of the comonad $T^{\geqslant1}$. The same reason and explicit formula (6.4.1) for τ imply that

$$\begin{aligned}&[\bigoplus_{i=1}^{n} \boxtimes^{\mathbf{n}}((T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j<i}, s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i), (T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j>i}) \hookrightarrow \boxtimes_u^{i\in\mathbf{n}} s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)\\ &\qquad \xrightarrow{\boxtimes_u^{i\in\mathbf{n}}\mathrm{in}_1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\underline{\mathsf{Q}}(1;f)} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})]\\ &= [\bigoplus_{i=1}^{n} \boxtimes^{\mathbf{n}}((T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j<i}, s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i), (T^0s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j))_{j>i}) \hookrightarrow \boxtimes_u^{i\in\mathbf{n}} s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)\\ &\xrightarrow{\boxtimes_u^{i\in\mathbf{n}}\mathrm{in}_1} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\boxtimes_u^{i\in\mathbf{n}}\Delta} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\tau} T^{\geqslant1}\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)\\ &\qquad\qquad \xrightarrow{T^{\geqslant1}\underline{\mathsf{Q}}(1;f)} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})].\end{aligned}$$

Hint: the inequalities $m_j \leqslant m \leqslant \sum_{i=1}^{n} m_i$ for all $j\in\mathbf{n}$ imply for $(m_j)_{j\in\mathbf{n}} = e_i$ that $m=1$.

Taking these substitutions into account we obtain an equation in $\mathscr{Q}$, equivalent to commutation condition (8.9.3):

$$\begin{aligned}&\bigl[\mathbb{1}_p[-1] \xrightarrow{\hat b} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\Delta} T^{\geqslant1}T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{T^{\geqslant1}\underline{\mathsf{Q}}(f;1)} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr]\\ &= \bigl[\mathbb{1}_p[-1] \xrightarrow{(\lambda_u^{i:\mathbf{1}\to\mathbf{n}})_i} \bigoplus_{i=1}^{n}\boxtimes_u^{\mathbf{n}}((\mathbb{1}_u)_{j<i},\mathbb{1}_p[-1],(\mathbb{1}_u)_{j>i}) \xrightarrow{\oplus_{i=1}^n\boxtimes_u^{\mathbf{n}}((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j}\cdot\mathrm{in}_1)_{j<i},\hat b,(\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j}\cdot\mathrm{in}_1)_{j>i})}\\ &\quad \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\boxtimes_u^{i\in\mathbf{n}}\Delta} \boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\tau} T^{\geqslant1}\boxtimes_u^{i\in\mathbf{n}} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)\\ &\qquad\qquad \xrightarrow{T^{\geqslant1}\underline{\mathsf{Q}}(1;f)} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr].\end{aligned}$$

Notice that composition of quiver morphisms $\mathbb{1}_u \xrightarrow{\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j}} s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j) \xrightarrow{\mathrm{in}_1} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j)$, which is a $T^{\geqslant1}$-coalgebra morphism, represents a morphism $(\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_j})\hat{}: () \to T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j)$ in $\widehat{\mathscr{Q}_{uT^{\geqslant1}}} = \widehat{\mathscr{Q}}_{u\widehat{T^{\geqslant1}}}$. Thus, a morphism $f\in\mathsf{Q}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$ is an A_∞-functor if and only if the following equation between compositions in $\widehat{\mathscr{Q}_{uT^{\geqslant1}}}$ holds:

$$\begin{aligned}&\bigl[\mathbb{1}_p[-1] \xrightarrow{\hat b} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\underline{\mathsf{Q}}(f;1)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr]\\ &= \sum_{j=1}^{n}\bigl[\mathbb{1}_p[-1] \xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat b,((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j}} (T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{\underline{\mathsf{Q}}(1;f)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})\bigr]. \quad (8.9.4)\end{aligned}$$

Notice that all summands are $T^{\geqslant 1}$-coalgebra morphisms, therefore by Remark 6.10 the sum is also a $T^{\geqslant 1}$-coalgebra morphism. If $n = 0$, then $\boxtimes_u^{i\in\varnothing} T^{\geqslant 1} s\mathcal{A}_i = \mathbb{1}_u$ and elements $f \in \mathsf{Q}(\,;\mathcal{B})$ are identified with objects of $\mathcal{B}$. In this case commutation with differentials holds true automatically.

8.10 Remark. According to Remark 5.21 equation (8.9.4) is equivalent to the following equation in $\widehat{\mathscr{Q}}_{uT^{\geqslant 1}}$:

$$\big[\mathbb{1}_p[-1] \xrightarrow{\hat{b}} T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{B}; s\mathcal{B}) \xrightarrow{\underline{\widehat{\mathscr{Q}}_{uT^{\geqslant 1}}^{f}}(\hat{f};1)} T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; s\mathcal{B})\big]$$
$$= \sum_{j=1}^{n} \big[\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{pr}}_1)_{i<j},\hat{b},(\dot{\mathrm{pr}}_1)_{i>j}} (T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{A}_i; s\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{\underline{\widehat{\mathscr{Q}}_{uT^{\geqslant 1}}^{f}}(1;\hat{f})} T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; s\mathcal{B})\big],$$

where $\dot{\mathrm{pr}}_1 : () \to T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{A}_i; s\mathcal{A}_i)$ is $(\varphi^{\widehat{\mathscr{Q}}_{uT^{\geqslant 1}}^{f}})^{-1}(\mathrm{id}_{s\mathcal{A}_i}^{\widehat{\mathscr{Q}}_{uT^{\geqslant 1}}^{f}})$. Composing both sides of the equation with $\theta : T^{\geqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; s\mathcal{B}) \to \underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B})$ and taking into account equations (5.19.2), (4.22.1) and (4.23.1) we obtain an equation in $\widehat{\mathscr{Q}}_u$:

$$\big[\mathbb{1}_p[-1] \xrightarrow{\hat{b}\theta} \underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{B}; T^{\geqslant 1}s\mathcal{B}) \xrightarrow{\underline{\widehat{\mathscr{Q}}_u}(\hat{f};1)} \underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B})\big]$$
$$= \sum_{j=1}^{n} \big[\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{id}}_{T^{\geqslant 1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(\dot{\mathrm{id}}_{T^{\geqslant 1}s\mathcal{A}_i})_{i>j}} (\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{A}_i; T^{\geqslant 1}s\mathcal{A}_i))_{i\in\mathbf{n}}$$
$$\xrightarrow{\underline{\widehat{\mathscr{Q}}_u}(1;\hat{f})} \underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B})\big].$$

Applying the multifunctor $T^{\leqslant 1} : \widehat{\mathscr{Q}}_u \to \widehat{\mathscr{Q}}_p$ to both sides and composing the source with $\mathrm{in}_1 : \mathbb{1}_p[-1] \to T^{\leqslant 1}(\mathbb{1}_p[-1])$ and the target with the closing transformation $\underline{T^{\leqslant 1}} : T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B}) \to \underline{\widehat{\mathscr{Q}}_p}((Ts\mathcal{A}_i)_{i\in\mathbf{n}}; Ts\mathcal{B})$ we get an equation in $\widehat{\mathscr{Q}}_p$:

$$\big[\mathbb{1}_p[-1] \xrightarrow{(\hat{b}\theta)\cdot\mathrm{in}_1} T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{B}; T^{\geqslant 1}s\mathcal{B}) \xrightarrow{T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}(\hat{f};1)}$$
$$T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B}) \xrightarrow{\underline{T^{\leqslant 1}}} \underline{\widehat{\mathscr{Q}}_p}((Ts\mathcal{A}_i)_{i\in\mathbf{n}}; Ts\mathcal{B})\big]$$
$$= \sum_{j=1}^{n} \big[\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{id}}_{T^{\geqslant 1}s\mathcal{A}_i})_{i<j},(\hat{b}\theta)\cdot\mathrm{in}_1,(\dot{\mathrm{id}}_{T^{\geqslant 1}s\mathcal{A}_i})_{i>j}} (T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}(T^{\geqslant 1}s\mathcal{A}_i; T^{\geqslant 1}s\mathcal{A}_i))_{i\in\mathbf{n}}$$
$$\xrightarrow{T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}(1;\hat{f})} T^{\leqslant 1}\underline{\widehat{\mathscr{Q}}_u}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in\mathbf{n}}; T^{\geqslant 1}s\mathcal{B}) \xrightarrow{\underline{T^{\leqslant 1}}} \underline{\widehat{\mathscr{Q}}_p}((Ts\mathcal{A}_i)_{i\in\mathbf{n}}; Ts\mathcal{B})\big].$$

Using equations (4.22.1) and (4.23.1) we transform the above to:

$$\big[\mathbb{1}_p[-1] \xrightarrow{\tilde b} \widehat{\underline{\mathscr{Q}}_p}(Ts\mathcal{B};Ts\mathcal{B}) \xrightarrow{\widehat{\underline{\mathscr{Q}}_p}(\widehat{T^{\leqslant 1}}\hat f;1)} \widehat{\underline{\mathscr{Q}}_p}((Ts\mathcal{A}_i)_{i\in\mathbf{n}};Ts\mathcal{B})\big]$$
$$= \sum_{j=1}^n \big[\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{id}}_{Ts\mathcal{A}_i})_{i<j},\tilde b,(\dot{\mathrm{id}}_{Ts\mathcal{A}_i})_{i>j}} (\widehat{\underline{\mathscr{Q}}_p}(Ts\mathcal{A}_i;Ts\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{\widehat{\underline{\mathscr{Q}}_p}(1;\widehat{T^{\leqslant 1}}\hat f)} \widehat{\underline{\mathscr{Q}}_p}((Ts\mathcal{A}_i)_{i\in\mathbf{n}};Ts\mathcal{B})\big].$$

Due to (7.9.4) $\widehat{T^{\leqslant 1}}\hat f = \tilde f$. Using Lemma 4.27 we can write the above equation in $\mathscr{Q}_p$ via symmetric Monoidal category data:

$$\big[\mathbb{1}_p[-1] \xrightarrow{\tilde b} \underline{\mathscr{Q}_p}(Ts\mathcal{B},Ts\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(\tilde f,1)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i,Ts\mathcal{B})\big]$$
$$= \sum_{j=1}^n \big[\mathbb{1}_p[-1] \xrightarrow{\lambda^{j:\mathbf{1}\to\mathbf{n}}} \boxtimes^{i\in\mathbf{n}}\,[(\mathbb{1}_p)_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_p)_{i>j}] \xrightarrow{\boxtimes^{i\in\mathbf{n}}[(\dot{\mathrm{id}}_{Ts\mathcal{A}_i})_{i<j},\tilde b,(\dot{\mathrm{id}}_{Ts\mathcal{A}_i})_{i>j}]}$$
$$\boxtimes^{i\in\mathbf{n}}\,\underline{\mathscr{Q}_p}(Ts\mathcal{A}_i,Ts\mathcal{A}_i) \xrightarrow{\underline{\boxtimes}^{\mathbf{n}}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i,\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i) \xrightarrow{\underline{\mathscr{Q}_p}(1,\tilde f)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i,Ts\mathcal{B})\big].$$

Another form of this equation in $\underline{\mathscr{Q}_p}$ is the following:

$$\big(\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i \xrightarrow{\dot{\tilde f}} Ts\mathcal{B} \xrightarrow{\tilde b} Ts\mathcal{B}\big) = \big(\boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i \xrightarrow{\sum_{i=1}^n 1^{\boxtimes(i-1)}\underline{\boxtimes}\tilde b\underline{\boxtimes}1^{\boxtimes(n-i)}} \boxtimes^{i\in\mathbf{n}}\,Ts\mathcal{A}_i \xrightarrow{\dot{\tilde f}} Ts\mathcal{B}\big). \tag{8.10.1}$$

An element $f \in \mathsf{Q}((\mathcal{A}_i))_{i\in I};\mathcal{B})$ is an A_∞-functor if and only if $\tilde f$ satisfies the above equation. Indeed, we have seen that equation (8.8.1) implies (8.10.1). On the other hand, due to (7.10.1) composing equation (8.10.1) with $\mathrm{pr}_1 : Ts\mathcal{B} \to s\mathcal{B}$ we obtain (8.8.1), which means that f is an A_∞-functor.

8.11 Lemma. *Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be A_∞-categories, and let $f \in \mathsf{Q}((\mathcal{A}_i)_{i\in I};\mathcal{B})$. Then both quiver morphisms*

$$L^f, R^f : \boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]] \to T^{\geqslant 1}s\mathcal{B},$$

defined (via restrictions) as compositions in $\mathscr{Q}$:

$$L^f = \big\{\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]] \xrightarrow{\lambda_u^{\triangleright\sqcup 1:I\sqcup\mathbf{1}\to\mathbf{1}\sqcup\mathbf{1}}} (\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i)\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{\hat f\boxtimes_u 1} T^{\geqslant 1}s\mathcal{B}\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\varphi(\hat b)} T^{\geqslant 1}s\mathcal{B}\big\},$$

$$R^f\big|_{\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]]} = \sum_{j\in I}\big\{\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]] \hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]]$$
$$\xrightarrow{\lambda_u^{h_j}} \boxtimes_u^{I}\,[(T^{\geqslant 1}s\mathcal{A}_i)_{i<j}^{i\in I},T^{\geqslant 1}s\mathcal{A}_j\boxtimes_u\mathbb{1}_p[-1],(T^{\geqslant 1}s\mathcal{A}_i)_{i>j}^{i\in I}]$$
$$\xrightarrow{\boxtimes_u^I[(1)_{i<j},\varphi(\hat b),(1)_{i>j}]} \boxtimes_u^{i\in I}\,T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\hat f} T^{\geqslant 1}s\mathcal{B}\big\},$$

$$R^f\big|_{(\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i)\boxtimes T^0\mathbb{1}_p[-1]} = \big\{(\boxtimes_u^{i\in I}T^{\geqslant 1}s\mathcal{A}_i)\boxtimes T^0\mathbb{1}_p[-1] \xrightarrow{(\lambda^{\mathsf{I}\,\cdot\,})^{-1}} \boxtimes_u^{i\in I}\,T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\hat f} T^{\geqslant 1}s\mathcal{B}\big\}$$

are $T^{\geqslant 1}$-coalgebra morphisms. Here $h_j : I \sqcup \mathbf{1} \to I$ is determined by $h_j\big|_I = \mathrm{id}$, $h_j(1) = j$.

The morphism f is an A_∞-functor if and only if $L^f = R^f$. The restriction of this equation to $(\boxtimes_u^{i\in I} T^{\geqslant 1} s\mathcal{A}_i) \boxtimes T^0 \mathbb{1}_p[-1]$ always holds.

Proof. The element f is an A_∞-functor if and only if condition (8.9.4) holds. An equivalent condition can be obtained by applying bijection $\varphi = \varphi^{\widehat{\mathscr{Q}}'_{uT^{\geqslant 1}}}$ to both sides of this equation. Denote by l (resp. r) the left (resp. right) hand side of (8.9.4). Define the $T^{\geqslant 1}$-coalgebra morphisms $L^f = \varphi(l)$ (resp. $R^f = \varphi(r)$). Thus, the morphism f is an A_∞-functor if and only if $l = r$, which is equivalent to equation $L^f = R^f$.

Let us prove that L^f, R^f can be presented by expressions given in the statement of the lemma. Indeed, by Remark 5.21 L^f equals the composition in $\mathscr{Q}_{uT^{\geqslant 1}}$

$$
\begin{aligned}
&\bigl\{\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, \mathbb{1}_p[-1]] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat{b}]} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B})] \\
&\qquad \xrightarrow[\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}(\hat{f};1)]]{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\underline{\mathsf{Q}}(f;1)\hat{}\,] \;\Vert} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\bigr\} \\
&= \bigl\{\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, \mathbb{1}_p[-1]] \xrightarrow{\lambda_u^{\triangleright\sqcup 1:I\sqcup\mathbf{1}\to\mathbf{1}\sqcup\mathbf{1}}} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \mathbb{1}_p[-1] \\
&\qquad \xrightarrow{\hat{f}\boxtimes_u\hat{b}} T^{\geqslant 1}s\mathcal{B} \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\bigr\},
\end{aligned}
$$

which is the expression given in lemma.

Computing the restriction $R^f\big|_{\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},\mathbb{1}_p[-1]]}$ we get by additivity of ev' in the second argument and by Remark 5.21 the composition in $\mathscr{Q}$

$$
\begin{aligned}
&\sum_{j\in I}\bigl\{\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I}, \mathbb{1}_p[-1]] \hookrightarrow (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \mathbb{1}_p[-1] \\
&\qquad \xrightarrow{1\boxtimes_u\lambda_u^{j:\mathbf{1}\to I}} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \boxtimes_u^I[(\mathbb{1}_u)_{i<j}, \mathbb{1}_p[-1], (\mathbb{1}_u)_{i>j}] \\
&\qquad \xrightarrow{1\boxtimes_u\boxtimes_u^I[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j}]} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u (\boxtimes_u^{i\in I} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)) \\
&\qquad \xrightarrow[1\boxtimes_u\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}(\mathrm{id};\hat{f})]{1\boxtimes_u\underline{\mathsf{Q}}(\mathrm{id};f)\hat{}\;\Vert} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\bigr\} \\
&= \sum_{j\in I}\bigl\{\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I}, \mathbb{1}_p[-1]] \hookrightarrow (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \mathbb{1}_p[-1] \\
&\qquad \xrightarrow{1\boxtimes_u\lambda_u^{j:\mathbf{1}\to I}} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u \boxtimes_u^I[(\mathbb{1}_u)_{i<j}, \mathbb{1}_p[-1], (\mathbb{1}_u)_{i>j}] \\
&\qquad \xrightarrow{1\boxtimes_u\boxtimes_u^I[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j}]} (\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes_u (\boxtimes_u^{i\in I} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)) \\
&\qquad \xrightarrow{\sigma_{(12)}} \boxtimes_u^{i\in I}\,[T^{\geqslant 1}s\mathcal{A}_i \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i)] \xrightarrow{\boxtimes_u^I \mathrm{ev}^{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}} \boxtimes_u^{i\in I}\, T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{\hat{f}} T^{\geqslant 1}s\mathcal{B}\bigr\},
\end{aligned}
$$

which is the expression given in lemma.

The restriction

$$R^f\big|_{(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i)\boxtimes T^0\mathbb{1}_p[-1]} = \big\{(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i)\boxtimes T^0\mathbb{1}_p[-1] \xrightarrow{1\boxtimes(T^0r\cdot\theta)} (\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i)\boxtimes T^0\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B}) \xrightarrow{\mathrm{ev}''} T^{\geqslant1}s\mathcal{B}\big\}$$

equals the expression given in lemma due to (7.5.6), since r takes the object $*$ to f. Similarly, $L^f\big|_{(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i)\boxtimes T^0\mathbb{1}_p[-1]}$ reduces to the same expression.

The lemma is proven. □

8.12 Symmetric multicategory A_∞. Let us prove that subsets $\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})\subset \mathsf{Q}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ form a symmetric multicategory. If $\mathcal{A}$ is an A_∞-category, then $\mathrm{id}^{\mathsf{Q}}_{\mathcal{A}}\in\mathsf{Q}(\mathcal{A};\mathcal{A})$ is an A_∞-functor, see (8.8.1) or (8.9.4). Let us prove that A_∞-functors are closed with respect to compositions μ^{Q}_ϕ. Let $\phi: I\to J$ be a map in $\mathcal{S}$. Let $\mathcal{A}_i$, $i\in I$, $\mathcal{B}_j$, $j\in J$, $\mathcal{C}$ be A_∞-categories, and let $f^j:(\mathcal{A}_i)_{i\in\phi^{-1}j}\to\mathcal{B}_j$, $g:(\mathcal{B}_j)_{j\in J}\to\mathcal{C}$ be A_∞-functors. Their composition $h=\mu^{\mathsf{Q}}_\phi((f^j)_{j\in J},g):(\mathcal{A}_i)_{i\in I}\to\mathcal{C}$ satisfies the following equation in $\widehat{\mathscr{Q}_{uT^{\geqslant1}}}$

$$\begin{aligned}
&\big[\mathbb{1}_p[-1]\xrightarrow{\hat b} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{C};\mathcal{C})\xrightarrow{\underline{\mathsf{Q}}(h;1)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big]\\
&=\big[\mathbb{1}_p[-1]\xrightarrow{\hat b} T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{C};\mathcal{C})\xrightarrow{\underline{\mathsf{Q}}(g;1)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{B}_j)_{j\in J};\mathcal{C})\xrightarrow{\underline{\mathsf{Q}}((f^j)_{j\in J};1)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big]\\
&=\sum_{k\in J}\big[\mathbb{1}_p[-1]\xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j})\hat{})_{j<k},\hat b,((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j})\hat{})_{j>k}} (T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B}_j;\mathcal{B}_j))_{j\in J}\xrightarrow{\underline{\mathsf{Q}}(\mathrm{id}_J;g)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{B}_j)_{j\in J};\mathcal{C})\\
&\qquad\xrightarrow{\underline{\mathsf{Q}}((f^j)_{j\in J};1)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big]\\
&=\sum_{k\in J}\big[\mathbb{1}_p[-1]\xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j})\hat{})_{j<k},\hat b,((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j})\hat{})_{j>k}} (T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B}_j;\mathcal{B}_j))_{j\in J}\\
&\qquad\xrightarrow{(\underline{\mathsf{Q}}(f^j;1)\hat{})_{j\in J}} (T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j))_{j\in J}\xrightarrow{\underline{\mathsf{Q}}(\phi;g)\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big]
\end{aligned}\tag{8.12.1}$$

by Lemma 4.14, equation (8.9.4) and Lemma 4.13. We have

$$\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j}\cdot\underline{\mathsf{Q}}(f^j;1)=\dot f^j=(\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})_{i\in\phi^{-1}j}\cdot\underline{\mathsf{Q}}(\mathrm{id}_{\phi^{-1}j};f^j),$$

thus,

$$(\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{B}_j})\hat{}\cdot\underline{\mathsf{Q}}(f^j;1)\hat{}=\widehat{\dot f^j}=((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i\in\phi^{-1}j}\cdot\underline{\mathsf{Q}}(\mathrm{id}_{\phi^{-1}j};f^j)\hat{}.$$

Therefore, by (8.9.4) expression (8.12.1) can be written as

$$\sum_{k\in J}\sum_{l\in\phi^{-1}k}\Big[\mathbb{1}_p[-1]\xrightarrow{(((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i\in\phi^{-1}j})_{j<k},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})^{\phi i=k}_{i<l},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})^{\phi i=k}_{i>l},(((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i\in\phi^{-1}j})_{j>k}}$$

$$((T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in\phi^{-1}j})_{j\in J}\xrightarrow{(\underline{\mathsf{Q}}(\mathrm{id}_{\phi^{-1}j};f^j)\hat{})_{j\in J}}(T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j))_{j\in J}$$

$$\xrightarrow{\underline{\mathsf{Q}}(\phi;g)\hat{}}T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\Big]$$

$$=\sum_{m\in I}\Big[\mathbb{1}_p[-1]\xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<m},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>m}}(T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I}\xrightarrow{\underline{\mathsf{Q}}(\mathrm{id}_I;h)\hat{}}T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\Big]$$

due to Lemma 4.15. Hence h satisfies (8.9.4) and it is an A_∞-functor.

8.13 Closed multicategory A_∞. Let us prove that multicategory A_∞ is closed. First of all, we make a step towards constructing the internal homomorphisms objects.

8.14 Lemma. *Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be A_∞-categories. Then the quiver $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ has the following differential. Define for all $j\in I$ the maps $g_j:\mathbf{2}\to I\sqcup\mathbf{1}$, $g_j(1)=1\in\mathbf{1}$, $g_j(2)=j\in I$. There are $T^{\geqslant1}$-coalgebra morphisms*

$$^0\hat{B}=(\varphi^{\widehat{\mathscr{Q}_u}'_{T^{\geqslant1}}})^{-1}\big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]\xrightarrow{1\boxtimes_u\hat{b}}T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B})$$

$$\xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}}T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\},$$

$$^j\hat{B}=(\varphi^{\widehat{\mathscr{Q}_u}'_{T^{\geqslant1}}})^{-1}\big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]\xrightarrow{\lambda^{g_j}_u}$$

$$\boxtimes^{I\sqcup\mathbf{1}}_u\big[(\mathbb{1}_u)^{i\in I}_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_u)^{i\in I}_{i>j},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big]\xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}_u[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j},1]}$$

$$\boxtimes^{I\sqcup\mathbf{1}}_u\big[(T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big]\xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}}T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\}.$$

Their linear combination

$$\hat{B}={}^0\hat{B}-\sum_{j\in I}{}^j\hat{B}:\mathbb{1}_p[-1]\to T^{\geqslant1}s\underline{\mathsf{Q}}(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B});\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}))$$

is a $T^{\geqslant1}$-coalgebra morphism as well, see Remark 6.10. It is a differential in $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ which satisfies equivalent equations (8.2.1), (8.2.2) *and* (8.3.1). *The restriction of $T^{\geqslant1}$-coalgebra morphism*

$$\varphi^{\widehat{\mathscr{Q}_u}'_{T^{\geqslant1}}}(\hat{B}):T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]\to T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$$

to $T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^0\mathbb{1}_p[-1]$ *gives* $(\lambda^{\mathsf{I}\cdot})^{-1}$ *and the restriction to* $Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1]$ *gives*

$$\varphi(\hat{B}) \overset{\text{def}}{=} \bigl\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{1\boxtimes\hat{b}} Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \hookrightarrow T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}$$
$$- \sum_{j\in I}\bigl\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \hookrightarrow T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{\lambda^{g_j}_u} \boxtimes_u^{I\sqcup\mathbf{1}}[(\mathbb{1}_u)^{i\in I}_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_u)^{i\in I}_{i>j},T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j},1]}$$
$$\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\} \tag{8.14.1}$$

The morphism $\varphi(\hat{B})$ *satisfies equation* (8.5.1).

Proof. Additivity of ev' in the second argument implies formula (8.14.1) for $\varphi(\hat{B})$. Equation (8.5.1) for $\varphi(\hat{B})$ reads:

$$\bigl[Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-2] \xrightarrow{\sim} Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1]\boxtimes \mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{B})\boxtimes 1} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{B})} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr] = 0.$$

Substituting (8.14.1) for $\varphi(\hat{B})$ we shall take into account that $(\mu^{\underline{\mathsf{Q}}})\hat{} = (\mu^{\underline{\mathscr{Q}_u{}^{T^{\geqslant 1}}}})\hat{}$ is a particular case of $\mu^{\underline{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}}$ by (5.21.1). We get four summands (which are also sums).

The first summand is

$$\bigl\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-2] \xhookrightarrow{1\boxtimes\angle} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{1\boxtimes_u\hat{b}\boxtimes_u 1} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}\boxtimes_u 1} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{1\boxtimes_u\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}.$$

It vanishes due to associativity of $(\mu^{\underline{\mathsf{Q}}})\hat{} = \mu^{\underline{\widehat{\mathscr{Q}}^f_{uT^{\geqslant 1}}}}$ and due to equation (8.3.1).

The second and the third summands–sums are

$$- \sum_{j\in I}\bigl\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-2] \xhookrightarrow{1\boxtimes\angle} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{1\boxtimes_u\hat{b}\boxtimes_u 1} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}\boxtimes_u 1} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{\lambda^{g_j}_u} \boxtimes_u^{I\sqcup\mathbf{1}}[(\mathbb{1}_u)^{i\in I}_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_u)^{i\in I}_{i>j},T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j},1]}$$
$$\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}$$

$$
\begin{aligned}
-\sum_{j\in I}\bigl\{ & Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-2] \xhookrightarrow{1\boxtimes\angle} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1] \\
& \xrightarrow{\lambda_u^{g_j}\boxtimes_u 1} \boxtimes_u^{I\sqcup\mathbf{2}}[(\mathbb{1}_u)_{i<j}^{i\in I},\mathbb{1}_p[-1],(\mathbb{1}_u)_{i>j}^{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{2}}[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j},1,1]} \\
& \boxtimes_u^{I\sqcup\mathbf{2}}[(T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}\boxtimes_u 1} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1] \\
& \xrightarrow{1\boxtimes_u\hat{b}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{(\mu^{\mathsf{Q}}_{\triangleright})\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}.
\end{aligned}
$$

The second and the third summands cancel each other due to equation $\lambda^{(12):\mathbf{2}\to\mathbf{2}} = -1 : \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1]\to\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p[-1]$, which implies

$$
\begin{aligned}
\bigl\{\mathbb{1}_p[-2] \xhookrightarrow{\angle} \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1] \xrightarrow{\lambda_u^{(12):\mathbf{2}\to\mathbf{2}}} \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1]\bigr\} \\
= -\bigl\{\mathbb{1}_p[-2] \xhookrightarrow{\angle} \mathbb{1}_p[-1]\boxtimes_u\mathbb{1}_p[-1]\bigr\}, \quad (8.14.2)
\end{aligned}
$$

and due to associativity of $\underline{\mu} = (\mu^{\mathsf{Q}})\hat{} = \mu^{\widehat{\mathscr{Q}}^{f}_{uT^{\geqslant1}}}$. Namely, we use the particular case $((1)_I,\underline{\mu}_{\triangleright})\cdot\underline{\mu}_{\mathrm{id}} = (\underline{\mu}_{\mathrm{id}},1)\cdot\underline{\mu}_{\triangleright}$ of equation (4.2.2), written for the data $I\xrightarrow{\mathrm{id}} I\xrightarrow{\triangleright}\mathbf{1}$.

The fourth summand (double sum) is

$$
\begin{aligned}
+\sum_{j\in I}\sum_{k\in I}\bigl\{ & Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-2] \xhookrightarrow{1\boxtimes\angle} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1]\boxtimes_u \mathbb{1}_p[-1] \\
& \xrightarrow{\lambda_u^{g_k}\boxtimes_u 1} \boxtimes_u^{I\sqcup\mathbf{2}}[(\mathbb{1}_u)_{i<k}^{i\in I},\mathbb{1}_p[-1],(\mathbb{1}_u)_{i>k}^{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{2}}[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<k},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>k},1,1]} \\
& \boxtimes_u^{I\sqcup\mathbf{2}}[(T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}\boxtimes_u 1} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1] \\
& \xrightarrow{\lambda_u^{g_j}} \boxtimes_u^{I\sqcup\mathbf{1}}[(\mathbb{1}_u)_{i<j}^{i\in I},\mathbb{1}_p[-1],(\mathbb{1}_u)_{i>j}^{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j},1]} \\
& \boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{(\mu^{\mathsf{Q}}_{\mathrm{id}})\hat{}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}.
\end{aligned}
$$

This expression can be transformed with the use of the particular case $((1)_I,\underline{\mu}_{\mathrm{id}})\cdot\underline{\mu}_{\mathrm{id}} = (\underline{\mu}_{\mathbf{1}\to\mathbf{1}},1)\cdot\underline{\mu}_{\mathrm{id}}$ of equation (4.2.2), written for the data $I\xrightarrow{\mathrm{id}} I\xrightarrow{\mathrm{id}} I$. If $j\neq k$, the summands labeled by (j,k) and by (k,j) differ by the sign and cancel each other due to property (8.14.2). The summands labeled by (j,j) vanish due to equation (8.5.1) for the A_∞-category $\mathcal{A}_j$. Therefore, the double sum vanishes as well and the proposition is proven. □

8.15 Proposition. *Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be A_∞-categories. The differential $B : \mathbb{1}_p[-1]\to s\underline{\mathsf{Q}}(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B});\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}))$ from Lemma 8.14 has the following properties. A morphism $f\in\mathsf{Q}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ satisfies equation ${}_fB_0 = 0$ if and only if f is an A_∞-functor. Therefore, B gives an A_∞-category structure of the full subquiver $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\subset\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ consisting of A_∞-functors.*

Proof. Let us obtain explicit formulae for B. For this sake we write equation between (8.6.1) and (8.6.2) in $\widehat{\mathscr{Q}_u}$ for B. Its restriction to $Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1]$ coincides with the restriction of any of the following expressions for $\varphi^{\widehat{\mathscr{Q}_u}}(B)$:

$$\begin{aligned}
&\big\langle Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \\
&\qquad \hookrightarrow \big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{B})} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mathrm{pr}_1} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\}\big\rangle \\
&= \big\langle Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \\
&\hookrightarrow \big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1] \xrightarrow{1,\hat{b}} T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mu^{\mathsf{Q}}_{\triangleright}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\}\big\rangle \\
&- \big\langle Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \hookrightarrow \sum_{j\in I}\big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{id}}^{\mathscr{Q}_u T^{\geqslant1}}_{\mathcal{A}_i})_{i<j},\hat{b},(\dot{\mathrm{id}}^{\mathscr{Q}_u T^{\geqslant1}}_{\mathcal{A}_i})_{i>j},1} \\
&\qquad (T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mu^{\mathsf{Q}}_{\mathrm{id}}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\}\big\rangle \\
&= \big\langle Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \hookrightarrow \big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1] \\
&\qquad \xrightarrow{\theta,b} \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B}),\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B}) \xrightarrow{\mu^{\widehat{\mathscr{Q}_u}}_{\triangleright}} \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\}\big\rangle \\
&- \big\langle Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \hookrightarrow \sum_{j\in I}\big\{T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1] \xrightarrow{(\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i>j},\mathrm{pr}_1} \\
&\qquad (\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i\in I},\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}) \xrightarrow{\mu^{\widehat{\mathscr{Q}_u}}_{\mathrm{id}}} \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\}\big\rangle.
\end{aligned}$$

Compositions in braces are taken in $\widehat{\mathscr{Q}_{uT^{\geqslant1}}}$. Here composition for j-th term of the sum is indexed by the map $g_j : \mathbf{2}\to I\sqcup\mathbf{1}$, $g_j(1)=1\in\mathbf{1}$, $g_j(2)=j\in I$.

The above equation is the equation between (8.6.3) and (8.6.4) in $\mathscr{Q}$

$$\begin{aligned}
&\varphi^{\mathscr{Q}_p}(B) = \big[Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{1\boxtimes B} \\
&\qquad Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \underline{\mathscr{Q}_p}(Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big] \\
&= \big[Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \hookrightarrow T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\varphi^{\mathscr{Q}_u}(B)} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big] \\
&= \big\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{T^{\leqslant1}\theta\boxtimes b} T^{\leqslant1}\underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B})\boxtimes \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B}) \\
&\qquad \hookrightarrow \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B})\boxtimes_u \underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(\boxtimes^{i\in I}_u T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\} \\
&- \sum_{j\in I}\big\{Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{\lambda^{g_j}} \boxtimes^{I\sqcup\mathbf{1}}\,[(\mathbb{1}_p)^{i\in I}_{j<i},\mathbb{1}_p[-1],(\mathbb{1}_p)^{i\in I}_{j>i},Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \\
&\qquad \xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i>j},\mathrm{pr}_{0,1}]} \\
&\boxtimes^{I\sqcup\mathbf{1}}\,[(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i<j},\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j,T^{\geqslant1}s\mathcal{A}_j), \\
&\qquad (T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i>j},T^{\leqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})]
\end{aligned}$$

$$\hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}}[(\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i\in I},\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})] \xrightarrow{\mu_{\mathrm{id}}^{\widehat{\underline{\mathscr{Q}_u}}}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\}.$$

Due to Lemma 4.28 the multiplication in closed multicategory $\widehat{\underline{\mathscr{Q}_u}}$ is given by the composition in $\mathscr{Q}$

$$\begin{aligned} \mu_{\mathrm{id}}^{\widehat{\underline{\mathscr{Q}_u}}} = \big[&\boxtimes_u^{I\sqcup\mathbf{1}}[(\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i\in I},\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})] \\ &\xrightarrow{\lambda^{\triangleright\sqcup1:I\sqcup\mathbf{1}\to\mathbf{2}}} [\boxtimes_u^{i\in I}\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i)]\boxtimes_u\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}) \xrightarrow{\boxtimes_u^I\boxtimes_u1} \\ &\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i)\boxtimes_u\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big]. \end{aligned} \tag{8.15.1}$$

We may restrict the considered equation to $T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]$, $m\geqslant0$, and find the component B_m from it. Here we consider the case of $m=0$. Then the equation takes the form

$$\begin{aligned} &\big[T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes B_0} T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\underline{\mathscr{Q}_p}(T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})) \\ &\qquad\xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big] \\ &= \big\{T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1] \xrightarrow{T^0\theta\boxtimes b} T^0\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B})\boxtimes\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B}) \\ &\qquad\hookrightarrow \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{B})\boxtimes_u\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_u}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\} \\ &-\sum_{j\in I}\big\{T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1] \xrightarrow{\lambda^{g_j}} \boxtimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_p)_{j<i}^{i\in I},\mathbb{1}_p[-1],(\mathbb{1}_p)_{j>i}^{i\in I},T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \\ &\qquad\xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i>j},1]} \\ &\boxtimes^{I\sqcup\mathbf{1}}[(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i<j},\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j,T^{\geqslant1}s\mathcal{A}_j), \\ &\qquad(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i>j},T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \\ &\hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}}[(\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i\in I},\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})] \xrightarrow{\mu_{\mathrm{id}}^{\widehat{\underline{\mathscr{Q}_u}}}} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\}. \end{aligned}$$

Let $f\in\mathsf{Q}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ be a morphism. Then the value of

$$ {}_fB_0:\mathbb{1}_p[-1]\to s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})=\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}), \qquad *\mapsto f$$

is found from the above equation in the form:

$$\begin{aligned} {}_fB_0 &= \big\{\mathbb{1}_p[-1]\xrightarrow{b}\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{B},s\mathcal{B})\xrightarrow{\underline{\mathscr{Q}_u}(\hat{f},1)}\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\big\} \\ &-\sum_{j\in I}\big\{\mathbb{1}_p[-1]\xrightarrow{\lambda^{j:\mathbf{1}\to I}}\boxtimes^I[(\mathbb{1}_p)_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_p)_{i>j}]\xrightarrow{\boxtimes^I((T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i>j})} \\ &\boxtimes^I[(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i<j},\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j,T^{\geqslant1}s\mathcal{A}_j),(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i>j}] \end{aligned}$$

$$\hookrightarrow \boxtimes_u^{i\in I} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{A}_i) \xrightarrow{\boxtimes_u^I} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \xrightarrow{\underline{\mathscr{Q}_u}(1,f)} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big\}.$$

This is the difference of left and right hand sides of equation (8.9.2), which holds if and only if f is an A_∞-functor. Thus, the said equation can be written as condition ${}_fB_0 = 0$. This proves the proposition. □

8.16 Components of the differential B**.** We consider the differential B in the quiver $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$. The components B_m for $m > 1$ are obtained from the equation, obtained in the proof of Proposition 8.15:

$$\varphi^{\mathscr{Q}_p}(B_m) = \big[T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{1\boxtimes B_m} T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \underline{\mathscr{Q}_p}(T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}), s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})\big]$$
$$= \big\{T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{\theta\boxtimes b} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}) \boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{B}, s\mathcal{B}) \xrightarrow{\Upsilon\boxtimes\underline{\mathscr{Q}_p}(\mathrm{in}_1,1)} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}) \boxtimes \underline{\mathscr{Q}_p}(T^{\geqslant 1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_p}} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big\}.$$

Thus, B_m takes an element $r^1 \otimes \cdots \otimes r^m \in T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$ to the composition in $\underline{\mathscr{Q}_p}$

$$(r^1 \otimes \cdots \otimes r^m)B_m = \big[T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i \xrightarrow{(r^1\otimes\cdots\otimes r^m)\theta} T^{\geqslant 1}s\mathcal{B} \xrightarrow{b} s\mathcal{B}\big] \in s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}). \quad (8.16.1)$$

Now let us find the remaining component B_1 from the equation

$$\varphi^{\mathscr{Q}_p}(B_1) = \big\{s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{\theta\boxtimes b} \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}) \boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{B}, s\mathcal{B}) \xrightarrow{\Upsilon\boxtimes\underline{\mathscr{Q}_p}(\mathrm{in}_1,1)} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{B}) \boxtimes \underline{\mathscr{Q}_p}(T^{\geqslant 1}s\mathcal{B}, s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_p}} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big\}$$
$$- \sum_{j\in I} \big\{s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{\lambda^{(12):\mathbf{2}\to\mathbf{2}}} \mathbb{1}_p[-1] \boxtimes s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \xrightarrow{\lambda^{j:\mathbf{1}\to I}\boxtimes 1}$$
$$[\boxtimes^I((\mathbb{1}_p)^{i\in I}_{j<i}, \mathbb{1}_p[-1], (\mathbb{1}_p)^{i\in I}_{j>i})] \boxtimes s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \xrightarrow{[\boxtimes^I((T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant 1}s\mathcal{A}_i})_{i<j}, \hat{b}\theta, (T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant 1}s\mathcal{A}_i})_{i>j})]\boxtimes 1}$$
$$[\boxtimes^I((T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{A}_i))_{i<j}, \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}_j, T^{\geqslant 1}s\mathcal{A}_j), (T^0\underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{A}_i))_{i>j})] \boxtimes \boxtimes \underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})$$
$$\hookrightarrow [\boxtimes_u^{i\in I} \underline{\mathscr{Q}_u}(T^{\geqslant 1}s\mathcal{A}_i, T^{\geqslant 1}s\mathcal{A}_i)] \boxtimes \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \xrightarrow{\boxtimes_u^I\boxtimes 1}$$
$$\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i) \boxtimes \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B}) \xrightarrow{\underline{\mathscr{Q}_p}(1,\mathrm{in}_1\cdot\vartheta^{-1})\boxtimes\underline{\mathscr{Q}_p}(\vartheta,1)}$$
$$\underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, \boxtimes^{i\in I} Ts\mathcal{A}_i) \boxtimes \underline{\mathscr{Q}_p}(\boxtimes^{i\in I} Ts\mathcal{A}_i, s\mathcal{B}) \xrightarrow{\mu^{\mathscr{Q}_p}} \underline{\mathscr{Q}_p}(T^{\leqslant 1} \boxtimes_u^{i\in I} T^{\geqslant 1}s\mathcal{A}_i, s\mathcal{B})\big\}.$$

Here we have used expression (8.15.1) for $\mu^{\widehat{\mathscr{Q}_u}}_{\mathrm{id}}$. Using diagram (7.8.1) and the next after it we transform the above j-th term to

$$
\begin{aligned}
&\bigl\{s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes \mathbb{1}_p[-1] \xrightarrow{\lambda^{(12):\mathbf{2}\to\mathbf{2}}} \mathbb{1}_p[-1]\boxtimes s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\lambda^{j:\mathbf{1}\to I}\boxtimes 1} \\
&\quad [\boxtimes^I((\mathbb{1}_p)^{i\in I}_{j<i},\mathbb{1}_p[-1],(\mathbb{1}_p)^{i\in I}_{j>i})]\boxtimes s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{[\boxtimes^I((T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i<j},\hat{b}\theta,(T^0\dot{\mathrm{id}}^{\mathscr{Q}_u}_{T^{\geqslant1}s\mathcal{A}_i})_{i>j})]\boxtimes 1} \\
&[\boxtimes^I((T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i<j},\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_j,T^{\geqslant1}s\mathcal{A}_j),(T^0\underline{\mathscr{Q}_u}(T^{\geqslant1}s\mathcal{A}_i,T^{\geqslant1}s\mathcal{A}_i))_{i>j})]\boxtimes \\
&\qquad\qquad \boxtimes\underline{\mathscr{Q}_u}(\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}) \\
&\quad \xrightarrow{[\boxtimes^I \underline{T^{\leqslant1}}]\boxtimes\Upsilon} [\boxtimes^{i\in I}\underline{\mathscr{Q}_p}(Ts\mathcal{A}_i,Ts\mathcal{A}_i)]\boxtimes\underline{\mathscr{Q}_p}(T^{\leqslant1}\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B}) \\
&\xrightarrow{\boxtimes^I\boxtimes\underline{\mathscr{Q}_p}(\vartheta,1)} \underline{\mathscr{Q}_p}(\boxtimes^{i\in I}Ts\mathcal{A}_i,\boxtimes^{i\in I}Ts\mathcal{A}_i)\boxtimes\underline{\mathscr{Q}_p}(\boxtimes^{i\in I}Ts\mathcal{A}_i,s\mathcal{B}) \xrightarrow{\mu^{\underline{\mathscr{Q}_p}}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in I}Ts\mathcal{A}_i,s\mathcal{B}) \\
&\qquad\qquad \xrightarrow{\underline{\mathscr{Q}_p}(\vartheta^{-1},1)} \underline{\mathscr{Q}_p}(T^{\leqslant1}\boxtimes_u^{i\in I}T^{\geqslant1}s\mathcal{A}_i,s\mathcal{B})\bigr\}.
\end{aligned}
$$

Therefore, the map $B_1 : s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$ of degree 1 takes an element r to the difference of compositions in $\underline{\mathscr{Q}_p}$

$$
\begin{aligned}
(r)B_1 = &\bigl[T^{\leqslant1}\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i \xrightarrow{(r)\theta} T^{\geqslant1}s\mathcal{B} \xrightarrow{b} s\mathcal{B}\bigr] \\
&-(-)^r\bigl[T^{\leqslant1}\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i \xrightarrow{\vartheta^{-1}} \boxtimes^{i\in I}Ts\mathcal{A}_i \xrightarrow{\sum_{j=1}^n 1^{\boxtimes(j-1)}\boxtimes(\hat{b}\theta)\cdot\mathrm{in}_1\boxtimes 1^{\boxtimes(n-j)}} \boxtimes^{i\in\mathbf{n}}Ts\mathcal{A}_i \\
&\qquad\qquad \xrightarrow{\vartheta} T^{\leqslant1}\boxtimes_u^{i\in\mathbf{n}}T^{\geqslant1}s\mathcal{A}_i \xrightarrow{r} s\mathcal{B}\bigr].
\end{aligned}
$$

8.17 Proposition. *The evaluations*

$$
\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i);\mathcal{B}} = \bigl[(\mathcal{A}_i)_{i\in I},\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \hookrightarrow (\mathcal{A}_i)_{i\in I},\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mathrm{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}}} \mathcal{B}\bigr] \tag{8.17.1}
$$

are A_∞-functors. Moreover, B is the only differential in $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ such that $\mathrm{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}}$: $(\mathcal{A}_i)_{i\in I},\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathcal{B}$ satisfies equation $L^{\mathrm{ev}^{\mathsf{Q}}} = R^{\mathrm{ev}^{\mathsf{Q}}}$ from Lemma 8.11.

Proof. Indeed, equation $L^{\mathrm{ev}^{\mathsf{Q}}} = R^{\mathrm{ev}^{\mathsf{Q}}}$ restricted to $\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I},Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]]$ takes the form in $\widehat{\mathscr{Q}_{u}}_{T^{\geqslant1}}$

$$
\begin{aligned}
&\bigl\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I},Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \hookrightarrow \\
&\bigl[\boxtimes_u^{I\sqcup\mathbf{1}}((T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}))\bigr]\boxtimes_u\mathbb{1}_p[-1] \xrightarrow{(\mathrm{ev}^{\mathsf{Q}})\hat{}\boxtimes_u 1} T^{\geqslant1}s\mathcal{B}\boxtimes_u\mathbb{1}_p[-1] \xrightarrow{\varphi(\hat{b})} T^{\geqslant1}s\mathcal{B}\bigr\} \\
&\ = \sum_{j\in I}\bigl\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I},Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]] \\
&\qquad\qquad \hookrightarrow \boxtimes_u^{I\sqcup\mathbf{2}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}),\mathbb{1}_p[-1]]
\end{aligned}
$$

$$\xrightarrow{\lambda_u^{k_j}} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i<j}^{i\in I}, T^{\geqslant 1}s\mathcal{A}_j \boxtimes_u \mathbb{1}_p[-1], (T^{\geqslant 1}s\mathcal{A}_i)_{i>j}^{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_{i<j},\varphi(\hat{b}),(1)_{i>j},1]} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{(\mathrm{ev}^{\mathsf{Q}})\hat{}} T^{\geqslant 1}s\mathcal{B}\big\}$$
$$+ \big\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]] \xrightarrow{\lambda^{\mathrm{id}_I\sqcup\triangleright: I\sqcup\mathbf{2}\to I\sqcup\mathbf{1}}}$$
$$\boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]] \xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[(1)_I,\varphi(\hat{B})]} \boxtimes^{I\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})]$$
$$\hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{(\mathrm{ev}^{\mathsf{Q}})\hat{}} T^{\geqslant 1}s\mathcal{B}\big\}. \quad (8.17.2)$$

Here the j-th term is obtained with the composition $\mu_{k_j}^{\widehat{\mathscr{Q}_{u}{}_{T^{\geqslant 1}}}}$, where $k_j : I \sqcup \mathbf{2} \to I \sqcup \mathbf{1}$ is determined by $k_j\big|_I = \mathrm{id}$, $k_j(1) = 1$, $k_j(2) = j$. Invertibility of $\varphi^{\widehat{\mathscr{Q}_{u}{}_{T^{\geqslant 1}}}'}$ implies that there exists no more than one $T^{\geqslant 1}$-coalgebra morphism $\varphi(\hat{B})$ which satisfies the above equation. Let us verify that morphism (8.14.1) solves the above equation.

The contribution of the first term of (8.14.1) cancels with the left hand side of (8.17.2). Indeed, this contribution

$$\big\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]]$$
$$\hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \boxtimes_u \mathbb{1}_p[-1]]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,1\boxtimes_u\hat{b}]} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B})]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\mu_{\triangleright}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}]} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\big\}$$
$$= \big\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]]$$
$$\hookrightarrow [\boxtimes_u^{I\sqcup\mathbf{1}}((T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}))] \boxtimes_u \mathbb{1}_p[-1]$$
$$\xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}\boxtimes_u\hat{b}} T^{\geqslant 1}s\mathcal{B} \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\big\}$$

is equal to the left hand side of (8.17.2).

The contribution of the j-th term of (8.14.1) cancels with the j-th term of the sum in (8.17.2). Indeed, this contribution

$$-\big\{\boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]]$$
$$\hookrightarrow \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \boxtimes_u \mathbb{1}_p[-1]]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\lambda_u^{g_j}]} \boxtimes_u^{I\sqcup\mathbf{1}} \big[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{I\sqcup\mathbf{1}}[(\mathbb{1}_u)_{i<j}^{i\in I}, \mathbb{1}_p[-1], (\mathbb{1}_u)_{i>j}^{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\big]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\boxtimes_u^{I\sqcup\mathbf{1}}[(\dot{\mathrm{id}}_{s\mathcal{A}_i}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}})_{i<j},\hat{b},(\dot{\mathrm{id}}_{s\mathcal{A}_i}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}})_{i>j},1]]}$$
$$\boxtimes_u^{I\sqcup\mathbf{1}} \big[(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\big]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\mu_{\mathrm{id}}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}]} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant 1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant 1}}}} T^{\geqslant 1}s\mathcal{B}\big\}$$

$$
\begin{aligned}
&= -\bigl\{ \boxtimes^{I\sqcup\mathbf{2}}[(Ts\mathcal{A}_i)_{i\in I}, Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]] \\
&\qquad \hookrightarrow \boxtimes_u^{I\sqcup\mathbf{2}} [(T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}), \mathbb{1}_p[-1]] \xrightarrow{\lambda_u^{l_j}} \\
&\boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant1}s\mathcal{A}_i \boxtimes_u \mathbb{1}_u)_{i<j}^{i\in I}, T^{\geqslant1}s\mathcal{A}_j \boxtimes_u \mathbb{1}_p[-1], (T^{\geqslant1}s\mathcal{A}_i \boxtimes_u \mathbb{1}_u)_{i>j}^{i\in I}, T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1\boxtimes_u \dot{\mathrm{id}}_{s\mathcal{A}_i}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}})_{i<j}, 1\boxtimes_u\hat{b}, (1\boxtimes_u \dot{\mathrm{id}}_{s\mathcal{A}_i}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}})_{i>j}, 1]} \\
&\qquad \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant1}s\mathcal{A}_i \boxtimes_u T^{\geqslant1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I}, T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}})_I, 1]} \boxtimes_u^{I\sqcup\mathbf{1}} [(T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, T^{\geqslant1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}}} T^{\geqslant1}s\mathcal{B} \bigr\}
\end{aligned}
$$

differs only by sign from the j-th term in the right hand side of (8.17.2). The map $l_j : I \sqcup \mathbf{2} \to I \sqcup \mathbf{1}$ is given by its restrictions $l_j\big|_I = \mathrm{id}_I$, $l_j\big|_{\mathbf{2}} = g_j : \mathbf{2} \to I \sqcup \mathbf{1}$. Thus, equation (8.17.2) is verified. As a corollary, $\mathrm{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i);\mathcal{B}}$ is an A_∞-functor. $\square$

8.18 Restriction of an A_∞-functor to a subset of arguments. Let $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be an A_∞-functor, and let $J \subset I$ be a subset. Choose a family of objects $(X_i)_{i\in I\smallsetminus J} \in \prod_{i\in I\smallsetminus J} \mathrm{Ob}\,\mathcal{A}_i$. We view them as A_∞-functors $X_i : () \to \mathcal{A}_i$. Define an A_∞-functor $f|_J^{(X_i)_{i\in I\smallsetminus J}} : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$, the restriction of f to arguments in J as the image of $((\mathrm{id}_{\mathcal{A}_j})_{j\in J}, (X_i)_{i\in I\smallsetminus J}, f)$ under the composition map

$$
\mu^{\mathsf{A}_\infty}_{J\hookrightarrow I} : \prod_{j\in J} \mathsf{A}_\infty(\mathcal{A}_j;\mathcal{A}_j) \times \prod_{i\in I\smallsetminus J} \mathsf{A}_\infty(\,;\mathcal{A}_i) \times \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathsf{A}_\infty((\mathcal{A}_j)_{j\in J};\mathcal{B}). \tag{8.18.1}
$$

The A_∞-functor $f|_J^{(X_i)_{i\in I\smallsetminus J}}$ takes an object $(X_j)_{j\in J}$ to the object $((X_i)_{i\in I})f \in \mathrm{Ob}\,\mathcal{B}$. Its components are

$$
\bigl(f|_J^{(X_i)_{i\in I\smallsetminus J}}\bigr)_k : \otimes^{j\in J} T^{k_j} s\mathcal{A}_j(X_j, Y_j) \xrightarrow{\lambda^{J\hookrightarrow I}} \otimes^{i\in I} T^{\overline{k}_i} s\mathcal{A}_i(X_i, Y_i) \xrightarrow{f_{\overline{k}}} s\mathcal{B}\bigl(((X_i)_{i\in I})f, ((Y_i)_{i\in I})f\bigr), \tag{8.18.2}
$$

where $k = (k_j)_{j\in J} \in \mathbb{Z}^J_{\geqslant0}$, $\overline{k} = (\overline{k}_i)_{i\in I} \in \mathbb{Z}^I_{\geqslant0}$, $\overline{k}_i = k_i$ if $i \in J$, $\overline{k}_i = 0$ and $Y_i = X_i$ if $i \in I \smallsetminus J$.

In particular, for $J = \{j\} \hookrightarrow I = \mathbf{n}$ we have an A_∞-functor $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j \to \mathcal{B}$, the restriction of f to j-th argument. It takes an object $X_j \in \mathrm{Ob}\,\mathcal{A}_j$ to the object $((X_i)_{i\in I})f \in \mathrm{Ob}\,\mathcal{B}$. The k-th component is

$$
\begin{aligned}
\bigl(f|_j^{(X_i)_{i\neq j}}\bigr)_k &: T^k s\mathcal{A}_j(X_j, Y_j) \\
&\simeq \otimes^{i\in I}\bigl[(T^0 s\mathcal{A}_i(X_i, X_i))_{i<j}, T^k s\mathcal{A}_j(X_j, Y_j), (T^0 s\mathcal{A}_i(X_i, X_i))_{i>j}\bigr] \\
&\qquad \xrightarrow{f_{ke_j}} s\mathcal{B}\bigl((X_1, \dots, X_n)f, (X_1, \dots, X_{j-1}, Y_j, X_{j+1}, \dots, X_n)f\bigr),
\end{aligned}
$$

where $ke_j = (0, \dots, 0, k, 0, \dots, 0) \in \mathbb{Z}^n$ has k on j-th place. The first component $\bigl(f|_j^{(X_i)_{i\neq j}}\bigr)_1 = f_{e_j}$ commutes with b_1 and for all families of objects $(X_j)_{j\in I}, (Y_j)_{j\in I} \in$

$\prod_{j\in I}\operatorname{Ob}\mathcal{A}_j$ there are chain maps

$$sf_{e_j}s^{-1}:\mathsf{k}\mathcal{A}_j(X_j,Y_j)\\ \to\mathsf{k}\mathcal{B}\bigl((Y_1,\dots,Y_{j-1},X_j,X_{j+1},\dots,X_n)f,(Y_1,\dots,Y_{j-1},Y_j,X_{j+1},\dots,X_n)f\bigr). \quad (8.18.3)$$

8.19 Theorem. *The A_∞-categories $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ together with evaluations $\operatorname{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i);\mathcal{B}}$ turn A_∞ into a closed multicategory.*

Proof. Let $(\mathcal{A}_i)_{i\in I}$, $(\mathcal{B}_j)_{j\in J}$, $\mathcal{C}$ be families of A_∞-categories. There is a mapping

$$\varphi^{\mathsf{A}_\infty}:\mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}))\to\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}),\quad g\mapsto[(\mathrm{id}_{\mathcal{A}_i})_{i\in I},g]\cdot\operatorname{ev}^{\mathsf{A}_\infty}_{(\mathcal{A}_i);\mathcal{B}},$$

where the composition is that of A_∞. This is a restriction of φ^{Q}. Let us construct an inverse mapping to $\varphi^{\mathsf{A}_\infty}$.

An arbitrary element $f\in\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C})\subset\mathsf{Q}((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C})$ determines the element $g=(f)(\varphi^{\mathsf{Q}})^{-1}\in\mathsf{Q}((\mathcal{B}_j)_{j\in J};\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}))$. Let us study the range of values of $\operatorname{Ob}g$.

A family of objects Y_j of A_∞-categories $\mathcal{B}_j$ can be viewed as a family of A_∞-functors $Y_j:()\to\mathcal{B}_j$. The object $((Y_j)_{j\in J})g$ of $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})$ can be obtained as the composition $()\xrightarrow{(Y_j)_{j\in J}}(\mathcal{B}_j)_{j\in J}\xrightarrow{g}\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})$ in Q. This is the element $[((Y_j)_{j\in J},g)\mu^{\mathsf{Q}}_{\varnothing\to J}]\varphi^{\mathsf{Q}}_{();(\mathcal{A}_i)_{i\in I};\mathcal{C}}\in\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})$. By Proposition 4.11 applied to the map $\varnothing\to J$ the same element can be written as

$$[(\mathrm{id}_{\mathcal{A}_i})_{i\in I},(Y_j)_{j\in J},(g)\varphi^{\mathsf{Q}}_{(\mathcal{B}_j)_{j\in J};(\mathcal{A}_i)_{i\in I};\mathcal{C}}]\mu^{\mathsf{Q}}_{I\hookrightarrow I\sqcup J}=\bigl(f\big|_I^{(Y_j)_{j\in J}}:(\mathcal{A}_i)_{i\in I}\to\mathcal{C}\bigr).$$

Being a restriction of the A_∞-functor f this is an A_∞-functor as well. Thus, g is an element of $\mathsf{Q}((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C}))$. Let us prove that it is an A_∞-functor.

We start with equation $L^f=R^f$ from Lemma 8.11, which holds for the A_∞-functor $f=(g)\varphi^{\mathsf{Q}}=[(\mathrm{id}_{\mathcal{A}_i})_{i\in I},g]\cdot\operatorname{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}}$. In particular, it has a restriction:

$$\begin{aligned}
&\bigl\{\boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\hookrightarrow\langle\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j]\rangle\boxtimes_u\mathbb{1}_p[-1]\\
&\xrightarrow{\langle\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat g]\rangle\boxtimes_u1}\langle\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})]\rangle\boxtimes_u\mathbb{1}_p[-1]\\
&\xrightarrow{(\operatorname{ev}^{\mathsf{A}_\infty})\hat{}\boxtimes_u1}T^{\geqslant1}s\mathcal{C}\boxtimes_u\mathbb{1}_p[-1]\xrightarrow{\varphi(\hat b)}T^{\geqslant1}s\mathcal{C}\bigr\}\\
&=\sum_{k\in I}\bigl\{\boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\hookrightarrow\boxtimes_u^{I\sqcup J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\\
&\xrightarrow{\lambda_u^{pk}}\boxtimes_u^{I\sqcup J}[(T^{\geqslant1}s\mathcal{A}_i)^{i\in I}_{i<k},T^{\geqslant1}s\mathcal{A}_k\boxtimes_u\mathbb{1}_p[-1],(T^{\geqslant1}s\mathcal{A}_i)^{i\in I}_{i>k},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J}]\xrightarrow{\boxtimes_u^{I\sqcup J}[(1)_{i<k},\varphi(\hat b),(1)_{i>k},(1)_J]}\\
&\boxtimes_u^{I\sqcup J}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J}]\xrightarrow{\lambda_u^{\mathrm{id}\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}}\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j]\\
&\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat g]}\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})]\xrightarrow{(\operatorname{ev}^{\mathsf{A}_\infty})\hat{}}T^{\geqslant1}s\mathcal{C}\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l\in J} \bigl\{ \boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \hookrightarrow \\
&\qquad \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\rangle \\
&\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\lambda_u^{q_l}]} \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J}[(T^{\geqslant1}s\mathcal{B}_j)_{j<l}^{j\in J}, T^{\geqslant1}s\mathcal{B}_l\boxtimes_u\mathbb{1}_p[-1],(T^{\geqslant1}s\mathcal{B}_j)_{j>l}^{j\in J}]\rangle \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}\langle (1)_I,\boxtimes_u^J[(1)_{j<l},\varphi(\hat b),(1)_{j>l}]\rangle} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat g]} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})] \xrightarrow{(\mathrm{ev}^{\mathsf{A}_\infty})\hat{}} T^{\geqslant1}s\mathcal{C}\bigr\}.
\end{aligned}
$$

Here $p_k : I\sqcup J\sqcup\mathbf{1}\to I\sqcup J$ is determined by $p_k\big|_{I\sqcup J}=\mathrm{id}$, $p_k(1)=k\in I$, and $q_l : J\sqcup\mathbf{1}\to J$ is determined by $q_l\big|_J=\mathrm{id}$, $q_l(1)=l$. Plug in expression (8.17.2) for $((\mathrm{ev}^{\mathsf{A}_\infty})\hat{}\boxtimes_u 1)\cdot\varphi(\hat b)$. Then sums over $i\in I$ cancel each other and we obtain the following corollary in $\mathscr{Q}$:

$$
\begin{aligned}
&\bigl\{ \boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \hookrightarrow \boxtimes_u^{I\sqcup J\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \\
&\qquad \xrightarrow{\lambda_u^{1\sqcup\triangleright\sqcup1:I\sqcup J\sqcup\mathbf{1}\to I\sqcup\mathbf{1}\sqcup\mathbf{1}}\cdot\lambda_u^{1\sqcup\triangleright:I\sqcup\mathbf{2}\to I\sqcup\mathbf{1}}} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j)\boxtimes_u\mathbb{1}_p[-1]] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat g\boxtimes_u 1]} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\boxtimes_u\mathbb{1}_p[-1]] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\varphi(\hat B)]} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}}} T^{\geqslant1}s\mathcal{C}\bigr\} \\
&= \sum_{l\in J} \bigl\{ \boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \hookrightarrow \boxtimes_u^{I\sqcup J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \\
&\qquad \xrightarrow{\lambda_u^{\mathrm{id}_I\sqcup\triangleright:I\sqcup(J\sqcup\mathbf{1})\to I\sqcup\mathbf{1}}} \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\rangle \\
&\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\lambda_u^{q_l}]} \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J}[(T^{\geqslant1}s\mathcal{B}_j)_{j<l}^{j\in J}, T^{\geqslant1}s\mathcal{B}_l\boxtimes_u\mathbb{1}_p[-1],(T^{\geqslant1}s\mathcal{B}_j)_{j>l}^{j\in J}]\rangle \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}\langle (1)_I,\boxtimes_u^J[(1)_{j<l},\varphi(\hat b),(1)_{j>l}]\rangle} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j] \\
&\qquad \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat g]} \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}}} T^{\geqslant1}s\mathcal{C}\bigr\}.
\end{aligned}
$$

Notice that this is the restriction of the equation

$$
\begin{aligned}
&\bigl\{ \boxtimes_u^{I\sqcup J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \xrightarrow{\lambda_u^{\mathrm{id}_I\sqcup\triangleright:I\sqcup(J\sqcup\mathbf{1})\to I\sqcup\mathbf{1}}} \\
&\qquad \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\rangle \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,L^g]} \\
&\qquad\qquad \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}}} T^{\geqslant1}s\mathcal{C}\bigr\} \\
&= \bigl\{ \boxtimes_u^{I\sqcup J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]] \xrightarrow{\lambda_u^{\mathrm{id}_I\sqcup\triangleright:I\sqcup(J\sqcup\mathbf{1})\to I\sqcup\mathbf{1}}} \\
&\qquad \boxtimes_u^{I\sqcup\mathbf{1}}\,\langle (T^{\geqslant1}s\mathcal{A}_i)_{i\in I}, \boxtimes_u^{J\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]\rangle \xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,R^g]} \\
&\qquad\qquad \boxtimes_u^{I\sqcup\mathbf{1}}\,[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})] \xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}u}^f_{T^{\geqslant1}}}} T^{\geqslant1}s\mathcal{C}\bigr\} \qquad (8.19.1)
\end{aligned}
$$

to $\boxtimes^{I\sqcup J\sqcup\mathbf{1}}[(Ts\mathcal{A}_i)_{i\in I},(Ts\mathcal{B}_j)_{j\in J},\mathbb{1}_p[-1]]$, where L^g, R^g are defined as in Lemma 8.11. The restriction of this equation to the direct complement $\langle\boxtimes_u^{I\sqcup J}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J}]\rangle\boxtimes T^0(\mathbb{1}_p[-1])$ holds as well. Indeed, both sides can be reduced to

$$\bigl\{\langle\boxtimes_u^{I\sqcup J}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J}]\rangle\boxtimes T^0(\mathbb{1}_p[-1])\xrightarrow{(\lambda^{\mathsf{I}\cdot})^{-1}}\boxtimes_u^{I\sqcup J}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},(T^{\geqslant1}s\mathcal{B}_j)_{j\in J}]$$
$$\xrightarrow{\lambda_u^{\mathrm{id}_I\sqcup\triangleright:I\sqcup J\to I\sqcup\mathbf{1}}}\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},\boxtimes_u^{j\in J}T^{\geqslant1}s\mathcal{B}_j]$$
$$\xrightarrow{\boxtimes_u^{I\sqcup\mathbf{1}}[(1)_I,\hat{g}]}\boxtimes_u^{I\sqcup\mathbf{1}}[(T^{\geqslant1}s\mathcal{A}_i)_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})]\xrightarrow{\mathrm{ev}^{\widehat{\mathscr{Q}}_{uT^{\geqslant1}}^f}}T^{\geqslant1}s\mathcal{C}\bigr\}.$$

Thus, equation (8.19.1) is verified. Since $\widehat{\mathscr{Q}}_{uT^{\geqslant1}}$ is closed, we obtain $L^g=R^g$. By Lemma 8.11 g is an A_∞-functor. The obtained mapping

$$\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C})\to\mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})),\quad f\mapsto g=(f)(\varphi^{\mathsf{Q}})^{-1},$$

which is a restriction of $(\varphi^{\mathsf{Q}})^{-1}$, is inverse to $\varphi^{\mathsf{A}_\infty}$, which is a restriction of φ^{Q}. The theorem is proven. □

8.20 A coalgebra approach to the multicategory A_∞. General statements of Chapters 6 and 7 hold, in particular, for the category of cochain complexes $\mathcal{V}=\mathbf{dg}$. Therefore, the categories ${}^d\mathscr{Q}_p=({}^d\mathscr{Q},\boxtimes^I,\lambda^f)$ and ${}^d\mathscr{Q}_u=({}^d\mathscr{Q},\boxtimes_u^I,\lambda_u^f)$ of differential graded quivers are symmetric closed Monoidal by Propositions 7.2 and 7.5. The data $((T^{\geqslant1},\tau^I),\Delta,\varepsilon):{}^d\mathscr{Q}_u\to{}^d\mathscr{Q}_u$ constitute a lax symmetric Monoidal comonad by Proposition 6.16. The category ${}^d\mathscr{Q}_{uT^{\geqslant1}}=({}^d\mathscr{Q}_{T^{\geqslant1}},\boxtimes_u^I,\lambda_u^f)$ of $T^{\geqslant1}$-coalgebras in ${}^d\mathscr{Q}$ is symmetric Monoidal by Remark 5.12. The category $\mathrm{ac}{}^d\mathscr{Q}_p=(\mathrm{ac}{}^d\mathscr{Q},\boxtimes^I,\lambda^f)$ of augmented counital coassociative coalgebras in ${}^d\mathscr{Q}$ is symmetric Monoidal by Proposition 6.14. The full and faithful functor $T^{\leqslant1}:{}^d\mathscr{Q}_{T^{\geqslant1}}\to\mathrm{ac}{}^d\mathscr{Q}$ gives rise to a symmetric Monoidal functor $(T^{\leqslant1},\vartheta^I):{}^d\mathscr{Q}_{uT^{\geqslant1}}\to\mathrm{ac}{}^d\mathscr{Q}_p$, see (6.1.3).

There is a symmetric Monoidal functor $F:\mathbf{dg}\to\mathbf{gr}$, $(X,d)\mapsto X$, which forgets the differential d in a graded $\Bbbk$-module X. Clearly, the forgetful functor F is faithful and commutes with arbitrary $\mathscr{U}$-small limits and colimits. The functor F induces faithful symmetric Monoidal functors ${}^F\mathscr{Q}_p:{}^d\mathscr{Q}_p\to\mathscr{Q}_p$, ${}^F\mathscr{Q}_u:{}^d\mathscr{Q}_u\to\mathscr{Q}_u$ and $\mathrm{ac}{}^F\mathscr{Q}_p:\mathrm{ac}{}^d\mathscr{Q}_p\to\mathrm{ac}\mathscr{Q}_p$.

The correspondence $\operatorname{Ob}G:\mathcal{A}\mapsto(Ts\mathcal{A},\Delta_0,\mathrm{pr}_0,\mathrm{in}_0,\tilde{b})$ is a mapping $\operatorname{Ob}\mathsf{A}_\infty\to\operatorname{Ob}\mathrm{ac}{}^d\mathscr{Q}_p$. Indeed, $\tilde{b}$ is a $(1,1)$-coderivation with respect to Δ_0, and the necessary equation $\tilde{b}^2=0$ in $\underline{\mathscr{Q}_p}$ holds by Remark 8.4. Notice that any codifferential $d:Ts\mathcal{A}\to Ts\mathcal{A}$ for the augmented counital coalgebra $(Ts\mathcal{A},\Delta_0,\mathrm{pr}_0,\mathrm{in}_0)$ is necessarily of the form $\left(\begin{smallmatrix}0&0\\0&\hat{b}\theta\end{smallmatrix}\right)=\tilde{b}$ in the decomposition $Ts\mathcal{A}=T^0s\mathcal{A}\oplus T^{\geqslant1}s\mathcal{A}$ for some differential b. Indeed, equations $\tilde{b}^2=0$ and $(\hat{b}\theta)\cdot\mathrm{in}_1\cdot b=0$ are equivalent by Remark 8.4. As augmentation $\mathrm{in}_0:T^0s\mathcal{A}\to Ts\mathcal{A}$ is a chain map, it implies that $d\big|_{T^0s\mathcal{A}}=0$, hence, $b_0=0$.

Denote by $\mathcal{T}$ the composition of fully faithful symmetric multifunctors

$$\mathcal{T}=\bigl(\mathsf{Q}\xrightarrow{\sim}\widehat{\mathscr{Q}}_u^{T^{\geqslant1}}\xrightarrow{\sim}\widehat{\mathscr{Q}}_{uT^{\geqslant1}}^f\xrightarrow{\widehat{T^{\geqslant1}}'}\widehat{\mathscr{Q}}_{uT^{\geqslant1}}\bigr).$$

The above statements imply that there is a pull-back square

$$\begin{tikzcd}
\operatorname{Ob}\mathsf{A}_\infty \arrow[rrr, "\operatorname{Ob}G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & & \operatorname{Ob}\widehat{\mathrm{ac}^d\mathscr{Q}_p} \arrow[d, "\operatorname{Ob}\widehat{\mathrm{ac}^F\mathscr{Q}_p}"'] \\
\operatorname{Ob}\mathsf{Q} \arrow[r, "\operatorname{Ob}\mathfrak{T}"] & \operatorname{Ob}\widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, equal] & \operatorname{Ob}\widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, "\operatorname{Ob}\widehat{T^{\leqslant 1}}"] & \operatorname{Ob}\widehat{\mathrm{ac}\mathscr{Q}_p}
\end{tikzcd} \tag{8.20.1}$$

We claim that there is a unique symmetric multifunctor $G : \mathsf{A}_\infty \to \widehat{\mathrm{ac}^d\mathscr{Q}_p}$, given by $\operatorname{Ob}G : \mathcal{A} \mapsto (Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0, \tilde{b})$ on objects, which makes the following diagram of symmetric multicategories and multifunctors commutative

$$\begin{tikzcd}
\mathsf{A}_\infty \arrow[rrr, "G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & & \widehat{\mathrm{ac}^d\mathscr{Q}_p} \arrow[d, "\widehat{\mathrm{ac}^F\mathscr{Q}_p}"] \\
\mathsf{Q} \arrow[r, "\mathfrak{T}"] & \widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, equal] & \widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, "\widehat{T^{\leqslant 1}}"] & \widehat{\mathrm{ac}\mathscr{Q}_p}
\end{tikzcd} \tag{8.20.2}$$

Indeed, the vertical arrows are faithful multifunctors, therefore G can be given on morphisms only by the same correspondence $(f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}) \mapsto (\tilde{f} : \boxtimes^{i\in I} Ts\mathcal{A}_i \to Ts\mathcal{B})$ as the lower row, computed in Proposition 7.12. The lower row multifunctor is fully faithful. It is proven in Remark 8.10 that an element $f \in \mathsf{Q}((\mathcal{A}_i))_{i\in I}; \mathcal{B})$ is an A_∞-functor if and only if $\tilde{f}$ satisfies equation (8.10.1), that is, $\tilde{f}$ is a chain map, equivalently, $\tilde{f} \in \widehat{\mathrm{ac}^d\mathscr{Q}_p}((Ts\mathcal{A}_i)_{i\in I}; Ts\mathcal{B})$. Thus, multifunctor $G : f \mapsto \tilde{f}$ exists and is fully faithful. Diagram (8.20.2) commutes and its rows are fully faithful. On objects this diagram gives pull-back square (8.20.1). Therefore, diagram (8.20.2) is a pull-back square itself.

Other pull-back squares of symmetric Monoidal categories and functors and of symmetric multicategories and multifunctors follow:

$$\begin{tikzcd}
{}^d\mathscr{Q}_{uT^{\geqslant 1}} \arrow[r, "T^{\leqslant 1}"] \arrow[d, "{}^F\mathscr{Q}_{uT^{\geqslant 1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{ac}^d\mathscr{Q}_p \arrow[d, "\mathrm{ac}^F\mathscr{Q}_p"'] \\
\mathscr{Q}_{uT^{\geqslant 1}} \arrow[r, "T^{\leqslant 1}"] & \mathrm{ac}\mathscr{Q}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{{}^d\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, "\widehat{T^{\leqslant 1}}"] \arrow[d, "\widehat{{}^F\mathscr{Q}_{uT^{\geqslant 1}}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{\mathrm{ac}^d\mathscr{Q}_p} \arrow[d, "\widehat{\mathrm{ac}^F\mathscr{Q}_p}"'] \\
\widehat{\mathscr{Q}_{uT^{\geqslant 1}}} \arrow[r, "\widehat{T^{\leqslant 1}}"] & \widehat{\mathrm{ac}\mathscr{Q}_p}
\end{tikzcd} \tag{8.20.3}$$

The vertical arrows come from the functor $F : {}^d\mathscr{Q} \to \mathscr{Q}$ that forgets the differential. Indeed, on objects these give the pasting of pull-back squares

$$\begin{tikzcd}
\operatorname{Ob}{}^d\mathscr{Q}_{T^{\geqslant 1}} \arrow[r, hook] \arrow[d, "\operatorname{Ob}{}^F\mathscr{Q}_{T^{\geqslant 1}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Ob}\mathrm{c}^d\mathscr{Q} \arrow[r, "\operatorname{Ob}T^{\leqslant 1}"] \arrow[d, "\operatorname{Ob}\mathrm{c}^F\mathscr{Q}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Ob}\mathrm{ac}^d\mathscr{Q} \arrow[d, "\operatorname{Ob}\mathrm{ac}^F\mathscr{Q}"] \\
\operatorname{Ob}\mathscr{Q}_{T^{\geqslant 1}} \arrow[r, hook] & \operatorname{Ob}\mathrm{c}\mathscr{Q} \arrow[r, "\operatorname{Ob}T^{\leqslant 1}"] & \operatorname{Ob}\mathrm{ac}\mathscr{Q}
\end{tikzcd}$$

due to Proposition 6.8, since F commutes with kernels and countable colimits. On morphisms all horizontal arrows of (8.20.3) are bijective.

8.21 Proposition. *There is a unique fully faithful symmetric multifunctor* $E : \mathsf{A}_\infty \hookrightarrow \widehat{{}^d\mathcal{Q}}_{uT^{\geqslant 1}}$ *such that the composite full embedding*

$$\mathsf{A}_\infty \xrightarrow{E} \widehat{{}^d\mathcal{Q}}_{uT^{\geqslant 1}} = \widehat{{}^d\mathcal{Q}_{uT^{\geqslant 1}}} \xrightarrow{\widehat{T^{\leqslant 1}}} \widehat{\mathrm{ac}^d\mathcal{Q}_p}$$

coincides with $G : \mathcal{A} \mapsto (Ts\mathcal{A}, \Delta_0, \mathrm{pr}_0, \mathrm{in}_0, \tilde{b})$, $G : (f : \mathcal{A} \to \mathcal{B}) \mapsto (\tilde{f} : Ts\mathcal{A} \to Ts\mathcal{B})$. *The diagram*

$$\begin{tikzcd}
\mathsf{A}_\infty \arrow[r, "E"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{{}^d\mathcal{Q}}_{uT^{\geqslant 1}} \arrow[d, "\Psi = {}^F\mathcal{Q}_{uT^{\geqslant 1}}"] \\
\mathsf{Q} \arrow[r, "\mathcal{T}"] & \widehat{\mathcal{Q}}_{uT^{\geqslant 1}}
\end{tikzcd} \tag{8.21.1}$$

is a pull-back square.

Proof. Comparing two pull-back squares (namely, (8.20.2) and the right of (8.20.3)) in the category $\mathcal{SM}\mathrm{ulticat}$ of symmetric multicategories and multifunctors we deduce the existence of E. Since both multifunctors $\widehat{T^{\leqslant 1}} : \widehat{{}^d\mathcal{Q}_{uT^{\geqslant 1}}} \to \widehat{\mathrm{ac}^d\mathcal{Q}_p}$ and G are fully faithful, so is E. Moreover, $\widehat{T^{\leqslant 1}}$ is a monomorphism in $\mathcal{SM}\mathrm{ulticat}$ and general category theory implies that (8.21.1) is a pull-back square as well. □

8.22 Differential B. Being a closed symmetric multicategory $\underline{\mathsf{A}_\infty}$ has the composition morphism (an A_∞-functor)

$$\mu_\phi^{\mathsf{A}_\infty} : (\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, \underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C}) \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{C})$$

for any map $\phi : I \to J \in \mathcal{S}$. The corresponding morphisms $(\mu_\phi^{\mathsf{A}_\infty})^\sim$ of augmented coalgebras in $\mathcal{Q}_p$ are denoted

$$M_\phi = \bigl\{\boxtimes^{J\sqcup\mathbf{1}}[(Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, Ts\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C})] \xrightarrow[\sim]{\vartheta}$$
$$T^{\leqslant 1}\boxtimes_u^{J\sqcup\mathbf{1}}[(T^{\geqslant 1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, T^{\geqslant 1}s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C})] \xrightarrow{T^{\leqslant 1}((\mu_\phi^{\mathsf{A}_\infty})\hat{})} Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{C})\bigr\}.$$

When $\phi : I \to J \in \mathcal{S}$ is non-decreasing, this map can be restored unambiguously from the list of arguments, and we abbreviate M_ϕ to M.

In particular, we have augmented coalgebra homomorphisms

$$M = M_\triangleright : Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{B}; \mathcal{B}) \to Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B}),$$
$$M = M_{\mathrm{id}} : \boxtimes^{I\sqcup\mathbf{1}}[(Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_i; \mathcal{A}_i))_{i\in I}, Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B})] \to Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B}).$$

8.23 Proposition. *Restriction of $\varphi(\hat{B})$ can be expressed via composition M as follows:*

$$
\begin{aligned}
&\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\hookrightarrow T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]\xrightarrow{\varphi(\hat{B})}T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\\
&\qquad\qquad\hookrightarrow Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&=\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xrightarrow{1\boxtimes\hat{b}\,\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{B})\\
&\qquad\qquad\xrightarrow{M_\triangleright}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&-\sum_{j\in I}\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xrightarrow{\lambda^{g_j}}\boxtimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_p)^{i\in I}_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_p)^{i\in I}_{i>j},Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\\
&\qquad\xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})^\sim)_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})^\sim)_{i>j},1]}\boxtimes^{I\sqcup\mathbf{1}}[(Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\\
&\qquad\qquad\xrightarrow{M_{\mathrm{id}}}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}. \qquad (8.23.1)
\end{aligned}
$$

Proof. Substitute $\varphi(\hat{B})$ from (8.14.1) in the left hand side:

$$
\begin{aligned}
&\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xhookrightarrow{1\boxtimes\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\leqslant1}\mathbb{1}_p[-1]\\
&\qquad\xrightarrow{\vartheta}T^{\leqslant1}[T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]]\xrightarrow{T^{\leqslant1}\varphi(\hat{B})}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&=\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xhookrightarrow{1\boxtimes\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\leqslant1}\mathbb{1}_p[-1]\\
&\qquad\xrightarrow{1\boxtimes T^{\leqslant1}\hat{b}}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{B})\\
&\qquad\xrightarrow{\vartheta}T^{\leqslant1}[T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u T^{\geqslant1}s\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{B})]\xrightarrow{T^{\leqslant1}((\mu^{\underline{\mathsf{A}_\infty}}_{\triangleright})\hat{})}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&-\sum_{j\in I}\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xhookrightarrow{1\boxtimes\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\leqslant1}\mathbb{1}_p[-1]\\
&\qquad\xrightarrow{\vartheta}T^{\leqslant1}[T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes_u\mathbb{1}_p[-1]]\xrightarrow{T^{\leqslant1}\lambda^{g_j}_u}\\
&\quad T^{\leqslant1}\boxtimes^{I\sqcup\mathbf{1}}_u[(\mathbb{1}_u)^{i\in I}_{i<j},\mathbb{1}_p[-1],(\mathbb{1}_u)^{i\in I}_{i>j},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\xrightarrow{T^{\leqslant1}\boxtimes^{I\sqcup\mathbf{1}}_u[((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})\hat{})_{i>j},1]}\\
&\qquad T^{\leqslant1}\boxtimes^{I\sqcup\mathbf{1}}_u[(T^{\geqslant1}s\underline{\mathsf{A}_\infty}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},T^{\geqslant1}s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\xrightarrow{T^{\leqslant1}(\mu^{\underline{\mathsf{A}_\infty}}_{\mathrm{id}})\hat{}}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&=\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xrightarrow{1\boxtimes\hat{b}}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\geqslant1}s\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{B})\\
&\qquad\xhookrightarrow{1\boxtimes\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{B})\xrightarrow{M_\triangleright}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}\\
&-\sum_{j\in I}\bigl\{Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes\mathbb{1}_p[-1]\xhookrightarrow{1\boxtimes\mathrm{in}_1}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\boxtimes T^{\leqslant1}\mathbb{1}_p[-1]\xrightarrow{\lambda^{g_j}}\\
&\quad\boxtimes^{I\sqcup\mathbf{1}}[(\mathbb{1}_p)^{i\in I}_{i<j},T^{\leqslant1}\mathbb{1}_p[-1],(\mathbb{1}_p)^{i\in I}_{i>j},Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})^\sim)_{i<j},T^{\leqslant1}\hat{b},((\dot{\mathrm{id}}^{\mathsf{A}_\infty}_{\mathcal{A}_i})^\sim)_{i>j},1]}\\
&\qquad\boxtimes^{I\sqcup\mathbf{1}}[(Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_i;\mathcal{A}_i))_{i\in I},Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})]\xrightarrow{M_{\mathrm{id}}}Ts\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})\bigr\}.
\end{aligned}
$$

This is the right hand side of (8.23.1). □

8.24 Remark. The formulas of the previous proposition are valid also in Q. The $(1,1)$-coderivation $\tilde{B} = \hat{B}\theta \cdot \mathrm{in}_1 : Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B})$ of degree 1 can be presented as the commutator in $\underline{\mathsf{Q}}$

$$
\begin{aligned}
\tilde{B} &= (1 \boxtimes \hat{b}\,\mathrm{in}_1)M - (-)^{\deg} \sum_{i=1}^{n} (1^{\boxtimes(i-1)} \boxtimes \hat{b}\,\mathrm{in}_1 \boxtimes 1^{\boxtimes(n-i)} \boxtimes 1)M, \\
r\tilde{B} &= (r^1 \otimes \cdots \otimes r^m \boxtimes \hat{b}\,\mathrm{in}_1)M - (-)^r \sum_{i=1}^{n} (1^{\boxtimes(i-1)} \boxtimes \hat{b}\,\mathrm{in}_1 \boxtimes 1^{\boxtimes(n-i)} \boxtimes r^1 \otimes \cdots \otimes r^m)M,
\end{aligned}
\tag{8.24.1}
$$

for a diagram $r = \bigl(f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{m-1} \xrightarrow{r^m} f^m\bigr)$ in $\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}))$.

8.25 Components of the composition M. Sometimes we use the presentation of A_∞-functors $f \in \mathsf{A}_\infty(\mathcal{A}_1,\dots,\mathcal{A}_k;\mathcal{B})$ as augmented coalgebra homomorphisms $f : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \to Ts\mathcal{B}$ (see Remark 7.10) of degree 0, commuting with the differentials:

$$
f\tilde{b} = \sum_{i=1}^{k} \bigl(1^{\boxtimes(i-1)} \boxtimes \tilde{b} \boxtimes 1^{\boxtimes(k-i)}\bigr) f : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \to Ts\mathcal{B},
$$

such that the restriction of f to $T^0s\mathcal{A}_1 \boxtimes \cdots \boxtimes T^0s\mathcal{A}_k$ vanishes. Here the cut comultiplication Δ_0 applied to x includes the terms $1 \otimes x$ and $x \otimes 1$. This coalgebra homomorphism, written also as $f : (\mathcal{A}_1,\dots,\mathcal{A}_k) \to \mathcal{B}$, can be expressed uniquely via its components, which form a quiver map $\check{f} = f\,\mathrm{pr}_1 : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \to s\mathcal{B}$ of degree 0. By assumption, the restriction of $\check{f}$ to $T^0s\mathcal{A}_1 \boxtimes \cdots \boxtimes T^0s\mathcal{A}_k$ vanishes. The homomorphism f is recovered from $\check{f}$ via the exterior of the following commutative diagram:

$$
\begin{tikzcd}
Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \arrow[rr, "f"] \arrow[dd, "\Delta_0^{(l)} \boxtimes \cdots \boxtimes \Delta_0^{(l)}"'] \arrow[ddrr, "\Delta_0^{(l)}"] & & Ts\mathcal{B} \arrow[d, "\Delta_0^{(l)}"'] \arrow[dr, "\mathrm{pr}_l"] & \\
& & (Ts\mathcal{B})^{\otimes l} \arrow[r, "\mathrm{pr}_1^{\otimes l}"] & (s\mathcal{B})^{\otimes l} \\
(Ts\mathcal{A}_1)^{\otimes l} \boxtimes \cdots \boxtimes (Ts\mathcal{A}_k)^{\otimes l} \arrow[rr, "\sigma_{k,l}", "\varkappa^{-1}"'] & & (Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k)^{\otimes l} \arrow[u, "f^{\otimes l}"'] \arrow[ur, "\check{f}^{\otimes l}"'] &
\end{tikzcd}
\tag{8.25.1}
$$

The symmetry $\sigma_{k,l} = c_{s_{k,l}}$ corresponds to the permutation $s_{k,l}$ of the set $\{1,2,\dots,kl\}$,

$$
s_{k,l}(1 + t + nl) = 1 + n + tk \quad \text{for} \quad 0 \leqslant t < l, 0 \leqslant n < k. \tag{8.25.2}
$$

The morphisms $r : f \to g$ of $\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B})(f, g)$ called *A_∞-transformations* are identified in Proposition 7.15 with (f, g)-coderivations, written also as $r : f \to g : (\mathcal{A}_1, \dots, \mathcal{A}_k) \to \mathcal{B}$. The coderivation r is recovered from its components, which form a quiver map $\check{r} = r\,\mathrm{pr}_1 : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \to s\mathcal{B}$, via the exterior of the following commutative diagram:

$$
\begin{tikzcd}[column sep=large]
Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \arrow[rr, "r"] \arrow[dd, "\Delta_0^{(l)} \boxtimes \cdots \boxtimes \Delta_0^{(l)}"'] \arrow[ddrr, "\Delta_0^{(l)}"] & & Ts\mathcal{B} \arrow[d, "\Delta_0^{(l)}"'] \arrow[dr, "\mathrm{pr}_l"] & \\
& & (Ts\mathcal{B})^{\otimes l} \arrow[r, "\mathrm{pr}_1^{\otimes l}"] & (s\mathcal{B})^{\otimes l} \\
(Ts\mathcal{A}_1)^{\otimes l} \boxtimes \cdots \boxtimes (Ts\mathcal{A}_k)^{\otimes l} \arrow[rr, "\sigma_{k,l}", "\varkappa^{-1}"'] & & (Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k)^{\otimes l} \arrow[u, "\sum f^{\otimes q} \otimes r \otimes g^{\otimes t}"] \arrow[ur, "\sum_{q+1+t=l} \check{f}^{\otimes q} \otimes \check{r} \otimes \check{g}^{\otimes t}"'] &
\end{tikzcd}
$$

which is nothing else but diagram (7.15.1), written for the $T^{\geqslant 1}$-coalgebra $\mathcal{C} = \boxtimes_u^{i \in I} T^{\geqslant 1} s\mathcal{A}_i$. This presentation of r implies that

$$
r = \bigl(Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \xrightarrow{\Delta_0^{(3)}} (Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k)^{\otimes 3} \xrightarrow{f \otimes \check{r} \otimes g} Ts\mathcal{B} \otimes s\mathcal{B} \otimes Ts\mathcal{B} \xrightarrow{\mu} Ts\mathcal{B}\bigr),
$$

where μ is the multiplication in tensor algebra of a $\Bbbk$-linear quiver. This is a general form of a (f, g)-coderivation.

Given a system of coderivations

$$
r = \bigl(f^0 \xrightarrow{r^1} f^1 \xrightarrow{r^2} \dots f^{m-1} \xrightarrow{r^m} f^m\bigr) : (\mathcal{A}_1, \dots, \mathcal{A}_k) \to \mathcal{B},
$$

we consider $(r^1 \otimes \cdots \otimes r^m)\theta \equiv \theta_{\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B}}(r)$ given by (7.16.3), or, equivalently, by

$$
\begin{aligned}
(r^1 \otimes \cdots \otimes r^m)\theta = \bigl(Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \xrightarrow{\Delta_0^{(2m+1)}} (Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k)^{\otimes(2m+1)} \\
\xrightarrow{f^0 \otimes \check{r}^1 \otimes f^1 \otimes \cdots \otimes \check{r}^m \otimes f^m} Ts\mathcal{B} \otimes s\mathcal{B} \otimes Ts\mathcal{B} \otimes \cdots \otimes s\mathcal{B} \otimes Ts\mathcal{B} \xrightarrow{\mu} Ts\mathcal{B}\bigr).
\end{aligned}
$$

The properties of μ with respect to Δ_0 imply that for each $m \geqslant 0$ the map θ satisfies the equation

$$
(r^1 \otimes r^2 \otimes \cdots \otimes r^m)\theta\Delta_0 = \Delta_0 \sum_{p=0}^{m} (r^1 \otimes \cdots \otimes r^p)\theta \otimes (r^{p+1} \otimes \cdots \otimes r^m)\theta. \tag{8.25.3}
$$

Let us write down explicitly the action A_∞-functor $\alpha = \mathrm{ev}^{\mathsf{A}_\infty}_{\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B}}$ from (8.17.1):

$$
\alpha : Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B}) \to Ts\mathcal{B}, \tag{8.25.4}
$$

$$
a \boxtimes (r^1 \otimes \cdots \otimes r^m) \mapsto a[(r^1 \otimes \cdots \otimes r^m)\theta] = a\Delta_0^{(2m+1)}(f^0 \otimes \check{r}^1 \otimes f^1 \otimes \cdots \otimes \check{r}^m \otimes f^m)\mu,
$$

where $a \in Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k$ and $r^1 \otimes \cdots \otimes r^m \in T^m s\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_k; \mathcal{B})$. One can deduce that α is an augmented coalgebra homomorphism directly from equation (8.25.3).

Closedness of multicategory A_∞ implies, in particular, that for each A_∞-functor $f : (\mathcal{A}_1, \dots, \mathcal{A}_k) \to \mathcal{B}$ there exists an A_∞-functor $\psi : \mathcal{A}_k \to \underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_{k-1}; \mathcal{B})$ such that

$$f = \bigl(Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_k \xrightarrow{1\otimes\psi} Ts\mathcal{A}_1 \boxtimes \cdots \boxtimes Ts\mathcal{A}_{k-1} \boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_1, \dots, \mathcal{A}_{k-1}; \mathcal{B}) \xrightarrow{\alpha} Ts\mathcal{B}\bigr).$$

A similar statement holds in Q for graded $\Bbbk$-quivers $\mathcal{A}_1$, $\dots$, $\mathcal{A}_k$, $\mathcal{B}$ and augmented coalgebra homomorphisms f, ψ.

Universality of the action map gives, in particular, the components of the composition M in the considered multicategories. Let $\mathcal{A}_i^j$, $\mathcal{B}_i$, $\mathcal{C}$ be graded $\Bbbk$-linear quivers. Then

$$\begin{aligned} M : \boxtimes^{\mathbf{n}\sqcup\mathbf{1}}\bigl[\bigl(Ts\underline{\mathsf{Q}}(\mathcal{A}_i^i, \dots, \mathcal{A}_i^{m_i}; \mathcal{B}_i)\bigr)_{i\in\mathbf{n}}, Ts\underline{\mathsf{Q}}(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{C})\bigr] \\ \to Ts\underline{\mathsf{Q}}(\mathcal{A}_1^1, \dots, \mathcal{A}_1^{m_1}, \dots, \mathcal{A}_n^1, \dots, \mathcal{A}_n^{m_n}; \mathcal{C}), \end{aligned}$$

is the only augmented coalgebra morphism such that the following equation holds:

$$\begin{aligned} &\bigl[(\boxtimes^{i\in\mathbf{n}} \boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j) \boxtimes \bigl(\boxtimes^{i\in\mathbf{n}} Ts\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)\bigr) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \\ &\qquad \xrightarrow{\sigma_{(12)}\boxtimes 1} \bigl(\boxtimes^{i\in\mathbf{n}}[(\boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)]\bigr) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \\ &\qquad\qquad \xrightarrow{(\boxtimes^{\mathbf{n}}\alpha)\boxtimes 1} (\boxtimes^{i\in\mathbf{n}} Ts\mathcal{B}_i) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \xrightarrow{\alpha} Ts\mathcal{C}\bigr] \\ &= \bigl[(\boxtimes^{i\in\mathbf{n}} \boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j) \boxtimes \bigl(\boxtimes^{i\in\mathbf{n}} Ts\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)\bigr) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \\ &\qquad \xrightarrow{1\boxtimes M} (\boxtimes^{i\in\mathbf{n}} \boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j) \boxtimes Ts\underline{\mathsf{Q}}(\mathcal{A}_1^1, \dots, \mathcal{A}_1^{m_1}, \dots, \mathcal{A}_n^1, \dots, \mathcal{A}_n^{m_n}; \mathcal{C}) \xrightarrow{\alpha} Ts\mathcal{C}\bigr]. \end{aligned}$$

This equation is equivalent to the following:

$$\begin{aligned} &\bigl[\bigl(\boxtimes^{i\in\mathbf{n}} Ts\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)\bigr) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \xrightarrow{\theta^{\boxtimes(n+1)}} \\ &\qquad (\boxtimes^{i\in\mathbf{n}} \underline{\mathscr{Q}_p}(\boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j, Ts\mathcal{B}_i)) \boxtimes \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} Ts\mathcal{B}_i, Ts\mathcal{C}) \\ &\qquad\qquad \xrightarrow{\mu^{\widehat{\mathscr{Q}_p}}} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} \boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j, Ts\mathcal{C})\bigr] \\ &= \bigl[\bigl(\boxtimes^{i\in\mathbf{n}} Ts\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)\bigr) \boxtimes Ts\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \\ &\qquad \xrightarrow{M} Ts\underline{\mathsf{Q}}(\mathcal{A}_1^1, \dots, \mathcal{A}_1^{m_1}, \dots, \mathcal{A}_n^1, \dots, \mathcal{A}_n^{m_n}; \mathcal{C}) \xrightarrow{\theta} \underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{n}} \boxtimes^{j\in\mathbf{m_i}} Ts\mathcal{A}_i^j, Ts\mathcal{C})\bigr]. \end{aligned}$$

By abuse of notations, θ denotes here the composition $\overline{\theta} \cdot \underline{\mathscr{Q}_p}(1, \mathrm{in}_1)$.

Being a augmented coalgebra morphism, M is determined in a unique way by its components:

$$\begin{aligned} M_{k_1\dots k_n l} : \bigl(\boxtimes^{i\in\mathbf{n}} T^{k_i} s\underline{\mathsf{Q}}((\mathcal{A}_i^j)_{j\in\mathbf{m_i}}; \mathcal{B}_i)\bigr) \boxtimes T^l s\underline{\mathsf{Q}}((\mathcal{B}_i)_{i\in\mathbf{n}}; \mathcal{C}) \\ \to s\underline{\mathsf{Q}}(\mathcal{A}_1^1, \dots, \mathcal{A}_1^{m_1}, \dots, \mathcal{A}_n^1, \dots, \mathcal{A}_n^{m_n}; \mathcal{C}). \end{aligned}$$

Composing the above equation with $\mathrm{pr}_1 : Ts\mathcal{C} \to s\mathcal{C}$ we get the following equation

$$\begin{aligned}\big[(r^{11} \otimes \dots \otimes r^{1k_1})\theta \boxtimes \dots \boxtimes (r^{n1} \otimes \dots \otimes r^{nk_n})\theta\big] \cdot (t^1 \otimes \dots \otimes t^l)\theta\,\mathrm{pr}_1 \\ = (r^{11} \otimes \dots \otimes r^{1k_1} \boxtimes \dots \boxtimes r^{n1} \otimes \dots \otimes r^{nk_n} \boxtimes t^1 \otimes \dots \otimes t^l) M_{k_1\dots k_n l}\,\mathrm{pr}_1 : \\ Ts\mathcal{A}_1^1 \boxtimes \dots \boxtimes Ts\mathcal{A}_1^{m_1} \boxtimes \dots \boxtimes Ts\mathcal{A}_n^1 \boxtimes \dots \boxtimes Ts\mathcal{A}_n^{m_n} \to s\mathcal{C}, \quad (8.25.5)\end{aligned}$$

valid for an arbitrary element

$$\begin{aligned}r^{11} \otimes \dots \otimes r^{1k_1} \boxtimes \dots \boxtimes r^{n1} \otimes \dots \otimes r^{nk_n} \boxtimes t^1 \otimes \dots \otimes t^l \\ \in \boxtimes^{\mathbf{n}\sqcup\mathbf{1}}\big[\big(T^{k_i} s\underline{\mathsf{Q}}(\mathcal{A}_i^i, \dots, \mathcal{A}_i^{m_i}; \mathcal{B}_i)\big)_{i\in\mathbf{n}}, T^l s\underline{\mathsf{Q}}(\mathcal{B}_1, \dots, \mathcal{B}_n; \mathcal{C})\big]. \quad (8.25.6)\end{aligned}$$

Since $(r^{11} \otimes \dots \otimes r^{1k_1} \boxtimes \dots \boxtimes r^{n1} \otimes \dots \otimes r^{nk_n} \boxtimes t^1 \otimes \dots \otimes t^l) M_{k_1\dots k_n l}$ is a coderivation, it is determined in a unique way by its composition with pr_1, that is by the left hand side of equation (8.25.5). For $l > 1$ the map $M_{k_1\dots k_n l}$ vanishes due to $(t^1 \otimes \dots \otimes t^l)\theta\,\mathrm{pr}_1 = 0$. For $l = 1$ we have $(t^1)\theta = t^1$, and for $l = 0$ the map $()\theta = f^0$ is an augmented coalgebra homomorphism.

The general recipe of diagram (8.25.1) gives the following formula for matrix coefficients of M_ϕ for a map $\phi : I \to J \in \mathcal{S}$ indexed by $(k_j)_{j\in J}$, l and m:

$$\begin{aligned}(M_\phi)^m_{(k_j)_{j\in J};l} = \sum_{\substack{f_j:\mathbf{k_j}\to\mathbf{m}\\ g:\mathbf{l}\to\mathbf{m}}} \big\{\boxtimes^{J\sqcup\mathbf{1}}[(T^{k_j} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, T^l s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C})] \\ \xrightarrow{\boxtimes^{J\sqcup\mathbf{1}}[(\lambda^{f_j})_{j\in J}, \lambda^g]} \boxtimes^{J\sqcup\mathbf{1}}\,[(\otimes^{p\in\mathbf{m}} \otimes^{f_j^{-1}p} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, \otimes^{p\in\mathbf{m}} \otimes^{g^{-1}p} s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C})] \\ \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{J\sqcup\mathbf{1}}[(\otimes^{f_j^{-1}p} s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\phi^{-1}j}; \mathcal{B}_j))_{j\in J}, \otimes^{g^{-1}p} s\underline{\mathsf{A}_\infty}((\mathcal{B}_j)_{j\in J}; \mathcal{C})] \\ \xrightarrow{\otimes^{p\in\mathbf{m}} M_{(|f_j^{-1}p|)_{j\in J};|g^{-1}p|}} T^m s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{C})\big\},\end{aligned}$$

where summation goes over all families of maps $f_j, g \in \mathcal{O}$, $j \in J$. Conditions on components of M imply that only summands with injective $g : \mathbf{l} \hookrightarrow \mathbf{m}$ have to considered. If the condition $\operatorname{Im} g \cup \bigcup_{j\in J} \operatorname{Im} f_j = \mathbf{m}$ does not hold, then the summand corresponding to $(f_j)_{j\in J}$, g vanishes.

8.26 Remark. By Lemma 8.6 we can find from (8.23.1) the component B_m of the differential B for any $m \geqslant 0$. Namely,

$$\begin{aligned}&\varphi^{\mathcal{Q}_p}(B_m) \\ &= \big\{T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{1\boxtimes b} T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes T^1 s\underline{\mathsf{Q}}(\mathcal{B}; \mathcal{B}) \xrightarrow{M_{m1}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})\big\} \\ &- \sum_{j\in I} \big\{T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \boxtimes \mathbb{1}_p[-1] \xrightarrow{\lambda^{g_j}} \boxtimes^{I\sqcup\mathbf{1}}\,[(\mathbb{1}_p)^{i\in I}_{i<j}, \mathbb{1}_p[-1], (\mathbb{1}_p)^{i\in I}_{i>j}, T^m s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})]\end{aligned}$$

$$\xrightarrow{\boxtimes^{I\sqcup\mathbf{1}}[(T^0\mathrm{id}^{\mathsf{Q}}_{\mathcal{A}_i})_{i<j},b,(T^0\mathrm{id}^{\mathsf{Q}}_{\mathcal{A}_i})_{i>j},1]}$$

$$\boxtimes^{I\sqcup\mathbf{1}}\left[(T^0s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i<j},T^1s\underline{\mathsf{Q}}(\mathcal{A}_j;\mathcal{A}_j),(T^0s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i>j},Ts\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\right] \xrightarrow{M_{0\dots010\dots0m}} s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})\big\}.$$

The sum over $j\in I$ does not vanish only if $m=0,1$.

In short form, for $r=r^1\otimes\dots\otimes r^m$ we have

$$rB=(r^1\otimes\dots\otimes r^m\boxtimes b)M_{m1}-(-)^r\sum_{i=1}^n(1^{\boxtimes(i-1)}\boxtimes b\boxtimes 1^{\boxtimes(n-i)}\boxtimes r^1\otimes\dots\otimes r^m)M_{0\dots010\dots0m}.$$

For $m=0$ we have $(f^0)B_0=0$ iff f_0 is an A_∞-functor. In other cases

$$\begin{aligned} rB_1&=(r\boxtimes b)M_{11}-(-)^r\textstyle\sum_{i=1}^n(1^{\boxtimes i-1}\boxtimes b\boxtimes 1^{\boxtimes n-i}\boxtimes r)M_{0\dots010\dots01} &&\text{for } m=1,\\ (r^1\otimes\dots\otimes r^m)B_m&=(r^1\otimes\dots\otimes r^m\boxtimes b)M_{m1} &&\text{for } m>1.\end{aligned}$$

8.27 Examples. 1. Let $f^i:(\mathcal{A}_i^i,\dots,\mathcal{A}_i^{m_i})\to\mathcal{B}_i$, $i\in\mathbf{n}$ be A_∞-functors and let $r:g\to h:(\mathcal{B}_1,\dots,\mathcal{B}_n)\to\mathcal{C}$ be a coderivation. The augmented coalgebra homomorphism

$$\begin{aligned} M:T^0s\underline{\mathsf{A}_\infty}(\mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1};\mathcal{B}_1)\boxtimes\dots\boxtimes T^0s\underline{\mathsf{A}_\infty}(\mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n};\mathcal{B}_n)\boxtimes T^1s\underline{\mathsf{A}_\infty}(\mathcal{B}_1,\dots,\mathcal{B}_n;\mathcal{C})\\ \to Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1},\dots,\mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n};\mathcal{C})\end{aligned}$$

applied to $f^1\boxtimes\dots\boxtimes f^n\boxtimes r$ gives only one summand $(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01}\in T^1s\underline{\mathsf{A}_\infty}(\mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1},\dots,\mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n};\mathcal{C})$. Equation (8.25.5) in the form

$$(f^1\boxtimes\dots\boxtimes f^n)\cdot r\,\mathrm{pr}_1=(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01}\,\mathrm{pr}_1$$

implies that the $\big((f^1\boxtimes\dots\boxtimes f^n)\cdot g,(f^1\boxtimes\dots\boxtimes f^n)\cdot h\big)$-coderivations $(f^1\boxtimes\dots\boxtimes f^n)\cdot r$ and $(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01}$ coincide. Thus,

$$(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M=(f^1\boxtimes\dots\boxtimes f^n\boxtimes r)M_{0\dots01}=(f^1\boxtimes\dots\boxtimes f^n)\cdot r.$$

2. Let $f^1:(\mathcal{A}_1^1,\dots,\mathcal{A}_1^{m_1})\to\mathcal{B}_1$, …, $f^n:(\mathcal{A}_n^1,\dots,\mathcal{A}_n^{m_n})\to\mathcal{B}_n$, $g:(\mathcal{B}_1,\dots,\mathcal{B}_n)\to\mathcal{C}$ and $h^i:(\mathcal{A}_i^1,\dots,\mathcal{A}_i^{m_i})\to\mathcal{B}_i$ for some i, $1\leqslant i\leqslant n$, be A_∞-functors. Suppose that $p:f^i\to h^i:(\mathcal{A}_i^1,\dots,\mathcal{A}_i^{m_i})\to\mathcal{B}_i$ is a coderivation. Then, similarly to the previous example

$$\begin{aligned}&(f^1\boxtimes\dots\boxtimes f^{i-1}\boxtimes p\boxtimes f^{i+1}\boxtimes\dots\boxtimes f^n\boxtimes g)M_{0\dots010\dots00}\\ &=(f^1\boxtimes\dots\boxtimes f^{i-1}\boxtimes p\boxtimes f^{i+1}\boxtimes\dots\boxtimes f^n\boxtimes g)M=(f^1\boxtimes\dots\boxtimes f^{i-1}\boxtimes p\boxtimes f^{i+1}\boxtimes\dots\boxtimes f^n)\cdot g\end{aligned}$$

is a $\big((f^1\boxtimes\dots\boxtimes f^i\boxtimes\dots\boxtimes f^n)\cdot g,(f^1\boxtimes\dots\boxtimes h^i\boxtimes\dots\boxtimes f^n)\cdot g\big)$-coderivation.

3. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$ be A_∞-categories. Let $g : \mathcal{Y} \to \mathcal{W}$ be an A_∞-functor. It induces the A_∞-functor

$$A_\infty(\mathcal{X}, g) = A_\infty(1, g) \overset{\text{def}}{=} \underline{\mathsf{A}_\infty}(1; g) = (1, \dot{g}) \cdot_{\mathsf{A}_\infty} \mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}} : A_\infty(\mathcal{X}, \mathcal{Y}) \to A_\infty(\mathcal{X}, \mathcal{W})$$

defined via (4.12.2) and computed in Lemma 4.16. As usual, we identify it with $\underline{\mathsf{A}_\infty}(1; g)\widehat{\ }= (1 \boxtimes_u \hat{\dot{g}}) \cdot_{\mathcal{Q}} (\mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}})\widehat{\ }: T^{\geqslant 1} sA_\infty(\mathcal{X}, \mathcal{Y}) \to T^{\geqslant 1} sA_\infty(\mathcal{X}, \mathcal{W})$, or due to Section 8.22 with

$$(1 \boxtimes g)M \overset{\text{def}}{=} T^{\leqslant 1}(\underline{\mathsf{A}_\infty}(1; g)\widehat{\ }) = (1 \boxtimes (T^0 \dot{g}) \operatorname{in}_0) \cdot_{\mathcal{Q}} M_{\mathbf{1}\to\mathbf{1}} : TsA_\infty(\mathcal{X}, \mathcal{Y}) \to TsA_\infty(\mathcal{X}, \mathcal{W}).$$

It takes an A_∞-functor h to $h(1 \boxtimes g)M = \hat{h}g$. The first component maps an A_∞-transformation r to $r[(1 \boxtimes g)M]_1 = \hat{r}g$.

4. An A_∞-functor $f : \mathcal{X} \to \mathcal{Y}$ induces the *strict* A_∞-functor

$$A_\infty(f, \mathcal{W}) = A_\infty(f, 1) \overset{\text{def}}{=} \underline{\mathsf{A}_\infty}(f; 1) = (\dot{f}, 1) \cdot_{\mathsf{A}_\infty} \mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}} : A_\infty(\mathcal{Y}, \mathcal{W}) \to A_\infty(\mathcal{X}, \mathcal{W})$$

defined via (4.12.3) and computed in Lemma 4.17. We shall identify it with $\underline{\mathsf{A}_\infty}(f; 1)\widehat{\ }= (\hat{\dot{f}} \boxtimes_u 1) \cdot_{\mathcal{Q}} (\mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}})\widehat{\ }: T^{\geqslant 1} sA_\infty(\mathcal{Y}, \mathcal{W}) \to T^{\geqslant 1} sA_\infty(\mathcal{X}, \mathcal{W})$, or due to Section 8.22 with

$$(f \boxtimes 1)M \overset{\text{def}}{=} T^{\leqslant 1}(\underline{\mathsf{A}_\infty}(f; 1)\widehat{\ }) = ((T^0 \dot{f}) \operatorname{in}_0 \boxtimes 1) \cdot_{\mathcal{Q}} M_{\mathbf{1}\to\mathbf{1}} : TsA_\infty(\mathcal{Y}, \mathcal{W}) \to TsA_\infty(\mathcal{X}, \mathcal{W}).$$

It takes an A_∞-functor h to $h(f \boxtimes 1)M = \hat{f}h$. The first component maps an A_∞-transformation r to $r[(f \boxtimes 1)M]_1 = \hat{f}r$.

8.28 Cohomological Hochschild complex. Particular cases of operations M_{k1} were introduced and used by Kadeishvili [Kad88], Getzler [Get93], Getzler and Jones [GJ94], and by Gerstenhaber and Voronov [GV95, VG95]. They considered an A_∞-algebra A, its identity endomorphism and A_∞-transformations $c, c^1, \dots, c^k \in sC(A) \overset{\text{def}}{=} sA_\infty(A, A)(\mathrm{id}, \mathrm{id}) = \prod_{n\geqslant 0} \underline{\mathsf{C}}_{\Bbbk}((sA)^{\otimes n}, sA)$. When A is a differential graded algebra, this complex is nothing else but the Hochschild cochain complex of A. If A is a non-unital associative algebra, or an A_∞-algebra, such interpretation might be doubtful, nevertheless, the complex $(sC(A), B_1)$ still might be *called* the Hochschild cochain complex [Get93]. The mentioned authors defined operations $c\{c^1, \dots, c^k\} = m_{1,k}(c; c^1, \dots, c^k)$ in this complex called, in particular, braces which in our conventions are written as $M_{k1} : T^k sC(A) \otimes sC(A) \to sC(A)$, $c^1 \otimes \dots \otimes c^k \otimes c \mapsto (c^1 \otimes \dots \otimes c^k \boxtimes c)M_{k1}$, where

$$\begin{aligned}
[(c^1 \otimes \dots \otimes c^k \boxtimes c)M_{k1}]_n &= \sum_{l\geqslant 0} (c^1 \otimes \dots \otimes c^k)\theta_{nl} c_l \\
&= \sum_{i_0+\dots+i_k+j_1+\dots+j_k=n} (1^{\otimes i_0} \otimes c^1_{j_1} \otimes 1^{\otimes i_1} \otimes \dots \otimes c^k_{j_k} \otimes 1^{\otimes i_k}) c_{i_0+\dots+i_k+k}. \quad (8.28.1)
\end{aligned}$$

The operation $\circ = M_{11} : sC(A) \otimes sC(A) \to sC(A)$ was originally introduced by Gerstenhaber and $[,] : sC(A) \otimes sC(A) \to sC(A)$, $[p,q] = (p \boxtimes q)M_{11} - (-)^{pq}(q \boxtimes p)M_{11}$ is precisely his bracket [Ger63]. It turns $sC(A)$ into a graded Lie algebra. Together with isomorphisms $M_{10} = (\lambda^{\mathsf{I}\cdot})^{-1} : sC(A) \otimes T^0sC(A) \xrightarrow{\sim} sC(A)$ the operations M_{k1} are the only non-vanishing components of the differential coalgebra homomorphism $M : TsC(A) \otimes TsC(A) \to TsC(A)$, as noticed by Getzler and Jones [GJ94, Section 5.2]. It makes $TsC(A)$ into a differential graded bialgebra. Graded $\Bbbk$-modules $C(\mathcal{A})$ with this property are called B_∞-algebras by definition 5.2 of Getzler and Jones [GJ94] based on work of Baues [Bau81]. See Tamarkin [Tam98] for the usage of B_∞-algebras. Vanishing of M_{kl} for $l > 1$ makes $C(A)$ a special kind of a B_∞-algebra, namely, a homotopy Gerstenhaber algebra (see Voronov and Gerstenhaber [VG95] and Voronov [Vor00], who clarified the history of the subject).

Instead of an A_∞-algebra A one can consider an A_∞-category $\mathcal{A}$ without too many changes. Again the complex $sC(\mathcal{A}) \overset{\text{def}}{=} (sA_\infty(\mathcal{A},\mathcal{A})(\mathrm{id},\mathrm{id}), B_1)$ might be *called* the *Hochschild cochain complex* of $\mathcal{A}$ (when $\mathcal{A}$ is differential graded category this terminology is well motivated). The cut comultiplication Δ_0, the counit $\varepsilon = \mathrm{pr}_0$ and the differential $B : TsC(\mathcal{A}) \to TsC(\mathcal{A})$ turn $C(\mathcal{A})$ into an A_∞-algebra. Indeed, for an arbitrary A_∞-category $\mathcal{C}$ the endomorphism complexes $\mathcal{C}(X,X)$ are A_∞-algebras, in particular, for $\mathcal{C} = A_\infty(\mathcal{A},\mathcal{A})$ and for $X = \mathrm{id}_{\mathcal{A}}$. The graded $\Bbbk$-module $C(\mathcal{A})$ carries operations M_{k1} given by (8.28.1) and isomorphisms M_{10}, which are the only non-vanishing components of the differential coalgebra homomorphism $M : TsC(\mathcal{A}) \otimes TsC(\mathcal{A}) \to TsC(\mathcal{A})$. This is an associative multiplication with the unit $\eta = \mathrm{in}_0 : \Bbbk = T^0sC(\mathcal{A}) \to TsC(\mathcal{A})$. Together these structures make $(TsC(\mathcal{A}), \Delta_0, \mathrm{pr}_0, M, \mathrm{in}_0, B)$ into a differential graded bialgebra. Thus, $C(\mathcal{A})$ is a B_∞-algebra, and even a homotopy Gerstenhaber algebra.

These statements follow immediately from the described enriched multicategory picture. In fact, the multicategory $\underline{\mathsf{A}_\infty}$ is enriched in the multicategory A_∞ which is a full submulticategory of $\widehat{\mathrm{ac}^d\mathscr{Q}_p}$, where $\mathrm{ac}^d\mathscr{Q}_p$ stands for the symmetric Monoidal category of augmented differential counital coassociative coalgebras.

Thus, $\underline{\mathsf{A}_\infty}$ is a $\widehat{\mathrm{ac}^d\mathscr{Q}_p}$-multicategory which is precisely the same as a $\mathrm{ac}^d\mathscr{Q}_p$-multicategory. In particular, $Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})$ is a unital associative algebra in $\mathrm{ac}^d\mathscr{Q}_p$ for any A_∞-category $\mathcal{A}$. Its unit is given by the morphism $\eta = \bigl(\mathbb{1}_p \xrightarrow{T^0\dot{\mathrm{id}}} T^0s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{in}_0} Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})\bigr)$, $* \mapsto \mathrm{id}_{\mathcal{A}}$ of $\mathrm{ac}^d\mathscr{Q}_p$. This quiver map acts as

$$\begin{aligned}
&\bigl\langle \Bbbk \Longrightarrow [T^0s\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})](\mathrm{id},\mathrm{id}) \xrightarrow{\mathrm{in}_0} [Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})](\mathrm{id},\mathrm{id})\bigr\rangle \\
&= \bigl\langle \Bbbk \Longrightarrow T^0s[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})] \xrightarrow{\mathrm{in}_0} Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})] \hookrightarrow [Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})](\mathrm{id},\mathrm{id})\bigr\rangle
\end{aligned}$$

on morphisms. The graded $\Bbbk$-module $Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})]$ has the induced differential coalgebra structure. This $\Bbbk$-submodule is also closed under associative multiplication M

which is a differential coalgebra homomorphism:

$$M : Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})] \otimes Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})] \to Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})].$$

This $\Bbbk$-subalgebra contains also the two-sided unit $\eta = \mathrm{in}_0 : \Bbbk \to Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})]$. Thus, $TsC(\mathcal{A}) = Ts[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})]$ is a differential graded $\Bbbk$-bialgebra.

Let us denote by $\mathcal{K}$ the symmetric closed Monoidal category of differential graded $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy. It is a quotient of the symmetric closed Monoidal category $\mathsf{C}_\Bbbk = \mathbf{dg}$ of complexes (see Example 3.27) by homotopy equivalence relation. Thus, the tensor product in $\mathcal{K}$ is the tensor product of complexes, the unit object is $\Bbbk$, viewed as a complex concentrated in degree 0, and the symmetry is the standard symmetry $c : X \otimes Y \to Y \otimes X$, $x \otimes y \mapsto (-)^{xy} y \otimes x$. For each pair of complexes X and Y, the inner hom-object $\underline{\mathcal{K}}(X,Y)$ is the complex $\underline{\mathsf{C}}_\Bbbk(X,Y)$. The evaluation morphism $\mathrm{ev}^{\mathcal{K}} : X \otimes \underline{\mathcal{K}}(X,Y) \to Y$ and the coevaluation morphism $\mathrm{coev}^{\mathcal{K}} : Y \to \underline{\mathcal{K}}(X, X \otimes Y)$ in $\mathcal{K}$ are the homotopy classes of the evaluation morphism $\mathrm{ev}^{\mathsf{C}_\Bbbk} : X \otimes \underline{\mathsf{C}}_\Bbbk(X,Y) \to Y$ and the coevaluation morphism $\mathrm{coev}^{\mathsf{C}_\Bbbk} : Y \to \underline{\mathsf{C}}_\Bbbk(X, X \otimes Y)$ in $\mathsf{C}_\Bbbk$, respectively. Therefore, $\mathcal{K}$-category $\underline{\mathcal{K}}$ coincides with $\underline{\mathsf{C}}_\Bbbk$ viewed as a $\mathcal{K}$-category via the projection functor $\mathbf{dg} \to \mathcal{K}$.

Gerstenhaber algebras with commutative or non-commutative associative operation are defined e.g. by Ginzburg [Gin05, Section 6.6].

The following proposition is essentially proven by Getzler and Jones in [GJ94, Section 5.2], although these authors do not formulate it in the same way as we do.

8.29 Proposition. *For any graded $\Bbbk$-module C and any differential bialgebra structure of the form $(TsC, \Delta_0, \mathrm{pr}_0, M, \mathrm{in}_0, B)$ with the cut comultiplication Δ_0, the operations $B_2 : \Bbbk[-1] \to \underline{\mathsf{C}}_\Bbbk(T^2sC, sC)$ and $[\,,] : \Bbbk \to \underline{\mathsf{C}}_\Bbbk(T^2sC, sC)$ (that is, $B_2 = \mathrm{in}_2 \cdot B \cdot \mathrm{pr}_1 : T^2sC \to sC$, $[\,,] = M - (12)^\sim M : T^2sC \to sC$ has degree 1, respectively, 0) turn sC into a non-unital Gerstenhaber algebra in the $\mathcal{K}$-category $\underline{\mathcal{K}}$ with associative multiplication $m_2 = (s \otimes s)B_2 s^{-1} : T^2C \to C$ of degree 0 which is commutative.*

Proof. First of all, the equation $1B = 0$ implies that the component B_0 vanishes, and B turns C into an A_∞-algebra. Since $\mathrm{in}_0 : \Bbbk \to TsC$ is the unit of the associative algebra (TsC, M), we have, obviously, $(1 \otimes 1)M = 1$, $(1 \otimes x)M = x$ and $(x \otimes 1)M = x$ for all $x \in T^n sC$, $n \geqslant 0$. This imposes the following constraints on the components $M_{kl} = M \cdot \mathrm{pr}_1 : T^k sC \otimes T^l sC \to sC$ of the coalgebra homomorphism M:

$$M_{00} = 0, \quad M_{01} = (\lambda^{\cdot \mathsf{I}})^{-1}, \quad M_{10} = (\lambda^{\mathsf{I} \cdot})^{-1}, \quad M_{0n} = M_{n0} = 0 \;\text{ for }\; n > 1.$$

Since (TsC, M, B) is a differential graded associative algebra, the commutator $[\,,] = M - (12)^\sim M : (TsC)^{\otimes 2} \to TsC$, $[x,y] = (x \boxtimes y)M - (-)^{xy}(y \boxtimes x)M$, $x, y \in TsC$ turns $(TsC, [\,,], B)$ into a differential graded Lie algebra. In particular, $[\,,]B = (1 \otimes B + B \otimes 1)[\,,]$, that is, $[x,y]B = [x, yB] + (-)^y [xB, y]$.

The graded $\Bbbk$-submodule sC is closed under the bracket. Indeed, for $x, y \in sC$ we have $(x \boxtimes y)M = x \otimes y + (-)^{xy} y \otimes x + (x \boxtimes y)M_{11}$, or in other notation

$$M = 1 + (12)^\sim + M_{11} : sC \otimes sC \to TsC. \tag{8.29.1}$$

This implies the equation $(12)^\sim M = (12)^\sim + 1 + (12)^\sim M_{11} : sC \otimes sC \to TsC$. Hence, the commutator $[\,,] = M - (12)^\sim M = M_{11} - (12)^\sim M_{11} : sC \otimes sC \to TsC$ takes values in sC, $[x, y] = (x \boxtimes y)M_{11} - (-)^{xy}(y \boxtimes x)M_{11} \in sC$. Thus sC is a graded Lie subalgebra of TsC. The differential B restricts to sC as $B = B_1 : sC \to sC$, and $(sC, [\,,], B_1)$ is a differential graded Lie subalgebra of TsC.

Being an A_∞-algebra C has the binary multiplication $m_2 = (s \otimes s)B_2 s^{-1} : C \otimes C \to C$ (a chain map with respect to the differential $m_1 = sB_1 s^{-1} : C \to C$), which is associative in $\mathcal{K}$. Let us prove that it is also commutative in $\mathcal{K}$. Equation (8.29.1) implies that

$$\begin{aligned} MB &= 1 \otimes B_1 + B_1 \otimes 1 + B_2 + (12)^\sim(1 \otimes B_1 + B_1 \otimes 1 + B_2) + M_{11}B_1 \\ &= (1 \otimes B_1 + B_1 \otimes 1)(M - M_{11}) + B_2 + (12)^\sim B_2 + M_{11}B_1 \\ &= (1 \otimes B + B \otimes 1)M + B_2 + (12)^\sim B_2 + M_{11}B_1 - (1 \otimes B_1 + B_1 \otimes 1)M_{11} : \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad sC \otimes sC \to TsC. \end{aligned}$$

Since M is a chain map we have $MB = (1 \otimes B + B \otimes 1)M$. Therefore, $B_2 + (12)^\sim B_2 : sC \otimes sC \to TsC$ is homotopic to 0 (with the homotopy $-M_{11}$). This implies that

$$m_2 - (12)^\sim m_2 + (s \otimes s)M_{11}s^{-1}m_1 + (1 \otimes m_1 + m_1 \otimes 1)(s \otimes s)M_{11}s^{-1} = 0 : C \otimes C \to C,$$

in other words, m_2 is homotopy commutative (with the homotopy $-(s \otimes s)M_{11}s^{-1}$).

Let us prove a relation between the multiplication and the bracket in sC. Restriction of M to the summands $T^2sC \otimes T^1sC$ and $T^1sC \otimes T^2sC$ equals

$$\begin{aligned} M &= 1 + (23)^\sim + (123)^\sim + 1 \otimes M_{11} + (23)^\sim(M_{11} \otimes 1) + M_{21} : T^2sC \otimes T^1sC \to TsC, \\ M &= 1 + (12)^\sim + (321)^\sim + M_{11} \otimes 1 + (12)^\sim(1 \otimes M_{11}) + M_{12} : T^1sC \otimes T^2sC \to TsC. \end{aligned}$$

Composing the second line with $(123)^\sim$ and subtracting it from the first line we get the formula

$$M - (123)^\sim M = 1 \otimes [\,,] + (23)^\sim([\,,] \otimes 1) + M_{21} - (123)^\sim M_{12} : T^2sC \otimes T^1sC \to TsC. \tag{8.29.2}$$

Composed with B it gives

$$\begin{aligned} &MB - (123)^\sim MB \\ &= (1 \otimes [\,,])(1 \otimes B_1 + B_1 \otimes 1 + B_2) + (23)^\sim([\,,] \otimes 1)(1 \otimes B_1 + B_1 \otimes 1 + B_2) + M_{21}B_1 - (123)^\sim M_{12}B_1 \\ &= (1 \otimes 1 \otimes B_1 + 1 \otimes B_1 \otimes 1 + B_1 \otimes 1 \otimes 1)\{1 \otimes [\,,] + (23)^\sim([\,,] \otimes 1)\} + (1 \otimes [\,,])B_2 \\ &\qquad + (23)^\sim([\,,] \otimes 1)B_2 + M_{21}B_1 - (123)^\sim M_{12}B_1 : T^2sC \otimes T^1sC \to TsC. \end{aligned} \tag{8.29.3}$$

Replacing MB with $(1 \otimes B + B \otimes 1)M$ we transform the left hand side to

$$[1\otimes 1\otimes B_1+(1\otimes B_1+B_1\otimes 1+B_2)\otimes 1]M-(123)^\sim[1\otimes(1\otimes B_1+B_1\otimes 1+B_2)+B_1\otimes 1\otimes 1]M$$
$$= (1\otimes 1\otimes B_1+1\otimes B_1\otimes 1+B_1\otimes 1\otimes 1)(M-(123)^\sim M)+(B_2\otimes 1)(M-(12)^\sim M)$$
$$= (1\otimes 1\otimes B_1+1\otimes B_1\otimes 1+B_1\otimes 1\otimes 1)\{1\otimes[\,,]+(23)^\sim([\,,]\otimes 1)+M_{21}-(123)^\sim M_{12}\}+(B_2\otimes 1)[\,,].$$

Comparing this with the last expression from (8.29.3) we get the equation between quiver maps (cycles of degree 1)

$$(B_2\otimes 1)[\,,]-(1\otimes[\,,])B_2-(23)^\sim([\,,]\otimes 1)B_2$$
$$=(M_{21}-(123)^\sim M_{12})B_1-(1\otimes 1\otimes B_1+1\otimes B_1\otimes 1+B_1\otimes 1\otimes 1)(M_{21}-(123)^\sim M_{12}): T^3sC\to sC.$$

Thus, the left hand side is the boundary of $M_{21}-(123)^\sim M_{12}$ in $\underline{\mathsf{C}}_\Bbbk(T^3sC,sC)$. In other words, the corresponding chain map $\Bbbk[-1]\to\underline{\mathsf{C}}_\Bbbk(T^3sC,sC)$ is null-homotopic. □

8.30 Corollary. *The Hochschild cohomology $H^\bullet(C)$ is a non-unital Gerstenhaber algebra with associative multiplication which is commutative.*

For any graded $\Bbbk$-module C we have defined in Example 6.13 mappings $\Delta'^{(r)}: TsC\to(TsC)^{\otimes r}$ for $r\geqslant 0$. Their matrix coefficients are given by the isomorphisms $(\Delta'^{(r)})_n^{m_1\dots m_r}=\lambda^f: T^nsC\to\otimes^{i\in\mathbf{r}}T^{m_i}sC$, if f is surjective, where $(f:\mathbf{n}\to\mathbf{r})\in\mathcal{O}$ has $|f^{-1}i|=m_i$ for all $i\in\mathbf{r}$. If f is not surjective, the matrix coefficient $(\Delta'^{(r)})_n^{m_1\dots m_r}$ vanishes. The comultiplication $\Delta'=\Delta'^{(2)}:TsC\to(TsC)^{\otimes 2}$ is coassociative.

Let $M^{\mathbf{r}}:(TsC)^{\otimes r}\to TsC$, $r\geqslant 0$, be an associative unital graded algebra structure of TsC with the unit $M^{\mathbf{0}}=\mathrm{in}_0$. Assume that the multiplication $M=M^{\mathbf{2}}$ and a differential B make $H=(TsC,\Delta_0,\mathrm{pr}_0,M,\mathrm{in}_0,B)$ into a differential graded bialgebra.

8.31 Proposition. *The differential graded bialgebra H admits an antipode*

$$\gamma=\sum_{r\geqslant 0}(-1)^r\Delta'^{(r)}\cdot M^{\mathbf{r}},$$

and a skew antipode (an antipode for the graded bialgebra $H^{\mathrm{coop}}=(TsC,\Delta_0^{\mathrm{op}},\mathrm{pr}_0,M,\mathrm{in}_0)$ with the opposite comultiplication)

$$\gamma'=\sum_{r\geqslant 0}(-1)^r\Delta'^{\mathrm{op}(r)}\cdot M^{\mathbf{r}}.$$

Thus,

$$(1)\gamma=1,\quad (x_1\dots x_k)\gamma=\sum_{r=1}^k(-1)^r(x_1\dots x_k)\Delta'^{(r)}\cdot M^{\mathbf{r}}\qquad\text{for } k>0,$$
$$(1)\gamma'=1,\quad (x_1\dots x_k)\gamma'=\sum_{r=1}^k(-1)^r(x_1\dots x_k)\Delta'^{\mathrm{op}(r)}\cdot M^{\mathbf{r}}\qquad\text{for } k>0.$$

Proof. The degree 0 map $\Delta' = \Delta_0 - \mathrm{id}_{\mathcal{A}} \otimes \eta - \eta \otimes \mathrm{id}_{\mathcal{A}} + \varepsilon\eta \otimes \eta$ is a chain map. Let us prove one of antipode axioms $\Delta_0(\mathrm{id} \otimes \gamma)M = \varepsilon\eta$. On $T^0 sC = \Bbbk$ both parts are identity maps. And on $T^{\geqslant 1} sC$ we have

$$\begin{aligned}
\Delta_0(\mathrm{id} \otimes \gamma)M &= (\eta \otimes \mathrm{id} + \Delta' + \mathrm{id} \otimes \eta)(\mathrm{id} \otimes \gamma)M \\
&= \gamma + \Delta'\Big(\mathrm{id} \otimes \sum_{r \geqslant 1} (-1)^r \Delta'^{(r)} \cdot M^{\mathbf{r}}\Big) M + \mathrm{id} \\
&= \sum_{r \geqslant 1} (-1)^r \Delta'^{(r)} \cdot M^{\mathbf{r}} + \sum_{r \geqslant 1} (-1)^r \Delta'^{(r+1)} \cdot M^{\mathbf{r+1}} + \mathrm{id} \\
&= -\Delta'^{(1)} \cdot M^{\mathbf{1}} + \mathrm{id} = 0.
\end{aligned}$$

The other axiom for γ and axioms for γ' are proved similarly. □

The maps $\gamma : H \to H$, $\gamma' : H \to H$ are inverse to each other by general bialgebra theory. Thus the considered H is a differential graded Hopf algebra. In particular, $Ts\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{A})$ is a differential graded Hopf algebra.

Chapter 9

Multicategory of unital A_∞-categories

In this chapter we add the missing notion of unit morphisms to the picture of A_∞-categories. Ordinary categories are required to have unit (or identity) morphisms. This requirement is imposed also on unital A_∞-categories. However, unit morphisms in A_∞-category are units only up to homotopy. Similarly, unital functors have to preserve unit morphisms up to a boundary.

Before introducing weak notions, we recall the notion of strictly unital A_∞-category. We introduce also strictly unital A_∞-functors (with many entries) and the corresponding A_∞-transformations. We shall prove that these form a closed symmetric multicategory.

Unital A_∞-categories and unital A_∞-functors form a symmetric multicategory $\mathsf{A}^{\mathrm{u}}_\infty$. There is a multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$, where $\mathcal{K}$ is the homotopy category of complexes of $\Bbbk$-modules. We prove that the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ is closed.

With an A_∞-category $\mathcal{A}$ a strictly unital category $\mathcal{A}^{\mathsf{su}}$ and an embedding $e_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathsf{su}}$ are associated in a natural way. We prove that the A_∞-functor $A_\infty(e_{\mathcal{A}}, \mathcal{C}) = (e_{\mathcal{A}} \boxtimes 1)M : A^u_\infty(\mathcal{A}^{\mathsf{su}}, \mathcal{C}) \to A_\infty(\mathcal{A}, \mathcal{C})$ is an A_∞-equivalence for an arbitrary unital A_∞-category $\mathcal{C}$. In other words, the pair $(\mathcal{A}^{\mathsf{su}}, e_{\mathcal{A}})$ is a unital envelope of an A_∞-category $\mathcal{A}$.

9.1 Strictly unital A_∞-categories and A_∞-functors. Recall that an A_∞-category $\mathcal{A}$ is called *strictly unital* if for each object $X \in \operatorname{Ob}\mathcal{A}$ there is a strict unit, that is, a $\Bbbk$-linear map ${}_X\mathbf{i}^{\mathcal{A}}_0 : \Bbbk \to (s\mathcal{A})^{-1}(X, X)$ such that ${}_X\mathbf{i}^{\mathcal{A}}_0 b_1 = 0$ and the following conditions are satisfied: for all pairs X, Y of objects of $\mathcal{A}$ the chain maps $(1 \otimes {}_Y\mathbf{i}^{\mathcal{A}}_0)b_2, -({}_X\mathbf{i}^{\mathcal{A}}_0 \otimes 1)b_2 : s\mathcal{A}(X, Y) \to s\mathcal{A}(X, Y)$ are equal to the identity map and $(\cdots \otimes \mathbf{i}^{\mathcal{A}}_0 \otimes \cdots)b_n = 0$ if $n \geqslant 3$ (cf. [Fuk02, Kel01]). For example, differential graded categories are strictly unital. Strictly unital A_∞-categories are unital in the sense of [Lyu03, Definition 7.3] and Section 9.10.

Recall the notation

$$e_i = e^I_i = (\chi(i' = i))_{i' \in I} = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^I.$$

9.2 Definition. An A_∞-functor $f : (\mathcal{A}_i)_{i \in I} \to \mathcal{B}$ between strictly unital A_∞-categories is *strictly unital*, when all its components vanish if any of its entries is $\mathbf{i}^{\mathcal{A}_i}_0$, except $\mathbf{i}^{\mathcal{A}_i}_0 f_{e_i} = \mathbf{i}^{\mathcal{B}}_0$.

For an arbitrary strictly unital A_∞-functor $f : (\mathcal{A}_i)_{i \in I} \to \mathcal{B}$ all its restrictions $f\big|^{(X_i)_{i \in I \smallsetminus J}}_J$ are strictly unital as well. In particular, the restriction $g = f\big|^{(X_i)_{i \neq j}}_j : \mathcal{A}_j \to \mathcal{B}$ of f to j-th argument is strictly unital. By definition above it is unital in the sense of [Lyu03, Definition 8.1] (see also Definition 9.11). As we shall see, this implies that a strictly unital A_∞-functor f is unital.

9.3 Proposition. *Composition in A_∞ of strictly unital A_∞-functors is strictly unital.*

Proof. Let $\phi : I \to J$ be a map in $\mathcal{S}$. Let $f^j : (\mathcal{A}_i)_{i\in\phi^{-1}j} \to \mathcal{B}_j$, $g : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ be strictly unital A_∞-functors. Then the components h_k of the composition $h = (f^j)\cdot g : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ with $|k| > 1$ are polynomials of components f^j_p, g_p, where necessarily $|p| > 1$ for some of them. Therefore, h_k vanish if any of its entries is $\mathbf{i}_0$. Let $i \in I$ and let $j = \phi i$. The component $h_{e^I_i} = f^j_{e_i^{\phi^{-1}j}} \cdot g_{e^J_j}$ gives

$$\mathbf{i}_0^{\mathcal{A}_i} h_{e^I_i} = \mathbf{i}_0^{\mathcal{A}_i} f^j_{e_i^{\phi^{-1}j}} g_{e^J_j} = \mathbf{i}_0^{\mathcal{B}_j} g_{e^J_j} = \mathbf{i}_0^{\mathcal{C}}.$$

Hence, h is strictly unital. □

9.4 Corollary. *There is a symmetric submulticategory $\mathsf{A}_\infty^{\mathrm{su}} \subset \mathsf{A}_\infty$, whose objects are strictly unital A_∞-categories and morphisms are strictly unital A_∞-functors.*

9.5 Proposition. *The multicategory $\mathsf{A}_\infty^{\mathrm{su}}$ is closed with the inner object of morphisms $\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$, whose objects are strictly unital A_∞-functors $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ and morphisms $r \in s\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$ are $r \in s\underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$ such that all components of r vanish if any of its entries is $\mathbf{i}_0^{\mathcal{A}_i}$. Evaluation is obtained by restriction:*

$$\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{su}}} = \bigl[(\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xhookrightarrow{(1)_I,\iota} (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty}} \mathcal{B}\bigr]. \tag{9.5.1}$$

Proof. First of all we show that so defined subquiver $\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is closed under A_∞-operations.

9.6 Lemma. *The quiver $\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is a strictly unital A_∞-subcategory of the strictly unital A_∞-category $\underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$.*

Proof. For a strictly unital A_∞-category $\mathcal{B}$ and arbitrary A_∞-categories $\mathcal{A}_i$, $i \in I$, the A_∞-category $\underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is strictly unital. Indeed, for $I = \varnothing$ we have an isomorphism $\underline{\mathsf{A}}_\infty(;\mathcal{B}) \simeq \mathcal{B}$, for 1-element I this is shown in [Lyu03, Section 8.11], for other I this follows by induction from the isomorphism

$$\underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in\mathbf{1}\sqcup I};\mathcal{B}) \simeq \underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\underline{\mathsf{A}}_\infty(\mathcal{A}_1;\mathcal{B})).$$

Consider the element

$$r^1 \otimes \cdots \otimes r^m \in s\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^0,f^1) \otimes \cdots \otimes s\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^{m-1},f^m)$$

for $m \geqslant 1$. We are going to prove that $(r^1 \otimes \cdots \otimes r^m)B_m \in s\underline{\mathsf{A}}_\infty^{\mathrm{su}}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f^0,f^m)$. Suppose first that $m \geqslant 2$. Then

$$(r^1 \otimes \cdots \otimes r^m)B_m = \bigl[\boxtimes^{i\in I} Ts\mathcal{A}_i \xrightarrow{(r^1\otimes\cdots\otimes r^m)\theta} T^{\geqslant 1}s\mathcal{B} \xrightarrow{b} s\mathcal{B}\bigr]$$

due to (8.16.1). Explicit formula (7.16.3) for $(r^1 \otimes \cdots \otimes r^m)\theta$ gives for $I = \mathbf{n}$ that $(r^1 \otimes \cdots \otimes r^m)B_m$ is a sum of terms $(f^0_{k^0,p_0} \otimes r^1_{j_1} \otimes f^1_{k^1,p_1} \otimes \cdots \otimes r^m_{j_m} \otimes f^m_{k^m,p_m})b_N$, where $p_q \geqslant 0$, $k^q, j_q \in (\mathbb{Z}_{\geqslant 0})^I$, $N = p_0+\cdots+p_m+m$, and $f^q_{k^q p_q} : \boxtimes^{i\in I} T^{k^q_i} s\mathcal{A}_i \to T^{p_q} s\mathcal{B}$, $k^q \neq 0$, are matrix coefficients of $\overline{f^q}$. If $\mathbf{i}_0$ occurs as entry of $r^q_{j_q}$ the term vanishes. If $\mathbf{i}_0$ is an entry of $f^q_{k^q p_q}$, the output of it vanishes or gives $\mathbf{i}_0$, in which case b_N vanishes, since $N \geqslant 1+m \geqslant 3$.

Consider the case of $m = 1$:

$$
\begin{aligned}
(r)B_1 &= (r\theta)\cdot b - (-)^r \sum_{i=1}^{n} (1^{\boxtimes(i-1)} \boxtimes \tilde{b} \boxtimes 1^{\boxtimes(n-i)}) \cdot r \\
&= \begin{matrix}\text{signed permutation}\\ \text{\& insertion of units}\end{matrix} \times \sum_{\substack{p,q\geqslant 0\\ i_s,j,k_t\in(\mathbb{Z}_{\geqslant 0})^n;\, i_s,k_t\neq 0}} (f_{i_1} \otimes \cdots \otimes f_{i_p} \otimes r_j \otimes g_{k_1} \otimes \cdots \otimes g_{k_q}) b_{p+1+q} \\
&\qquad - (-)^r \sum_{i,a_i,t_i,c_i} (1^{\boxtimes(i-1)} \boxtimes (1^{\otimes a_i} \otimes b_{t_i} \otimes 1^{\otimes c_i}) \boxtimes 1^{\boxtimes(n-i)}) \cdot r_{l^1,\dots,l^{i-1},a_i+1+c_i,l^{i+1},\dots,l^n}
\end{aligned}
$$

due to (7.16.4).

Composing it with $1^{\boxtimes(i-1)} \boxtimes (1^{\otimes a} \otimes \mathbf{i}_0 \otimes 1^{\otimes c}) \boxtimes 1^{\boxtimes(n-i)}$ we get no more than two terms from the first and the second sums:

$$
\begin{aligned}
&\chi(c=0)[1 \otimes (1^{\boxtimes(i-1)} \boxtimes \mathbf{i}_0 \boxtimes 1^{\boxtimes(n-i)})](r_{l-e_i} \otimes g_{e_i})b_2 \\
&\qquad + \chi(a=0)[(1^{\boxtimes(i-1)} \boxtimes \mathbf{i}_0 \boxtimes 1^{\boxtimes(n-i)}) \otimes 1](f_{e_i} \otimes r_{l-e_i})b_2 \\
&\qquad\qquad - (-)^r \chi(a>0) r_{l-e_i} + (-)^r \chi(c>0) r_{l-e_i} \\
&\qquad = (-)^r[-\chi(a=0) + \chi(c=0) - \chi(a>0) + \chi(c>0)] r_{l-e_i} = 0,
\end{aligned}
$$

because insertion of unit and signed permutation for the two terms of the first sum produce the same sign. Therefore, $\underline{\mathsf{A}}^{\mathrm{su}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is an A_∞-subcategory of $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$. Since the strict unit elements $f\mathbf{i}^{\mathcal{B}}$ of $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,f)$ have only zeroth components, they belong to $\underline{\mathsf{A}}^{\mathrm{su}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,f)$. Thus, the A_∞-category $\underline{\mathsf{A}}^{\mathrm{su}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is strictly unital. $\square$

Let us prove that evaluation (9.5.1) is a strictly unital A_∞-functor. Indeed, its components $\mathrm{ev}_{n,m}$ vanish if $m > 1$. The component

$$
\mathrm{ev}_{n,0} : \boxtimes^{i\in I} T^{n_i} s\mathcal{A}_i(X_i,Y_i) \to s\mathcal{B}((X_i)f,(Y_i)f), \qquad \boxtimes^{i\in I} a_i \mapsto (\boxtimes^{i\in I} a_i) f_{(n_i)_I}
$$

vanishes for $|n| > 1$ if one of the factors of a_i is $\mathbf{i}^{\mathcal{A}_i}_0$ for some $i \in I$, because f is strictly unital. For the same reason the output is $\mathbf{i}^{\mathcal{B}}$, when $|n| = 1$, one of a_i is $\mathbf{i}^{\mathcal{A}_i}_0$ and other a_i equal 1.

The component

$$
\begin{aligned}
\mathrm{ev}_{n,1} : \boxtimes^{i\in I} T^{n_i} s\mathcal{A}_i(X_i,Y_i) \boxtimes s\underline{\mathsf{A}}^{\mathrm{su}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) &\to s\mathcal{B}((X_i)f,(Y_i)g), \\
\boxtimes^{i\in I} a_i \boxtimes r &\mapsto (\boxtimes^{i\in I} a_i) r_{(n_i)_I}
\end{aligned}
$$

vanishes for $n \neq 0$ if one of the factors of a_i is $\mathbf{i}_0^{\mathcal{A}_i}$ for some $i \in I$, because $r \in \underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$. When $n=0$, all $a_i = 1$, $f = g$ and $r = f\mathbf{i}^{\mathcal{B}}$ is the strict unit, then the output is ${}_{(X_i)f}\mathbf{i}_0^{\mathcal{B}}$, as required. Thus, evaluation (9.5.1) is a strictly unital A_∞-functor.

Since $\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{su}}}$ is obtained by restriction from $\mathrm{ev}^{\mathsf{A}_\infty}$, we have a commutative diagram

$$\begin{tikzcd}
\mathsf{A}_\infty^{\mathrm{su}}\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[r, "\varphi^{\mathsf{A}_\infty^{\mathrm{su}}}"] \arrow[d, hook, "\iota"'] & \mathsf{A}_\infty^{\mathrm{su}}\big((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\big) \arrow[dd, hook, "\iota"] \\
\mathsf{A}_\infty\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[d, hook, "\mathsf{A}_\infty(1;\iota)"'] & \\
\mathsf{A}_\infty\big((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})\big) \arrow[r, "\varphi^{\mathsf{A}_\infty}", "\sim"'] & \mathsf{A}_\infty\big((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\big),
\end{tikzcd}$$

which implies that $\varphi^{\mathsf{A}_\infty^{\mathrm{su}}}$ is injective.

To prove that it is surjective, consider a strictly unital A_∞-functor $f : (\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J} \to \mathcal{C}$. It is the image $(g)\varphi^{\mathsf{A}_\infty}$ of a unique A_∞-functor $g : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})$. It takes objects Y_j of $\mathcal{B}_j$, $j \in J$ to the A_∞-functor $(Y_j)g = f|_I^{(Y_j)_{j\in J}} : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$. Due to (8.18.2) its components are $[(Y_j)g]_m = \big(f|_I^{(Y_j)_{j\in J}}\big)_m = f_{m,0}$, $m \in (\mathbb{Z}_{\geqslant 0})^I$. Thus, $(Y_j)g$ is strictly unital. The n-th component, $n = (n_j)_{j\in J}$, $n \neq 0$,

$$g_n : \otimes^{j\in J} T^{n_j} s\mathcal{B}_j(X_j,Y_j) \to s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{C})((X_j)g,(Y_j)g)$$

takes an element $\beta \in \otimes^{j\in J} T^{n_j} s\mathcal{B}_j(X_j,Y_j)$ to $(\beta)g_n$ with the components

$$[(\beta)g_n]_m = (1 \boxtimes \beta) f_{m,n}.$$

Thus, $(\beta)g_n \in s\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{C})((X_j)g,(Y_j)g)$. Therefore, g lifts to an A_∞-functor $g : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{C})$.

Let us prove that g is strictly unital. If $|n| > 1$ and $\mathbf{i}_0$ is a factor of β, then $[(\beta)g_n]_m = (1 \boxtimes \beta) f_{m,n}$ vanish for all m, that is, $(\beta)g_n = 0$. If $n = e_j$, and $\beta = \mathbf{i}_0^{\mathcal{B}_j}$, then all m-th components of $(\beta)g_n$ vanish except $[(\beta)g_{e_j}]_0 = (\mathbf{i}_0^{\mathcal{B}_j}) f_{0,e_j} = \mathbf{i}_0^{\mathcal{C}}$. Therefore, $(\beta)g_{e_j} = \mathbf{i}_0^{\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{C})}$. This defines a map $f \mapsto g$ inverse to $\varphi^{\mathsf{A}_\infty^{\mathrm{su}}}$. The proposition is proven. □

We have a multicategory embedding $\mathsf{A}_\infty^{\mathrm{su}} \hookrightarrow \mathsf{A}_\infty$ (on objects and on morphisms). It induces the closing transformation

$$\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \hookrightarrow \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}),$$

which is a natural embedding.

9.7 Multifunctor k. Now we are going to define the main subject of this chapter, unital A_∞-categories and unital A_∞-functors. In order to do this, we begin by constructing a (multi)functor k which forgets all components of the differential of an A_∞-category except the first and the second ones.

We denote by $\mathcal{K}$ the symmetric closed Monoidal category of differential graded $\Bbbk$-modules, whose morphisms are chain maps modulo homotopy. See discussion preceding Proposition 8.29. We shall consider non-unital categories and functors, enriched in $\mathcal{K}$. They form a category $\mathcal{K}\text{-}\mathcal{C}at^{nu}$. Unital $\mathcal{K}$-categories and $\mathcal{K}$-functors form a smaller category $\mathcal{K}\text{-}\mathcal{C}at$. Thus, $\mathcal{K}\text{-}\mathcal{C}at(\mathcal{A},\mathcal{B}) \subset \mathcal{K}\text{-}\mathcal{C}at^{nu}(\mathcal{A},\mathcal{B})$ for all unital $\mathcal{K}$-categories $\mathcal{A}$, $\mathcal{B}$. There is a functor $\mathsf{k} : A_\infty \to \mathcal{K}\text{-}\mathcal{C}at^{nu}$, constructed in [Lyu03, Proposition 8.6]. It assigns to an A_∞-category $\mathcal{C}$ the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ with the same set of objects $\operatorname{Ob}\mathsf{k}\mathcal{C} = \operatorname{Ob}\mathcal{C}$, the same graded $\Bbbk$-module of morphisms $\mathsf{k}\mathcal{C}(X,Y) = \mathcal{C}(X,Y)$, equipped with the differential $m_1 = sb_1s^{-1}$. Composition $\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{C}}$ in $\mathsf{k}\mathcal{C}$ is given by (the homotopy equivalence class of) $m_2 = (s\otimes s)b_2s^{-1} : \mathcal{C}(X,Y)\otimes\mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$. We are going to extend the functor k to a sort of multifunctor $\mathsf{k} : \mathsf{A}_\infty \to \widehat{\mathcal{K}\text{-}\mathcal{C}at^{nu}}$. The mapping $\operatorname{Ob}\mathsf{k}$ which assigns the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ to an A_∞-category $\mathcal{C}$ is described above.

Let $f : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$ be an A_∞-functor. Define a (non-unital) $\mathcal{K}$-functor $\mathsf{k}f : \boxtimes^{j\in J}\mathsf{k}\mathcal{A}_j \to \mathsf{k}\mathcal{B}$ on objects as $\operatorname{Ob}\mathsf{k}f = \operatorname{Ob}f : \prod_{j\in J}\operatorname{Ob}\mathcal{A}_j \to \operatorname{Ob}\mathcal{B}$. On morphisms we set

$$\begin{aligned}\mathsf{k}f = \bigl[&\otimes^{j\in J}\mathsf{k}\mathcal{A}_j(X_j,Y_j) \xrightarrow{\otimes^{j\in J}(sf_{e_j}s^{-1})} \\ &\otimes^{j\in J}\mathsf{k}\mathcal{B}\bigl(((Y_i)_{i<j},(X_i)_{i\geqslant j})f,((Y_i)_{i\leqslant j},(X_i)_{i>j})f\bigr) \xrightarrow{\mu^J_{\mathsf{k}\mathcal{B}}} \mathsf{k}\mathcal{B}\bigl((X_i)_{i\in J}f,(Y_i)_{i\in J}f\bigr)\bigr],\end{aligned} \tag{9.7.1}$$

where chain maps $sf_{e_j}s^{-1}$ are given by (8.18.3), and $\mu^J_{\mathsf{k}\mathcal{B}}$ is the composition of $|J|$ composable arrows in $\mathsf{k}\mathcal{B}$.

9.8 Proposition. $\mathsf{k}f$ *is a (non-unital) $\mathcal{K}$-functor.*

Proof. Let X_i, Y_i, Z_i be objects of $\mathcal{A}_i$, $i\in I$. We must prove the following equation in $\mathcal{K}$:

$$\begin{aligned}\bigl[&\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}} \\ &\otimes^{i\in I}\mathcal{B}\bigl(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \\ &\qquad\otimes\otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(Y_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(Y_j)_{j>i})f\bigr) \\ &\xrightarrow{\mu^I_{\mathsf{k}\mathcal{B}}\otimes\mu^I_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i)_{i\in I}f,(Y_i)_{i\in I}f)\otimes\mathcal{B}((Y_i)_{i\in I}f,(Z_i)_{i\in I}f) \\ &\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}((X_i)_{i\in I}f,(Z_i)_{i\in I}f)\bigr] \\ = \bigl[&\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \xrightarrow{\sigma_{(12)}} \otimes^{i\in I}(\mathcal{A}_i(X_i,Y_i)\otimes\mathcal{A}_i(Y_i,Z_i)) \\ &\xrightarrow{\otimes^{i\in I}\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{A}_i}} \otimes^{i\in I}\mathcal{A}_i(X_i,Z_i)\end{aligned}$$

$$\xrightarrow{\otimes^{i\in I} sf_{e_i}s^{-1}} \otimes^{i\in I} \mathcal{B}\bigl(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \\ \xrightarrow{\mu^I_{\Bbbk\mathcal{B}}} \mathcal{B}\bigl((X_i)_{i\in I}f,(Z_i)_{i\in I}f\bigr)\bigr], \quad (9.8.1)$$

In particular case $I = \mathbf{1}$ it takes the form

$$\bigl[\mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z) \xrightarrow{sf_1s^{-1}\otimes sf_1s^{-1}} \mathcal{B}(Xf,Yf)\otimes\mathcal{B}(Yf,Zf) \xrightarrow{\mu^{\mathbf{2}}_{\Bbbk\mathcal{B}}} \mathcal{B}(Xf,Zf)\bigr] \\ = \bigl[\mathcal{A}(X,Y)\otimes\mathcal{A}(Y,Z) \xrightarrow{\mu^{\mathbf{2}}_{\Bbbk\mathcal{A}}} \mathcal{A}(X,Z) \xrightarrow{sf_1s^{-1}} \mathcal{B}(Xf,Zf)\bigr]. \quad (9.8.2)$$

This equation in $\mathcal{K}$ is proven in [Lyu03, Proposition 8.6]. In fact, the difference of the right and the left hand sides is the boundary of $(s\otimes s)f_2s^{-1}$.

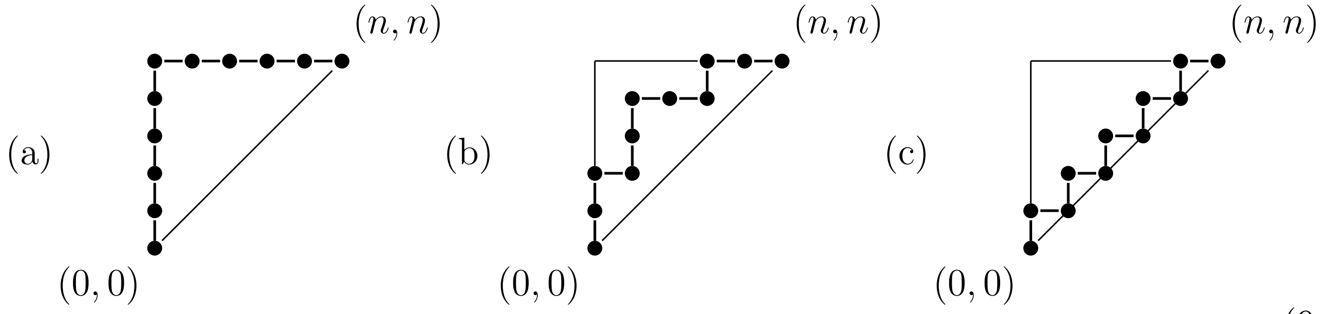
Since $f_{e_i} : \mathcal{A}_i \to \mathcal{B}$, $i \in I$, are A_∞-functors, they satisfy equation (9.8.2). Therefore, (9.8.1) is equivalent to the following equation in $\mathcal{K}$:

$$\bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}} \\ \otimes^{i\in I}\mathcal{B}\bigl(((Y_j)_{j<i},(X_j)_{j\geqslant i})f,((Y_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \\ \otimes\otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(Y_j)_{j\geqslant i})f,((Z_j)_{j\leqslant i},(Y_j)_{j>i})f\bigr) \\ \xrightarrow{\mu^I_{\Bbbk\mathcal{B}}\otimes\mu^I_{\Bbbk\mathcal{B}}} \mathcal{B}\bigl((X_i)_{i\in I}f,(Y_i)_{i\in I}f\bigr)\otimes\mathcal{B}\bigl((Y_i)_{i\in I}f,(Z_i)_{i\in I}f\bigr) \\ \xrightarrow{\mu^{\mathbf{2}}_{\Bbbk\mathcal{B}}} \mathcal{B}\bigl((X_i)_{i\in I}f,(Z_i)_{i\in I}f\bigr)\bigr] \\ = \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)\otimes\otimes^{i\in I}\mathcal{A}_i(Y_i,Z_i) \xrightarrow{\otimes^{i\in I}sf_{e_i}s^{-1}\otimes\otimes^{i\in I}sf_{e_i}s^{-1}} \\ \otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j<i},Y_i,(X_j)_{j>i})f\bigr) \\ \otimes\otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},Y_i,(X_j)_{j>i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f\bigr) \xrightarrow{\sigma_{(12)}} \\ \otimes^{i\in I}\bigl(\mathcal{B}\bigl(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j<i},Y_i,(X_j)_{j>i})f\bigr) \\ \otimes\mathcal{B}\bigl(((Z_j)_{j<i},Y_i,(X_j)_{j>i})f,((Z_j)_{j\leqslant i},(X_j)_{j>i})f\bigr)\bigr) \xrightarrow{\otimes^{i\in I}\mu^{\mathbf{2}}_{\Bbbk\mathcal{B}}} \\ \otimes^{i\in I}\mathcal{B}\bigl(((Z_j)_{j<i},(X_j)_{j\geqslant i})f,((Z_j)_{j\geqslant i},(X_j)_{j>i})f\bigr) \xrightarrow{\mu^I_{\Bbbk\mathcal{B}}} \mathcal{B}\bigl((X_i)_{i\in I}f,(Z_i)_{i\in I}f\bigr)\bigr], \quad (9.8.3)$$

which we are going to prove. We may assume that $I = \mathbf{n}$. Two parts of equation (9.8.3) are particular cases of the following construction.

Consider *staircases* defined as connected subsets S of the plane which are unions of $2n$ segments of the form $[(k-1,i-1),(k-1,i)]$ or $[(k-1,i),(k,i)]$ for integers $1 \leqslant k \leqslant i \leqslant n$.

We assume also that $(0,0) \in S$ and $(n,n) \in S$, see examples with $n = 5$ below.



(9.8.4)

Associate with a staircase S two non-decreasing functions $l, k : \mathbf{n} \to \mathbf{n}$. Namely, $l(p) = l_S(p)$ is the smallest l such that $(p,l) \in S$, and $k(i) = k_S(i)$ is the smallest k such that $(k-1,i) \in S$. Notice that $l_S(p) \geqslant p$ and $k_S(i) \leqslant i$. Moreover, any non-decreasing function $k : \mathbf{n} \to \mathbf{n}$ (resp. $l : \mathbf{n} \to \mathbf{n}$) such that $k(i) \leqslant i$ (resp. $l(p) \geqslant p$) determines a unique staircase S such that $k_S = k$ (resp. $l_S = l$). Thus, the sets of such functions k, staircases S, and such functions l are in bijection.

Let $(W_{p,l})_{0 \leqslant p \leqslant l \leqslant n}$ be objects of $\mathcal{B}$. The staircase S gives rise to a map

$$\otimes^{i\in\mathbf{n}} \mathcal{B}(W_{k(i)-1,i-1}, W_{k(i)-1,i}) \otimes \otimes^{p\in\mathbf{n}}\mathcal{B}(W_{p-1,l(p)}, W_{p,l(p)}) \\ \xrightarrow{\mathrm{sh}_S} \otimes^{\mathbf{2n}} \mathcal{B}(W_{\bullet\bullet}, W_{\bullet\bullet}) \xrightarrow{\mu^{\mathbf{2n}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{0,0}, W_{n,n}), \quad (9.8.5)$$

where the signed (n,n)-shuffle sh_S is associated with the staircase S. Namely, if the m-th segment of S, $1 \leqslant m \leqslant 2n$ (starting from the segment $[(0,0),(0,1)]$) is vertical (resp. horizontal), then the m-th factor of $\otimes^{\mathbf{2n}}\mathcal{B}(W_{\bullet\bullet}, W_{\bullet\bullet})$ comes from the first (resp. the last) n factors of the source. Thus the intermediate tensor product has the form $\otimes^{m\in\mathbf{2n}}\mathcal{B}(W_{v(m-1)}, W_{v(m)})$, where $(v(m))_{m=0}^{2n} \subset \mathbb{Z}^2$ is the list of all adjacent integer vertices belonging to S, $v(0) = (0,0)$, $v(2n) = (n,n)$. In particular, the composition mapping $\mu^{\mathbf{2n}}_{\mathsf{k}\mathcal{B}}$ makes sense.

Let us take $W_{p,l} = ((Z_j)_{j\leqslant p}, (Y_j)_{p<j\leqslant l}, (X_j)_{j>l})f$, $0 \leqslant p \leqslant l \leqslant n$. Given a staircase S, we extend mapping (9.8.5) to the following:

$$\varkappa(f,S) = \bigl[\otimes^{i\in\mathbf{n}}\mathcal{A}(X_i,Y_i) \otimes \otimes^{p\in\mathbf{n}}\mathcal{A}_p(Y_p,Z_p) \xrightarrow{\otimes^{i\in\mathbf{n}} sf_{e_i}s^{-1} \otimes \otimes^{p\in\mathbf{n}} sf_{e_p}s^{-1}} \\ \otimes^{i\in\mathbf{n}} \mathcal{B}\bigl(((Z_j)_{j<k(i)}, (Y_j)^{j<i}_{j\geqslant k(i)}, (X_j)_{j\geqslant i})f, ((Z_j)_{j<k(i)}, (Y_j)^{j\leqslant i}_{j\geqslant k(i)}, (X_i)_{j>i})f\bigr) \\ \otimes \otimes^{p\in\mathbf{n}}\mathcal{B}\bigl(((Z_j)_{j<p}, (Y_j)^{j\leqslant l(p)}_{j\geqslant p}, (X_j)_{j>l(p)})f, ((Z_j)_{j\leqslant p}, (Y_j)^{j\leqslant l(p)}_{j>p}, (X_j)_{j>l(p)})f\bigr) \\ = \otimes^{i\in\mathbf{n}}\mathcal{B}(W_{k(i)-1,i-1}, W_{k(i)-1,i}) \otimes \otimes^{p\in\mathbf{n}}\mathcal{B}(W_{p-1,l(p)}, W_{p,l(p)}) \\ \xrightarrow{\mathrm{sh}_S} \otimes^{\mathbf{2n}} \mathcal{B}(W_{v(m-1)}, W_{v(m)}) \xrightarrow{\mu^{\mathbf{2n}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{0,0}, W_{n,n}) = \mathcal{B}((X_j)_{j\in\mathbf{n}}f, (Z_j)_{j\in\mathbf{n}}f)\bigr].$$

The left hand side of (9.8.3) equals $\varkappa(f, S_a)$, where S_a from (9.8.4)(a) gives $\mathrm{sh}_{S_a} = \mathrm{id}$, $k(i) = 1$, $l(p) = n$. The right hand side of (9.8.3) equals $\varkappa(f, S_c)$, where S_c from (9.8.4)(c) gives $\mathrm{sh}_{S_c} = \varkappa$, $k(i) = i$, $l(p) = p$.

We claim that the composition $\varkappa(f,S)$ does not depend on the staircase S. Indeed, consider two staircases S, S' which coincide everywhere except in m-th and $(m+1)$-st segments, $0 < m < 2m-1$, as drawn:

$$\begin{tikzcd}[column sep=small] S \ni v(m) \arrow[r, "\bullet" description, phantom] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] & v(m+1) \\ v(m-1) & \bullet \arrow[r, dash] & \bullet & v'(m) \in S' \end{tikzcd}$$

Then the corresponding shuffles are related by the equation $\mathrm{sh}_{S'} = \mathrm{sh}_S \cdot (m, m+1)^\sim$. Let $i \in \mathbf{n}$ (resp. $p \in \mathbf{n}$) index the factors which come to m-th (resp $(m+1)$-st) place after application of sh_S. Expressions

$$\mathcal{B}(W_{k(i)-1,i-1}, W_{k(i)-1,i}) \otimes \mathcal{B}(W_{p-1,l(p)}, W_{p,l(p)})$$

and

$$\mathcal{B}(W_{v(m-1)}, W_{v(m)}) \otimes \mathcal{B}(W_{v(m)}, W_{v(m+1)})$$

are identical. This implies $p = k(i)$ and $i = l(p)$ and gives coordinates of the four points:

$$\begin{tikzcd} (p-1,i) = v(m) & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] & v(m+1) = (p,i) \\ (p-1,i-1) = v(m-1) & \bullet \arrow[r, dash] & \bullet & v'(m) = (p,i-1) \end{tikzcd}$$

In particular, $p \leqslant i-1$. The expression for $\varkappa(f,S')$ differs from that for $\varkappa(f,S)$ by an extra factor

$$\begin{aligned} (12)^\sim : \mathcal{B}(W_{v'(m)}, W_{v(m+1)}) \otimes \mathcal{B}(W_{v(m-1)}, W_{v'(m)}) \\ \to \mathcal{B}(W_{v(m-1)}, W_{v'(m)}) \otimes \mathcal{B}(W_{v'(m)}, W_{v(m+1)}). \end{aligned}$$

Thus, equation $\varkappa(f,S) = \varkappa(f,S')$ follows from the equation in $\mathcal{K}$

$$\begin{aligned} &\big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{sf_{e_i}s^{-1} \otimes sf_{e_p}s^{-1}} \\ &\qquad \mathcal{B}(W_{p-1,i-1}, W_{p-1,i}) \otimes \mathcal{B}(W_{p-1,i}, W_{p,i}) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{p-1,i-1}, W_{p,i})\big] \\ &= \big[\mathcal{A}_i(X_i,Y_i) \otimes \mathcal{A}_p(Y_p,Z_p) \xrightarrow{sf_{e_i}s^{-1} \otimes sf_{e_p}s^{-1}} \mathcal{B}(W_{p,i-1}, W_{p,i}) \otimes \mathcal{B}(W_{p-1,i-1}, W_{p,i-1}) \\ &\qquad \xrightarrow{(12)^\sim} \mathcal{B}(W_{p-1,i-1}, W_{p,i-1}) \otimes \mathcal{B}(W_{p,i-1}, W_{p,i}) \xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}} \mathcal{B}(W_{p-1,i-1}, W_{p,i})\big], \end{aligned}$$

which we are going to prove now.

Introduce the A_∞-functor of two variables

$$g = f|_{\{p,i\}}^{(Z_j)_{j<p},(Y_j)_{p<j<i},(X_j)_{j>i}} : \mathcal{A}_p, \mathcal{A}_i \to \mathcal{B}.$$

Recall that $p < i$. In terms of g the above equation in $\mathcal{K}$ can be rewritten as follows:

$$\begin{aligned}&\bigl[\mathcal{A}_i(X_i,Y_i)\otimes\mathcal{A}_p(Y_p,Z_p)\xrightarrow{sg_{01}s^{-1}\otimes sg_{10}s^{-1}}\\&\qquad\mathcal{B}((Y_p,X_i)g,(Y_p,Y_i)g)\otimes\mathcal{B}((Y_p,Y_i)g,(Z_p,Y_i)g)\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}}\mathcal{B}((Y_p,X_i)g,(Z_p,Y_i)g)\bigr]\\&=\bigl[\mathcal{A}_i(X_i,Y_i)\otimes\mathcal{A}_p(Y_p,Z_p)\xrightarrow{(12)^\sim}\mathcal{A}_p(Y_p,Z_p)\otimes\mathcal{A}_i(X_i,Y_i)\xrightarrow{sg_{10}s^{-1}\otimes sg_{01}s^{-1}}\\&\qquad\mathcal{B}((Y_p,X_i)g,(Z_p,X_i)g)\otimes\mathcal{B}((Z_p,X_i)g,(Z_p,Y_i)g)\xrightarrow{\mu^{\mathbf{2}}_{\mathsf{k}\mathcal{B}}}\mathcal{B}((Y_p,X_i)g,(Z_p,Y_i)g)\bigr].\end{aligned}\tag{9.8.6}$$

In order to prove it we recall that $\tilde{g}b = (1\boxtimes\tilde{b}+\tilde{b}\boxtimes 1)\overline{g}$ by (8.8.1). Restriction of this equation to $s\mathcal{A}_i\boxtimes s\mathcal{A}_p$ gives

$$\begin{aligned}&(g_{10}\otimes g_{01})b_2+(12)^\sim(g_{01}\otimes g_{10})b_2+g_{11}b_1=(1\otimes b_1+b_1\otimes 1)g_{11}:\\&\qquad\qquad s\mathcal{A}_i(X_i,Y_i)\otimes s\mathcal{A}_p(Y_p,Z_p)\to s\mathcal{B}((Y_p,X_i)g,(Z_p,Y_i)g).\end{aligned}$$

Thus, $(g_{10}\otimes g_{01})b_2+(12)^\sim(g_{01}\otimes g_{10})b_2$ is a boundary. Therefore,

$$(s\otimes s)(g_{01}\otimes g_{10})b_2s^{-1}=(12)^\sim(s\otimes s)(g_{10}\otimes g_{01})b_2s^{-1}$$

in $\mathcal{K}$. This implies equation (9.8.6).

Since any two staircases S' and S'' can be connected by a finite sequence of elementary modifications as above, we have $\varkappa(f,S')=\varkappa(f,S'')$. In particular, equation (9.8.3) holds and $\mathsf{k}f$ is a (non-unital) functor. ☐

9.9 Proposition. *The maps* $\operatorname{Ob}\mathsf{k}:\operatorname{Ob}\mathsf{A}_\infty\to\operatorname{Ob}\widehat{\mathcal{K}\text{-}\mathcal{C}at}^{nu}$, $\mathcal{C}\mapsto\mathsf{k}\mathcal{C}$, $\mathsf{k}:\mathsf{A}_\infty((\mathcal{A}_j)_{j\in J};\mathcal{B})\to\widehat{\mathcal{K}\text{-}\mathcal{C}at}^{nu}(\boxtimes^{j\in J}\mathsf{k}\mathcal{A}_j,\mathsf{k}\mathcal{B})$, $f\mapsto\mathsf{k}f$ *(given by (9.7.1) for non-empty J only!) are compatible with composition and identity functors, and define a kind of plain multifunctor* $\mathsf{k}:\mathsf{A}_\infty\to\widehat{\mathcal{K}\text{-}\mathcal{C}at}^{nu}$.

Proof. Let $\phi:I\to J$ be a isotonic surjection with non-empty J. We shall prove that k is compatible with the composition μ_ϕ. Let $f^j:(\mathcal{A}_i)_{i\in\phi^{-1}j}\to\mathcal{B}_j$, $g:(\mathcal{B}_j)_{j\in J}\to\mathcal{C}$ be A_∞-functors. They are taken by k to

$$\begin{aligned}\mathsf{k}f^j=\bigl[&\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i)\xrightarrow{\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})}\\&\otimes^{i\in\phi^{-1}j}\mathcal{B}_j\bigl(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\bigr)\\&\qquad\xrightarrow{\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{B}_j}}\mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr)\bigr],\\\mathsf{k}g=\bigl[&\otimes^{j\in J}\mathcal{B}_j(U_j,W_j)\xrightarrow{\otimes^{j\in J}(sg_{e_j^J}s^{-1})}\\&\otimes^{j\in J}\mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}}\mathcal{C}\bigl((U_j)_{j\in J}g,(W_j)_{j\in J}g\bigr)\bigr].\end{aligned}$$

Composition μ_ϕ of these morphisms in $\widehat{\mathcal{K}\text{-}\mathcal{C}at^{nu}}$ is

$$
\begin{aligned}
(\mathsf{k}f^j)_{j\in J}\cdot\mathsf{k}g = \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i) \\
&\xrightarrow{\otimes^{j\in J}\mathsf{k}f^j} \otimes^{j\in J}\mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr) \\
&\xrightarrow{\mathsf{k}g} \mathcal{C}\bigl(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\bigr)\bigr] \\
= \bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) &\xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})} \\
&\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{B}_j\bigl(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\bigr) \\
&\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{B}_j}} \otimes^{j\in J}\mathcal{B}_j\bigl((X_i)_{i\in\phi^{-1}j}f^j,(Y_i)_{i\in\phi^{-1}j}f^j\bigr) \\
&\xrightarrow{\otimes^{j\in J}(sg_{e^J_j}s^{-1})} \otimes^{j\in J}\mathcal{C}\bigl((((Y_i)_{i\in\phi^{-1}l}f^l)_{l<j},((X_i)_{i\in\phi^{-1}l}f^l)_{l\geqslant j})g, \\
&\qquad\qquad (((Y_i)_{i\in\phi^{-1}l}f^l)_{l\leqslant j},((X_i)_{i\in\phi^{-1}l}f^l)_{l>j})g\bigr) \\
&\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\bigl(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\bigr)\bigr].
\end{aligned}
\tag{9.9.1}
$$

Since $\mathsf{k}g\big|_j^{(W_l)_{l<j},(U_l)_{l>j}} : \mathcal{B}_j \to \mathcal{C}$, $V \mapsto ((W_l)_{l<j},V,(U_l)_{l>j})g$, given on morphisms by $sg_{e^J_j}s^{-1} : \mathcal{B}_j(U_j,W_j) \to \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)$ is a $\mathcal{K}$-functor, we have an equation in $\mathcal{K}$:

$$
\begin{aligned}
&\bigl[\otimes^{i\in P}\mathcal{B}_j(V^j_{i-1},V^j_i) \xrightarrow{\mu^P_{\mathcal{B}_j}} \mathcal{B}_j(U_j,W_j) \xrightarrow{sg_{e^J_j}s^{-1}} \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)\bigr] \\
&= \bigl[\otimes^{i\in P}\mathcal{B}_j(V^j_{i-1},V^j_i) \xrightarrow{\otimes^{i\in P}sg_{e^J_j}s^{-1}} \otimes^{i\in P}\mathcal{C}\bigl(((W_l)_{l<j},V^j_{i-1},(U_l)_{l>j})g,((W_l)_{l<j},V^j_i,(U_l)_{l>j})g\bigr) \\
&\qquad\xrightarrow{\mu^P_{\mathcal{C}}} \mathcal{C}\bigl(((W_l)_{l<j},(U_l)_{l\geqslant j})g,((W_l)_{l\leqslant j},(U_l)_{l>j})g\bigr)\bigr],
\end{aligned}
\tag{9.9.2}
$$

where $P = \mathbf{p}$, $U_j = V^j_0$, $W_j = V^j_p$. Applying this identity to $P = \phi^{-1}j$, we turn (9.9.1) into:

$$
\begin{aligned}
&\bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\lambda^\phi} \otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sf^j_{e_i^{\phi^{-1}j}}s^{-1})} \\
&\quad\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{B}_j\bigl(((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j\bigr) \xrightarrow{\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}(sg_{e^J_j}s^{-1})} \\
&\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{C}\bigl\{\bigl(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\bigr)g, \\
&\qquad\bigl(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\bigr)g\bigr\} \\
&\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{C}}} \otimes^{j\in J}\mathcal{C}\bigl((((Y_k)_{\phi k=l}f^l)_{l<j},((X_k)_{\phi k=l}f^l)_{l\geqslant j})g,(((Y_k)_{\phi k=l}f^l)_{l\leqslant j},((X_k)_{\phi k=l}f^l)_{l>j})g\bigr) \\
&\qquad\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}} \mathcal{C}\bigl(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\bigr)\bigr].
\end{aligned}
\tag{9.9.3}
$$

Denote by h the composition $(f^j)_{j\in J}\cdot g:(\mathcal{A}_i)_{i\in I}\to\mathcal{C}$ in A_∞. We have $f^j_{e_i^{\phi^{-1}j}}\cdot g_{e_j^J}=h_{e_i^I}$ if $\phi i=j$. Furthermore, $\lambda^\phi\cdot\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}h_{e_i^I}=\otimes^{i\in I}h_{e_i^I}\cdot\lambda^\phi$ by naturality of λ^ϕ. Thus, (9.9.3) equals

$$\begin{aligned}
\big[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)&\xrightarrow{\otimes^{i\in I}(sh_{e_i^I}s^{-1})}\otimes^{i\in I}\,\mathcal{C}\big(((Y_k)_{k<i},(X_k)_{k\geqslant i})h,((Y_k)_{k\leqslant i},(X_k)_{k>i})h\big)\\
&=\otimes^{i\in I}\mathcal{C}\big\{\big(((Y_k)_{\phi k=l}f^l)_{l<\phi i},((Y_k)^{\phi k=\phi i}_{k<i},(X_k)^{\phi k=\phi i}_{k\geqslant i})f^{\phi i},((X_k)_{\phi k=l}f^l)_{l>\phi i}\big)g,\\
&\qquad\big(((Y_k)_{\phi k=l}f^l)_{l<\phi i},((Y_k)^{\phi k=\phi i}_{k\leqslant i},(X_k)^{\phi k=\phi i}_{k>i})f^{\phi i},((X_k)_{\phi k=l}f^l)_{l>\phi i}\big)g\big\}\xrightarrow{\lambda^\phi}\\
&\otimes^{j\in J}\otimes^{i\in\phi^{-1}j}\mathcal{C}\big\{\big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k<i},(X_k)^{\phi k=j}_{k\geqslant i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g,\\
&\qquad\big(((Y_k)_{\phi k=l}f^l)_{l<j},((Y_k)^{\phi k=j}_{k\leqslant i},(X_k)^{\phi k=j}_{k>i})f^j,((X_k)_{\phi k=l}f^l)_{l>j}\big)g\big\}\xrightarrow{\otimes^{j\in J}\mu^{\phi^{-1}j}_{\mathsf{k}\mathcal{C}}}\\
&\otimes^{j\in J}\,\mathcal{C}\big(\big(((Y_k)_{\phi k=l}f^l)_{l<j},((X_k)_{\phi k=l}f^l)_{l\geqslant j}\big)g,\big(((Y_k)_{\phi k=l}f^l)_{l\leqslant j},((X_k)_{\phi k=l}f^l)_{l>j}\big)g\big)\\
&\xrightarrow{\mu^J_{\mathsf{k}\mathcal{C}}}\mathcal{C}\big(((X_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g,((Y_i)_{i\in\phi^{-1}j}f^j)_{j\in J}g\big)\big].
\end{aligned}\tag{9.9.4}$$

Any $\Bbbk$-linear category is an algebra in the monoidal category $\mathscr{Q}_p$. In particular, equation (2.25.1) holds for $\mathcal{C}$. Hence, we may replace the last three arrows in (9.9.4) with one and get

$$\begin{aligned}
\big[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i)&\xrightarrow{\otimes^{i\in I}(sh_{e_i^I}s^{-1})}\otimes^{i\in I}\,\mathcal{C}\big(((Y_k)_{k<i},(X_k)_{k\geqslant i})h,((Y_k)_{k\leqslant i},(X_k)_{k>i})h\big)\\
&\xrightarrow{\mu^I_{\mathsf{k}\mathcal{C}}}\mathcal{C}\big((X_i)_{i\in I}h,(Y_i)_{i\in I}h\big)\big]=\mathsf{k}h.
\end{aligned}$$

This proves compatibility of k with compositions. Compatibility with identity functors is obvious. □

When $n=1$, an A_∞-functor $f:\mathcal{A}\to\mathcal{B}$ is mapped to the $\mathcal{K}$-functor $\mathsf{k}f:\mathsf{k}\mathcal{A}\to\mathsf{k}\mathcal{B}$ with $\mathsf{k}f=sf_1s^{-1}:\mathcal{A}(X,Y)\to\mathcal{B}(Xf,Yf)$ as was defined in [Lyu03, Proposition 8.6].

9.10 Unital A_∞-categories and A_∞-functors. An A_∞-category $\mathcal{C}$ is called *unital* if $\mathsf{k}\mathcal{C}$ is unital, that is, for each object X of $\mathcal{C}$ there is a unit element $1_X:\Bbbk\to\mathcal{C}(X,X)\in\mathcal{K}$ such that equations $(\mathrm{id}\otimes 1_Y)m_2=1=(1_X\otimes\mathrm{id})m_2:\mathcal{C}(X,Y)\to\mathcal{C}(X,Y)$ hold in $\mathcal{K}$. In other terms, for each object X of $\mathcal{C}$ there is a *unit element* ${}_X\mathbf{i}_0^{\mathcal{C}}:\Bbbk\to(s\mathcal{C})^{-1}(X,X)$ – a cycle defined up to a boundary, such that the chain maps $(1\otimes{}_Y\mathbf{i}_0^{\mathcal{C}})b_2$, $-({}_X\mathbf{i}_0^{\mathcal{C}}\otimes 1)b_2:s\mathcal{C}(X,Y)\to s\mathcal{C}(X,Y)$ are homotopic to the identity map. This definition is equivalent to unitality in the sense of [Lyu03, Definition 7.3], see also [*ibid*, Lemma 7.4].

If $\mathcal{B}$ is a unital A_∞-category and $f\in\mathsf{A}_\infty(;\mathcal{B})$ (such f is identified with an object $X=()f$ of $\mathcal{B}$), then we define $\mathsf{k}f:\boxtimes^\varnothing()\to\mathsf{k}\mathcal{B}$, $()_{i\in\varnothing}\mapsto()f=X$, on morphisms via (9.7.1) for $n=0$. That is,

$$\mathsf{k}f=\big[\otimes^\varnothing()=\Bbbk\xrightarrow[1_X]{\mu^{\mathbf{0}}_{\mathsf{k}\mathcal{B}}}\mathsf{k}\mathcal{B}(X,X)\big],\qquad 1\mapsto 1_X.\tag{9.10.1}$$

9.11 Definition. Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be unital A_∞-categories. An A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is called *unital* if the $\mathcal{K}$-functor $\mathsf{k}f : \boxtimes^{i\in I}\mathsf{k}\mathcal{A}_i \to \mathsf{k}\mathcal{B}$ is unital. The set of unital A_∞-functors is denoted $\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \subset \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$.

Due to (9.10.1) for $I = \varnothing$ any A_∞-functor $f : () \to \mathcal{B}$ is unital (if $\mathcal{B}$ is unital). For one-element I an A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ between unital A_∞-categories is unital if and only if ${}_X\mathbf{i}^{\mathcal{A}}_0 f_1 - {}_{Xf}\mathbf{i}^{\mathcal{B}}_0 \in \operatorname{Im} b_1$ for all objects X of $\mathcal{A}$. This criterion coincides with [Lyu03, Definition 8.1]. Since $\widehat{\mathcal{K}\text{-}\mathcal{C}at}$ is a submulticategory of $\widehat{\mathcal{K}\text{-}\mathcal{C}at^{nu}}$, the subsets $\mathsf{A}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \subset \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$ form a submulticategory $\mathsf{A}^{\mathrm{u}}_\infty \subset \mathsf{A}_\infty$ with unital A_∞-categories as objects, and unital A_∞-functors as multimorphisms.

In unital case the statement of Proposition 9.9 extends to empty set J of arguments, and to compositions μ_ϕ for arbitrary isotonic maps $\phi : I \to J$. Proof repeats the proof of Proposition 9.9 word by word.

Summary. There is a plain multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$.

Consider a change of symmetric Monoidal base, the lax symmetric Monoidal functor $H^0 : \mathcal{K} \to \Bbbk\text{-mod}$, $C \mapsto \mathcal{K}(\Bbbk, C) = \operatorname{Ker}(d : C^0 \to C^1)/\operatorname{Im}(d : C^{-1} \to C^0) = H^0(C)$, cf. (2.7.2). Here $\Bbbk = \mathbb{1}_{\mathcal{K}}$ is the graded $\Bbbk$-module $\Bbbk$ concentrated in degree 0. According to [Man07] H^0 provides a lax symmetric Monoidal $\mathcal{C}at$-functor $H^0_* : \mathcal{K}\text{-}\mathcal{C}at \to \Bbbk\text{-}\mathcal{C}at$, which we have denoted also $\mathcal{B} \mapsto \overline{\mathcal{B}}$ in Section 2.7. Thus, $\operatorname{Ob}\overline{\mathcal{B}} = \operatorname{Ob}\mathcal{B}$ and $\overline{\mathcal{B}}(X,Y) = H^0(\mathcal{B}(X,Y))$. Hence, there is a symmetric multifunctor $\widehat{H^0_*} : \widehat{\mathcal{K}\text{-}\mathcal{C}at} \to \widehat{\Bbbk\text{-}\mathcal{C}at}$. Composing it with $\Bbbk : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$ we get a plain multifunctor, denoted

$$H^0 = \widehat{H^0_*} \circ \mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\Bbbk\text{-}\mathcal{C}at}$$

by abuse of notation. It assigns to a unital A_∞-category $\mathcal{C}$ the $\Bbbk$-linear category $H^0(\mathcal{C}) = \overline{\mathsf{k}\mathcal{C}}$ with the same set of objects $\operatorname{Ob} H^0(\mathcal{C}) = \operatorname{Ob}\mathcal{C}$, the $\Bbbk$-module of morphisms $H^0(\mathcal{C})(X,Y) = H^0\big(\mathcal{C}(X,Y), m_1\big)$, with the composition induced by m_2.

9.12 Definition (Fukaya)**.** Objects X, Y of a unital A_∞-category $\mathcal{C}$ are called *isomorphic* if they are isomorphic as objects of the ordinary category $H^0(\mathcal{C})$. In detail, X and Y are isomorphic if there are elements $q \in s\mathcal{C}(X,Y)$, $t \in s\mathcal{C}(Y,X)$ of degree -1 (isomorphisms) such that $qb_1 = 0$, $tb_1 = 0$, $(q\otimes t)b_2 - {}_X\mathbf{i}^{\mathcal{C}}_0 \in \operatorname{Im} b_1$, $(t\otimes q)b_2 - {}_Y\mathbf{i}^{\mathcal{C}}_0 \in \operatorname{Im} b_1$.

Assume that $\mathcal{C}$ is an A_∞-category and $\Bbbk$ is a field. Then unitality of graded $\Bbbk$-linear category $H(\mathcal{C})$ (*cohomological unitality* [Kel01, LH03, Sei08]) is equivalent to unitality of A_∞-category $\mathcal{C}$ itself. Indeed, any chain complex of $\Bbbk$-vector spaces is homotopy isomorphic to its homology, the graded $\Bbbk$-vector space equipped with zero differential. Therefore, any two chain maps inducing the same map in homology are homotopic. Certainly, this does not hold for arbitrary complexes of modules over an arbitrary commutative ring $\Bbbk$.

9.13 Proposition. *Let $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ be unital A_∞-categories, and let $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be an A_∞-functor. It is unital if and only if A_∞-functors $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j \to \mathcal{B}$ are unital for all $j \in I$.*

Proof. If $I = \varnothing$, the statement holds true. Let $j \in I$, and let $X_i \in \operatorname{Ob}\mathcal{A}_i$ for $i \in I$, $i \neq j$. The composition map

$$\mu^{\mathsf{A}_\infty}_{j\hookrightarrow I} : \mathsf{A}_\infty(\mathcal{A}_j;\mathcal{A}_j) \times \prod_{i\neq j} \mathsf{A}_\infty(\,;\mathcal{A}_i) \times \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathsf{A}_\infty(\mathcal{A}_j;\mathcal{B})$$

takes $((X_i)_{i\neq j}, \mathrm{id}_{\mathcal{A}_j}, f)$ to $f|_j^{(X_i)_{i\neq j}}$. We have seen that A_∞-functors $X_i : () \to \mathcal{A}_i$ and $\mathrm{id} : \mathcal{A}_j \to \mathcal{A}_j$ are unital. If f is unital, then $f|_j^{(X_i)_{i\neq j}} = ((X_i)_{i\neq j}, \mathrm{id}_{\mathcal{A}_j}, f)\mu^{\mathsf{A}_\infty}_{j\hookrightarrow I}$ is unital due to $\mathsf{A}^{\mathrm{u}}_\infty$ being a submulticategory.

Assume that A_∞-functors $f|_j^{(X_i)_{i\neq j}} : \mathcal{A}_j \to \mathcal{B}$ are unital for all $j \in \mathbf{n}$. Then for any family $(X_i)_{i\in\mathbf{n}}$, $X_i \in \operatorname{Ob}\mathcal{A}_i$, we have

$$\begin{aligned} 1_{(X_j)_{j\in\mathbf{n}}}.\mathsf{k}f &= (\otimes^{j\in\mathbf{n}} 1_{X_j} s f_{e_j} s^{-1})\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}} = [\otimes^{j\in\mathbf{n}} 1_{X_j} s (f|_j^{(X_i)_{i\neq j}})_1 s^{-1}]\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}} \\ &= [\otimes^{j\in\mathbf{n}} \langle 1_{X_j}.(\mathsf{k}f|_j^{(X_i)_{i\neq j}})\rangle]\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}} = (\otimes^{j\in\mathbf{n}} 1_{(X_1,\dots,X_n)f})\mu^{\mathbf{n}}_{\mathsf{k}\mathcal{B}} = 1_{(X_1,\dots,X_n)\mathsf{k}f}. \end{aligned}$$

Thus $\mathsf{k}f$ is unital, hence, f is unital. □

9.14 Corollary. *If $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ is a unital A_∞-functor, and $J \subset I$, then $f|_J^{(X_i)_{i\in I\smallsetminus J}} : (\mathcal{A}_j)_{j\in J} \to \mathcal{B}$ is unital for all families of objects $(X_i)_{i\in I\smallsetminus J} \in \prod_{i\in I\smallsetminus J}\operatorname{Ob}\mathcal{A}_i$.*

Indeed, restrictions of $f|_J^{(X_i)_{i\in I\smallsetminus J}}$ to j-th argument coincide with those of f.

9.15 Closedness of multicategory of unital A_∞-categories.

Let $\mathcal{B}$ be a unital A_∞-category. Then $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is unital as well. Indeed, for $I = \varnothing$ this follows from the isomorphism $\underline{\mathsf{A}_\infty}(\,;\mathcal{B}) = \underline{\mathscr{Q}_p}(\Bbbk, s\mathcal{B})[-1] \simeq \mathcal{B}$, due to closedness of A_∞, or to (7.11.4). For $I = \mathbf{1}$ the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) = A_\infty(\mathcal{A};\mathcal{B})$ is unital due to [Lyu03, Proposition 7.7]. For $I = \mathbf{n}$, $n > 1$, we have an isomorphism of Proposition 4.12

$$\underline{\varphi} : \underline{\mathsf{A}_\infty}\big((\mathcal{A}_i)_{i=2}^n; \underline{\mathsf{A}_\infty}(\mathcal{A}_1;\mathcal{B})\big) \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i=1}^n;\mathcal{B}). \tag{9.15.1}$$

By induction assumption we may assume that the source is unital. By [Lyu03, Theorem 8.8] the target is unital and $\underline{\varphi}$ is a unital A_∞-equivalence. The claim follows by induction on n. For what follows, we need to flesh out the unit elements.

9.16 Proposition. *Let $\mathcal{A}_i$, $\mathcal{B}$, $i \in I$, be A_∞-categories. Suppose that $\mathcal{B}$ is unital. Then the A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is unital with unit elements*

$${}_f\mathbf{i}_0^{\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})} = f\mathbf{i}^{\mathcal{B}} \in s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,f),$$

for each A_∞-functor $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$.

The proof of the proposition repeats the proof of [Lyu03, Proposition 7.7] mutatis mutandis and hence is omitted here.

According to general definition, A_∞-functors $f, g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ with values in a unital A_∞-category $\mathcal{B}$ are isomorphic if they are isomorphic as objects of the unital A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$.

9.17 Lemma. *Let* $r : f \to g : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ *be a* natural A_∞-transformation, *that is,* $\deg r = -1$ *and* $rB_1 = 0$, *and let* $\mathcal{B}$ *be unital. Assume that for all families* $(X_i)_{i\in I}$ *of objects* $X_i \in \operatorname{Ob} \mathcal{A}_i$ *there are elements* ${}_{(X_i)_{i\in I}}p_0 : \Bbbk \to (s\mathcal{B})^{-1}\big(((X_i)_{i\in I})g, ((X_i)_{i\in I})f\big)$ *such that* ${}_{(X_i)_{i\in I}}p_0 b_1 = 0$ *and*

$$\big({}_{(X_i)_{i\in I}}r_0 \otimes {}_{(X_i)_{i\in I}}p_0\big)b_2 - {}_{((X_i)_{i\in I})f}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1,$$
$$\big({}_{(X_i)_{i\in I}}p_0 \otimes {}_{(X_i)_{i\in I}}r_0\big)b_2 - {}_{((X_i)_{i\in I})g}\mathbf{i}_0^{\mathcal{B}} \in \operatorname{Im} b_1.$$

Then given ${}_{(X_i)_{i\in I}}p_0$ *extend to a natural A_∞-transformation* $p : g \to f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ *inverse to* r, *that is,*

$$(r \otimes p)B_2 - f\mathbf{i}^{\mathcal{B}} \in \operatorname{Im} B_1, \qquad (p \otimes r)B_2 - g\mathbf{i}^{\mathcal{B}} \in \operatorname{Im} B_1.$$

Proof. If $I = \varnothing$, then $p = p_0$ is inverse to $r = r_0$. If $|I| = 1$, the statement is proven in [Lyu03, Proposition 7.15]. For $I = \mathbf{n}$, $n > 1$, isomorphism $\underline{\varphi}_1$ from (9.15.1) takes some coderivation $\hat{t} : \boxtimes_{i=2}^n Ts\mathcal{A}_i \to Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_1; \mathcal{B})$ to the given transformation $r = \big[\boxtimes^{i\in\mathbf{n}} Ts\mathcal{A}_i \xrightarrow{1\boxtimes\hat{t}} Ts\mathcal{A}_1 \boxtimes Ts\underline{\mathsf{A}_\infty}(\mathcal{A}_1; \mathcal{B}) \xrightarrow{\mathrm{ev}} s\mathcal{B}\big]$. Restricting $\hat{t}$ to $1 \in \boxtimes_{i=2}^n T^0 s\mathcal{A}_i$ we get $q \overset{\text{def}}{=} t_0 \in T^1 s\underline{\mathsf{A}_\infty}(\mathcal{A}_1; \mathcal{B})$. Restricting r to $1 \in \boxtimes^{i\in\mathbf{n}} T^0 s\mathcal{A}_i$ we get $r_0 = q_0$ by (8.25.4). Therefore, invertibility of $q_0 = r_0$ implies invertibility of $t_0 = q$ with some inverse w, $w_0 = p_0$. By induction on $n > 0$ it implies invertibility of t with an inverse z such that $z_0 = w$.

Since the A_∞-functor $\underline{\varphi}$ is unital, its first component $\underline{\varphi}_1$ takes a pair of inverse to each other cycles t, z to a pair of inverse to each other cycles $t\underline{\varphi}_1 = r$, $p \overset{\text{def}}{=} z\underline{\varphi}_1$. The 0-th component $p_0 = (z_0)_0 = w_0$ of p coincides with the given collection p_0. □

Let us show that multicategory $\mathsf{A}_\infty^{\mathrm{u}}$ is closed. If $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ are unital A_∞-categories, we set $\underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \subset \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$ to be the full A_∞-subcategory, whose objects are unital A_∞-functors $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$. Since $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$ is unital, so is $\underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$. The evaluation A_∞-functor $\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{u}}} : (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \to \mathcal{B}$ is taken to be the restriction of $\mathrm{ev}^{\mathsf{A}_\infty} : (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \to \mathcal{B}$. We have to show that the A_∞-functor $\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{u}}}$ is unital.

If $I = \varnothing$, then $\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{u}}} : \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\,;\mathcal{B}) \to \mathcal{B}$ is the natural isomorphism, hence, it is unital by [Lyu03, Corollary 8.9]. If $j \in I = \mathbf{n}$, $X_i \in \operatorname{Ob} \mathcal{A}_i$ for $i \neq j$, and $g \in \mathsf{A}_\infty^{\mathrm{u}}((\mathcal{A}_i)_{i\in\mathbf{n}}; \mathcal{B})$, then due to (8.25.4)

$$\mathrm{ev}\big|_j^{(X_i)_{i\neq j}, g} = g\big|_j^{(X_i)_{i\neq j}} : Ts\mathcal{A}_j \simeq \big(\boxtimes^{i\in\mathbf{n}\smallsetminus\{j\}} T^0 s\mathcal{A}_i\big) \boxtimes Ts\mathcal{A}_j \boxtimes T^0 s\underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in\mathbf{n}}; \mathcal{B}) \to s\mathcal{B}$$

is unital, since $g|_j^{(X_i)_{i\neq j}}$ is unital by Proposition 9.13. If $X_i \in \operatorname{Ob}\mathcal{A}_i$ for $i \in \mathbf{n}$, the first component $\bigl(\operatorname{ev}|_{n+1}^{(X_i)_{i\in\mathbf{n}}}\bigr)_1 : T^1 s\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to s\mathcal{B}$ of A_∞-functor $\operatorname{ev}|_{n+1}^{(X_i)_{i\in\mathbf{n}}}$ takes, due to (8.25.4), an A_∞-transformation $r : g \to h : (\mathcal{A}_i)_{i\in\mathbf{n}} \to \mathcal{B}$ to its 0-th component $r_{0\dots0} \in s\mathcal{B}((X_1,\dots,X_n)g,(X_1,\dots,X_n)h)$. In particular, the unit element $g\mathbf{i}^{\mathcal{B}} : g \to g$ of g goes to the unit element ${}_{(X_1,\dots,X_n)g}\mathbf{i}_0^{\mathcal{B}} \in s\mathcal{B}((X_1,\dots,X_n)g,(X_1,\dots,X_n)g)$. By Proposition 9.13 we conclude that $\operatorname{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$ is unital.

9.18 Proposition. *So defined $\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$, $\operatorname{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}$ turn $\mathsf{A}^{\mathrm{u}}_\infty$ into a closed multicategory.*

Proof. Let $(\mathcal{A}_i)_{i\in I}$, $(\mathcal{B}_j)_{j\in J}$, $\mathcal{C}$ be unital A_∞-categories. Denote by $e : \underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C}) \hookrightarrow \underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})$ the full embedding. Mappings (4.7.1) for $\mathsf{A}^{\mathrm{u}}_\infty$ and A_∞ are related by embeddings:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty\bigl((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigr) \arrow[r, "\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}"] \arrow[d, hook] & \mathsf{A}^{\mathrm{u}}_\infty\bigl((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\bigr) \arrow[dd, hook] \\
\mathsf{A}_\infty\bigl((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigr) \arrow[d, hook, "\mathsf{A}_\infty((\mathcal{B}_j)_{j\in J};e)"'] & \\
\mathsf{A}_\infty\bigl((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigr) \arrow[r, "\varphi^{\mathsf{A}_\infty}", "\sim"'] & \mathsf{A}_\infty\bigl((\mathcal{A}_i)_{i\in I},(\mathcal{B}_j)_{j\in J};\mathcal{C}\bigr)
\end{tikzcd}$$

Therefore, $\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}$ is injective. Let us prove its surjectivity.

Let $g : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})$ be an A_∞-functor such that $f = \varphi^{\mathsf{A}_\infty}(g) : (\mathcal{A}_i)_{i\in I}, (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ is unital. Let $(Y_j)_{j\in J}$ be a family of objects $Y_j \in \operatorname{Ob}\mathcal{B}_j$. Then A_∞-functor $\bigl((Y_j)_{j\in J}\bigr)g = f|_I^{(Y_j)_{j\in J}} : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ is unital by Corollary 9.14. Therefore, $g = he$ for an A_∞-functor $h : (\mathcal{B}_j)_{j\in J} \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})$.

Let us prove that h is unital. This is obvious if $J = \varnothing$. Assume that $J \neq \varnothing$ and consider an arbitrary $k \in J$, a family $(X_i)_{i\in I} \in \prod_{i\in I}\operatorname{Ob}\mathcal{A}_i$ and a family $(Y_j)_{j\neq k} \in \prod_{j\neq k}\operatorname{Ob}\mathcal{B}_j$. The restriction of

$$\overline{f} = \bigl[(\boxtimes^{i\in I}Ts\mathcal{A}_i)\boxtimes(\boxtimes^{j\in J}Ts\mathcal{B}_j) \xrightarrow{(\boxtimes^I \mathrm{id})\boxtimes\tilde{h}} (\boxtimes^{i\in I}Ts\mathcal{A}_i)\boxtimes Ts\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C}) \xrightarrow{\operatorname{ev}} s\mathcal{C}\bigr]$$

to the k-th argument

$$\overline{f}|_k^{(X_i)_{i\in I},(Y_j)_{j\neq k}} \\ = \bigl[(\boxtimes^{i\in I}T^0s\mathcal{A}_i)\boxtimes Ts\mathcal{B}_k \xrightarrow{(\boxtimes^I \mathrm{id})\boxtimes(h|_k^{(Y_j)_{j\neq k}})^\sim} (\boxtimes^{i\in I}T^0s\mathcal{A}_i)\boxtimes Ts\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C}) \xrightarrow{\operatorname{ev}} s\mathcal{C}\bigr]$$

is unital by Proposition 9.13. It depends only on restriction $h|_k^{(Y_j)_{j\neq k}} : \mathcal{B}_k \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})$. Its first component $\bigl(h|_k^{(Y_j)_{j\neq k}}\bigr)_1 : s\mathcal{B}_k(Y_k,Y_k) \to s\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigl(((Y_j)_{j\in J})h,((Y_j)_{j\in J})h\bigr)$

takes the unit element ${}_{Y_k}\mathbf{i}_0^{\mathcal{B}_k}$ to some element r^k. Applying k we find that r^k is a cycle, idempotent modulo boundary:

$$(r^k)B_1 = 0, \qquad (r^k \otimes r^k)B_2 - r^k \in \operatorname{Im} B_1.$$

The first component

$$\bigl(f|_k^{(X_i)_{i\in I},(Y_j)_{j\neq k}}\bigr)_1 = \bigl[s\mathcal{B}_k(Y_k,Y_k) \xrightarrow{(h|_k^{(Y_j)_{j\neq k}})_1} s\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigl(((Y_j)_{j\in J})h,((Y_j)_{j\in J})h\bigr)$$
$$\xrightarrow{(\operatorname{ev}|_{|I|+1}^{(X_i)_{i\in I}})_1} s\mathcal{C}\bigl(((X_i)_{i\in I})((Y_j)_{j\in J})h,((X_i)_{i\in I})((Y_j)_{j\in J})h\bigr)\bigr]$$

takes ${}_{Y_k}\mathbf{i}_0^{\mathcal{B}_k}$ to ${}_{(X_i)_{i\in I}}r^k_{0\dots0} \in s\mathcal{C}\bigl(((X_i)_{i\in I},(Y_j)_{j\in J})f,((X_i)_{i\in I},(Y_j)_{j\in J})f\bigr)$ due to (8.25.4). Unitality of $f|_k$ implies that

$${}_{(X_i)_{i\in I}}r^k_{0\dots0} - {}_{((X_i)_{i\in I},(Y_j)_{j\in J})f}\mathbf{i}_0^{\mathcal{C}} \in \operatorname{Im} b_1.$$

By Lemma 9.17 this implies invertibility of r^k. Being also idempotent, r^k is equal to the unit transformation $((Y_j)_{j\in J})h\mathbf{i}^{\mathcal{C}}$ of the A_∞-functor $((Y_j)_{j\in J})h : (\mathcal{A}_i)_{i\in I} \to \mathcal{C}$ modulo boundary. Thus, $h|_k^{(Y_j)_{j\neq k}}$ is unital. By Proposition 9.13 $h \in \mathsf{A}^{\mathrm{u}}_\infty\bigl((\mathcal{B}_j)_{j\in J};\underline{\mathsf{A}}^{\mathrm{u}}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{C})\bigr)$, hence, $\varphi^{\mathsf{A}^{\mathrm{u}}_\infty}$ is surjective, and, moreover, bijective. Therefore, the multicategory $\mathsf{A}^{\mathrm{u}}_\infty$ is closed. □

9.19 Remark. If $\mathcal{A}$ is a unital A_∞-category, then the Gerstenhaber algebra $C = C(\mathcal{A}) = \underline{\mathsf{A}}_\infty(\mathcal{A};\mathcal{A})(\mathrm{id},\mathrm{id})$ in the $\mathcal{K}$-category $\underline{\mathcal{K}}$ from Proposition 8.29 has a homotopy unit $\mathbf{i}^{\mathcal{A}}s^{-1}$. Also the Gerstenhaber algebra $H^\bullet(C(\mathcal{A}))$ from Corollary 8.30 is unital in this case.

9.20 A^u_∞-2-functors and A^u_∞-2-transformations. Here we relate the 2-category approach to A_∞-categories of [Lyu03, LM06a, LM08c] with the closed multicategory approach.

From the point of view of [Lyu03] unital A_∞-categories form a kind of category, enriched in itself. For practical use this recursive picture is truncated at the second step. So from this point of view A^u_∞ is a kind of category, whose objects are unital A_∞-categories. Instead of sets of homomorphisms it has unital A_∞-categories $A^u_\infty(\mathcal{A},\mathcal{B}) = \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\mathcal{B})$. Composition in A^u_∞ is given by the unital A_∞-functor

$$\mu^{\mathsf{A}^{\mathrm{u}}_\infty}_{\mathbf{1}\to\mathbf{1}} : \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{B};\mathcal{C}) \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{A};\mathcal{C}),$$

or, equivalently, by the differential augmented coalgebra morphism

$$M : TsA^u_\infty(\mathcal{A},\mathcal{B}) \boxtimes TsA^u_\infty(\mathcal{B},\mathcal{C}) \to TsA^u_\infty(\mathcal{A},\mathcal{C}),$$

see [Lyu03, Sections 4, 6]. Associativity of $\mu^{\mathsf{A}^{\mathrm{u}}_\infty}_{\mathbf{1}\to\mathbf{1}}$ presented by equation (4.2.2) for the maps $\mathbf{1} \to \mathbf{1} \to \mathbf{1}$ implies associativity of the A_∞-functor M in the sense of [Lyu03,

Proposition 4.1]. The units are identity A_∞-functors $\mathrm{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A}$, corresponding to the quiver morphism $\mathrm{pr}_1 : T^{\geqslant 1}s\mathcal{A} \to s\mathcal{A}$. Thus, the A^u_∞-2-category structure of A^u_∞ coincides with the $\mathsf{A}^\mathrm{u}_\infty$-category structure of $\underline{\mathsf{A}^\mathrm{u}_\infty}$.

Functors $F : A^u_\infty \to A^u_\infty$ compatible with these structures are called strict A^u_∞-2-functors in [LM06a]. They are precisely $\mathsf{A}^\mathrm{u}_\infty$-functors $\underline{\mathsf{A}^\mathrm{u}_\infty} \to \underline{\mathsf{A}^\mathrm{u}_\infty}$ in the sense of Definition 4.3.

Some A^u_∞-2-functors come from augmented multifunctors as shown in Proposition 4.30. Let $F : \mathsf{A}^\mathrm{u}_\infty \to \mathsf{A}^\mathrm{u}_\infty$ be a multifunctor together with a multinatural transformation $u_F : \mathrm{Id} \to F$. This gives an element as in Section 4.29

$$F' = \bigl[\underline{\mathsf{A}^\mathrm{u}_\infty}(\mathcal{A};\mathcal{B}) \xrightarrow{u_F} F\underline{\mathsf{A}^\mathrm{u}_\infty}(\mathcal{A};\mathcal{B}) \xrightarrow{F} \underline{\mathsf{A}^\mathrm{u}_\infty}(F\mathcal{A};F\mathcal{B})\bigr],$$

that is, a unital A_∞-functor $F' : A^u_\infty(\mathcal{A},\mathcal{B}) \to A^u_\infty(F\mathcal{A};F\mathcal{B})$ which satisfies equation

$$\begin{tikzcd}
A^u_\infty(\mathcal{A},\mathcal{B}), A^u_\infty(\mathcal{B},\mathcal{C}) \arrow[r, "{F',F'}"] \arrow[d, "\mu_{\mathbf{1}\to\mathbf{1}}"'] & A^u_\infty(F\mathcal{A},F\mathcal{B}), A^u_\infty(F\mathcal{B},F\mathcal{C}) \arrow[d, "\mu_{\mathbf{1}\to\mathbf{1}}"] \\
A^u_\infty(\mathcal{A},\mathcal{C}) \arrow[r, "F'"] & A^u_\infty(F\mathcal{A},F\mathcal{C}),
\end{tikzcd}$$

due to diagram (4.30.1). This is nothing else but equation (3.1.1) of [LM06a], which is one of the conditions of F' being an A^u_∞-2-functor. Also $F' : \underline{\mathsf{A}^\mathrm{u}_\infty}(\mathcal{A};\mathcal{B}) \to \underline{\mathsf{A}^\mathrm{u}_\infty}(F\mathcal{A};F\mathcal{B})$ maps the unit $\mathrm{id}_\mathcal{A}$ to the unit $\mathrm{id}_{F\mathcal{A}}$ due to Proposition 4.30. Hence, a multifunctor $F : \mathsf{A}^\mathrm{u}_\infty \to \mathsf{A}^\mathrm{u}_\infty$ together with a multinatural transformation $u_F : \mathrm{Id} \to F$ produces an A^u_∞-2-functor $F' : A^u_\infty \to A^u_\infty$.

Thus A^u_∞-2-notions are the same as $\mathsf{A}^\mathrm{u}_\infty$-notions. This holds not only for categories and functors, but also for strict A^u_∞-2-transformations $\lambda : G \to F : A^u_\infty \to A^u_\infty$ defined in [LM06a, Definition 3.2]. They are collections of unital A_∞-functors $\lambda_\mathcal{A} : G\mathcal{A} \to F\mathcal{A}$, $\mathcal{A} \in \operatorname{Ob} A^u_\infty$, such that the following equation holds:

$$\begin{tikzcd}
A^u_\infty(\mathcal{C},\mathcal{D}) \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & A^u_\infty(G\mathcal{C},G\mathcal{D}) \arrow[d, "(1\boxtimes\lambda_\mathcal{D})M \,=\, \underline{\mathsf{A}^\mathrm{u}_\infty}(1;\lambda_\mathcal{D})"] \\
A^u_\infty(F\mathcal{C},F\mathcal{D}) \arrow[r, "(\lambda_\mathcal{C}\boxtimes 1)M", "\underline{\mathsf{A}^\mathrm{u}_\infty}(\lambda_\mathcal{C};1)"'] & A^u_\infty(G\mathcal{C},F\mathcal{D})
\end{tikzcd} \tag{9.20.1}$$

Here different expressions give the same arrows due to Lemmata 4.16 and 4.17 as shown in Examples 8.27.3–4. Thus, an A^u_∞-2-transformation is precisely a natural transformation of $\mathsf{A}^\mathrm{u}_\infty$-functors in the sense of Definition 4.4.

Let $\lambda : G \to F : \mathsf{A}^\mathrm{u}_\infty \to \mathsf{A}^\mathrm{u}_\infty$ be a natural transformation of multifunctors (not necessarily a multinatural transformation!). This is a collection of elements $\lambda_\mathcal{A} \in \mathsf{A}^\mathrm{u}_\infty(G\mathcal{A};F\mathcal{A})$,

or simply of unital A_∞-functors $\lambda_{\mathcal{A}} : G\mathcal{A} \to F\mathcal{A}$. Under condition (4.35.1) in the form

$$\begin{tikzcd} F\mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "1{,}u_F"] \arrow[d, "\lambda_{\mathcal{C}},1"'] & F\mathcal{C}, F\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "F\,\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] \arrow[d, phantom, "="] & F\mathcal{D} \arrow[d, "\lambda_{\mathcal{D}}"] \\ G\mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "1{,}u_G"] & G\mathcal{C}, G\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "G\,\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] & G\mathcal{D} \end{tikzcd}$$

this collection is also a strict A^u_∞-2-transformation $\lambda : G' \to F' : A^u_\infty \to A^u_\infty$. Proposition 10.20 gives an example in which the above condition holds true.

As follows from Proposition 4.5 $\mathsf{A}^{\mathrm{u}}_\infty$-functors $\underline{\mathsf{A}}^{\mathrm{u}}_\infty \to \underline{\mathsf{A}}^{\mathrm{u}}_\infty$ and their natural transformations form a strict monoidal category, the endomorphisms category in a 2-category. Another way to describe this category is to say that its objects are strict A^u_∞-2-functors and morphisms are strict A^u_∞-2-transformations. Thus we can denote this strict monoidal category by A^u_∞-2. The composition of A^u_∞-2-transformations $\lambda_{\mathcal{C}} : F\mathcal{C} \to G\mathcal{C}$ and $\mu_{\mathcal{C}} : G\mathcal{C} \to H\mathcal{C}$ is $(\lambda\mu)_{\mathcal{C}} = \bigl(F\mathcal{C} \xrightarrow{\lambda_{\mathcal{C}}} G\mathcal{C} \xrightarrow{\mu_{\mathcal{C}}} H\mathcal{C}\bigr)$. The monoidal product of objects of A^u_∞-2 is given by the composition of A^u_∞-2-functors

$$F \cdot G = G \circ F = \bigl(A^u_\infty(\mathcal{C},\mathcal{D}) \xrightarrow{F} A^u_\infty(F\mathcal{C},F\mathcal{D}) \xrightarrow{G} A^u_\infty(GF\mathcal{C},GF\mathcal{D})\bigr).$$

The monoidal products of an object H with a morphism $\lambda : F \to G$ are given by the following formulas:

$$\begin{aligned} H\cdot\lambda &: H\cdot F \to H\cdot G, & (H\cdot\lambda)_{\mathcal{C}} &= \lambda_{H\mathcal{C}} : FH\mathcal{C} \to GH\mathcal{C}, \\ \lambda\cdot H &: F\cdot H \to G\cdot H, & (\lambda\cdot H)_{\mathcal{C}} &= H(\lambda_{\mathcal{C}}) : HF\mathcal{C} \to HG\mathcal{C}. \end{aligned}$$

Algebras in the monoidal category A^u_∞-2 are called *A^u_∞-2-monads.*

One way to obtain A^u_∞-2-monads is described in Section 4.35. Assume that (F, m_F, u_F) is a monad in the ordinary category $\mathsf{A}^{\mathrm{u}}_\infty$, where F is a multifunctor and $u_F : \mathrm{Id} \to F$ is multinatural. Suppose that for all unital A_∞-categories $\mathcal{C}$ and $\mathcal{D}$ the exterior of the following diagram commutes:

$$\begin{tikzcd} F^2\mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "1{,}u_F"] \arrow[d, "m_F{,}1"'] & F^2\mathcal{C}, F\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "1{,}Fu_F"] \arrow[d, dotted, "m_F{,}1"] & F^2\mathcal{C}, F^2\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "F^2\,\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] & F^2\mathcal{D} \arrow[d, "m_F"] \\ F\mathcal{C}, \underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[r, "1{,}u_F"] & F\mathcal{C}, F\underline{\mathsf{A}}^{\mathrm{u}}_\infty(\mathcal{C};\mathcal{D}) \arrow[rr, "F\,\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] & & F\mathcal{D} \end{tikzcd}$$

These assumptions imply that (F', m_F, u_F) is an A^u_∞-2-monad. The pentagon containing the dotted arrow is not required to commute, however, it does commute in our main example of the shift multifunctor $F = S = -^{[\,]}$, see Section 10.30.

9.21 2-category structure of A^u_∞. View the multifunctor $H^0 : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\Bbbk\text{-}\mathcal{C}at}$, introduced in Section 9.11, as a change of multicategory base. It turns the $\mathsf{A}^{\mathrm{u}}_\infty$-category A^u_∞ into a $\Bbbk$-$\mathcal{C}at$-category $\overline{A^u_\infty}$, that is, a 2-category, whose 2-morphism sets are $\Bbbk$-modules, and vertical and horizontal compositions of 2-morphisms are (poly)linear. Namely, objects of $\overline{A^u_\infty}$ are unital A_∞-categories, 1-morphisms are unital A_∞-functors, 2-morphisms rs^{-1} are equivalence classes of natural A_∞-transformations r. An A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$, $r \in A_\infty(\mathcal{A},\mathcal{B})(f,g)[1]$ is *natural*, if $\deg r = -1$ and $rB_1 = 0$. Natural A_∞-transformations $p, r : f \to g : \mathcal{A} \to \mathcal{B}$ are *equivalent* ($p \equiv r$) if they are homologous, that is, differ by a B_1-boundary. If A_∞-transformations $f \xrightarrow{r} g \xrightarrow{p} h : \mathcal{A} \to \mathcal{B}$ are natural, the vertical composition of 2-morphisms rs^{-1} and ps^{-1} is represented by $(r \otimes p)B_2 s^{-1}$. Unit transformations $\mathbf{i}^{\mathcal{B}} : \mathrm{id} \to \mathrm{id} : \mathcal{B} \to \mathcal{B}$ (unit elements in $sA_\infty(\mathcal{B},\mathcal{B})(\mathrm{id},\mathrm{id})$) provide for any unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ the natural A_∞-transformations $f\mathbf{i}^{\mathcal{B}} \equiv \mathbf{i}^{\mathcal{A}} f : f \to f : \mathcal{A} \to \mathcal{B}$, representing the identity 2-morphism 1_f, cf. [Lyu03, Sections 7, 8].

The following proposition generalizes theorem 8.8 of [Lyu03], see also [Fuk02, Theorem 8.6]. It means that a 'partial' quasi-inverse to an A_∞-equivalence defined on a full A_∞-subcategory can be extended to a quasi-inverse on the whole A_∞-category.

9.22 Proposition. *Let $\mathcal{C}$, $\mathcal{B}$ be unital A_∞-categories, let $\iota : \mathcal{D} \hookrightarrow \mathcal{B}$ be embedding of a full A_∞-subcategory, let $\phi : \mathcal{C} \to \mathcal{B}$ be an A_∞-equivalence, let $w : \mathcal{D} \to \mathcal{C}$ be an A_∞-functor, and let $q : \iota \to w\phi : \mathcal{D} \to \mathcal{B}$ be an invertible natural A_∞-transformation. Let $h : \operatorname{Ob} \mathcal{B} \to \operatorname{Ob} \mathcal{C}$ be a mapping such that $X\iota h = Xw$ for all $X \in \operatorname{Ob} \mathcal{D}$. Assume that for each object X of $\mathcal{B}$ a cycle ${}_X r_0 \in \mathcal{B}(X, Xh\phi)[1]^{-1}$ is given such that the morphism $[{}_X r_0 s^{-1}] : X \to Xh\phi$ of $H^0(\mathcal{B})$ is invertible, and ${}_{X\iota} r_0 = {}_X q_0$ whenever $X \in \operatorname{Ob} \mathcal{D}$.*

Then there is an A_∞-equivalence $\psi : \mathcal{B} \to \mathcal{C}$ such that $\operatorname{Ob} \psi = h$ and $\psi\big|_{\mathcal{D}} = \iota\psi = w$, there is an invertible natural A_∞-transformation $r : \mathrm{id}_{\mathcal{B}} \to \psi\phi : \mathcal{B} \to \mathcal{B}$ such that its 0-th component consists of the given elements ${}_X r_0$ and such that $r\big|_{\mathcal{D}} = \iota r = q$. In particular, ψ is quasi-inverse to ϕ.

In the case of empty subcategory $\mathcal{D} = \varnothing \subset \mathcal{B}$ the A_∞-functor w and the A_∞-transformation q do not appear. Proof of the above proposition follows the proof of [Lyu03, Theorem 8.8] very closely. We give full details for the sake of completeness.

Proof. We have to satisfy the equations

$$\iota\psi = w, \qquad \psi b = b\psi, \qquad \iota r = q, \qquad rb + br = 0.$$

We already know the map $\operatorname{Ob} \psi$ and the component r_0. Let us construct the remaining components of ψ and r by induction. Given a positive integer n, assume that we have

already found components ψ_m, r_m of the sought ψ, r for $m < n$, such that the equations

$$\psi_m = w_m : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 h, X_m h), \forall X_i \in \operatorname{Ob}\mathcal{D}, \quad (9.22.1)$$

$$(\psi b)_m + (b\psi)_m = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{C}(X_0 h, X_m h), \quad (9.22.2)$$

$$r_m = q_m : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{B}(X_0, X_m h\phi), \forall X_i \in \operatorname{Ob}\mathcal{D}, \quad (9.22.3)$$

$$(rb + br)_m = 0 : s\mathcal{B}(X_0, X_1) \otimes \cdots \otimes s\mathcal{B}(X_{m-1}, X_m) \to s\mathcal{B}(X_0, X_m h\phi) \quad (9.22.4)$$

are satisfied for all $m < n$. Introduce a coalgebra homomorphism $\tilde{\psi} : Ts\mathcal{B} \to Ts\mathcal{C}$ of degree 0 by its components $(\psi_1, \dots, \psi_{n-1}, 0, 0, \dots)$ and an $(\mathrm{id}_\mathcal{B}, \tilde{\psi}\phi)$-coderivation $\tilde{r} : Ts\mathcal{B} \to Ts\mathcal{B}$ of degree -1 by its components $(r_0, r_1, \dots, r_{n-1}, 0, 0, \dots)$. Define a $(\tilde{\psi}, \tilde{\psi})$-coderivation $\lambda = \tilde{\psi}b - b\tilde{\psi}$ of degree 1 and a map $\nu = -\tilde{r}b - b\tilde{r} + (\tilde{r} \otimes \lambda\phi)\theta : Ts\mathcal{B} \to Ts\mathcal{B}$ of degree 0. The commutator $\tilde{r}b + b\tilde{r}$ has the following property:

$$(\tilde{r}b + b\tilde{r})\Delta_0 = \Delta_0\big[1 \otimes (\tilde{r}b + b\tilde{r}) + (\tilde{r}b + b\tilde{r}) \otimes \tilde{\psi}\phi + \tilde{r} \otimes \lambda\phi\big].$$

As follows from (8.25.3) (see also [Lyu03, Proposition 3.1]) the map $(\tilde{r} \otimes \lambda\phi)\theta$ has a similar property

$$(\tilde{r} \otimes \lambda\phi)\theta\Delta_0 = \Delta_0\big[1 \otimes (\tilde{r} \otimes \lambda\phi)\theta + (\tilde{r} \otimes \lambda\phi)\theta \otimes \tilde{\psi}\phi + \tilde{r} \otimes \lambda\phi\big].$$

Taking the difference we find that ν is an $(\mathrm{id}_\mathcal{B}, \tilde{\psi}\phi)$-coderivation. Equations (9.22.2), (9.22.4) imply that $\lambda_m = 0$, $\nu_m = 0$ for $m < n$ (the image of $(\tilde{r} \otimes \lambda\phi)\theta$ is contained in $T^{\geqslant 2}s\mathcal{B}$).

The identity $\lambda b + b\lambda = 0$ implies that

$$\lambda_n d \overset{\text{def}}{=} \lambda_n b_1 + \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\lambda_n = 0. \quad (9.22.5)$$

The identity

$$\nu b - b\nu = (\tilde{r} \otimes \lambda\phi)\theta b - b(\tilde{r} \otimes \lambda\phi)\theta$$

implies that

$$\nu_n d \overset{\text{def}}{=} \nu_n b_1 - \sum_{\alpha+1+\beta=n} (1^{\otimes\alpha} \otimes b_1 \otimes 1^{\otimes\beta})\nu_n = -(r_0 \otimes \lambda_n\phi_1)b_2 = -\lambda_n\phi_1(r_0 \otimes 1)b_2. \quad (9.22.6)$$

Consider a sequence $\xi = (X_0, X_1, \dots, X_{n-1}, X_n)$ of objects of $\mathcal{B}$. If all of them are contained in $\operatorname{Ob}\mathcal{D}$, we define ψ_m and r_m for this sequence of objects by (9.22.1) and (9.22.3) respectively. Since w is an A_∞-functor, equation (9.22.2) for ξ follows. Since q is a natural A_∞-transformation, equation (9.22.4) for ξ follows.

Suppose that some object X_i from the sequence ξ is not contained in $\operatorname{Ob}\mathcal{D}$. Consider the complex of $\Bbbk$-modules $N = s\mathcal{B}(X_0, X_1) \otimes_\Bbbk \cdots \otimes_\Bbbk s\mathcal{B}(X_{n-1}, X_n)$, and introduce a chain map

$$u = \underline{\mathsf{C}}_\Bbbk(N, \phi_1(r_0 \otimes 1)b_2) : \underline{\mathsf{C}}_\Bbbk(N, s\mathcal{C}(X_0 h, X_n h)) \to \underline{\mathsf{C}}_\Bbbk(N, s\mathcal{B}(X_0, X_n h\phi)).$$

The differential d in the source and target of this map is given by formulas generalizing (9.22.5) and (9.22.6) in an obvious way. Since the $\mathcal{K}$-functor $\mathsf{k}\phi$ is full and faithful and the morphisms $[{}_Xr_0s^{-1}]$ are invertible, the chain maps ϕ_1 and $(r_0 \otimes 1)b_2$ are homotopy invertible (see [Lyu03, Lemma 7.14] for details). Hence, the map u is homotopy invertible as well. Therefore, the complex $\operatorname{Cone}(u)$ is contractible (e.g. by [Lyu03, Lemma B.1]). Equations (9.22.5) and (9.22.6) in the form $-\lambda_n d = 0$, $\nu_n d + \lambda_n u = 0$ imply that

$$(\lambda_n, \nu_n) \in \underline{\mathsf{C}}^1_{\Bbbk}(N, s\mathcal{C}(X_0h, X_nh)) \oplus \underline{\mathsf{C}}^0_{\Bbbk}(N, s\mathcal{B}(X_0, X_nh\phi)) = \operatorname{Cone}^0(u)$$

is a cycle. Hence, it is a boundary of some element

$$(\psi_n, r_n) \in \underline{\mathsf{C}}^0_{\Bbbk}(N, s\mathcal{C}(X_0h, X_nh)) \oplus \underline{\mathsf{C}}^{-1}_{\Bbbk}(N, s\mathcal{B}(X_0, X_nh\phi)) = \operatorname{Cone}^{-1}(u),$$

that is, $-\psi_n d = \lambda_n$ and $r_n d + \psi_n\phi_1(r_0 \otimes 1)b_2 = \nu_n$. In other words, equations (9.22.2), (9.22.4) are satisfied for $m = n$, and we prove the existence of ψ and r by induction.

Since the elements $[{}_Xr_0s^{-1}]$ are invertible, the constructed natural A_∞-transformation r is invertible. Therefore, the constructed A_∞-functor ψ is isomorphic to some quasi-inverse of ϕ. Hence, ψ is unital and quasi-inverse to ϕ itself. □

9.23 Remark. Due to A_∞-version of Yoneda Lemma cited in Corollary 1.4 any $\mathscr{U}$-small unital A_∞-category $\mathcal{C}$ is A_∞-equivalent to a $\mathscr{U}$-small differential graded category $\mathcal{D}$. Moreover, one can assume that $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathcal{C}$. Then there exist A_∞-equivalences $\phi : \mathcal{C} \to \mathcal{D}$, $\psi : \mathcal{D} \to \mathcal{C}$, quasi-inverse to each other, such that $\operatorname{Ob}\phi = \operatorname{Ob}\psi = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$.

9.24 Unital envelopes of A_∞-categories. Given an A_∞-category $\mathcal{A}$, we associate a strictly unital A_∞-category $\mathcal{A}^{\mathsf{su}}$ with it. It has the same set of objects and for any pair of objects $X, Y \in \operatorname{Ob}\mathcal{A}$ the graded $\Bbbk$-module $s\mathcal{A}^{\mathsf{su}}(X,Y)$ is given by

$$s\mathcal{A}^{\mathsf{su}}(X,Y) = \begin{cases} s\mathcal{A}(X,Y), & X \neq Y, \\ s\mathcal{A}(X,X) \oplus \Bbbk\,{}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}, & X = Y, \end{cases}$$

where ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a new generator of degree -1. The element ${}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{su}}}$ is a strict unit by definition, and the canonical embedding $e_{\mathcal{A}} = u_{\mathsf{su}} : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathsf{su}}$ is a strict A_∞-functor. We call the A_∞-category $\mathcal{A}^{\mathsf{su}}$ the *strictly unital envelope* of $\mathcal{A}$. Our goal is to prove that it is also a unital envelope of $\mathcal{A}$, where the unitality is understood in a weak sense.

9.25 Lemma. *Let $(\mathcal{A}_i)_{i\in I}$ be A_∞-categories, and let $\mathcal{B}$ be a strictly unital A_∞-category. Then the restriction map $\mathsf{A}^{\mathsf{su}}_\infty((\mathcal{A}^{\mathsf{su}}_i)_{i\in I};\mathcal{B}) \to \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$, $f \mapsto f\big|_{(\mathcal{A}_i)_I}$ is a bijection.*

Proof. Given $f \in \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B})$, denote by g the corresponding strictly unital extension of f, which is a morphism of $\mathsf{Q}^{\mathsf{su}}((\mathcal{A}^{\mathsf{su}}_i)_{i\in I};\mathcal{B}) \subset \mathsf{Q}((\mathcal{A}^{\mathsf{su}}_i)_{i\in I};\mathcal{B})$. This subset consists of morphisms f in Q, whose components vanish if any of its entries is $\mathbf{i}_0^{\mathcal{A}_i}$, except $\mathbf{i}_0^{\mathcal{A}_i} f_{e_i} = \mathbf{i}_0^{\mathcal{B}}$.

Let us prove that g is an A_∞-functor. Let its arguments be indexed by $I = \mathbf{m}$. Restriction of equation $\tilde{g}b = \sum_{i=1}^{m}(1^{\boxtimes(i-1)} \boxtimes \tilde{b} \boxtimes 1^{\boxtimes(n-i)})\overline{g}$ to $\boxtimes^{i\in I} T^{n_i} s\mathcal{A}_i$ gives:

$$
\begin{aligned}
&\begin{matrix}\text{signed permutation}\\ \text{\& insertion of units}\end{matrix} \times \sum_{l^1+\dots+l^k=n\in(\mathbb{Z}_{\geqslant 0})^m} (g_{l^1}\otimes\dots\otimes g_{l^k})b_k \\
&= \sum_{a_i+t_i+c_i=n_i}^{1\leqslant i\leqslant m} (1^{\boxtimes(i-1)} \boxtimes (1^{\otimes a_i}\otimes b_{t_i}\otimes 1^{\otimes c_i}) \boxtimes 1^{\boxtimes(m-i)}) \cdot g_{n_1,\dots,n_{i-1},a_i+1+c_i,n_{i+1},\dots,n_m}. \qquad (9.25.1)
\end{aligned}
$$

Suppose first that $|n| = \sum_{i\in I} n_i > 2$ and $\mathbf{i}_0^{\mathcal{A}_i}$, $\mathbf{i}_0^{\mathcal{A}_j}$ enter the arguments for some $i < j$. We claim that in this case summands of both sums vanish. In the left hand side summands might not vanish only if $\mathbf{i}_0^{\mathcal{A}_j}$ resp. $\mathbf{i}_0^{\mathcal{A}_j}$ are arguments of some g_{e_i} resp. g_{e_j}. Since $|n| > 2$, it follows that $k \geqslant 3$, thus the corresponding summand vanishes due to $\mathbf{i}_0^{\mathcal{B}}$ being a strict unit. In the right hand side at least one $\mathbf{i}_0$ is an argument of $g_{n'}$, $n' = (n_1,\dots,n_{i-1},a_i+1+c_i,n_{i+1},\dots,n_m)$ with $|n'| > 1$. The corresponding summand vanishes by definition of g.

Suppose now that $|n| = 2$ and $\mathbf{i}_0^{\mathcal{A}_i}$, $\mathbf{i}_0^{\mathcal{A}_j}$ enter the arguments for some $i < j$. The right hand vanishes by the same argument as above. In the left hand side we have only two non-vanishing summands:

$$
\begin{aligned}
&(\mathbf{i}_0^{\mathcal{A}_i}\boxtimes\mathbf{i}_0^{\mathcal{A}_j})(g_{e_i}\otimes g_{e_j})b_2 + (\mathbf{i}_0^{\mathcal{A}_i}\boxtimes\mathbf{i}_0^{\mathcal{A}_j})(12)^{\sim}(g_{e_j}\otimes g_{e_i})b_2 \\
&= (\mathbf{i}_0^{\mathcal{A}_i}g_{e_i}\otimes\mathbf{i}_0^{\mathcal{A}_j}g_{e_j})b_2 - (\mathbf{i}_0^{\mathcal{A}_j}g_{e_j}\otimes\mathbf{i}_0^{\mathcal{A}_i}g_{e_i})b_2 \\
&= (\mathbf{i}_0^{\mathcal{B}}\otimes\mathbf{i}_0^{\mathcal{B}})b_2 - (\mathbf{i}_0^{\mathcal{B}}\otimes\mathbf{i}_0^{\mathcal{B}})b_2 = 0,
\end{aligned}
$$

where $(12)^{\sim}$ means the signed permutation. Thus, they cancel each other.

Suppose that $\mathbf{i}_0^{\mathcal{A}_i}$ enters the arguments for a single $i \in I$. Composing (9.25.1) with a $\Bbbk$-linear map

$$
1^{\boxtimes(i-1)} \boxtimes (1^{\otimes a_i}\otimes\mathbf{i}_0^{\mathcal{A}_i}\otimes 1^{\otimes c_i}) \boxtimes 1^{\boxtimes(m-i)} : \boxtimes^{i\in I} Ts\mathcal{A}_i^{\mathsf{su}} \to \boxtimes^{i\in I} Ts\mathcal{A}_i^{\mathsf{su}}
$$

we get the following equation:

$$
\chi(a_i=0)(\mathbf{i}_0^{\mathcal{A}_i}g_{e_i}\otimes g_{n-e_i})b_2 + \chi(c_i=0)(g_{n-e_i}\otimes\mathbf{i}_0^{\mathcal{A}_i}g_{e_i})b_2 = \chi(a_i>0)g_{n-e_i} - \chi(c_i>0)g_{n-e_i}.
$$

All the other summands vanish by definition of g or due to $\mathbf{i}_0^{\mathcal{B}}$ being a strict unit. The above equation transforms into the identity

$$
\big[-\chi(a_i=0) - \chi(a_i>0) + \chi(c_i=0) + \chi(c_i>0)\big]g_{n-e_i} = 0.
$$

This proves the lemma. $\square$

Now we extend the map $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{su}}$ to a multifunctor $-^{\mathsf{su}} : \mathsf{A}_\infty \to \mathsf{A}_\infty^{\mathrm{su}}$. On morphisms it is given by the map

$$\mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{\mathsf{A}_\infty(1;e)} \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{su}}) \xrightarrow{\sim} \mathsf{A}_\infty^{\mathrm{su}}((\mathcal{A}_i^{\mathsf{su}})_{i\in I};\mathcal{B}^{\mathsf{su}}), \qquad f \mapsto f^{\mathsf{su}},$$

where the isomorphism is inverse to that of Lemma 9.25. In other terms, f^{su} is the only strictly unital A_∞-functor which makes commutative the following diagram in A_∞:

$$\begin{tikzcd} (\mathcal{A}_i)_{i\in I} \arrow[r, "f"] \arrow[d, "(e)_I"'] & \mathcal{B} \arrow[d, "e"] \\ (\mathcal{A}_i^{\mathsf{su}})_{i\in I} \arrow[r, "f^{\mathsf{su}}"] & \mathcal{B}^{\mathsf{su}} \end{tikzcd} \tag{9.25.2}$$

Notice that $(\mathrm{id}_{\mathcal{A}})^{\mathsf{su}} = \mathrm{id}_{\mathcal{A}^{\mathsf{su}}}$. The map $-^{\mathsf{su}}$ agrees with multiplication in multicategories. Indeed, for A_∞-functors $f_j : (\mathcal{A}_i)_{i\in\phi^{-1}j} \to \mathcal{B}_j$, $j \in J$, $g : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$, $\phi : I \to J$, we have a commutative diagram in A_∞

$$\begin{tikzcd} (\mathcal{A}_i)_{i\in I} \arrow[r, "(f_j)_{j\in J}"] \arrow[d, "(e)_I"'] & (\mathcal{B}_j)_{j\in J} \arrow[r, "g"] \arrow[d, "(e)_J"'] & \mathcal{C} \arrow[d, "e"] \\ (\mathcal{A}_i^{\mathsf{su}})_{i\in I} \arrow[r, "(f_j^{\mathsf{su}})_{j\in J}"] & (\mathcal{B}_j^{\mathsf{su}})_{j\in J} \arrow[r, "g^{\mathsf{su}}"] & \mathcal{C}^{\mathsf{su}}. \end{tikzcd}$$

Uniqueness implies that

$$[(f_j)_{j\in J}\cdot g]^{\mathsf{su}} = (f_j^{\mathsf{su}})_{j\in J}\cdot g^{\mathsf{su}}.$$

Therefore, $-^{\mathsf{su}} : \mathsf{A}_\infty \to \mathsf{A}_\infty^{\mathrm{su}}$ is a multifunctor. The associated closing transformation

$$\underline{\mathsf{su}} : \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{su}} \to \underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i^{\mathsf{su}})_{i\in I};\mathcal{B}^{\mathsf{su}})$$

satisfies the equation

$$(\mathrm{ev}^{\mathsf{A}_\infty})^{\mathsf{su}} = \bigl[(\mathcal{A}_i^{\mathsf{su}})_{i\in I}, \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{su}} \xrightarrow{(1)_I,\underline{\mathsf{su}}} (\mathcal{A}_i^{\mathsf{su}})_{i\in I}, \underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i^{\mathsf{su}})_{i\in I};\mathcal{B}^{\mathsf{su}}) \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{su}}}} \mathcal{B}^{\mathsf{su}}\bigr].$$

Corollary 4.20 implies that

$$\mathrm{Ob}\,\underline{\mathsf{su}} = -^{\mathsf{su}} : \mathsf{A}_\infty((\mathcal{A}_i)_{i\in I};\mathcal{B}) \to \mathsf{A}_\infty^{\mathrm{su}}((\mathcal{A}_i^{\mathsf{su}})_{i\in I};\mathcal{B}^{\mathsf{su}}).$$

Components of $\underline{\mathsf{su}}$ are found from the equation

$$[(\boxtimes^{i\in I}1^{\otimes n_i})\boxtimes \underline{\mathsf{su}}_m]\,\mathrm{ev}_{n,1}^{\mathsf{A}_\infty^{\mathrm{su}}} = [(\mathrm{ev}^{\mathsf{A}_\infty})^{\mathsf{su}}]_{n,m}.$$

It implies that $\underline{\mathsf{su}}_m = 0$ for $m > 1$, thus $\underline{\mathsf{su}}$ is a strict A_∞-functor. The first component $\underline{\mathsf{su}}_1$ has to map ${}_f\mathbf{i}^{\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{su}}}$ to $f^{\mathsf{su}}\mathbf{i}_0^{\mathcal{B}^{\mathsf{su}}}$. Its restriction

$$\underline{\mathsf{su}}_1 : s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g) \to s\underline{\mathsf{A}_\infty^{\mathrm{su}}}((\mathcal{A}_i^{\mathsf{su}})_{i\in I};\mathcal{B}^{\mathsf{su}})(f^{\mathsf{su}},g^{\mathsf{su}})$$

takes $r \in s\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})(f,g)$ to the only $r^{\mathsf{su}} \in s\underline{\mathsf{A}^{\mathsf{su}}_\infty}((\mathcal{A}^{\mathsf{su}}_i)_{i\in I};\mathcal{B}^{\mathsf{su}})(f^{\mathsf{su}},g^{\mathsf{su}})$ such that $r^{\mathsf{su}}\big|_{(\mathcal{A}_i)_{i\in I}} = r$. In particular, a given A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ goes to the $(f^{\mathsf{su}},g^{\mathsf{su}})$-coderivation $r^{\mathsf{su}} : Ts\mathcal{A}^{\mathsf{su}} \to Ts\mathcal{B}^{\mathsf{su}}$. Its components are given by

$$\begin{aligned} r^{\mathsf{su}}_n\big|_{(s\mathcal{A})^{\otimes n}} &= r_n e_{\mathcal{B}1} : (s\mathcal{A})^{\otimes n} \to s\mathcal{B}^{\mathsf{su}}, \qquad & n \geqslant 0,\\ (1^{\otimes a}\otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes 1^{\otimes c})r^{\mathsf{su}}_n &= 0 : (s\mathcal{A}^{\mathsf{su}})^{\otimes a+c} \to s\mathcal{B}^{\mathsf{su}}, \qquad & n = a+1+c \geqslant 1. \end{aligned}$$

9.26 Remark. It follows from the above formulas that

$$re_{\mathcal{B}} = e_{\mathcal{A}}r^{\mathsf{su}} : fe_{\mathcal{B}} = e_{\mathcal{A}}f^{\mathsf{su}} \to ge_{\mathcal{B}} = e_{\mathcal{A}}g^{\mathsf{su}} : \mathcal{A} \to \mathcal{B}^{\mathsf{su}}.$$

The composition $\mathsf{A}_\infty \xrightarrow{-^{\mathsf{su}}} \mathsf{A}^{\mathsf{su}}_\infty \hookrightarrow \mathsf{A}_\infty$ is again denoted $-^{\mathsf{su}}$ by abuse of notation. Diagram (9.25.2) implies that $u_{\mathsf{su}} = e : \mathrm{Id} \to -^{\mathsf{su}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$, $e : \mathcal{A} \hookrightarrow \mathcal{A}^{\mathsf{su}}$ is a multinatural transformation. Thus, $(-^{\mathsf{su}}, u_{\mathsf{su}}) : \mathsf{A}_\infty \to \mathsf{A}_\infty$ is an augmented multifunctor.

9.27 Corollary. *The multifunctor $-^{\mathsf{su}}$ gives rise to an A_∞-2-functor $-^{\mathsf{su}\prime} : A_\infty \to A_\infty$, that is, an A_∞-functor*

$$-^{\mathsf{su}\prime} = \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \xhookrightarrow{e} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})^{\mathsf{su}} \xrightarrow{\underline{\mathsf{su}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{su}};\mathcal{B}^{\mathsf{su}})\big]$$

by Corollary 4.35, see also Section 9.20. It is denoted also $-^{\mathsf{su}} : A_\infty \to A_\infty$ by abuse of notation.

9.28 Corollary. *For arbitrary A_∞-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ the following diagram commutes:*

$$\begin{tikzcd} TsA_\infty(\mathcal{A},\mathcal{B})\boxtimes TsA_\infty(\mathcal{B},\mathcal{C}) \arrow[r, "M"] \arrow[d, "-^{\mathsf{su}}\boxtimes-^{\mathsf{su}}"'] \arrow[dr, phantom, "="] & TsA_\infty(\mathcal{A},\mathcal{C}) \arrow[d, "-^{\mathsf{su}}"] \\ TsA^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{B}^{\mathsf{su}})\boxtimes TsA^u_\infty(\mathcal{B}^{\mathsf{su}},\mathcal{C}^{\mathsf{su}}) \arrow[r, "M"] & TsA^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}^{\mathsf{su}}) \end{tikzcd}$$

9.29 Remark. The A_∞-2-functor $-^{\mathsf{su}} : A^u_\infty \to A^u_\infty$ is not an A^u_∞-2-functor. Indeed, for unital A_∞-categories $\mathcal{A}$, $\mathcal{B}$ the composite A_∞-functor

$$A^u_\infty(\mathcal{A},\mathcal{B}) \hookrightarrow \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \xhookrightarrow{e} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})^{\mathsf{su}} \xrightarrow{\underline{\mathsf{su}}} \underline{\mathsf{A}^{\mathsf{su}}_\infty}(\mathcal{A}^{\mathsf{su}};\mathcal{B}^{\mathsf{su}}) \hookrightarrow A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{B}^{\mathsf{su}})$$

is not unital, in general, because the cycles $\mathbf{i}^{\mathcal{B}}_0$ and $\mathbf{i}^{\mathcal{B}^{\mathsf{su}}}_0$ are not homologous in $s\mathcal{B}^{\mathsf{su}}$.

Suppose $\mathcal{C}$ is an A_∞-category unital in the sense of Section 9.10. Then it is also unital in the sense of Kontsevich and Soibelman [KS09, Definition 4.2.3] by [LM06b]. That is, there is an A_∞-functor $U = U^{\mathcal{C}}_{\mathsf{su}} : \mathcal{C}^{\mathsf{su}} \to \mathcal{C}$ such that $e_{\mathcal{C}}U = U|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$. By Proposition 1.13 the A_∞-functor U is unital. Let us introduce the following A_∞-functors:

$$\begin{aligned} \Phi = A_\infty(u_{\mathsf{su}},\mathcal{C}) &= \big[A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}) \xrightarrow{(e_{\mathcal{A}}\boxtimes 1)M} A_\infty(\mathcal{A},\mathcal{C})\big],\\ \Psi = F_{\mathsf{su}} &= \big[A_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{-^{\mathsf{su}}} A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}^{\mathsf{su}}) \xrightarrow[A^u_\infty(1,U)]{(1\boxtimes U)M} A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C})\big]. \end{aligned} \tag{9.29.1}$$

Note that the A_∞-functor Φ is strict. For an arbitrary A_∞-functor $g : \mathcal{A} \to \mathcal{C}$ we have $g\Psi\Phi = e_\mathcal{A} g^{\mathsf{su}} U = g e_\mathcal{C} U = g$ by (9.25.2). Similarly, for an arbitrary A_∞-transformation $r : g \to h : \mathcal{A} \to \mathcal{C}$ we get $r\Psi_1\Phi_1 = e_\mathcal{A} r^{\mathsf{su}} U = r e_\mathcal{C} U = r$ by Remark 9.26. Therefore, $\Psi\Phi = \mathrm{id} : A_\infty(\mathcal{A}, \mathcal{C}) \to A_\infty(\mathcal{A}, \mathcal{C})$.

Let us compute $\Phi\Psi$. We have:

$$\begin{aligned}\Phi\Psi = (e_\mathcal{A} \boxtimes 1)M -^{\mathsf{su}} (1 \boxtimes U)M &= -^{\mathsf{su}}(e^{\mathsf{su}}_\mathcal{A} \boxtimes 1)M(1 \boxtimes U)M \\ &= -^{\mathsf{su}}(1 \boxtimes U)M(e^{\mathsf{su}}_\mathcal{A} \boxtimes 1)M : A^u_\infty(\mathcal{A}^{\mathsf{su}}, \mathcal{C}) \to A^u_\infty(\mathcal{A}^{\mathsf{su}}, \mathcal{C})\end{aligned}$$

by Corollary 9.28 and due to associativity of M. Note that

$$-^{\mathsf{su}}(1 \boxtimes U)M(e_{\mathcal{A}^{\mathsf{su}}} \boxtimes 1)M = \mathrm{id} : A^u_\infty(\mathcal{A}^{\mathsf{su}}, \mathcal{C}) \to A^u_\infty(\mathcal{A}^{\mathsf{su}}, \mathcal{C}).$$

The verification is similar to that we did above for $\Psi\Phi$.

Let us introduce an A_∞-transformation $r : e_{\mathcal{A}^{\mathsf{su}}} \to e^{\mathsf{su}}_\mathcal{A} : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}^{\mathsf{su}\,\mathsf{su}}$ by its components: ${}_X r_0 = {}_X \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \in s\mathcal{A}^{\mathsf{su}\,\mathsf{su}}(X, X)$, $r_n = 0$ if $n \geqslant 1$.

9.30 Lemma. *The A_∞-transformation $r : e_{\mathcal{A}^{\mathsf{su}}} \to e^{\mathsf{su}}_\mathcal{A} : \mathcal{A}^{\mathsf{su}} \to \mathcal{A}^{\mathsf{su}\,\mathsf{su}}$ is natural.*

Proof. The n-th component of the equation $rB_1 = 0$ reads as follows:

$$\sum_{a+c=n} ((e_{\mathcal{A}^{\mathsf{su}}})^{\otimes a}_1 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes (e^{\mathsf{su}}_\mathcal{A})^{\otimes c}_1) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_{n+1} = 0 : (s\mathcal{A}^{\mathsf{su}})^{\otimes n} \to s\mathcal{A}^{\mathsf{su}\,\mathsf{su}}.$$

This equation holds obviously true if $n \neq 1$. Indeed, either in the product $\dots \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \dots$ some $\mathbf{i}^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_0$ occurs, then $b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_{n+1}$ vanishes by definition, or $\dots \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \dots \in (s\mathcal{A}^{\mathsf{su}})^{\otimes n+1}$. Since $b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_{n+1}\big|_{(s\mathcal{A})^{\otimes n+1}} = b^{\mathcal{A}^{\mathsf{su}}}_{n+1}$, it follows that $(\dots \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \dots) b^{\mathcal{A}^{\mathsf{su}}}_{n+1} = 0$. In the case $n = 1$ we have:

$$((e_{\mathcal{A}^{\mathsf{su}}})_1 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 + (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes (e^{\mathsf{su}}_\mathcal{A})_1) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 = 0 : s\mathcal{A}^{\mathsf{su}} \to s\mathcal{A}^{\mathsf{su}\,\mathsf{su}}.$$

Restricting the left hand side of this equation to $s\mathcal{A}$ we get

$$(1 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 + (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes 1) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 = (1 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}}}_2 + (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes 1) b^{\mathcal{A}^{\mathsf{su}}}_2 = 0 : s\mathcal{A} \to s\mathcal{A}^{\mathsf{su}\,\mathsf{su}}$$

by definition of $b^{\mathcal{A}^{\mathsf{su}}}$. Finally,

$$\begin{aligned}&\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \big[((e_{\mathcal{A}^{\mathsf{su}}})_1 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 + (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes (e^{\mathsf{su}}_\mathcal{A})_1) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2\big] \\ &\qquad = (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 - (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}\,\mathsf{su}}}_2 = (\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 \otimes \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0) b^{\mathcal{A}^{\mathsf{su}}}_2 - \mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 = 0.\end{aligned}$$

The lemma is proven. □

Notice that $(r \boxtimes 1)M$ is an $((e_{\mathcal{A}^{\mathsf{su}}} \boxtimes 1)M, (e^{\mathsf{su}}_\mathcal{A} \boxtimes 1)M)$-coderivation, since M is an augmented coalgebra homomorphism. The identity $(1 \boxtimes B + B \boxtimes 1)M = MB$ implies that

$(r\boxtimes 1)M$ is a natural A_∞-transformation. Composing it with the A_∞-functor $-^{\mathsf{su}}(1\boxtimes U)M$ we get a natural A_∞-transformation

$$\alpha = -^{\mathsf{su}}(1\boxtimes U)M(r\boxtimes 1)M : \mathrm{id} = -^{\mathsf{su}}(1\boxtimes U)M(e_{\mathcal{A}^{\mathsf{su}}}\boxtimes 1)M \\ \longrightarrow -^{\mathsf{su}}(1\boxtimes U)M(e^{\mathsf{su}}_{\mathcal{A}}\boxtimes 1)M = \Phi\Psi : A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}) \to A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}).$$

We claim that α is invertible. By [Lyu03, Proposition 7.15] it suffices to show that the 0-th component of α is invertible. For an arbitrary unital A_∞-functor $f:\mathcal{A}^{\mathsf{su}}\to\mathcal{C}$ we have

$${}_f\alpha_0 = rf^{\mathsf{su}}U : e_{\mathcal{A}^{\mathsf{su}}}f^{\mathsf{su}}U = f \to e^{\mathsf{su}}_{\mathcal{A}}f^{\mathsf{su}}U : \mathcal{A}^{\mathsf{su}}\to\mathcal{C}.$$

It is a natural A_∞-transformation whose 0-th component equals ${}_X\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 f^{\mathsf{su}}_1U_1 = {}_X\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 f_1$. Since f is unital, it follows that ${}_X\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 f_1 - {}_{Xf}\mathbf{i}^{\mathcal{C}}_0 \in \operatorname{Im} b_1$, in particular ${}_X\mathbf{i}^{\mathcal{A}^{\mathsf{su}}}_0 f_1$ is invertible. By [Lyu03, Proposition 7.15] the A_∞-transformation ${}_f\alpha_0$ is invertible for any unital A_∞-functor $f:\mathcal{A}^{\mathsf{su}}\to\mathcal{C}$. By the same proposition α is invertible. Note that r itself is not invertible.

It follows from the above considerations that Φ and Ψ are quasi-inverse A_∞-equivalences. Note that this implies that both are unital (see [Lyu03, Corollary 8.9]). We have proven

9.31 Theorem. *For an arbitrary unital A_∞-category $\mathcal{C}$ the restriction strict A_∞-functor $A_\infty(u_{\mathsf{su}},\mathcal{C}) = (u_{\mathsf{su}}\boxtimes 1)M : A^u_\infty(\mathcal{A}^{\mathsf{su}},\mathcal{C}) \to A_\infty(\mathcal{A},\mathcal{C})$ is an A_∞-equivalence with a one-sided inverse $F_{\mathsf{su}} = \Psi$ (which is also an A_∞-equivalence), namely, $F_{\mathsf{su}}\cdot A_\infty(u_{\mathsf{su}},\mathcal{C}) = \mathrm{id}$. Consequently, the pair $(\mathcal{A}^{\mathsf{su}}, u_{\mathsf{su}}:\mathcal{A}\hookrightarrow\mathcal{A}^{\mathsf{su}})$ unitally represents the A^u_∞-2-functor $A^u_\infty\to A^u_\infty$, $\mathcal{C}\mapsto A_\infty(\mathcal{A},\mathcal{C})$ in the sense of [LM06a, Section 3.5] and [LM08c, Section 1.5].*

9.32 Free A_∞-categories and free dg-categories. The free A_∞-category $\mathcal{F}\mathcal{Q}$ is associated with a differential graded quiver $\mathcal{Q}$ in [LM06a], see also Section 1.20. On the other hand, the quiver $\mathcal{Q}$ gives rise to a non-unital free differential graded category generated by $\mathcal{Q}$, the differential graded category of paths in $\mathcal{Q}$. Its underlying differential graded quiver is the restricted tensor quiver $T^{\geqslant1}\mathcal{Q} = \bigoplus_{n=1}^\infty T^n\mathcal{Q}$, where $T^n\mathcal{Q}$ is the n-th tensor power of $\mathcal{Q}$ in the monoidal category of differential graded quivers with the fixed set of objects $\operatorname{Ob}\mathcal{Q}$. The embedding $\mathcal{Q}\hookrightarrow T^{\geqslant1}\mathcal{Q}$ is a chain quiver map, therefore by [LM06a, Corollary 2.4] it extends to a strict A_∞-functor $g:\mathcal{F}\mathcal{Q}\to T^{\geqslant1}\mathcal{Q}$. Note that by construction $\operatorname{Ob} g = \mathrm{id}_{\operatorname{Ob}\mathcal{Q}}$.

9.33 Proposition. *The first component $sg_1s^{-1}:\mathcal{F}\mathcal{Q}(X,Y)\to T^{\geqslant1}\mathcal{Q}(X,Y)$ identifies with the map*

$$\mathcal{F}\mathcal{Q}(X,Y) \cong \bigoplus_{n=1}^\infty T^n\mathcal{Q}(X,Y)\otimes A_\infty(n) \xrightarrow{\sum 1\otimes\varepsilon} \bigoplus_{n=1}^\infty T^n\mathcal{Q}(X,Y)\otimes Ass(n) \cong T^{\geqslant1}\mathcal{Q}(X,Y).$$

In particular, it is homotopy invertible for each pair of objects $X,Y\in\operatorname{Ob}\mathcal{Q}$.

Proof. In Section 1.20 we have constructed an invertible strict A_∞-functor $f = \widehat{\mathrm{in}_1} : \mathcal{F}\mathcal{Q} \to \widetilde{\mathcal{F}}\mathcal{Q}$. Let us now check that the morphisms $g_1 : s\mathcal{F}\mathcal{Q} \to sT^{\geqslant 1}\mathcal{Q}$, $f_1 : s\mathcal{F}\mathcal{Q} \to s\widetilde{\mathcal{F}}\mathcal{Q}$, and

$$h = \sum 1 \otimes \varepsilon : \widetilde{\mathcal{F}}\mathcal{Q} = \bigoplus_{n\geqslant 1} (s\mathcal{Q})^{\otimes n} \otimes A_\infty(n) \to \bigoplus_{n\geqslant 1} (s\mathcal{Q})^{\otimes n} \otimes Ass(n) = T^{\geqslant 1}\mathcal{Q}$$

satisfy the relation $g_1 = f_1 \cdot s^{-1}hs$. Indeed, $g_1|_{s\mathcal{F}_t\mathcal{Q}}$ is given by the composite

$$\bigl[s\mathcal{F}_t\mathcal{Q} = (s\mathcal{Q})^{\otimes n}[-|t|] \xrightarrow{s^{|t|}} (s\mathcal{Q})^{\otimes n} \xrightarrow{\mathrm{in}_1^{\otimes n}} (sT^{\geqslant 1}\mathcal{Q})^{\otimes n} \xrightarrow{b_t^{T^{\geqslant 1}\mathcal{Q}}} sT^{\geqslant 1}\mathcal{Q}\bigr].$$

Therefore, it suffices to check the commutativity of the diagram

$$\begin{tikzcd}
(s\mathcal{Q})^{\otimes n} \arrow[r, "\mathrm{in}_1^{\otimes n}"] \arrow[d, "\mathrm{in}_1^{\otimes n}"'] & (sT^{\geqslant 1}\mathcal{Q})^{\otimes n} \arrow[r, "b_t^{T^{\geqslant 1}\mathcal{Q}}"] & sT^{\geqslant 1}\mathcal{Q} \\
(s\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \arrow[rr, "b_t^{\widetilde{\mathcal{F}}\mathcal{Q}}"] & & s\widetilde{\mathcal{F}}\mathcal{Q} \arrow[u, "s^{-1}hs"']
\end{tikzcd}$$

It is equivalent to the commutativity of the exterior of the diagram

$$\begin{tikzcd}
\mathcal{Q}^{\otimes n} \arrow[rr, "\mathrm{in}_1^{\otimes n}"] \arrow[dr, "s^{\otimes n}"] \arrow[ddd, "\mathrm{in}_1^{\otimes n}"] & & (T^{\geqslant 1}\mathcal{Q})^{\otimes n} \arrow[rr, "(-)^\sigma m_t^{T^{\geqslant 1}\mathcal{Q}}"] \arrow[d, "s^{\otimes n}"] & & T^{\geqslant 1}\mathcal{Q} \\
& (s\mathcal{Q})^{\otimes n} \arrow[r, "\mathrm{in}_1^{\otimes n}"] \arrow[d, "\mathrm{in}_1^{\otimes n}"] & (sT^{\geqslant 1}\mathcal{Q})^{\otimes n} \arrow[r, "b_t^{T^{\geqslant 1}\mathcal{Q}}"] & sT^{\geqslant 1}\mathcal{Q} \arrow[ur, "s^{-1}"] & \\
& (s\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \arrow[rr, "b_t^{\widetilde{\mathcal{F}}\mathcal{Q}}"] & & s\widetilde{\mathcal{F}}\mathcal{Q} \arrow[u, "s^{-1}hs"'] \arrow[dr, "s^{-1}"] & \\
(\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \arrow[ur, "s^{\otimes n}"] \arrow[rrrr, "(-)^\sigma m_t^{\widetilde{\mathcal{F}}\mathcal{Q}}"] & & & & \widetilde{\mathcal{F}}\mathcal{Q} \arrow[uuu, "h"']
\end{tikzcd}$$

Dropping the common sign $(-)^\sigma$, we end up with the following equation to be checked:

$$\bigl[\mathcal{Q}^{\otimes n} \xrightarrow{\mathrm{in}_1^{\otimes n}} (T^{\geqslant 1}\mathcal{Q})^{\otimes n} \xrightarrow{m_t^{T^{\geqslant 1}\mathcal{Q}}} T^{\geqslant 1}\mathcal{Q}\bigr] = \bigl[\mathcal{Q}^{\otimes n} \xrightarrow{\mathrm{in}_1^{\otimes n}} (\widetilde{\mathcal{F}}\mathcal{Q})^{\otimes n} \xrightarrow{m_t^{\widetilde{\mathcal{F}}\mathcal{Q}}} \widetilde{\mathcal{F}}\mathcal{Q} \xrightarrow{h} T^{\geqslant 1}\mathcal{Q}\bigr].$$

Note that the composite of the first and the second arrows in the right hand side coincides with (1.20.2). Clearly, both sides vanish if t has an internal vertex adjacent to more than 3 edges. If each internal vertex of t is adjacent to exactly 3 vertices, then both sides coincide with the embedding of $\mathcal{Q}^{\otimes n} = T^n\mathcal{Q}$ into $T^{\geqslant 1}\mathcal{Q}$. Thus $g_1 = f_1 \cdot shs^{-1}$. Since the chain map h is homotopy invertible and f_1 is an isomorphism, it follows that the chain map g_1 is homotopy invertible as well. $\square$

9.34 Corollary. *The A_∞-functor $g^{\mathsf{su}} : (\mathcal{F}\mathcal{Q})^{\mathsf{su}} \to (T^{\geqslant 1}\mathcal{Q})^{\mathsf{su}} = T\mathcal{Q}$ is an A_∞-equivalence.*

Proof. Clearly, $\operatorname{Ob} g^{\mathsf{su}} = \mathrm{id}_{\operatorname{Ob}\mathcal{Q}}$. Furthermore, the first component g_1^{su} is obviously homotopy invertible since so is g_1. It follows that the A_∞-functor g^{su} is an A_∞-equivalence by [Lyu03, Theorem 8.8]. □

Clearly, $(T^{\geqslant 1}\mathcal{Q})^{\mathsf{su}}$ is isomorphic to the genuine (unital) free differential graded category $T\mathcal{Q}$ generated by the differential graded quiver $\mathcal{Q}$. The following corollary implies that $T\mathcal{Q}$ represents the A_∞^u-2-functor $\mathcal{C} \mapsto A_1(\mathcal{Q}, \mathcal{C})$.

9.35 Corollary. *The restriction A_∞-functor* $\mathrm{restr} : A_\infty^u(T\mathcal{Q}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{C})$ *is an A_∞-equivalence, for each unital A_∞-category $\mathcal{C}$.*

Proof. The A_∞-functor $\mathrm{restr} : A_\infty^u(T\mathcal{Q}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{C})$ is equal to the composite

$$A_\infty^u(T\mathcal{Q}, \mathcal{C}) \xrightarrow{A_\infty^u(g^{\mathsf{su}},1)} A_\infty^u((\mathcal{F}\mathcal{Q})^{\mathsf{su}}, \mathcal{C}) \xrightarrow{(u_{\mathsf{su}}\boxtimes 1)M} A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{C}) \xrightarrow{\mathrm{restr}} A_1(\mathcal{Q}, \mathcal{C}).$$

The first A_∞-functor is an A_∞-equivalence since g^{su} is an A_∞-equivalence by Corollary 9.34 and because $A_\infty^u(-, \mathcal{C})$ preserves A_∞-equivalences, as an arbitrary A_∞^u-2-functor does. The second A_∞-functor is an A_∞-equivalence by Theorem 9.31, and the last A_∞-functor is an A_∞-equivalence by [LM06a, Theorem 2.12]. Hence, the entire composite is an A_∞-equivalence. □

Chapter 10

A_∞-categories closed under shifts

In this chapter we construct the monad of shifts first on quivers, then for A_∞-categories. The functor of shifts takes a graded quiver $\mathcal{A}$ to the graded quiver $\mathcal{A}^{[]}$ obtained by adding formal shifts of objects. This functor exists in several versions: a lax Monoidal functor $-^{[]} : \mathscr{Q}_p \to \mathscr{Q}_p$, a lax Monoidal functor $-^{[]} : \mathscr{Q}_u \to \mathscr{Q}_u$, a multifunctor $-^{[]} : \mathsf{Q} \to \mathsf{Q}$, and a multifunctor $-^{[]} : \mathsf{A}_\infty \to \mathsf{A}_\infty$. The functor of shifts gives rise to a monad for which the unit $u_{[]} : \mathrm{Id} \to -^{[]}$ is a Monoidal (resp. multinatural) transformation, and the multiplication $m_{[]} : -^{[][]} \to -^{[]}$ is a natural transformation. The results for $-^{[]} : \mathsf{Q} \to \mathsf{Q}$ are obtained in two different ways. On the first way the properties of $-^{[]} : \mathscr{Q}_u \to \mathscr{Q}_u$ are lifted to Q via commutation of the monad $-^{[]}$ and the comonad $T^{\geqslant 1}$. The second approach uses the action of the category of graded categories on the Kleisli multicategory of graded quivers, described in Appendix C. We define an algebra $\mathcal{Z}$ in the category of differential graded categories, i.e., a strictly monoidal differential graded category. Tensoring with $\mathcal{Z}$ (or equivalently, the action of $\mathcal{Z}$) is precisely the functor of shifts.

The monad of shifts constructed for graded quivers extends to A_∞-categories. This can be achieved directly, but we prefer the second way: the above mentioned action is extended in Appendix C to an action of the category of differential graded categories on the multicategory of A_∞-categories. The action of $\mathcal{Z}$ is the monad of shifts. It assigns to an A_∞-category $\mathcal{A}$ a certain A_∞-category $\mathcal{A}^{[]}$ with formally added shifted objects. The unit of the monad $u_{[]} : \mathcal{A} \to \mathcal{A}^{[]}$ is a full embedding. A_∞-categories for which this embedding is an A_∞-equivalence are *closed under shifts*. For instance, $\mathcal{A}^{[]}$ is always closed under shifts. Thus, the monad $-^{[]}$ is a kind of completion. We prove that closedness under shifts is preserved under taking quotients and A_∞-categories of A_∞-functors.

10.1 An algebra in the category of differential graded categories. The category of differential graded categories $\mathbf{dg}\text{-}\mathcal{C}at$ equipped with the tensor product $\boxtimes$ is a symmetric monoidal category. Let us study an algebra in this category, which will be used to define the functor of shifts. Let $\mathcal{Z}$ be a differential graded quiver with $\operatorname{Ob}\mathcal{Z} = \mathbb{Z}$, $\mathcal{Z}(m,n) = \Bbbk[n-m]$ and zero differential. Consider an arbitrary $\Bbbk$-linear graded category structure of $\mathcal{Z}$ given by isomorphisms

$$\begin{aligned} \mu' = \phi'(l,m,n) : \mathcal{Z}(l,m) \otimes_\Bbbk \mathcal{Z}(m,n) = \Bbbk[m-l] \otimes \Bbbk[n-m] \to \Bbbk[n-l] = \mathcal{Z}(l,n), \\ 1s^{m-l} \otimes 1s^{n-m} \mapsto \phi'(l,m,n)s^{n-l}. \end{aligned}$$

It is specified by a function $\phi' : \mathbb{Z}^3 \to \Bbbk^\times$ with values in the multiplicative group $\Bbbk^\times$. The associativity of composition

$$(1 \otimes \mu')\mu' = (\mu' \otimes 1)\mu' : \mathcal{Z}(k,l) \otimes \mathcal{Z}(l,m) \otimes \mathcal{Z}(m,n) \to \mathcal{Z}(k,n)$$

implies that ϕ' is a cycle:

$$\phi'(l,m,n) \cdot \phi'(k,l,n) = \phi'(k,l,m) \cdot \phi'(k,m,n).$$

An arbitrary such cycle ϕ' is a boundary

$$\phi'(l,m,n) = \xi(l,m) \cdot \xi(m,n) \cdot \xi(l,n)^{-1} \tag{10.1.1}$$

of a function $\xi : \mathbb{Z}^2 \to \Bbbk^\times$, namely, $\xi(l,m) = \phi'(0,l,m)$. The graded quiver automorphism $\xi : \mathcal{Z} \to \mathcal{Z}$, $n \mapsto n$, $\xi(m,n) : \mathcal{Z}(m,n) \to \mathcal{Z}(m,n)$ equips $\mathcal{Z}$ with another (isomorphic) category structure $(\mathcal{Z}, \mu)$ with $\phi(l,m,n) = 1$, so that $\xi : (\mathcal{Z}, \mu') \to (\mathcal{Z}, \mu)$ is a category isomorphism. In the following we shall consider only composition μ in $\mathcal{Z}$ specified by the function $\phi(l,m,n) = 1$. The elements $1 \in \Bbbk = \mathcal{Z}(n,n)$ are identity morphisms of $\mathcal{Z}$.

We equip the object $\mathcal{Z}$ of $(\mathbf{dg}\text{-}\mathcal{C}at, \boxtimes)$ with an algebra structure, given by multiplication – differential graded functor

$$\begin{gathered}\otimes_\psi : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z}, \qquad m \times n \mapsto m+n, \\ \otimes_\psi = \psi(n,m,k,l) : (\mathcal{Z} \boxtimes \mathcal{Z})(n \times m, k \times l) = \mathcal{Z}(n,k) \otimes \mathcal{Z}(m,l) \to \mathcal{Z}(n+m, k+l), \\ 1s^{k-n} \otimes 1s^{l-m} \mapsto \psi(n,m,k,l)s^{k+l-n-m}.\end{gathered}$$

We assume that the function $\psi : \mathbb{Z}^4 \to \Bbbk$ takes values in $\Bbbk^\times$. Being a functor, $\otimes_\psi$ has to satisfy the equation:

$$\begin{tikzcd}
\mathcal{Z}(a,c) \otimes \mathcal{Z}(b,d) \otimes \mathcal{Z}(c,e) \otimes \mathcal{Z}(d,f) \arrow[r, "(1\otimes c\otimes 1)(\mu\otimes\mu)"] \arrow[d, equal] \arrow[dr, phantom, "="] & \mathcal{Z}(a,e) \otimes \mathcal{Z}(b,f) \arrow[d, equal] \\
(\mathcal{Z} \boxtimes \mathcal{Z})(a \times b, c \times d) \otimes (\mathcal{Z} \boxtimes \mathcal{Z})(c \times d, e \times f) \arrow[r, "\mu"] \arrow[d, "\otimes_\psi \otimes \otimes_\psi"'] \arrow[dr, phantom, "="] & (\mathcal{Z} \boxtimes \mathcal{Z})(a \times b, e \times f) \arrow[d, "\otimes_\psi"] \\
\mathcal{Z}(a+b, c+d) \otimes \mathcal{Z}(c+d, e+f) \arrow[r, "\mu"] & \mathcal{Z}(a+b, e+f).
\end{tikzcd}$$

This equation in the form

$$\psi(a,b,c,d) \cdot \psi(c,d,e,f) = (-1)^{(d-b)(e-c)}\psi(a,b,e,f)$$

specifies the boundary of the 2-cochain $\psi : \mathbb{Z}^4 \to \Bbbk^\times$. Generic solution to this equation is

$$\psi(a,b,c,d) = (-1)^{c(b-d)}\chi(a,b) \cdot \chi(c,d)^{-1} \tag{10.1.2}$$

for some function $\chi : \mathbb{Z}^2 \to \Bbbk^\times$ (take $\chi(a,b) = \psi(a,b,0,0)$).

Associativity of the algebra $(\mathcal{Z}, \otimes_\psi)$

$$\begin{tikzcd}
(\mathcal{Z} \boxtimes \mathcal{Z} \boxtimes \mathcal{Z})(a \times b \times c, d \times e \times f) \arrow[r, "1\boxtimes\otimes_\psi"] \arrow[d, "\otimes_\psi\boxtimes 1"'] \arrow[dr, phantom, "="] & (\mathcal{Z} \boxtimes \mathcal{Z})(a \times (b+c), d \times (e+f)) \arrow[d, "\otimes_\psi"] \\
(\mathcal{Z} \boxtimes \mathcal{Z})((a+b) \times c, (d+e) \times f) \arrow[r, "\otimes_\psi"] & \mathcal{Z}(a+b+c, d+e+f)
\end{tikzcd}$$

is expressed by the equation

$$\psi(a,b,d,e) \cdot \psi(a+b,c,d+e,f) = \psi(b,c,e,f) \cdot \psi(a,b+c,d,e+f). \tag{10.1.3}$$

It means that the function $\overline{\psi} : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \Bbbk^\times$, $\overline{\psi}(a,b,c,d) = \psi(a,c,b,d)$ is a 2-cocycle on the group $\mathbb{Z}^2$. For a graded category $\mathcal{Z}$ with multiplication (10.1.1) related to $(\mathcal{Z}, \mu)$ via the category isomorphism $\xi : (\mathcal{Z}, \mu') \to (\mathcal{Z}, \mu)$, $n \mapsto n$, $\xi(m,n) : \mathcal{Z}(m,n) \to \mathcal{Z}(m,n)$, we would get the 2-cocycle

$$\overline{\psi}'(a,b,c,d) = \xi(a,b) \cdot \xi(c,d) \cdot \overline{\psi}(a,b,c,d) \cdot \xi(a+c,b+d)^{-1},$$

cohomologous to $\overline{\psi}$. Thus, the algebra structure of $\mathcal{Z}$ possesses an invariant – the cohomology class $[\overline{\psi}] \in H^2(\mathbb{Z}^2, \Bbbk^\times)$.

Plugging expression (10.1.2) into equation (10.1.3), we reduce the latter to

$$\chi(a,b) \cdot \chi(a,b+c)^{-1} \cdot \chi(a+b,c) \cdot \chi(b,c)^{-1} = \chi(d,e) \cdot \chi(d,e+f)^{-1} \cdot \chi(d+e,f) \cdot \chi(e,f)^{-1}.$$

The common value of the left and right hand sides does not depend on the arguments $a,b,c,d,e,f \in \mathbb{Z}$. Setting $d = e = f = 0$ we find that this constant equals 1. Thus, $\chi : \mathbb{Z}^2 \to \Bbbk^\times$ is a 2-cocycle on the group $\mathbb{Z}$ with values in $\Bbbk^\times$. Since $\mathbb{Z}$ is a free group, its cohomology group $H^2(\mathbb{Z}, \Bbbk^\times)$ vanishes. Therefore, the cocycle χ has the form

$$\chi(a,b) = \lambda(a) \cdot \lambda(b) \cdot \lambda(a+b)^{-1}$$

for some function $\lambda : \mathbb{Z} \to \Bbbk^\times$. The corresponding function $\psi : \mathbb{Z}^4 \to \Bbbk^\times$ is given by the formula

$$\psi_\lambda(a,b,c,d) = (-1)^{c(b-d)} \lambda(a) \cdot \lambda(b) \cdot \lambda(a+b)^{-1} \cdot \lambda(c)^{-1} \cdot \lambda(d)^{-1} \cdot \lambda(c+d). \tag{10.1.4}$$

Denote the multiplication functor $\otimes_{\psi_\lambda}$ also by $\otimes^\lambda$. Define an automorphism of the graded category $\mathcal{Z}$ as $\overline{\lambda} : \mathcal{Z} \to \mathcal{Z}$, $n \mapsto n$, $\overline{\lambda} = \lambda(m)^{-1} \cdot \lambda(n) : \mathcal{Z}(m,n) \to \mathcal{Z}(m,n)$. We claim that $\overline{\lambda} : (\mathcal{Z}, \otimes^1) \to (\mathcal{Z}, \otimes^\lambda)$ is an algebra isomorphism, where the first algebra uses

$$\lambda_1(a) = 1, \qquad \psi_1(a,b,c,d) = (-1)^{c(b-d)}.$$

This follows from the computation

$$\begin{tikzcd}
(\mathcal{Z}\boxtimes\mathcal{Z})(a\times b,c\times d) \arrow[r, equal] \arrow[d, "\overline{\lambda}\boxtimes\overline{\lambda}"'] & \mathcal{Z}(a,c)\otimes\mathcal{Z}(b,d) \arrow[r, "\otimes^1", "\psi_1(a,b,c,d)"'] \arrow[d, "\frac{\lambda(c)}{\lambda(a)}\otimes\frac{\lambda(d)}{\lambda(b)}"'] \arrow[dr, phantom, "="] & \mathcal{Z}(a+b,c+d) \arrow[d, "\frac{\lambda(c+d)}{\lambda(a+b)}"', "=\overline{\lambda}"] \\
(\mathcal{Z}\boxtimes\mathcal{Z})(a\times b,c\times d) \arrow[r, equal] & \mathcal{Z}(a,c)\otimes\mathcal{Z}(b,d) \arrow[r, "\otimes^\lambda", "\psi_\lambda(a,b,c,d)"'] & \mathcal{Z}(a+b,c+d)
\end{tikzcd}$$

which is equivalent to equation (10.1.4). The multiplicative cocycle $\overline{\psi}_1 \in Z^2(\mathbb{Z}^2,\Bbbk^\times)$ comes via the homomorphism $\mathbb{Z}/2\to\Bbbk^\times$, $a\mapsto(-1)^a$, from the additive cocycle $\mathbb{Z}^2\times\mathbb{Z}^2\to\mathbb{Z}/2$, $(a,c,b,d)\mapsto c(b-d)\pmod 2$, which represents the only non-trivial element of the cohomology group $H^2(\mathbb{Z}^2,\mathbb{Z}/2)=\mathbb{Z}/2$ with values in the trivial $\mathbb{Z}^2$-module $\mathbb{Z}/2$.

In the following we shall consider only multiplication $\otimes_\mathcal{Z}=\otimes^1$ in $\mathcal{Z}$ specified by the function $\psi_1(a,b,c,d)=(-1)^{c(b-d)}$. Clearly, the algebra $(\mathcal{Z},\otimes_\mathcal{Z})$ is unital with the unit $\eta_\mathcal{Z}:\mathbb{1}\to\mathcal{Z}$, $*\mapsto 0$, $\mathrm{id}:\mathbb{1}(*,*)=\Bbbk\to\mathcal{Z}(0,0)$. The same statement holds for an arbitrary $(\mathcal{Z},\otimes^\lambda)$, since $\psi=\psi_\lambda$ given by (10.1.4) satisfies $\psi(a,0,c,0)=0=\psi(0,b,0,d)$. Note that the algebra $\mathcal{Z}$ is not commutative.

10.2 Remark. Let $(\mathcal{C},\otimes,\mathbb{1})$ be a strict monoidal graded $\Bbbk$-linear category. We say that an arrow $\alpha:X\to Y$ in $\mathcal{C}$ is *bi-invertible* if it has both inverse morphism $\alpha^{-1}:Y\to X$ and a tensor inverse $\alpha^{\otimes-1}:X^{\otimes-1}\to Y^{\otimes-1}$, that is, $\alpha\cdot\alpha^{-1}=1_X$, $\alpha^{-1}\cdot\alpha=1_Y$, and $X\otimes X^{\otimes-1}=\mathbb{1}=X^{\otimes-1}\otimes X$, $Y\otimes Y^{\otimes-1}=\mathbb{1}=Y^{\otimes-1}\otimes Y$, $\alpha\otimes\alpha^{\otimes-1}=1_\mathbb{1}=\alpha^{\otimes-1}\otimes\alpha$. Let $y:\mathbb{1}\to Y$ be a bi-invertible arrow of degree -1. In assumption that all $Y^{\otimes n}$, $n\in\mathbb{Z}$ are distinct, y generates a strict monoidal graded $\Bbbk$-linear subcategory $\mathcal{J}$ of $\mathcal{C}$ with objects $Y^{\otimes n}$, $n\in\mathbb{Z}$, and with morphisms $\mathcal{J}(Y^{\otimes n},Y^{\otimes k})=\Bbbk y_{n,k}$, where

$$y_{n,k}=\left(Y^{\otimes n}\xrightarrow{(y^{-1})^{\otimes n}}\mathbb{1}^{\otimes n}=\mathbb{1}=\mathbb{1}^{\otimes k}\xrightarrow[((y^{-1})^{\otimes k})^{-1}]{(-1)^{\binom{k}{2}}y^{\otimes k}\ \|}Y^k\right),\qquad n,k\in\mathbb{Z},$$

has degree $n-k$. In fact,

$$y_{n,m}\cdot y_{m,k}=y_{n,k},\qquad y_{n,n}=1,$$
$$y_{n,k}\otimes y_{m,l}=\psi_1(n,m,k,l)y_{n+m,k+l},\qquad \psi_1(n,m,k,l)=(-1)^{k(m-l)}.$$

The latter equation is proved as follows:

$$\begin{aligned}
y_{n,k}\otimes y_{m,l}&=[(y^{-1})^{\otimes n}\cdot(-1)^{\binom{k}{2}}y^{\otimes k}]\otimes[(y^{-1})^{\otimes m}\cdot(-1)^{\binom{l}{2}}y^{\otimes l}]\\
&=(-1)^{km+\binom{k}{2}+\binom{l}{2}}[(y^{-1})^{\otimes n}\otimes(y^{-1})^{\otimes m}]\cdot(y^{\otimes k}\otimes y^{\otimes l})\\
&=(-1)^{km+\binom{k}{2}+\binom{l}{2}-\binom{k+l}{2}}(y^{-1})^{\otimes n+m}\cdot(-1)^{\binom{k+l}{2}}y^{\otimes k+l}\\
&=(-1)^{k(m-l)}y_{n+m,k+l}.
\end{aligned}$$

There is an isomorphism of categories $\mathcal{Z}\to\mathcal{J}$, $n\mapsto Y^{\otimes n}$, $\mathcal{Z}(n,k)\ni 1s^{k-n}\mapsto y_{n,k}\in\mathcal{J}(n,k)$. It shows that $\mathcal{Z}$ is the universal (initial) strict monoidal graded $\Bbbk$-linear category freely generated by a bi-invertible morphism y as above.

10.3 The functor of shifts. Given a graded quiver $\mathcal{A}$, we produce another one $\mathcal{A}^{[]} = \mathcal{A} \boxtimes \mathcal{Z}$, obtained by adding formal shifts of objects. Here $\mathcal{Z}$ is the differential graded category of Section 10.1. The set of objects is

$$\operatorname{Ob} \mathcal{A}^{[]} = \{X[n] = (X, n) \mid X \in \operatorname{Ob} \mathcal{A}, n \in \mathbb{Z}\} = \operatorname{Ob} \mathcal{A} \times \mathbb{Z}.$$

The graded $\Bbbk$-module of morphisms is $\mathcal{A}^{[]}(X[n], Y[m]) = \mathcal{A}(X,Y) \otimes \Bbbk[m-n]$. We identify it with $\mathcal{A}(X,Y)[m-n]$ via the isomorphism

$$\mathcal{A}(X,Y)[m-n] \xrightarrow{s^{n-m}} \mathcal{A}(X,Y) \xrightarrow{\lambda^{\mathsf{I}} \cdot} \mathcal{A}(X,Y) \otimes \Bbbk \xrightarrow{1 \otimes s^{m-n}} \mathcal{A}(X,Y) \otimes \Bbbk[m-n]$$
$$= \mathcal{A}(X,Y) \otimes \mathcal{Z}(n,m) = \mathcal{A}^{[]}(X[n], Y[m]).$$

Given a morphism of graded quivers $f : \mathcal{A} \to \mathcal{B}$, we define another one $f^{[]} = f \boxtimes 1_{\mathcal{Z}} : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$. On objects it acts as $\operatorname{Ob} f^{[]} : X[n] \mapsto (Xf)[n]$. This defines a functor $-^{[]} = - \boxtimes \mathcal{Z} : \mathscr{Q} \to \mathscr{Q}$. Commutative diagram

$$\begin{tikzcd}
\mathcal{A}(X,Y)[m-n] \arrow[r, "s^{n-m}"] \arrow[d, "f^{[m-n]}"'] & \mathcal{A}(X,Y) \arrow[r, "\lambda^{\mathsf{I}} \cdot"] \arrow[d, "f"'] & \mathcal{A}(X,Y) \otimes \Bbbk \arrow[r, "1 \otimes s^{m-n}"] \arrow[d, "f \otimes 1"] & \mathcal{A}(X,Y) \otimes \mathcal{Z}(n,m) \arrow[d, "f \otimes 1"] \\
\mathcal{B}^{[]}(Xf, Yf)[m-n] \arrow[r, "s^{n-m}"] & \mathcal{B}(Xf, Yf) \arrow[r, "\lambda^{\mathsf{I}} \cdot"] & \mathcal{B}(Xf, Yf) \otimes \Bbbk \arrow[r, "1 \otimes s^{m-n}"] & \mathcal{B}(Xf, Yf) \otimes \mathcal{Z}(n,m)
\end{tikzcd}$$

describes the action of $-^{[]}$ on morphisms in another presentation:

$$f^{[m-n]} = s^{n-m} f s^{m-n} : \mathcal{A}(X,Y)[m-n] \to \mathcal{B}(Xf, Yf)[m-n].$$

Let us make $-^{[]}$ into a lax Monoidal functor $(-^{[]}, \sigma^I) : (\mathscr{Q}, \boxtimes^I_u, \lambda^f_u) \to (\mathscr{Q}, \boxtimes^I_u, \lambda^f_u)$. The functorial morphism of graded quivers

$$\sigma^I : \boxtimes^{i \in I}_u (\mathcal{A}_i^{[]}) \to (\boxtimes^{i \in I}_u \mathcal{A}_i)^{[]}$$

acts on objects via $(X_i[n_i])_{i \in I} \mapsto (X_i)_{i \in I}[\sum_{i \in I} n_i]$. Its action on morphisms is defined below. First of all we describe its plain version $(-^{[]}, \widetilde{\sigma}^I) : (\mathscr{Q}, \boxtimes^I, \lambda^f), \to (\mathscr{Q}, \boxtimes^I, \lambda^f)$

$$\widetilde{\sigma}^I = \bigl[\boxtimes^{i \in I} (\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i \in I} \mathcal{A}_i) \boxtimes (\boxtimes^I \mathcal{Z}) \xrightarrow{1 \boxtimes \otimes^I_{\mathcal{Z}}} (\boxtimes^{i \in I} \mathcal{A}_i) \boxtimes \mathcal{Z}\bigr].$$

10.4 Proposition. *The pair* $(-^{[]}, \widetilde{\sigma}^I) : \mathscr{Q}_p = (\mathscr{Q}, \boxtimes^I, \lambda^f) \to (\mathscr{Q}, \boxtimes^I, \lambda^f) = \mathscr{Q}_p$ *is a lax Monoidal functor.*

Proof. Lax Monoidality of $(-^{[]}, \widetilde{\sigma}^I)$ expressed by equation (2.17.2) takes the form

$$\bigl[\boxtimes^{i \in I} (\mathcal{A}_i^{[]}) \xrightarrow{\lambda^f} \boxtimes^{j \in J} \boxtimes^{i \in f^{-1} j} (\mathcal{A}_i^{[]}) \xrightarrow{\boxtimes^{j \in J} \widetilde{\sigma}^{f^{-1} j}} \boxtimes^{j \in J} (\boxtimes^{i \in f^{-1} j} \mathcal{A}_i)^{[]} \xrightarrow{\widetilde{\sigma}^J} (\boxtimes^{j \in J} \boxtimes^{i \in f^{-1} j} \mathcal{A}_i)^{[]}\bigr]$$
$$= \bigl[\boxtimes^{i \in I} (\mathcal{A}_i^{[]}) \xrightarrow{\widetilde{\sigma}^I} (\boxtimes^{i \in I} \mathcal{A}_i)^{[]} \xrightarrow{(\lambda^f)^{[]}} (\boxtimes^{j \in J} \boxtimes^{i \in f^{-1} j} \mathcal{A}_i)^{[]}\bigr]. \quad (10.4.1)$$

This equation coincides with the exterior of the following diagram:

$$
\begin{tikzcd}[column sep=small]
& \boxtimes^{j\in J}\big[(\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes(\boxtimes^{f^{-1}j}\mathcal{Z})\big] \arrow[dr, "\boxtimes^{j\in J}(1\boxtimes\otimes_{\mathcal{Z}}^{f^{-1}j})"] \arrow[dd, "\sigma_{(12)}"'] & \\
\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}(\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[ur, "\boxtimes^{j\in J}\sigma_{(12)}"] & & \boxtimes^{j\in J}\big[(\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes\mathcal{Z}\big] \arrow[dd, "\sigma_{(12)}"] \\
& (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes(\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\mathcal{Z}) \arrow[dr, "1\boxtimes(\boxtimes^{j\in J}\otimes_{\mathcal{Z}}^{f^{-1}j})"] & \\
\boxtimes^{i\in I}(\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[uu, "\lambda^f"] \arrow[dd, "\sigma_{(12)}"'] & & (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes(\boxtimes^{J}\mathcal{Z}) \arrow[dd, "1\boxtimes\otimes_{\mathcal{Z}}^{J}"] \\
& (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes(\boxtimes^{I}\mathcal{Z}) \arrow[uu, "1\boxtimes\lambda^f"'] \arrow[dr, "1\boxtimes\otimes_{\mathcal{Z}}^{I}"] & \\
(\boxtimes^{i\in I}\mathcal{A}_i)\boxtimes(\boxtimes^{I}\mathcal{Z}) \arrow[ur, "\lambda^f\boxtimes 1"] \arrow[r, "1\boxtimes\otimes_{\mathcal{Z}}^{I}"] & (\boxtimes^{i\in I}\mathcal{A}_i)\boxtimes\mathcal{Z} \arrow[r, "\lambda^f\boxtimes 1"] & (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{A}_i)\boxtimes\mathcal{Z}
\end{tikzcd}
$$

The hexagon follows from coherence principle of Remark 2.34. The top right square is due to naturality of $\sigma_{(12)}$. The middle right square expresses the fact that $(\mathcal{Z},\otimes_{\mathcal{Z}}^{I})$ is an algebra, cf. (2.25.1). The bottom square commutes for obvious reasons. Therefore, $(-^{[]},\widetilde{\sigma}^I)$ is a lax Monoidal functor. $\square$

Since the algebra $\mathcal{Z}$ is not commutative, the lax Monoidal functor $(-^{[]},\widetilde{\sigma})$ is not symmetric.

The unital version of $\widetilde{\sigma}$ is $\sigma^I:\boxtimes_u^{i\in I}(\mathcal{C}_i^{[]})\to(\boxtimes_u^{i\in I}\mathcal{C}_i)^{[]}$, defined by

$$
\begin{aligned}
\sigma^I = \Big[\boxtimes_u^{i\in I}(\mathcal{C}_i\boxtimes\mathcal{Z}) = \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}T^{\chi(i\in S)}(\mathcal{C}_i\boxtimes\mathcal{Z}) \\
\xrightarrow[\sim]{\oplus\boxtimes^I\overline{\varkappa}} \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}(T^{\chi(i\in S)}\mathcal{C}_i\boxtimes T^{\chi(i\in S)}\mathcal{Z}) \\
\xrightarrow[\sim]{\oplus\sigma_{(12)}} \bigoplus_{\varnothing\neq S\subset I}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i)\boxtimes(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{Z}) \\
\xrightarrow{\oplus 1\boxtimes(\boxtimes^I\mu_{\mathcal{Z}})} \bigoplus_{\varnothing\neq S\subset I}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i)\boxtimes(\boxtimes^I\mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}^{I}} (\boxtimes_u^{i\in I}\mathcal{C}_i)\boxtimes\mathcal{Z}\Big],
\end{aligned}
$$

where the composition $\mu_{\mathcal{Z}}$ is the identity morphism $\mu_{\mathcal{Z}}=\mathrm{id}_{\mathcal{Z}}:T^1\mathcal{Z}\to\mathcal{Z}$, or the unit $\mu_{\mathcal{Z}}=\eta_{\mathcal{Z}}:T^0\mathcal{Z}\to\mathcal{Z}$.

10.5 Lemma. *The morphisms σ^I and $\widetilde{\sigma}^I$ are related by the following embeddings:*

$$
\begin{tikzcd}
\boxtimes_u^{i\in I}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[rr, "\sigma^I"] \arrow[d, hook, "\mathrm{in}"'] \arrow[drr, phantom, "="] & & (\boxtimes_u^{i\in I}\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[d, hook, "\mathrm{in}\boxtimes 1"] \\
\boxtimes^{i\in I} T^{\leqslant 1}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, hook, "\boxtimes^I \xi'"] & \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\widetilde{\sigma}^I"] & (\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i) \boxtimes \mathcal{Z}
\end{tikzcd}
$$

where

$$\xi' = \bigl(T^{\leqslant 1}(\mathcal{C} \boxtimes \mathcal{Z}) \xrightarrow{\overline{\varkappa}} T^{\leqslant 1}\mathcal{C} \boxtimes T^{\leqslant 1}\mathcal{Z} \xrightarrow{1\boxtimes \mu_{\mathcal{Z}}} T^{\leqslant 1}\mathcal{C} \boxtimes \mathcal{Z}\bigr)$$

has restrictions $\xi'\big|_{\mathcal{C}\boxtimes\mathcal{Z}} = \mathrm{id}$, $\xi'\big|_{T^0(\mathcal{C}\boxtimes\mathcal{Z})} = \bigl(T^0(\mathcal{C} \boxtimes \mathcal{Z}) \xrightarrow[\sim]{\overline{\varkappa}} T^0\mathcal{C} \boxtimes T^0\mathcal{Z} \xhookrightarrow{1\boxtimes\eta_{\mathcal{Z}}} T^0\mathcal{C} \boxtimes \mathcal{Z}\bigr)$, *and the embedding* in *is the natural transformation*

$$\mathrm{in} = \bigl[\boxtimes_u^{i\in I}\mathcal{C}_i \xhookrightarrow{\mathrm{in}_1} T^{\leqslant 1} \boxtimes_u^{i\in I} \mathcal{C}_i \xrightarrow[\sim]{(\vartheta^I)^{-1}} \boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i\bigr].$$

Proof. This is the exterior of the following diagram:

$$
\begin{tikzcd}
\boxtimes_u^{i\in I}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "(\boxtimes^I\overline{\varkappa})\sigma_{(12)}"] \arrow[d, hook, "\mathrm{in}"'] & \bigoplus_{\varnothing\ne S\subset I}(\boxtimes^{i\in I} T^{\chi(i\in S)}\mathcal{C}_i) \boxtimes (\boxtimes^{i\in I} T^{\chi(i\in S)}\mathcal{Z}) \arrow[r, "1\boxtimes(\mu\cdot\otimes_{\mathcal{Z}}^I)"] \arrow[d, hook, "\mathrm{in}"'] & (\boxtimes_u^{i\in I}\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[d, hook, "\mathrm{in}\boxtimes 1"] \\
\boxtimes^{i\in I} T^{\leqslant 1}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "(\boxtimes^I\overline{\varkappa})\sigma_{(12)}"] \arrow[d, hook, "\boxtimes^I\xi'"'] & \bigoplus_{S\subset I}(\boxtimes^{i\in I} T^{\chi(i\in S)}\mathcal{C}_i) \boxtimes (\boxtimes^{i\in I} T^{\chi(i\in S)}\mathcal{Z}) \arrow[r, "1\boxtimes(\mu\cdot\otimes_{\mathcal{Z}}^I)"] \arrow[d, "1\boxtimes(\boxtimes^{i\in I}\mu_{\mathcal{Z}})"'] & (\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[d, equal] \\
\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\sigma_{(12)}"] & (\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i) \boxtimes (\boxtimes^{i\in I}\mathcal{Z}) \arrow[r, "1\boxtimes\otimes_{\mathcal{Z}}^I"] & (\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i) \boxtimes \mathcal{Z}
\end{tikzcd}
\tag{10.5.1}
$$

in which all squares commute. □

10.6 Lemma. *The morphisms σ^I, $\widetilde{\sigma}^I$ and ξ' are related by the following equation for an arbitrary set I:*

$$
\begin{tikzcd}
\boxtimes^{i\in I} T^{\leqslant 1}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\vartheta^I"] \arrow[d, "\boxtimes^I\xi'"'] & T^{\leqslant 1} \boxtimes_u^{i\in I} (\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "T^{\leqslant 1}\sigma^I"] \arrow[d, phantom, "="] & T^{\leqslant 1}[(\boxtimes_u^{i\in I}\mathcal{C}_i) \boxtimes \mathcal{Z}] \arrow[d, "\xi'"] \\
\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\widetilde{\sigma}^I"] & (\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[r, "\vartheta^I\boxtimes 1"] & (T^{\leqslant 1} \boxtimes_u^{i\in I} \mathcal{C}_i) \boxtimes \mathcal{Z}
\end{tikzcd}
\tag{10.6.1}
$$

Proof. Indeed, on the summand $\boxtimes_u^{i\in I}(\mathcal{C}_i \boxtimes \mathcal{Z})$ of the left top corner the equation reduces to the statement of Lemma 10.5. On the summand $T^0 \boxtimes_u^{i\in I} (\mathcal{C}_i \boxtimes \mathcal{Z})$ the above equation

expands to

$$\begin{aligned}
&\bigl[T^0 \boxtimes^{i\in I} (\mathcal{C}_i \boxtimes \mathcal{Z}) \xrightarrow{T^0\widetilde{\sigma}^I} T^0[(\boxtimes^{i\in I}\mathcal{C}_i) \boxtimes \mathcal{Z}] \xrightarrow{\overline{\varkappa}} (T^0 \boxtimes^{i\in I} \mathcal{C}_i) \boxtimes T^0\mathcal{Z} \\
&\qquad \xrightarrow{1\boxtimes\eta_{\mathcal{Z}}} (T^0 \boxtimes^{i\in I} \mathcal{C}_i) \boxtimes \mathcal{Z} \xrightarrow{\overline{\varkappa}\boxtimes 1} (\boxtimes^{i\in I} T^0\mathcal{C}_i) \boxtimes \mathcal{Z}\bigr] \\
&= \bigl[T^0 \boxtimes^{i\in I} (\mathcal{C}_i \boxtimes \mathcal{Z}) \xrightarrow{\overline{\varkappa}} \boxtimes^{i\in I} T^0(\mathcal{C}_i \boxtimes \mathcal{Z}) \xrightarrow{\boxtimes^I\overline{\varkappa}} \boxtimes^{i\in I} (T^0\mathcal{C}_i \boxtimes T^0\mathcal{Z}) \\
&\qquad \xrightarrow{\boxtimes^I(1\boxtimes\eta_{\mathcal{Z}})} \boxtimes^{i\in I} (T^0\mathcal{C}_i \boxtimes \mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}T^0\mathcal{C}_i) \boxtimes (\boxtimes^I\mathcal{Z}) \xrightarrow{1\boxtimes\otimes^I_{\mathcal{Z}}} (\boxtimes^{i\in I}T^0\mathcal{C}_i) \boxtimes \mathcal{Z}\bigr].
\end{aligned}$$

This equation reduces to

$$\begin{tikzcd}
T^0 \boxtimes^I \mathcal{Z} \arrow[rr, "T^0\otimes^I_{\mathcal{Z}}"] \arrow[d, "\overline{\varkappa}"'] & & T^0\mathcal{Z} \arrow[d, "\eta_{\mathcal{Z}}"] \\
\boxtimes^I T^0\mathcal{Z} \arrow[r, "\boxtimes^I\eta_{\mathcal{Z}}"] & \boxtimes^I\mathcal{Z} \arrow[r, "\otimes^I_{\mathcal{Z}}"] & \mathcal{Z}
\end{tikzcd}$$

which means precisely the following: the tensor product of units $1_{n_i} \in \mathcal{Z}(n_i, n_i)$, $i \in I$ is equal to the unit $1_n \in \mathcal{Z}(n, n)$, $n = \sum_{i\in I} n_i$. This is certainly so, because $\otimes^I_{\mathcal{Z}}$ is a functor. □

10.7 Proposition. *The pair $(-^{[]}, \sigma^I) : \mathscr{Q}_u = (\mathscr{Q}, \boxtimes^I_u, \lambda^f_u) \to (\mathscr{Q}, \boxtimes^I_u, \lambda^f_u) = \mathscr{Q}_u$ is a lax Monoidal functor.*

Proof. To be a lax Monoidal functor $(-^{[]}, \sigma^I)$ has to satisfy the following equation:

$$\begin{aligned}
&\bigl[\boxtimes^{i\in I}_u(\mathcal{C}^{[]}_i) \xrightarrow{\lambda^f_u} \boxtimes^{j\in J}_u \boxtimes^{i\in f^{-1}j}_u(\mathcal{C}^{[]}_i) \xrightarrow{\boxtimes^{j\in J}_u\sigma^{f^{-1}j}} \boxtimes^{j\in J}_u (\boxtimes^{i\in f^{-1}j}_u\mathcal{C}_i)^{[]} \xrightarrow{\sigma^J} (\boxtimes^{j\in J}_u \boxtimes^{i\in f^{-1}j}_u \mathcal{C}_i)^{[]}\bigr] \\
&\qquad = \bigl[\boxtimes^{i\in I}_u(\mathcal{C}^{[]}_i) \xrightarrow{\sigma^I} (\boxtimes^{i\in I}_u\mathcal{C}_i)^{[]} \xrightarrow{(\lambda^f_u)^{[]}} (\boxtimes^{j\in J}_u \boxtimes^{i\in f^{-1}j}_u \mathcal{C}_i)^{[]}\bigr]. \qquad (10.7.1)
\end{aligned}$$

Its right hand side is related to the right hand side of (10.4.1) by the following embeddings:

$$\begin{tikzcd}
\boxtimes^{i\in I}_u(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\sigma^I"] \arrow[d, hook, "\mathrm{in}"'] & (\boxtimes^{i\in I}_u\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[r, "\lambda^f_u\boxtimes 1"] \arrow[dd, hook, "\mathrm{in}\boxtimes 1"] & (\boxtimes^{j\in J}_u \boxtimes^{i\in f^{-1}j}_u \mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[d, hook, "\mathrm{in}\boxtimes 1"] \\
\boxtimes^{i\in I}T^{\leqslant 1}(\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, phantom, "="] \arrow[d, hook, "\boxtimes^I\xi'"'] & \arrow[r, phantom, "="] & (\boxtimes^{j\in J}T^{\leqslant 1}\boxtimes^{i\in f^{-1}j}_u\mathcal{C}_i) \boxtimes \mathcal{Z} \arrow[d, "(\boxtimes^{j\in J}(\vartheta^{f^{-1}j})^{-1})\boxtimes 1"] \\
\boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i \boxtimes \mathcal{Z}) \arrow[r, "\widetilde{\sigma}^I"] & (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\boxtimes\mathcal{Z} \arrow[r, "\lambda^f\boxtimes 1"] & (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}T^{\leqslant 1}\mathcal{C}_i) \boxtimes \mathcal{Z}
\end{tikzcd} \qquad (10.7.2)$$

Indeed, the left pentagon is proven in Lemma 10.5, and the right pentagon follows from

the commutative diagram

$$
\begin{tikzcd}
\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, hook, "\mathrm{in}_1"] \arrow[d, "\lambda_u^f"'] \arrow[rr, bend left, "\mathrm{in}"] \arrow[dr, phantom, "="] & T^{\leqslant 1}\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, "(\vartheta^I)^{-1}"] \arrow[d, "T^{\leqslant 1}\lambda_u^f"] \arrow[drr, phantom, "="] & \boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \arrow[r, "\lambda^f"] & \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}T^{\leqslant 1}\mathcal{C}_i \arrow[d, "\boxtimes^{j\in J}\vartheta^{f^{-1}j}"] \\
\boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i \arrow[r, hook, "\mathrm{in}_1"] \arrow[rrr, bend right, "\mathrm{in}"'] & T^{\leqslant 1}\boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i \arrow[rr, "(\vartheta^J)^{-1}"] & & \boxtimes^{j\in J}T^{\leqslant 1}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i
\end{tikzcd}
\tag{10.7.3}
$$

which is an immediate consequence of definition (6.1.2) of λ_u^f.

The left hand side of (10.7.1) is related to the left hand side of (10.4.1) by the same embeddings as for right hand sides. Indeed, diagram on the following page is commutative by Lemma 10.5 and due to equation (10.6.1), which holds for an arbitrary subset $K \subset I$, in particular, for $K = f^{-1}j$, $j \in J$.

The bottom rows of diagrams (10.7.2) and (10.3) compose to the same quiver morphism due to equation (10.4.1). Their leftmost and rightmost columns pairwise coincide. Therefore, their top rows compose to the same quiver morphism. Thus equation (10.7.1) is proven and $(-^{[]}, \sigma^I) : \mathcal{Q}_u \to \mathcal{Q}_u$ is a lax Monoidal functor. □

10.8 Corollary. *ξ' is a Monoidal transformation*

$$
\begin{tikzcd}
\mathcal{Q}_u \arrow[r, "(-^{[]},\sigma)"] \arrow[d, "(T^{\leqslant 1},\vartheta)"'] & \mathcal{Q}_u \arrow[d, "(T^{\leqslant 1},\vartheta)"] \arrow[dl, Rightarrow, "\xi'"'] \\
\mathcal{Q}_p \arrow[r, "(-^{[]},\tilde{\sigma})"'] & \mathcal{Q}_p
\end{tikzcd}
$$

Indeed, this is claimed in equation (10.6.1).

10.9 The unit for the monad of shifts. For any graded $\Bbbk$-quiver $\mathcal{A}$ consider the morphism of quivers $u_{[]} : \mathcal{A} \to \mathcal{A}^{[]}$ given by the formulas $\operatorname{Ob} u_{[]} : X \mapsto X[0]$, $u_{[]} = \lambda^{\mathsf{I}\cdot} : \mathcal{A}(X,Y) \to \mathcal{A}(X,Y) \otimes \Bbbk = \mathcal{A}^{[]}(X[0], Y[0])$. One easily verifies that $u_{[]}$ determines a natural transformation $u_{[]} : \mathrm{Id} \to -^{[]} : \mathcal{Q} \to \mathcal{Q}$. Later it will become the unit of the monad $-^{[]}$.

10.10 Proposition. *The natural transformation $u_{[]}$ is Monoidal in the following two senses: $u_{[]} : \mathrm{Id} \to (-^{[]}, \tilde{\sigma}^I) : \mathcal{Q}_u \to \mathcal{Q}_u$ and $u_{[]} : \mathrm{Id} \to (-^{[]}, \sigma^I) : \mathcal{Q}_p \to \mathcal{Q}_p$.*

$$
\begin{tikzcd}
\boxtimes_u^{i\in I}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\lambda_u^f"] \arrow[dd, hook, "\mathbf{in}"'] & \boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes_u^{j\in J}\sigma^{f^{-1}j}"] \arrow[d, hook, "\mathbf{in}"'] \arrow[dr, phantom, "="] & \boxtimes_u^{j\in J}[(\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i)\boxtimes\mathcal{Z}] \arrow[r, "\sigma^J"] \arrow[d, hook, "\mathbf{in}"'] & (\boxtimes_u^{j\in J}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i)\boxtimes\mathcal{Z} \arrow[dd, hook, "\mathbf{in}\boxtimes1"'] \\
\arrow[r, phantom, "="] & \boxtimes^{j\in J}\boldsymbol{T}^{\leqslant1}\boxtimes_u^{i\in f^{-1}j}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes^{j\in J}\boldsymbol{T}^{\leqslant1}\sigma^{f^{-1}j}"] \arrow[d, "\boxtimes^{j\in J}(\vartheta^{f^{-1}j})^{-1}"', "\wr"] & \boxtimes^{j\in J}\boldsymbol{T}^{\leqslant1}[(\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i)\boxtimes\mathcal{Z}] \arrow[r, phantom, "="] \arrow[d, hook, "\boxtimes^J\xi'"'] & \\
\boxtimes^{i\in I}\boldsymbol{T}^{\leqslant1}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\lambda^f"] \arrow[d, hook, "\boxtimes^I\xi'"'] & \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\boldsymbol{T}^{\leqslant1}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[d, hook, "\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\xi'"'] & \boxtimes^{j\in J}[(\boldsymbol{T}^{\leqslant1}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i)\boxtimes\mathcal{Z}] \arrow[r, "\widetilde{\sigma}^J"] \arrow[d, "\boxtimes^{j\in J}[(\vartheta^{f^{-1}j})^{-1}\boxtimes1]"', "\wr"] & (\boxtimes^{j\in J}\boldsymbol{T}^{\leqslant1}\boxtimes_u^{i\in f^{-1}j}\mathcal{C}_i)\boxtimes\mathcal{Z} \arrow[d, "(\boxtimes^{j\in J}(\vartheta^{f^{-1}j})^{-1})\boxtimes1"', "\wr"] \\
\boxtimes^{i\in I}(\boldsymbol{T}^{\leqslant1}\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\lambda^f"] & \boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}(\boldsymbol{T}^{\leqslant1}\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes^{j\in J}\widetilde{\sigma}^{f^{-1}j}"] & \boxtimes^{j\in J}[(\boxtimes^{i\in f^{-1}j}\boldsymbol{T}^{\leqslant1}\mathcal{C}_i)\boxtimes\mathcal{Z}] \arrow[r, "\widetilde{\sigma}^J"] & (\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\boldsymbol{T}^{\leqslant1}\mathcal{C}_i)\boxtimes\mathcal{Z}
\end{tikzcd}
$$

Proof. The statement means the following two equations:

$$u_{[]} = \bigl\{\boxtimes^{i\in I}\mathcal{A}_i \xrightarrow{\boxtimes^I u_{[]}} \boxtimes^{i\in I}(\mathcal{A}_i^{[]}) \xrightarrow{\tilde\sigma^I} (\boxtimes^{i\in I}\mathcal{A}_i)^{[]}\bigr\}, \tag{10.10.1}$$

$$u_{[]} = \bigl\{\boxtimes_u^{i\in I}\mathcal{A}_i \xrightarrow{\boxtimes_u^I u_{[]}} \boxtimes_u^{i\in I}(\mathcal{A}_i^{[]}) \xrightarrow{\sigma^I} (\boxtimes_u^{i\in I}\mathcal{A}_i)^{[]}\bigr\}. \tag{10.10.2}$$

Let us prove the first one. On objects the right hand side gives the map

$$(X_i)_{i\in I} \xmapsto{\boxtimes^I(u_{[]})} (X_i[0])_{i\in I} \xmapsto{\tilde\sigma^I} (X_i)_{i\in I}[0],$$

which coincides with the map determined by $u_{[]}$ in the left hand side. On morphisms the right hand side gives the map

$$\Bigl[\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i) \xrightarrow{\otimes^I\lambda^{\mathsf{I}\cdot}} \otimes^{i\in I}\bigl[\mathcal{A}_i(X_i,Y_i)\otimes\mathcal{Z}(0,0)\bigr] \xrightarrow{\sigma_{(12)}} \\ (\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i))\otimes(\otimes^I\mathcal{Z}(0,0)) \xrightarrow[1\otimes(\lambda^{\varnothing\to I})^{-1}]{1\otimes\otimes^I_{\mathcal{Z}}} (\otimes^{i\in I}\mathcal{A}_i(X_i,Y_i))\otimes\mathcal{Z}(0,0)\Bigr] = \lambda^{\mathsf{I}\cdot}.$$

Equation follows by coherence principle of Lemma 2.33, since $\mathcal{Z}(0,0) = \Bbbk$ is the unit object, and the above equation holds in an arbitrary symmetric strictly monoidal category. Therefore, the above map coincides with the action of the left hand side of (10.10.1) on morphisms.

To prove (10.10.2) we consider the following diagram

$$\begin{tikzcd}
\boxtimes_u^{i\in I}\mathcal{C}_i \arrow[r, "\boxtimes_u^I u_{[]}"] \arrow[dd, hook, "\mathrm{in}"'] \arrow[dr, phantom, "="] & \boxtimes_u^{i\in I}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\sigma^I"] \arrow[d, hook, "\mathrm{in}"'] \arrow[dr, phantom, "="] & (\boxtimes_u^{i\in I}\mathcal{C}_i)\boxtimes\mathcal{Z} \arrow[dd, hook, "\mathrm{in}\boxtimes 1"'] \\
 & \boxtimes^{i\in I}T^{\leqslant 1}(\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[d, hook, "\boxtimes^I\xi'"] & \\
\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i \arrow[r, "\boxtimes^I u_{[]}"] \arrow[ur, "\boxtimes^I T^{\leqslant 1}u_{[]}"] & \boxtimes^{i\in I}(T^{\leqslant 1}\mathcal{C}_i\boxtimes\mathcal{Z}) \arrow[r, "\tilde\sigma^I"] & (\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{C}_i)\boxtimes\mathcal{Z}
\end{tikzcd} \tag{10.10.3}$$

The right pentagon is obtained in Lemma 10.5. The left quadrilateral expresses naturality of the embedding in. The triangle commutes due to the following equations:

$$\bigl[T^0\mathcal{C} \xrightarrow{T^0u_{[]}} T^0(\mathcal{C}\boxtimes\mathcal{Z}) \xrightarrow{\overline{\varkappa}} (T^0\mathcal{C})\boxtimes(T^0\mathcal{Z}) \xrightarrow{1_{T^0\mathcal{C}}\boxtimes\eta_{\mathcal{Z}}} T^0\mathcal{C}\boxtimes\mathcal{Z}\bigr] = u_{[]},$$

$$\bigl[\mathcal{C} \xrightarrow{u_{[]}} \mathcal{C}\boxtimes\mathcal{Z} \xrightarrow{\mathrm{id}} \mathcal{C}\boxtimes\mathcal{Z}\bigr] = u_{[]}.$$

The bottom row of diagram (10.10.3) composes to $u_{[]}$ due to (10.10.1). Naturality of $u_{[]}$ implies that the exterior of (10.10.3) would commute also if the top row is replaced with $u_{[]}$. Since the right vertical arrow is an embedding, we conclude that the composition in the top row is $u_{[]}$, that is, equation (10.10.2) holds. $\square$

10.11 The multiplication for the monad of shifts. Let us introduce a natural transformation $m_{[]} : -^{[][]} \to -^{[]} : \mathscr{Q} \to \mathscr{Q}$, which will be the multiplication for the monad $-^{[]}$. The morphism of graded $\Bbbk$-quivers $m_{[]} : \mathcal{A}^{[][]} \to \mathcal{A}^{[]}$ is defined by formula

$$m_{[]} = \big[(\mathcal{A} \boxtimes \mathcal{Z}) \boxtimes \mathcal{Z} \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} \mathcal{A} \boxtimes \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\lambda^{\mathsf{IV}}} \mathcal{A} \boxtimes (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1 \boxtimes \otimes_{\mathcal{Z}}} \mathcal{A} \boxtimes \mathcal{Z}\big],$$

which will be generalized to (C.5.1) in appendix. On objects it gives $\operatorname{Ob} m_{[]} : X[n][m] \mapsto X[n+m]$. Clearly, this is a natural transformation. It is not Monoidal, since the algebra $\mathcal{Z}$ is not commutative.

10.12 Commutation between the monad of shifts and the tensor comonad. The composition

$$\widetilde{\xi} \overset{\text{def}}{=} \big[T(\mathcal{A} \boxtimes \mathcal{Z}) \xrightarrow{\varkappa} T\mathcal{A} \boxtimes T\mathcal{Z} \xrightarrow{1 \boxtimes \mu_{\mathcal{Z}}} T\mathcal{A} \boxtimes \mathcal{Z}\big]$$

will be given an explicit presentation in (10.17.2). Its restriction

$$\xi \overset{\text{def}}{=} \big[T^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \xrightarrow{\varkappa} T^{\geqslant 1}\mathcal{A} \boxtimes T^{\geqslant 1}\mathcal{Z} \xrightarrow{1 \boxtimes \mu_{\mathcal{Z}}} T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}\big]$$

will become a commutation law between a comonad and a monad. There is an obvious commutative diagram

$$\begin{tikzcd}
T^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "\varkappa"] \arrow[d, hook, "\mathrm{in}"'] & T^{\geqslant 1}\mathcal{A} \boxtimes T^{\geqslant 1}\mathcal{Z} \arrow[r, "1 \boxtimes \mu_{\mathcal{Z}}"] \arrow[d, hook, "\mathrm{in} \boxtimes \mathrm{in}"'] & T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z} \arrow[d, hook, "\mathrm{in} \boxtimes 1"] \\
T(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "\varkappa"] & T\mathcal{A} \boxtimes T\mathcal{Z} \arrow[r, "1 \boxtimes \mu_{\mathcal{Z}}"] & T\mathcal{A} \boxtimes \mathcal{Z}
\end{tikzcd} \tag{10.12.1}$$

where the upper row composes to ξ and the lower row to $\widetilde{\xi}$.

10.13 Proposition. *The morphisms*

$$\begin{aligned}
\widetilde{\xi} &: (-^{[]}, \sigma) \cdot (T, \widetilde{\tau}) \quad \to (T, \widetilde{\tau}) \cdot (-^{[]}, \widetilde{\sigma}) \quad : \mathscr{Q}_u \to \mathscr{Q}_p, \\
\xi &: (-^{[]}, \sigma) \cdot (T^{\geqslant 1}, \tau) \to (T^{\geqslant 1}, \tau) \cdot (-^{[]}, \sigma) : \mathscr{Q}_u \to \mathscr{Q}_u
\end{aligned}$$

are Monoidal transformations.

Proof. The Monoidality of $\widetilde{\xi}$ is expressed by the left hexagon of diagram

$$\begin{tikzcd}
\boxtimes^{i\in I} T(\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\widetilde{\tau}"] \arrow[d, "\boxtimes^I \widetilde{\xi}"'] & T \boxtimes_u^{i\in I} (\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "T\sigma^I"] & T[(\boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z}] \arrow[r, hook, "T[\mathrm{in} \boxtimes 1]"] \arrow[d, "\widetilde{\xi}"'] \arrow[dr, phantom, "="] & T[(\boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i) \boxtimes \mathcal{Z}] \arrow[d, "\widetilde{\xi}"] \\
\boxtimes^{i\in I} (T\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\widetilde{\sigma}^I"] & (\boxtimes^{i\in I} T\mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[r, "\widetilde{\tau} \boxtimes 1"] & (T \boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[r, hook, "(T\,\mathrm{in}) \boxtimes 1"] & (T \boxtimes^{i\in I} T^{\leqslant 1}\mathcal{A}_i) \boxtimes \mathcal{Z}
\end{tikzcd}$$

It suffices to prove commutativity of its exterior. Using Lemma 10.5 we write the equation as follows:

$$
\begin{aligned}
&\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\widetilde{\tau}} T\boxtimes^{i\in I}_u(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{T\,\mathrm{in}} T\boxtimes^{i\in I}T^{\leqslant1}(\mathcal{A}_i\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{T\boxtimes^I\xi'} T\boxtimes^{i\in I}(T^{\leqslant1}\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{T\widetilde{\sigma}} T[(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}] \xrightarrow{\widetilde{\xi}} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big]\\
&=\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\boxtimes^I\widetilde{\xi}} \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\widetilde{\sigma}^I} (\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes\mathcal{Z}\\
&\qquad\xrightarrow{\widetilde{\tau}\boxtimes1} (T\boxtimes^{i\in I}_u\mathcal{A}_i)\boxtimes\mathcal{Z} \xhookrightarrow{(T\,\mathrm{in})\boxtimes1} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big].
\end{aligned}
$$

Substituting the definitions we get the equation

$$
\begin{aligned}
&\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\widetilde{\tau}} T\boxtimes^{i\in I}_u(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{T\,\mathrm{in}} T\boxtimes^{i\in I}T^{\leqslant1}(\mathcal{A}_i\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{T\boxtimes^I\varkappa} T\boxtimes^{i\in I}(T^{\leqslant1}\mathcal{A}_i\boxtimes T^{\leqslant1}\mathcal{Z}) \xrightarrow{T\boxtimes^I(1\boxtimes\mu_{\mathcal{Z}})} T\boxtimes^{i\in I}(T^{\leqslant1}\mathcal{A}_i\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{T\sigma_{(12)}} T[(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes(\boxtimes^I\mathcal{Z})] \xrightarrow{T[1\boxtimes\otimes^I_{\mathcal{Z}}]} T[(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}]\\
&\qquad\xrightarrow{\varkappa} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes T\mathcal{Z} \xrightarrow{1\boxtimes\mu_{\mathcal{Z}}} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big]\\
&=\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\boxtimes^I\varkappa} \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes T\mathcal{Z}) \xrightarrow{\boxtimes^I(1\boxtimes\mu_{\mathcal{Z}})} \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes(\boxtimes^I\mathcal{Z}) \xrightarrow{1\boxtimes\otimes^I_{\mathcal{Z}}} (\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes\mathcal{Z}\\
&\qquad\xrightarrow{\widetilde{\tau}\boxtimes1} (T\boxtimes^{i\in I}_u\mathcal{A}_i)\boxtimes\mathcal{Z} \xrightarrow{(T\,\mathrm{in})\boxtimes1} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big].
\end{aligned}
$$

By naturality we may rewrite this equation as follows:

$$
\begin{aligned}
&\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\widetilde{\tau}\cdot T\,\mathrm{in}} T\boxtimes^{i\in I}T^{\leqslant1}(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{T\boxtimes^I\varkappa} T\boxtimes^{i\in I}(T^{\leqslant1}\mathcal{A}_i\boxtimes T^{\leqslant1}\mathcal{Z})\\
&\qquad\xrightarrow{T\sigma_{(12)}} T[(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes(\boxtimes^I T^{\leqslant1}\mathcal{Z})] \xrightarrow{\varkappa} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes(T\boxtimes^I T^{\leqslant1}\mathcal{Z})\\
&\qquad\xrightarrow{1\boxtimes[(T\boxtimes^I\mu_{\mathcal{Z}})\cdot(T\otimes^I_{\mathcal{Z}})\cdot\mu_{\mathcal{Z}}]} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big]\\
&=\big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\boxtimes^I\varkappa} \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes T\mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes(\boxtimes^I T\mathcal{Z})\\
&\qquad\xrightarrow{(\widetilde{\tau}\cdot T\,\mathrm{in})\boxtimes1} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes(\boxtimes^I T\mathcal{Z}) \xrightarrow{1\boxtimes[(\boxtimes^I\mu_{\mathcal{Z}})\cdot\otimes^I_{\mathcal{Z}}]} (T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i)\boxtimes\mathcal{Z}\big].
\end{aligned}
$$

Let us prove this equation starting from the summand $\boxtimes^{i\in I}T^{m_i}(\mathcal{A}_i\boxtimes\mathcal{Z})$, $m_i\geqslant0$, of the source. Let $m\geqslant0$ be an integer. Consider a subset $S\subset I\times\mathbf{m}$ such that $\mathrm{pr}_2 S=\mathbf{m}$ and $S_i=\{p\in\mathbf{m}\mid(i,p)\in S\}$ has cardinality $|S_i|=m_i$. Expanding $\widetilde{\tau}$ we may present the above equation in the form

$$
\begin{aligned}
&\big[\boxtimes^{i\in I}T^{m_i}(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\alpha} (\otimes^{p\in\mathbf{m}}\boxtimes^{i\in I}T^{\chi((i,p)\in S)}\mathcal{A}_i)\boxtimes(\otimes^{p\in\mathbf{m}}\boxtimes^{i\in I}T^{\chi((i,p)\in S)}\mathcal{Z})\\
&\qquad\xrightarrow{1\boxtimes\beta} (\otimes^{p\in\mathbf{m}}\boxtimes^{i\in I}T^{\chi((i,p)\in S)}\mathcal{A}_i)\boxtimes\mathcal{Z}\big]
\end{aligned}
$$

$$= \bigl[\boxtimes^{i\in I} T^{m_i}(\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\gamma} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{m_i}\mathcal{Z}) \\ \xrightarrow{1\boxtimes\delta} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes \mathcal{Z}\bigr], \quad (10.13.1)$$

where

$$\alpha = \bigl[\boxtimes^{i\in I} T^{m_i}(\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}(\mathcal{A}_i \boxtimes \mathcal{Z}) \\ \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}(\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\otimes^{\mathbf{m}}\boxtimes^I\overline{\varkappa}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I}(T^{\chi((i,p)\in S)}\mathcal{A}_i \boxtimes T^{\chi((i,p)\in S)}\mathcal{Z}) \\ \xrightarrow{\otimes^{\mathbf{m}}\sigma_{(12)}} \otimes^{p\in\mathbf{m}} [(\boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{Z})] \\ \xrightarrow{\overline{\varkappa}} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{Z})\bigr],$$

$$\beta = \bigl[\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{T^m\boxtimes^I\mu_{\mathcal{Z}}} T^m \boxtimes^I \mathcal{Z} \xrightarrow{T^m\otimes^I_{\mathcal{Z}}} T^m\mathcal{Z} \xrightarrow{\mu^m_{\mathcal{Z}}} \mathcal{Z}\bigr],$$

$$\gamma = \bigl[\boxtimes^{i\in I} T^{m_i}(\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\boxtimes^I\overline{\varkappa}} \boxtimes^{i\in I} (T^{m_i}\mathcal{A}_i \boxtimes T^{m_i}\mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I} T^{m_i}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{m_i}\mathcal{Z}) \\ \xrightarrow{(\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}})\boxtimes 1} (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{m_i}\mathcal{Z}) \\ \xrightarrow{\overline{\varkappa}^{-1}\boxtimes 1} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{m_i}\mathcal{Z})\bigr],$$

$$\delta \overset{\text{def}}{=} \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{Z} \xrightarrow{\boxtimes^{i\in I}\mu^{m_i}_{\mathcal{Z}}} \boxtimes^I \mathcal{Z} \xrightarrow{\otimes^I_{\mathcal{Z}}} \mathcal{Z}\bigr] \\ = \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{Z} \xrightarrow{\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{\boxtimes^I T^m\mu_{\mathcal{Z}}} \boxtimes^I T^m\mathcal{Z} \xrightarrow{\boxtimes^I\mu^m_{\mathcal{Z}}} \boxtimes^I \mathcal{Z} \xrightarrow{\otimes^I_{\mathcal{Z}}} \mathcal{Z}\bigr].$$

The above equation is due to the fact similar to equation (2.25.1):

$$\mu^{m_i}_{\mathcal{Z}} = \bigl[T^{m_i}\mathcal{Z} \xrightarrow{\lambda^{S_i\hookrightarrow\mathbf{m}}} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{T^m\mu_{\mathcal{Z}}} T^m\mathcal{Z} \xrightarrow{\mu^m_{\mathcal{Z}}} \mathcal{Z}\bigr],$$

which means in words: any number of units inserted into a composition in $\mathcal{Z}$ will not change its value.

Being a functor $\otimes^I_{\mathcal{Z}} : \boxtimes^I\mathcal{Z} \to \mathcal{Z}$ satisfies the identity

$$\begin{tikzcd} T^m \boxtimes^I \mathcal{Z} \arrow[r, "\overline{\varkappa}"] \arrow[d, "T^m\otimes^I_{\mathcal{Z}}"'] \arrow[drr, phantom, "="] & \boxtimes^I T^m\mathcal{Z} \arrow[r, "\boxtimes^I\mu^m_{\mathcal{Z}}"] & \boxtimes^I\mathcal{Z} \arrow[d, "\otimes^I_{\mathcal{Z}}"] \\ T^m\mathcal{Z} \arrow[rr, "\mu^m_{\mathcal{Z}}"] & & \mathcal{Z} \end{tikzcd} \qquad (10.13.2)$$

Therefore,

$$\delta = \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{Z} \xrightarrow{\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{\boxtimes^I T^m\mu_{\mathcal{Z}}} \boxtimes^I T^m\mathcal{Z} \xrightarrow{\overline{\varkappa}^{-1}} T^m \boxtimes^I \mathcal{Z} \\ \xrightarrow{T^m\otimes^I_{\mathcal{Z}}} T^m\mathcal{Z} \xrightarrow{\mu^m_{\mathcal{Z}}} \mathcal{Z}\bigr] \\ = \bigl[\boxtimes^{i\in I} T^{m_i}\mathcal{Z} \xrightarrow{\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{\overline{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{Z} \xrightarrow{\beta} \mathcal{Z}\bigr].$$

We also have

$$
\begin{aligned}
\alpha = \bigl[\boxtimes^{i\in I} T^{m_i}(\mathcal{A}_i \boxtimes \mathcal{Z}) &\xrightarrow{\gamma} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T^{m_i}\mathcal{Z}) \\
&\xrightarrow{1\boxtimes(\boxtimes^{i\in I}\lambda^{S_i\hookrightarrow\mathbf{m}})} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)}\mathcal{Z}) \\
&\xrightarrow{1\boxtimes\overline{\varkappa}^{-1}} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{A}_i) \boxtimes (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)}\mathcal{Z})\bigr]
\end{aligned}
$$

due to coherence principle of Remark 2.34. This proves equation (10.13.1), thus, $\widetilde{\xi}$ is Monoidal.

Now we consider the case of ξ. In the diagram on the next page the floor, the ceiling, the left and the right walls–quadrilaterals clearly commute. Commutativity of the left triangle follows from the equation

$$
\widetilde{\xi} = \bigl[T(\mathcal{A}_i \boxtimes \mathcal{Z}) = T^{\leqslant 1}T^{\geqslant 1}(\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{T^{\leqslant 1}\xi} T^{\leqslant 1}(T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\xi'} T\mathcal{A}_i \boxtimes \mathcal{Z}\bigr],
$$

which is due to the fact that $\widetilde{\xi}$ coincides with ξ on $T^{\geqslant 1}(\mathcal{A}_i \boxtimes \mathcal{Z})$ and it coincides with ξ' on $T^0(\mathcal{A}_i \boxtimes \mathcal{Z})$. The front wall commutes due to $\widetilde{\xi}$ being Monoidal. Since $\mathrm{in}\boxtimes 1$ is an embedding, the back wall commutes as well, that is, ξ is Monoidal. □

10.14 Proposition. *ξ satisfies equation* (5.27.1) *of Proposition 5.27.*

Proof. Consider the following diagram

$$
\begin{tikzcd}
T^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[rrr, "\xi"] \arrow[dr, "\mathrm{in}"] \arrow[dd, "\Delta"'] & & & T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z} \arrow[dr, "\mathrm{in}\boxtimes 1"] \arrow[dd, "\Delta\boxtimes 1"'] & \\
& T(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[rrr, "\widetilde{\xi}"] \arrow[dd, "\widetilde{\Delta}"] & & & T\mathcal{A} \boxtimes \mathcal{Z} \arrow[dd, "\widetilde{\Delta}\boxtimes 1"] \\
T^{\geqslant 1}T^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[rr, "T^{\geqslant 1}\xi"] \arrow[dr, "\mathrm{in}"'] & & T^{\geqslant 1}(T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "\xi"] \arrow[d, "\mathrm{in}"] & T^{\geqslant 1}T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z} \arrow[dr, "\mathrm{in}\boxtimes 1"] & \\
& TT^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "T\xi"] & T(T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}) \arrow[rr, "\widetilde{\xi}"] & & TT^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}
\end{tikzcd}
$$

Here the floor, the ceiling, the left and the right walls clearly commute. Commutativity of the back wall has to be proven. We prove instead that the front wall commutes. Since $\mathrm{in}\boxtimes 1$ is an embedding, the back wall will have to commute as well.

The front wall expands to the equation

$$
\begin{aligned}
&\bigl[T(\mathcal{A} \boxtimes \mathcal{Z}) \xrightarrow{\varkappa} T\mathcal{A} \boxtimes T\mathcal{Z} \xrightarrow{1\boxtimes\mu_{\mathcal{Z}}} T\mathcal{A} \boxtimes \mathcal{Z} \xrightarrow{\widetilde{\Delta}\boxtimes 1} TT^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}\bigr] \\
&= \bigl[T(\mathcal{A} \boxtimes \mathcal{Z}) \xrightarrow{\widetilde{\Delta}} TT^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \xrightarrow{T\varkappa} T(T^{\geqslant 1}\mathcal{A} \boxtimes T^{\geqslant 1}\mathcal{Z}) \xrightarrow{T(1\boxtimes\mu_{\mathcal{Z}})} T(T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}) \\
&\qquad \xrightarrow{\varkappa} TT^{\geqslant 1}\mathcal{A} \boxtimes T\mathcal{Z} \xrightarrow{1\boxtimes\mu_{\mathcal{Z}}} TT^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}\bigr],
\end{aligned}
$$

$$
\begin{tikzcd}
\boxtimes_u^{i\in I} T^{\geqslant 1}(\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "\tau^I"] \arrow[dr, "\mathrm{in}"] \arrow[dd, "\boxtimes_u^I \xi"'] & & T^{\geqslant 1} \boxtimes_u^{i\in I} (\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "T^{\geqslant 1}\sigma^I"] \arrow[dr, "\mathrm{in}"] & & T^{\geqslant 1}[(\boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z}] \arrow[dr, "\mathrm{in}"] \arrow[dd, "\xi"'] & \\
& \boxtimes^{i\in I} T(\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "\widetilde{\tau}^I"] \arrow[dd, "\boxtimes^I \widetilde{\xi}"] \arrow[ddl, "\oplus^I T^{\leqslant 1}\xi"'] & & T \boxtimes_u^{i\in I} (\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "T\sigma^I"] & & T[(\boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z}] \arrow[dd, "\widetilde{\xi}"] \\
\boxtimes_u^{i\in I} (T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "\sigma^I"] \arrow[d, "\mathrm{in}"'] & & (\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[rr, "\tau^I \boxtimes 1"] \arrow[dr, "\mathrm{in} \boxtimes 1"] & & (T^{\geqslant 1} \boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[dr, "\mathrm{in} \boxtimes 1"] & \\
\boxtimes^{i\in I} T^{\leqslant 1}(T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\boxtimes^{i\in I}\xi'"] & \boxtimes^{i\in I} (T\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[rr, "\widetilde{\sigma}^I"] & & (\boxtimes^{i\in I} T\mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[rr, "\widetilde{\tau} \boxtimes 1"] & & (T \boxtimes_u^{i\in I} \mathcal{A}_i) \boxtimes \mathcal{Z}
\end{tikzcd}
$$

which we are going to verify. It is equivalent to the following equation between matrix elements:

$$
\begin{aligned}
&\big[T^m(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\overline{\varkappa}} T^m\mathcal{A}\boxtimes T^m\mathcal{Z} \xrightarrow{1\boxtimes\mu^m_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z} \xrightarrow{\lambda^g\boxtimes 1} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A})\boxtimes\mathcal{Z}\big] \\
&= \big[T^m(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\otimes^{\mathbf{n}}\overline{\varkappa}} \otimes^{p\in\mathbf{n}}(\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{g^{-1}p}\mathcal{Z}) \\
&\qquad \xrightarrow{\otimes^{\mathbf{n}}(1\boxtimes\mu^{g^{-1}p}_{\mathcal{Z}})} \otimes^{p\in\mathbf{n}}(\otimes^{g^{-1}p}\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\overline{\varkappa}} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A})\boxtimes(\otimes^{\mathbf{n}}\mathcal{Z}) \\
&\qquad \xrightarrow{1\boxtimes\mu^n_{\mathcal{Z}}} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A})\boxtimes\mathcal{Z}\big]
\end{aligned}
$$

for an arbitrary non-decreasing surjection $g:\mathbf{m}\twoheadrightarrow\mathbf{n}$. Since a graded category with objects set S is an algebra in $(\mathscr{Q}/S,\otimes^I_S,\lambda^f)$, it satisfies equation (2.25.1). This allows to rewrite the above equation in the form:

$$
\begin{aligned}
&\big[T^m(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\overline{\varkappa}} T^m\mathcal{A}\boxtimes T^m\mathcal{Z} \xrightarrow{\lambda^g\boxtimes\lambda^g} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{Z} \\
&\qquad \xrightarrow{1\boxtimes\otimes^{p\in\mathbf{n}}\mu^{g^{-1}p}_{\mathcal{Z}}} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{\mathbf{n}}\mathcal{Z} \xrightarrow{1\boxtimes\mu^n_{\mathcal{Z}}} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\mathcal{Z}\big] \\
&= \big[T^m(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}(\mathcal{A}\boxtimes\mathcal{Z}) \\
&\qquad \xrightarrow{\otimes^{\mathbf{n}}\overline{\varkappa}} \otimes^{p\in\mathbf{n}}(\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{g^{-1}p}\mathcal{Z}) \xrightarrow{\overline{\varkappa}} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{Z} \\
&\qquad \xrightarrow{1\boxtimes\otimes^{p\in\mathbf{n}}\mu^{g^{-1}p}_{\mathcal{Z}}} \otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{\mathbf{n}}\mathcal{Z} \xrightarrow{1\boxtimes\mu^n_{\mathcal{Z}}} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A})\boxtimes\mathcal{Z}\big].
\end{aligned}
$$

The last two arrows in both sides coincide. The previous compositions ending in $\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{A}\boxtimes\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p}\mathcal{Z}$ coincide due to coherence principle of Remark 2.34. Therefore, the considered diagram is commutative, and ξ satisfies equation (5.27.1). $\square$

10.15 Proposition. *The multifunctor of shifts $-^{[]} = M^{T^{\geqslant 1}} : \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \to \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ for $M = -^{[]}$ takes a quiver morphism $\overline{f} : \boxtimes^{i\in I}Ts\mathcal{A}_i \to s\mathcal{B}$ to the morphism*

$$
f^{[]} = \big[\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\boxtimes^I(\varkappa\cdot(1\boxtimes\mu_{\mathcal{Z}}))} \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}T\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}\mathcal{Z}) \xrightarrow{\overline{f}\boxtimes\otimes^I_{\mathcal{Z}}} \mathcal{B}\boxtimes\mathcal{Z}\big]. \tag{10.15.1}
$$

Proof. As in diagram (10.12.1) the upper two squares in

$$
\begin{tikzcd}
\boxtimes_u^{i\in I}T^{\geqslant 1}(\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes^I_u\varkappa"] \arrow[d, hook, "\mathrm{in}"'] & \boxtimes_u^{i\in I}(T^{\geqslant 1}\mathcal{A}_i\boxtimes T^{\geqslant 1}\mathcal{Z}) \arrow[r, "\boxtimes^I_u(1\boxtimes\mu_{\mathcal{Z}})"] \arrow[d, hook, "\mathrm{in}"'] & \boxtimes_u^{i\in I}(T^{\geqslant 1}\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[d, hook, "\mathrm{in}"'] \\
\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes^I(1_{T^0}\oplus\varkappa)"] \arrow[d, equal] & \boxtimes^{i\in I}T^{\leqslant 1}(T^{\geqslant 1}\mathcal{A}_i\boxtimes T^{\geqslant 1}\mathcal{Z}) \arrow[r, "\boxtimes^I T^{\leqslant 1}(1\boxtimes\mu_{\mathcal{Z}})"] \arrow[d, hook, "{\boxtimes^{i\in I}[\overline{\varkappa}(1_{T^0\mathcal{A}_i}\boxtimes 1_{T^0\mathcal{Z}})\oplus(\mathrm{in}\boxtimes\mathrm{in})]}"'] & \boxtimes^{i\in I}T^{\leqslant 1}(T^{\geqslant 1}\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[d, hook, "{\boxtimes^{i\in I}[\overline{\varkappa}(1_{T^0\mathcal{A}_i}\boxtimes\eta_{\mathcal{Z}})\oplus(\mathrm{in}\boxtimes 1_{\mathcal{Z}})]}"'] \\
\boxtimes^{i\in I}T(\mathcal{A}_i\boxtimes\mathcal{Z}) \arrow[r, "\boxtimes^I\varkappa"'] & \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes T\mathcal{Z}) \arrow[r, "\boxtimes^I(1\boxtimes\mu_{\mathcal{Z}})"'] & \boxtimes^{i\in I}(T\mathcal{A}_i\boxtimes\mathcal{Z})
\end{tikzcd}
$$

commute. The other two commute as well. Pasting this diagram together with diagram (10.5.1) for $\mathcal{C}_i = T^{\geqslant 1}\mathcal{A}_i$ we get the left and the central commutative squares in the following diagram

$$\begin{tikzcd}
\boxtimes_u^{i\in I} T^{\geqslant 1}(\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\boxtimes_u^I \xi"] \arrow[d, hook, "\mathrm{in}"'] & \boxtimes_u^{i\in I}(T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\sigma^I"] \arrow[d, hook, "\iota"] & (\boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[r, "f\boxtimes 1"] \arrow[d, hook, "\mathrm{in}\boxtimes 1"] & \mathcal{B} \boxtimes \mathcal{Z} \arrow[d, equal] \\
\boxtimes^{i\in I} T(\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\boxtimes^I \tilde{\xi}"] & \boxtimes^{i\in I}(T\mathcal{A}_i \boxtimes \mathcal{Z}) \arrow[r, "\tilde{\sigma}^I"] & (\boxtimes^{i\in I} T\mathcal{A}_i) \boxtimes \mathcal{Z} \arrow[r, "\overline{f}\boxtimes 1"] & \mathcal{B} \boxtimes \mathcal{Z}
\end{tikzcd}$$

where

$$\iota = \bigl[\boxtimes_u^{i\in I}(T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \xhookrightarrow{\mathrm{in}} \boxtimes^{i\in I} T^{\leqslant 1}(T^{\geqslant 1}\mathcal{A}_i \boxtimes \mathcal{Z}) \xhookrightarrow{\boxtimes^{i\in I}[\overline{\varkappa}(1_{T^0\mathcal{A}_i} \boxtimes \eta_{\mathcal{Z}}) \oplus (\mathrm{in} \boxtimes 1_{\mathcal{Z}})]} \boxtimes^{i\in I}(T\mathcal{A}_i \boxtimes \mathcal{Z})\bigr].$$

The right square also obviously commutes. The top row gives $fM^{T^{\geqslant 1}} = (\xi)_i \cdot_{\widehat{\mathscr{Q}}_u} (f\widehat{M})$. The left–bottom path gives $f^{[]}$ from (10.15.1). These maps are equal and the proposition is proven. □

10.16 Proposition. *The triple* $((-^{[]})^{T^{\geqslant 1}}, m_{[]}^{T^{\geqslant 1}}, u_{[]}^{T^{\geqslant 1}}) : \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \to \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ *is a monad in the Kleisli multicategory, where according to Theorem 5.32*

$$u_{[]}^{T^{\geqslant 1}} = \bigl(T^{\geqslant 1}\mathcal{A} \xrightarrow{\mathrm{pr}_1} \mathcal{A} \xrightarrow{u_{[]}} \mathcal{A}^{[]}\bigr),$$

$$m_{[]}^{T^{\geqslant 1}} = \bigl(T^{\geqslant 1}[(\mathcal{A} \boxtimes \mathcal{Z}) \boxtimes \mathcal{Z}] \xrightarrow{\mathrm{pr}_1} (\mathcal{A} \boxtimes \mathcal{Z}) \boxtimes \mathcal{Z} \xrightarrow{m_{[]}} \mathcal{A} \boxtimes \mathcal{Z}\bigr).$$

It induces the monad in Q, *which is denoted* $(-^{[]}, m_{[]}, u_{[]}) : \mathsf{Q} \to \mathsf{Q}$ *by abuse of notation. In detail,* $-^{[]} : \mathsf{Q} \to \mathsf{Q}$ *is a multifunctor, the multiplication* $m_{[]} : -^{[][]} \to -^{[]}$ *is a natural transformation, and the unit* $u_{[]} : \mathrm{Id} \to -^{[]}$ *is a multinatural transformation.*

Proof. We are going to verify that ξ together with $u_{[]}$, $m_{[]}$ satisfies the assumptions of Theorem 5.32.

Let us prove equation (5.32.2) for ξ and $u_{[]}$. This is the top equation in the couple of equations

$$\begin{tikzcd}
\bigl[T^{\geqslant 1}\mathcal{A} \arrow[r, "T^{\geqslant 1}u_{[]}"] \arrow[d, hook, "\mathrm{in}"'] & T^{\geqslant 1}(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "\xi"] \arrow[d, hook, "\mathrm{in}"'] & T^{\geqslant 1}\mathcal{A} \boxtimes \mathcal{Z}\bigr] = u_{[]}, \arrow[d, hook, "\mathrm{in}\boxtimes 1"] \\
\bigl[T\mathcal{A} \arrow[r, "Tu_{[]}"] & T(\mathcal{A} \boxtimes \mathcal{Z}) \arrow[r, "\tilde{\xi}"] & T\mathcal{A} \boxtimes \mathcal{Z}\bigr] = u_{[]}.
\end{tikzcd}$$

Due to embedding $\mathrm{in} \boxtimes 1$ it suffices to prove the bottom line. On objects both sides give the map $X \mapsto X[0]$. On morphisms we have an equation

$$\bigl[\otimes^{i\in\mathbf{n}}\mathcal{A}(X_{i-1}, X_i) \xrightarrow{\otimes^{\mathbf{n}}\lambda^{\mathsf{I}\,\cdot}} \otimes^{i\in\mathbf{n}}[\mathcal{A}(X_{i-1}, X_i) \otimes \mathcal{Z}(0,0)]$$
$$\xrightarrow{\overline{\varkappa}} [\otimes^{i\in\mathbf{n}}\mathcal{A}(X_{i-1}, X_i)] \otimes [\otimes^{\mathbf{n}}\mathcal{Z}(0,0)] \xrightarrow{1\otimes\mu_{\mathcal{Z}}^{n}} [\otimes^{i\in\mathbf{n}}\mathcal{A}(X_{i-1}, X_i)] \otimes \mathcal{Z}(0,0)\bigr] = \lambda^{\mathsf{I}\,\cdot},$$

since here $\mu_{\mathcal{Z}} = (\lambda^{\varnothing\to\mathbf{n}})^{-1} : \otimes^{\mathbf{n}}\Bbbk \to \Bbbk$. Equation (5.32.2) follows.

Let us prove equation (5.27.2) for ξ and the counit $\varepsilon = \mathrm{pr}_1$ of the comonad $T^{\geqslant 1}$. This is the top equation in the couple of equations

$$\begin{tikzcd}
\big[T^{\geqslant 1}(\mathcal{A}\boxtimes\mathcal{Z}) \arrow[r, "\xi"] \arrow[d, hook, "\mathrm{in}"'] & T^{\geqslant 1}\mathcal{A}\boxtimes\mathcal{Z} \arrow[r, "\mathrm{pr}_1\boxtimes 1"] \arrow[d, hook, "\mathrm{in}\boxtimes 1"] & \mathcal{A}\boxtimes\mathcal{Z}\big] = \mathrm{pr}_1, \arrow[d, equal] \\
\big[T(\mathcal{A}\boxtimes\mathcal{Z}) \arrow[r, "\tilde{\xi}"] & T\mathcal{A}\boxtimes\mathcal{Z} \arrow[r, "\mathrm{pr}_1\boxtimes 1"] & \mathcal{A}\boxtimes\mathcal{Z}\big] = \mathrm{pr}_1 .
\end{tikzcd}$$

It suffices to prove the bottom line. It expands to

$$\big[T^m(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\varkappa} T^m\mathcal{A}\boxtimes T^m\mathcal{Z} \xrightarrow{1\boxtimes\mu^m_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z} \xrightarrow{\mathrm{pr}_1\boxtimes 1} \mathcal{A}\boxtimes\mathcal{Z}\big] = \mathrm{pr}_1$$

for each $m \geqslant 0$. This equation holds since for all $m \neq 1$ both sides vanish, and for $m = 1$ both sides are identity morphisms. Equation (5.27.2) is proven.

Let us prove equation (5.32.1) for ξ and $m_{[]}$:

$$\begin{aligned}
&\big[T^{\geqslant 1}[(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z}] \xrightarrow{\xi} [T^{\geqslant 1}(\mathcal{A}\boxtimes\mathcal{Z})]\boxtimes\mathcal{Z} \xrightarrow{\xi\boxtimes 1} (T^{\geqslant 1}\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z} \\
&\qquad \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} T^{\geqslant 1}\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \xrightarrow{\lambda^{\mathsf{IV}}} T^{\geqslant 1}\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} T^{\geqslant 1}\mathcal{A}\boxtimes\mathcal{Z}\big] \\
&= \big[T^{\geqslant 1}[(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z}] \xrightarrow{T^{\geqslant 1}(\lambda^{\mathsf{VI}})^{-1}} T^{\geqslant 1}(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T^{\geqslant 1}\lambda^{\mathsf{IV}}} T^{\geqslant 1}[\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z})] \\
&\qquad \xrightarrow{T^{\geqslant 1}(1\boxtimes\otimes_{\mathcal{Z}})} T^{\geqslant 1}(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\xi} T^{\geqslant 1}\mathcal{A}\boxtimes\mathcal{Z}\big].
\end{aligned}$$

Due to embeddings this equation is a corollary of

$$\begin{aligned}
&\big[T[(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z}] \xrightarrow{\tilde{\xi}} [T(\mathcal{A}\boxtimes\mathcal{Z})]\boxtimes\mathcal{Z} \xrightarrow{\tilde{\xi}\boxtimes 1} (T\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z} \\
&\qquad \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} T\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \xrightarrow{\lambda^{\mathsf{IV}}} T\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} T\mathcal{A}\boxtimes\mathcal{Z}\big] \\
&= \big[T[(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z}] \xrightarrow{T(\lambda^{\mathsf{VI}})^{-1}} T(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T\lambda^{\mathsf{IV}}} T[\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z})] \xrightarrow{T(1\boxtimes\otimes_{\mathcal{Z}})} T(\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\tilde{\xi}} T\mathcal{A}\boxtimes\mathcal{Z}\big],
\end{aligned}$$

which we prove now. Using the definition of $\tilde{\xi}$ we get the equation to prove:

$$\begin{aligned}
&\big[T^m(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T^m\lambda^{\mathsf{VI}}} T^m[(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z}] \xrightarrow{\varkappa} T^m(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes T^m\mathcal{Z} \\
&\qquad \xrightarrow{1\boxtimes\mu^m_{\mathcal{Z}}} T^m(\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z} \xrightarrow{\varkappa\boxtimes 1} (T^m\mathcal{A}\boxtimes T^m\mathcal{Z})\boxtimes\mathcal{Z} \xrightarrow{(1\boxtimes\mu^m_{\mathcal{Z}})\boxtimes 1} (T^m\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathcal{Z} \\
&\qquad \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} T^m\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \xrightarrow{\lambda^{\mathsf{IV}}} T^m\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z}\big] \\
&= \big[T^m(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T^m\lambda^{\mathsf{IV}}} T^m[\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z})] \xrightarrow{T^m(1\boxtimes\otimes_{\mathcal{Z}})} T^m(\mathcal{A}\boxtimes\mathcal{Z}) \\
&\qquad \xrightarrow{\varkappa} T^m\mathcal{A}\boxtimes T^m\mathcal{Z} \xrightarrow{1\boxtimes\mu^m_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z}\big].
\end{aligned}$$

Naturality of $\overline{\varkappa}$ and the coherence principle of Remark 2.34 allows to rewrite this as follows:

$$
\begin{aligned}
&\bigl[T^m(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T^m\lambda^{\mathsf{IV}}} T^m[\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z})] \xrightarrow{\overline{\varkappa}} T^m\mathcal{A}\boxtimes T^m(\mathcal{Z}\boxtimes\mathcal{Z}) \\
&\qquad \xrightarrow{1\boxtimes\overline{\varkappa}} T^m\mathcal{A}\boxtimes(T^m\mathcal{Z}\boxtimes T^m\mathcal{Z}) \xrightarrow{1\boxtimes(\mu^m_{\mathcal{Z}}\boxtimes\mu^m_{\mathcal{Z}})} T^m\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&= \bigl[T^m(\mathcal{A}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{T^m\lambda^{\mathsf{IV}}} T^m[\mathcal{A}\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z})] \xrightarrow{\overline{\varkappa}} T^m\mathcal{A}\boxtimes T^m(\mathcal{Z}\boxtimes\mathcal{Z}) \\
&\qquad \xrightarrow{1\boxtimes T^m\otimes_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes T^m\mathcal{Z} \xrightarrow{1\boxtimes\mu^m_{\mathcal{Z}}} T^m\mathcal{A}\boxtimes\mathcal{Z}\bigr].
\end{aligned}
$$

This reduces to equation (10.13.2) for $I = \mathbf{2}$, which means that $\otimes_{\mathcal{Z}} : \mathcal{Z}\boxtimes\mathcal{Z}\to\mathcal{Z}$ is a functor. Thus, equation (5.32.1) is proven as well.

Applying Theorem 5.32 we deduce the claim of the proposition. □

10.17 The multifunctor of shifts. We are going to verify that the two ways to construct the multifunctor of shifts agree. The above approach was to construct a lax Monoidal functor of shifts and to transfer it to the Kleisli multicategory Q. The second way is to consider action $\boxdot$ of the category of graded categories on (Kleisli multicategory of) graded quivers. This is realized in Appendices A–C.

An algebra $\mathcal{Z}$ in the category of graded categories $\mathscr{D} = \mathbf{dg}-\mathcal{C}at$ was considered in Section 10.1. As explained in Section C.8 it gives rise to algebra $\widehat{\mathcal{Z}}$ in the multicategory $\widehat{\mathscr{D}}$ and to the multifunctor $-^{[]} = 1\boxdot\widehat{\mathcal{Z}} : \mathsf{Q}\to\mathsf{Q}$. We shall see immediately that this multifunctor coincides with the multifunctor of shifts from Proposition 10.16. That is why we use the same notation. The multifunctor $1\boxdot\widehat{\mathcal{Z}}$ takes a quiver $\mathcal{A}$ to another quiver $\mathcal{A}^{[]} = \mathcal{A}\boxdot\mathcal{Z}$, obtained by adding formal shifts of objects. Its set of objects is

$$\operatorname{Ob}\mathcal{A}^{[]} = \{X[n] = (X,n) \mid X\in\operatorname{Ob}\mathcal{A}, n\in\mathbb{Z}\} = (\operatorname{Ob}\mathcal{A})\times\mathbb{Z}.$$

The graded $\Bbbk$-modules of morphisms are

$$\mathcal{A}^{[]}(X[n],Y[m]) = \{\mathcal{A}(X,Y)[1]\otimes\Bbbk[m-n]\}[-1] \simeq \mathcal{A}(X,Y)[m-n].$$

In degree k the chosen isomorphism is $(\lambda^{\mathsf{I}\cdot})^{-1} : \mathcal{A}(X,Y)^{k+m-n}\otimes\Bbbk\to\mathcal{A}(X,Y)^{k+m-n}$.

A multimap $f : (\mathcal{A}_i)_{i\in I}\to\mathcal{B}$, which is a quiver map $\overline{f} : \boxtimes^{i\in I}Ts\mathcal{A}_i\to s\mathcal{B}$, whose restriction to $\boxtimes^{i\in I}T^0s\mathcal{A}_i$ vanishes, is mapped to the multimap $f^{[]} = f\boxdot\widehat{\mathcal{Z}} : (\mathcal{A}_i^{[]})_{i\in I}\to\mathcal{B}^{[]}$, described as follows. It takes an object $(X_i[n_i])_{i\in I}\in\operatorname{Ob}\boxtimes^{i\in I}(\mathcal{A}_i^{[]})$ to the object $((X_i)_{i\in I}f)[\sum_{i\in I}n_i]$. According to (C.5.1) and (C.8.1) the multimap $f\boxdot\widehat{\mathcal{Z}}$ can be presented as

$$
\begin{aligned}
f^{[]} = f\boxdot\widehat{\mathcal{Z}} = f\boxdot\otimes^I_{\mathcal{Z}} = \bigl[&\boxtimes^{i\in I}T(s\mathcal{A}_i\boxtimes\mathcal{Z}) \xrightarrow{\boxtimes^{i\in I}\varkappa} \boxtimes^{i\in I}(Ts\mathcal{A}_i\boxtimes T\mathcal{Z}) \\
&\xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}Ts\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}T\mathcal{Z}) \xrightarrow{\overline{f}\boxtimes(\boxtimes^I\mu_{\mathcal{Z}})} s\mathcal{B}\boxtimes(\boxtimes^{i\in I}\mathcal{Z}) \xrightarrow{1\boxtimes\otimes^I_{\mathcal{Z}}} s\mathcal{B}\boxtimes\mathcal{Z}\bigr]
\end{aligned}
\tag{10.17.1}
$$

$$= \bigl[\boxtimes^{i\in I} T(s\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\boxtimes^I(\varkappa\cdot(1\boxtimes\mu_{\mathcal{Z}}))} \boxtimes^{i\in I}(Ts\mathcal{A}_i \boxtimes \mathcal{Z}) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I} Ts\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I}\mathcal{Z}) \xrightarrow{\overline{f}\boxtimes\otimes^I_{\mathcal{Z}}} s\mathcal{B}\boxtimes\mathcal{Z}\bigr],$$

which agrees with the expression (10.15.1). Thus, the multifunctors $M^{T^{\geqslant 1}} : \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \to \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$ for $M = -^{[]}$ and $1 \boxdot \widehat{\mathcal{Z}} : \mathsf{Q} \to \mathsf{Q}$ are identified. Both shift multifunctors are denoted $-^{[]}$. In the particular case $I = \mathbf{1}$ a morphism $f : \mathcal{A} \to \mathcal{B}$ in Q is mapped to the composition

$$f \boxdot \widehat{\mathcal{Z}} = \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\varkappa} Ts\mathcal{A}\boxtimes T\mathcal{Z} \xrightarrow{1\boxtimes\mu_{\mathcal{Z}}} Ts\mathcal{A}\boxtimes\mathcal{Z} \xrightarrow{\overline{f}\boxtimes 1} s\mathcal{B}\boxtimes\mathcal{Z}\bigr].$$

Let us find its components explicitly.

Let $C_1, \dots, C_k$ be $\mathbb{Z}$-graded $\Bbbk$-modules, and let $n_1, \dots, n_k$ be integers. Put $n = n_1 + \dots + n_k$. Consider the following isomorphism (of degree 0) of shifted modules:

$$\begin{aligned}
\sigma^{+}_{(n_j)_j} &= (-1)^{\sum_{i<j} n_i n_j}(s^{-n_1}\otimes\dots\otimes s^{-n_k})s^n \\
&= (1\otimes\dots\otimes 1\otimes s^{-n_k})(1\otimes\dots\otimes s^{-n_{k-1}}\otimes 1)\dots(s^{-n_1}\otimes\dots\otimes 1\otimes 1)s^n \\
&= (s^{n_1}\otimes\dots\otimes s^{n_k})^{-1}s^n : C_1[n_1]\otimes\dots\otimes C_k[n_k] \to (C_1\otimes\dots\otimes C_k)[n].
\end{aligned}$$

It allows to represent $\widetilde{\xi} \overset{\text{def}}{=} \varkappa\cdot(1\boxtimes\mu_{\mathcal{Z}})$ as

$$\begin{aligned}
\varkappa\cdot(1\boxtimes\mu_{\mathcal{Z}}) &= \bigl[T^k(s\mathcal{A}\boxtimes\mathcal{Z})(X_0[n_0],X_k[n_k]) = \oplus_{X_j,n_j}\otimes^{j\in\mathbf{k}}(s\mathcal{A}\boxtimes\mathcal{Z})(X_{j-1}[n_{j-1}],X_j[n_j]) \\
&\simeq \oplus_{X_j,n_j}\otimes^{j\in\mathbf{k}}(s\mathcal{A}(X_{j-1},X_j)[n_j-n_{j-1}]) \xrightarrow{\sigma^{+}_{(n_j-n_{j-1})_j}} \oplus_{X_j}(\otimes^{j\in\mathbf{k}} s\mathcal{A}(X_{j-1},X_j))[n_k-n_0] \\
&\simeq (T^k s\mathcal{A}\boxtimes\mathcal{Z})(X_0[n_0],X_k[n_k])\bigr].
\end{aligned} \tag{10.17.2}$$

Let $f \in \underline{\mathsf{C}}_{\Bbbk}(C,D)$ be a homogeneous element (a $\Bbbk$-linear map $f : C \to D$ of certain degree $\deg f$). Define $f^{[n]} = (-)^{fn}s^{-n}fs^n : C[n] \to D[n]$, which is an element of $\underline{\mathsf{C}}_{\Bbbk}(C[n],D[n])$ of the same degree $\deg f$. Now we may write

$$\begin{aligned}
\overline{f}\boxtimes 1 = \bigl[&(T^k s\mathcal{A}\boxtimes\mathcal{Z})(X_0[n_0],X_k[n_k]) \simeq (\otimes^{j\in\mathbf{k}} s\mathcal{A}(X_{j-1},X_j))[n_k-n_0] \\
&\xrightarrow{f_k^{[n_k-n_0]}} s\mathcal{B}(X_0f,X_kf)[n_k-n_0] \simeq (s\mathcal{B}\boxtimes\mathcal{Z})(X_0f[n_0],X_kf[n_k])\bigr].
\end{aligned}$$

Thus, a morphism $f : \mathcal{A} \to \mathcal{B}$ in Q is mapped to

$$\begin{aligned}
f^{[]} = f\boxdot\widehat{\mathcal{Z}} = \bigl[&\otimes^{j\in\mathbf{k}} s\mathcal{A}^{[]}(X_{j-1}[n_{j-1}],X_j[n_j]) \\
&\simeq \otimes^{j\in\mathbf{k}}(s\mathcal{A}(X_{j-1},X_j)[n_j-n_{j-1}]) \xrightarrow{\sigma^{+}_{(n_j-n_{j-1})_j}} (\otimes^{j\in\mathbf{k}} s\mathcal{A}(X_{j-1},X_j))[n_k-n_0] \\
&\xrightarrow{f_k^{[n_k-n_0]}} s\mathcal{B}(X_0f,X_kf)[n_k-n_0] \simeq s\mathcal{B}^{[]}(X_0f[n_0],X_kf[n_k])\bigr].
\end{aligned} \tag{10.17.3}$$

In the particular case $I = \varnothing$ a multimap $f : () \to \mathcal{B}$ in Q is identified with a morphism $f : \mathbb{1}_u \to s\mathcal{B}$ in $\mathscr{Q}$, $\operatorname{Ob}\mathbb{1}_u = \{*\}$, $\mathbb{1}_u(*,*) = 0$, that is, with an object $X = f(*) \in \operatorname{Ob}\mathcal{B}$. The multimap $f^{[]} : () \to \mathcal{B}^{[]}$ takes $* \in \operatorname{Ob}\mathbb{1}_u$ to $X[0] \in \operatorname{Ob}\mathcal{B}^{[]}$ as (10.17.1) shows.

10.18 Lemma. *Let C_i, D_i be $\mathbb{Z}$-graded $\Bbbk$-modules for $1 \leqslant i \leqslant k$. Let $f_i \in \underline{\mathsf{C}}_\Bbbk(C_i, D_i)$ be homogeneous $\Bbbk$-linear maps. Then for any integers $n_1, \dots, n_k$ there is an equation between two elements of $\underline{\mathsf{C}}_\Bbbk\big(C_1[n_1] \otimes \cdots \otimes C_k[n_k], (D_1 \otimes \cdots \otimes D_k)[n_1 + \cdots + n_k]\big)$:*

$$\begin{tikzcd} C_1[n_1] \otimes \cdots \otimes C_k[n_k] \arrow[r, "\sigma^+_{(n_i)}"] \arrow[d, "f_1^{[n_1]} \otimes \cdots \otimes f_k^{[n_k]}"'] \arrow[dr, phantom, "="] & (C_1 \otimes \cdots \otimes C_k)[n_1 + \cdots + n_k] \arrow[d, "(f_1 \otimes \cdots \otimes f_k)^{[n_1 + \cdots + n_k]}"] \\ D_1[n_1] \otimes \cdots \otimes D_k[n_k] \arrow[r, "\sigma^+_{(n_i)}"] & (D_1 \otimes \cdots \otimes D_k)[n_1 + \cdots + n_k] . \end{tikzcd} \tag{10.18.1}$$

We have

$$(\sigma^+_{n_{11},\dots,n_{1k_1}} \otimes \cdots \otimes \sigma^+_{n_{r1},\dots,n_{rk_r}})\sigma^+_{n_{11}+\dots+n_{1k_1},\dots,\, n_{r1}+\dots+n_{rk_r}} = \sigma^+_{n_{11},\dots,n_{1k_1},\dots,\, n_{r1},\dots,n_{rk_r}}. \tag{10.18.2}$$

Proof. Denote $n = n_1 + \cdots + n_k$. Equation (10.18.1) is equivalent to

$$(s^{n_1} \otimes \cdots \otimes s^{n_k})(f_1^{[n_1]} \otimes \cdots \otimes f_k^{[n_k]})(s^{n_1} \otimes \cdots \otimes s^{n_k})^{-1} = s^n(f_1 \otimes \cdots \otimes f_k)^{[n]}s^{-n}.$$

Here both sides are equal to the same map $(-)^{(n_1+\dots+n_k)(f_1+\dots+f_k)} f_1 \otimes \cdots \otimes f_k \in \underline{\mathsf{C}}_\Bbbk(C_1 \otimes \cdots \otimes C_k, D_1 \otimes \cdots \otimes D_k)$, which proves the first claim. The second follows from the computation

$$\begin{aligned} &(s^{n_{11}} \otimes \cdots \otimes s^{n_{1k_1}} \otimes \cdots \otimes s^{n_{r1}} \otimes \cdots \otimes s^{n_{rk_r}})(\sigma^+_{n_{11},\dots,n_{1k_1}} \otimes \cdots \otimes \sigma^+_{n_{r1},\dots,n_{rk_r}})\sigma^+_{n_{11}+\dots+n_{1k_1},\dots,\, n_{r1}+\dots+n_{rk_r}} \\ &= (s^{n_{11}+\dots+n_{1k_1}} \otimes \cdots \otimes s^{n_{r1}+\dots+n_{rk_r}})\sigma^+_{n_{11}+\dots+n_{1k_1},\dots,\, n_{r1}+\dots+n_{rk_r}} = s^{n_{11}+\dots+n_{1k_1}+\dots+n_{r1}+\dots+n_{rk_r}}. \qquad \square \end{aligned}$$

10.19 The closing transformation for the multifunctor of shifts. In this section we study the closing transformation for the multifunctor of shifts.

Let us extend the multifunctor of shifts $S = -^{[]} : \mathsf{Q} \to \mathsf{Q}$ to the closing transformation $\underline{S}$ as described in Section 4.18. On objects $\underline{S}$ coincides with $\operatorname{Ob} S$. We want to find the only morphism

$$\underline{S}_{(\mathcal{A}_i);\mathcal{B}} : \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}}; \mathcal{B})^{[]} \to \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in\mathbf{k}}; \mathcal{B}^{[]}) \tag{10.19.1}$$

in Q such that the following equation holds in Q:

$$\begin{aligned} \Big[(\mathcal{A}_i^{[]})_{i\in\mathbf{k}}, \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}}; \mathcal{B})^{[]} \xrightarrow{(1_{\mathcal{A}_i^{[]}}),\underline{S}_{(\mathcal{A}_i);\mathcal{B}}} (\mathcal{A}_i^{[]})_{i\in\mathbf{k}}, \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in\mathbf{k}}; \mathcal{B}^{[]}) \xrightarrow{\operatorname{ev}^{\mathsf{Q}}_{(\mathcal{A}_i^{[]});\mathcal{B}^{[]}}} \mathcal{B}^{[]}\Big] \\ = (\operatorname{ev}^{\mathsf{Q}}_{(\mathcal{A}_i);\mathcal{B}})^{[]}. \end{aligned} \tag{10.19.2}$$

Corollary 4.20 shows that on objects morphism (10.19.1) takes $(f, 0)$, where f is an element of $\mathsf{Q}((\mathcal{A}_i)_{i\in\mathbf{k}}; \mathcal{B})$, to $f^{[]} : (\mathcal{A}_i^{[]})_{i\in\mathbf{k}} \to \mathcal{B}^{[]}$. On other objects $\underline{S}_{(\mathcal{A}_i);\mathcal{B}}$ can be found from the

following case of the above equations:

$$\begin{aligned}
&\big[(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes T^0s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]}) \\
&\qquad\xrightarrow{1\boxtimes T^0\underline{S}} (\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes T^0\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}),s\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}''} s\mathcal{B}^{[]}\big] \\
&=\big[(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i^{[]})\boxtimes T^0s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]}) \xrightarrow{(\boxtimes^{\mathbf{k}}\varkappa)\boxtimes\varkappa} \boxtimes^{i\in\mathbf{k}}(Ts\mathcal{A}_i\boxtimes T\mathcal{Z})\boxtimes(T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})\boxtimes T^0\mathcal{Z}) \\
&\qquad\xrightarrow{\sigma_{(12)}} ((\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i)\boxtimes T^0\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i,s\mathcal{B}))\boxtimes((\boxtimes^{\mathbf{k}}T\mathcal{Z})\boxtimes T^0\mathcal{Z}) \\
&\qquad\xrightarrow{\mathrm{ev}''\boxtimes((\boxtimes^{\mathbf{k}}\mu_{\mathcal{Z}})\boxtimes\mu^0_{\mathcal{Z}})} s\mathcal{B}\boxtimes(\boxtimes^{\mathbf{k+1}}\mathcal{Z}) \xrightarrow{1\boxtimes\otimes^{\mathbf{k+1}}_{\mathcal{Z}}} s\mathcal{B}\boxtimes\mathcal{Z}\big],
\end{aligned}$$

where $\mu^0_{\mathcal{Z}}:T^0\mathcal{Z}\to\mathcal{Z}$, $T^0\mathcal{Z}(p,p)=\Bbbk\ni 1\mapsto 1_p=1\in\Bbbk=\mathcal{Z}(p,p)$ defines the identity morphisms in $\mathcal{Z}$. Associativity of multiplication in $\mathcal{Z}$ implies that for an arbitrary multimap $f:(\mathcal{A}_i)_{i\in\mathbf{k}}\to\mathcal{B}$ and an arbitrary integer p the right hand side induces the map $(f,p)\underline{S}$

$$\begin{aligned}
&\boxtimes^{i\in\mathbf{k}}T(s\mathcal{A}_i\boxtimes\mathcal{Z})\simeq(\boxtimes^{i\in\mathbf{k}}T(s\mathcal{A}_i\boxtimes\mathcal{Z}))\otimes T^0s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})(f,f)\otimes T^0\mathcal{Z}(p,p) \\
&\qquad\xrightarrow{f^{[]}\otimes\mu^0} s\mathcal{B}\boxtimes\mathcal{Z}\otimes\mathcal{Z}(p,p)\xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} s\mathcal{B}\boxtimes\mathcal{Z}.
\end{aligned}$$

The above multimap $(f,p)\underline{S}=f^{[]}[p]$ takes an arbitrary object $(X_i[n_i])_{i\in\mathbf{k}}$ to the object $((X_i)_{i\in\mathbf{k}}f)[\sum_{i=1}^k n_i+p]$. Since $\psi_1(*,p,*,p)=0$, the multimap acts on morphisms via the map

$$\begin{aligned}
&\otimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]})(X_i[n_i],Y_i[m_i])\xrightarrow{f^{[]}} s\mathcal{B}^{[]}\Big(\big((X_i)_{i\in\mathbf{k}}f\big)[\textstyle\sum_{i=1}^k n_i],\big((Y_i)_{i\in\mathbf{k}}f\big)[\sum_{i=1}^k m_i]\Big) \\
&\qquad= s\mathcal{B}^{[]}\Big(\big((X_i)_{i\in\mathbf{k}}f\big)[\textstyle\sum_{i=1}^k n_i+p],\big((Y_i)_{i\in\mathbf{k}}f\big)[\sum_{i=1}^k m_i+p]\Big),
\end{aligned}$$

which explains the notation $f^{[]}[p]$. As we noticed above, $f^{[]}[0]=f^{[]}$.

Applying $T^{\leqslant1}$ to equations (10.19.2) and composing the result with the projection $\mathrm{pr}_1:T^{\leqslant1}s\mathcal{B}^{[]}\to s\mathcal{B}^{[]}$, we find the remaining part of equations in the expanded form

$$\begin{aligned}
&\big[(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes T^{\geqslant1}s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]}) \\
&\qquad\xrightarrow{1\boxtimes\underline{S}} (\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}),s\mathcal{B}^{[]})\xrightarrow{\mathrm{ev}'} s\mathcal{B}^{[]}\big] \\
&=\big[(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i^{[]})\boxtimes T^{\geqslant1}s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]}) \\
&\qquad\xrightarrow{(\boxtimes^{\mathbf{k}}\varkappa)\boxtimes\mathrm{pr}_1} \boxtimes^{i\in\mathbf{k}}(Ts\mathcal{A}_i\boxtimes T\mathcal{Z})\boxtimes(s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})\boxtimes\mathcal{Z}) \\
&\qquad\xrightarrow{\sigma_{(12)}} ((\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i)\boxtimes\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i,s\mathcal{B}))\boxtimes((\boxtimes^{\mathbf{k}}T\mathcal{Z})\boxtimes\mathcal{Z}) \\
&\qquad\xrightarrow{\mathrm{ev}'\boxtimes((\boxtimes^{\mathbf{k}}\mu_{\mathcal{Z}})\boxtimes1)} s\mathcal{B}\boxtimes(\boxtimes^{\mathbf{k+1}}\mathcal{Z})\xrightarrow{1\boxtimes\otimes^{\mathbf{k+1}}_{\mathcal{Z}}} s\mathcal{B}\boxtimes\mathcal{Z}\big].
\end{aligned}$$

These equations show that $\underline{S}$ is strict, that is, it can be presented as the composition

$$\underline{S}_{(\mathcal{A}_i);\mathcal{B}}=\big[T^{\geqslant1}s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]})\xrightarrow{\mathrm{pr}_1} s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]})\xrightarrow{\underline{S_1}} s\underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in\mathbf{k}};\mathcal{B}^{[]})\big]. \tag{10.19.3}$$

This allows to rewrite the above equation in the form

$$\begin{aligned}
&\big[(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]}) \xrightarrow{1\boxtimes\underline{S}_1} (\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}))\boxtimes\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts(\mathcal{A}_i^{[]}),s\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}'} s\mathcal{B}^{[]}\big]\\
&= \big[(\boxtimes^{i\in\mathbf{k}}T(s\mathcal{A}_i\boxtimes\mathcal{Z}))\boxtimes(\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i,s\mathcal{B})\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{\boxtimes^{\mathbf{k}}(\varkappa\cdot(1\boxtimes\mu))\boxtimes(1\boxtimes 1)} \boxtimes^{i\in\mathbf{k}}(Ts\mathcal{A}_i\boxtimes\mathcal{Z})\boxtimes(\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i,s\mathcal{B})\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{\sigma_{(12)}} ((\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i)\boxtimes\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i,s\mathcal{B}))\boxtimes(\boxtimes^{\mathbf{k}+\mathbf{1}}\mathcal{Z}) \xrightarrow{\mathrm{ev}'\boxtimes\otimes_{\mathcal{Z}}^{\mathbf{k}+\mathbf{1}}} s\mathcal{B}\boxtimes\mathcal{Z}\big].
\end{aligned}\tag{10.19.4}$$

Given an element $r\in s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})(f,g)$ of arbitrary degree and integers n, m, we can construct $r\otimes(1s^{m-n})\in s(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{k}};\mathcal{B})^{[]})((f,n),(g,m))$ and via the above recipe the element $(r\otimes(1s^{m-n}))\underline{S}_1\in\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i^{[]},s\mathcal{B}^{[]})(f^{[]}[n],g^{[]}[m])$. When $n=m=0$, we denote it by $r^{[]}=(r\otimes 1)\underline{S}_1\in\underline{\mathscr{Q}_p}(\boxtimes^{i\in\mathbf{k}}Ts\mathcal{A}_i^{[]},s\mathcal{B}^{[]})(f^{[]},g^{[]})$. In the particular case $k=1$ equation (10.19.4) reduces to

$$\begin{aligned}
&\big[Ts(\mathcal{A}^{[]})\boxtimes(\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{B})\boxtimes\mathcal{Z}) \xrightarrow{1\boxtimes\underline{S}_1} Ts(\mathcal{A}^{[]})\boxtimes\underline{\mathscr{Q}_p}(Ts(\mathcal{A}^{[]}),s\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{B}^{[]}\big]\\
&= \big[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes(\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{B})\boxtimes\mathcal{Z}) \xrightarrow{\varkappa\cdot(1\boxtimes\mu)\boxtimes(1\boxtimes1)} (Ts\mathcal{A}\boxtimes\mathcal{Z})\boxtimes(\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{B})\boxtimes\mathcal{Z})\\
&\qquad\xrightarrow{\sigma_{(12)}} (Ts\mathcal{A}\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{B}))\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}\boxtimes\otimes_{\mathcal{Z}}^{\mathbf{2}}} s\mathcal{B}\boxtimes\mathcal{Z}\big].
\end{aligned}$$

We describe the solution $(r\otimes(1s^{m-n}))\underline{S}_1: f^{[]}[n]\to g^{[]}[m]:\mathcal{A}^{[]}\to\mathcal{B}^{[]}$ by its components:

$$\begin{aligned}
&[(r\otimes(1s^{m-n}))\underline{S}_1]_p = (-)^{n_p(n-m)}(s^{n_1-n_0}\otimes\dots\otimes s^{n_p-n_{p-1}})^{-1}s^{n_p-n_0}r_p^{[n_p-n_0]}s^{m-n}:\\
&\otimes^{i\in\mathbf{p}}s\mathcal{A}^{[]}(X_{i-1}[n_{i-1}],X_i[n_i]) = \otimes^{i\in\mathbf{p}}(s\mathcal{A}(X_{i-1},X_i)[n_i-n_{i-1}]) \xrightarrow{\sigma^+_{(n_i-n_{i-1})_i}}\\
&\qquad(\otimes^{i\in\mathbf{p}}s\mathcal{A}(X_{i-1},X_i))[n_p-n_0] \xrightarrow{r_p^{[n_p-n_0]}} s\mathcal{B}(X_0f,X_pg)[n_p-n_0]\\
&\qquad\qquad\xrightarrow{(-)^{n_p(n-m)}s^{m-n}} s\mathcal{B}^{[]}((X_0,n_0)f^{[]}[n],(X_p,n_p)g^{[]}[m]).
\end{aligned}\tag{10.19.5}$$

10.20 Proposition. *The multifunctor $-^{[]}=-\boxdot\mathcal{Z}:\mathsf{Q}\to\mathsf{Q}$, the unit multinatural transformation $u_{[]}=-\boxdot(\eta_{\mathcal{Z}}:\mathbb{1}_p\to\mathcal{Z})$ and the multiplication natural transformation $m_{[]}=-\boxdot(\otimes_{\mathcal{Z}}:\mathcal{Z}\boxtimes\mathcal{Z}\to\mathcal{Z})$ form a monad which satisfies condition* (4.35.3) *in strong form: the right part of this diagram, namely,*

$$\begin{tikzcd}
\mathcal{X}^{[][]},\mathcal{W}^{[]} \arrow[r, "{1,u^{[]}_{[]}}"] \arrow[d, "{m_{[]},1}"'] & \mathcal{X}^{[][]},\mathcal{W}^{[][]} \arrow[r, "e^{[][]}"] & \mathcal{Y}^{[][]} \arrow[d, "m_{[]}"] \\
\mathcal{X}^{[]},\mathcal{W}^{[]} \arrow[rr, "e^{[]}"] & & \mathcal{Y}^{[]}
\end{tikzcd}\tag{10.20.1}$$

commutes for any morphism $e:\mathcal{X},\mathcal{W}\to\mathcal{Y}$ in Q.

Proof. Plugging the definitions of transformations in diagram (10.20.1), we have to prove

$$\begin{tikzcd}
(\mathcal{X}\boxdot(\mathcal{Z}\boxtimes\mathcal{Z}),\mathcal{W}\boxdot\mathcal{Z}) \arrow[r, "(1{,}1\boxdot(\eta_{\mathcal{Z}}\boxtimes 1))"] \arrow[d, "(1\boxdot\otimes_{\mathcal{Z}}{,}1)"'] & (\mathcal{X}\boxdot(\mathcal{Z}\boxtimes\mathcal{Z}),\mathcal{W}\boxdot(\mathcal{Z}\boxtimes\mathcal{Z})) \arrow[r, "e\boxdot\otimes_{\mathcal{Z}\boxtimes\mathcal{Z}}"] \arrow[d, phantom, "="] & \mathcal{Y}\boxdot(\mathcal{Z}\boxtimes\mathcal{Z}) \arrow[d, "1\boxdot\otimes_{\mathcal{Z}}"] \\
(\mathcal{X}\boxdot\mathcal{Z},\mathcal{W}\boxdot\mathcal{Z}) \arrow[rr, "e\boxdot\otimes_{\mathcal{Z}}"] & & \mathcal{Y}\boxdot\mathcal{Z}
\end{tikzcd}$$

Since $\boxdot$ is a multifunctor, this equation is equivalent to

$$e\boxdot\big((1_{\mathcal{Z}\boxtimes\mathcal{Z}}\boxtimes(\eta_{\mathcal{Z}}\boxtimes 1_{\mathcal{Z}}))\cdot\otimes_{\mathcal{Z}\boxtimes\mathcal{Z}}\cdot\otimes_{\mathcal{Z}}\big) = e\boxdot\big((\otimes_{\mathcal{Z}}\boxtimes 1_{\mathcal{Z}})\cdot\otimes_{\mathcal{Z}}\big) : (\mathcal{X}\boxdot(\mathcal{Z}\boxtimes\mathcal{Z}),\mathcal{W}\boxdot\mathcal{Z}) \to \mathcal{Y}\boxdot\mathcal{Z}.$$

The above equation follows from obvious commutativity of the diagram:

$$\begin{tikzcd}
\mathcal{Z}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \arrow[r, "1\boxtimes 1\boxtimes\eta_{\mathcal{Z}}\boxtimes 1"] \arrow[d, "\otimes_{\mathcal{Z}}\boxtimes 1"'] \arrow[drr, "\otimes^{\mathbf{3}}_{\mathcal{Z}}"] & \mathcal{Z}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \arrow[r, "1\boxtimes c\boxtimes 1"] \arrow[drr, "\otimes_{\mathcal{Z}\boxtimes\mathcal{Z}}"] & \mathcal{Z}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z}\boxtimes\mathcal{Z} \arrow[dr, "\otimes_{\mathcal{Z}}\boxtimes\otimes_{\mathcal{Z}}"] & \\
\mathcal{Z}\boxtimes\mathcal{Z} \arrow[r, "\otimes_{\mathcal{Z}}"] & \mathcal{Z} & & \mathcal{Z}\boxtimes\mathcal{Z} \arrow[ll, "\otimes_{\mathcal{Z}}"']
\end{tikzcd}$$

It implies commutativity of diagram (10.20.1). □

Consider the multifunctor $S = -\boxdot\mathcal{Z} = -^{[]} : \mathsf{Q} \to \mathsf{Q}$. Abusing the notation denote the corresponding Q-functor in $\underline{\mathsf{Q}}$ by

$$-^{[]} \overset{\text{def}}{=} S' = \big(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \xrightarrow{u_{[]}} \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{[]} \xrightarrow{\underline{S}} \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{B}^{[]})\big),\ \ f \mapsto f^{[]},\ \ r \mapsto r^{[]}.$$

10.21 Corollary. *The following diagram commutes:*

$$\begin{tikzcd}
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B}) \arrow[r, "-^{[][]}"] \arrow[d, "-^{[]}"'] \arrow[dr, phantom, "="] & \underline{\mathsf{Q}}(\mathcal{A}^{[][]};\mathcal{B}^{[][]}) \arrow[d, "\underline{\mathsf{Q}}(1;m^{\mathcal{B}}_{[]})"] \\
\underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \arrow[r, "\underline{\mathsf{Q}}(m^{\mathcal{A}}_{[]};1)"] & \underline{\mathsf{Q}}(\mathcal{A}^{[][]};\mathcal{B}^{[]})
\end{tikzcd}$$

This is diagram (4.35.2), proven in Corollary 4.35. Since not only exterior of diagram (4.35.3) commutes, but its right pentagon commutes as well, one can prove a stronger statement:

10.22 Corollary. *The following diagram*

$$\begin{tikzcd}
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B})^{[][]} \arrow[r, "\underline{S}^{[]}"] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{B}^{[]})^{[]} \arrow[r, "\underline{S}"] \arrow[d, phantom, "="] & \underline{\mathsf{Q}}(\mathcal{A}^{[][]};\mathcal{B}^{[][]}) \arrow[d, "\underline{\mathsf{Q}}(1;m^{\mathcal{B}}_{[]})"] \\
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B})^{[]} \arrow[u, "u^{[]}_{[]}"] \arrow[r, "\underline{S}"] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \arrow[r, "\underline{\mathsf{Q}}(m^{\mathcal{A}}_{[]};1)"] & \underline{\mathsf{Q}}(\mathcal{A}^{[][]};\mathcal{B}^{[]})
\end{tikzcd}$$

commutes for arbitrary graded $\Bbbk$*-quivers* $\mathcal{A}$, $\mathcal{B}$.

We shall neither supply the proof, nor use the above statement.

10.23 Proposition. *The following diagram commutes in* Q *for arbitrary quivers* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$*:*

$$\begin{tikzcd} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{B}),\underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "\_^{[]},\_^{[]}"] \arrow[d, "\mu^{\mathsf{Q}}"'] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{B}^{[]}),\underline{\mathsf{Q}}(\mathcal{B}^{[]};\mathcal{C}^{[]}) \arrow[d, "\mu^{\mathsf{Q}}"] \\ \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \arrow[r, "\_^{[]}"] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{C}^{[]}) \end{tikzcd} \tag{10.23.1}$$

Proof. This is the exterior of the diagram

$$\begin{tikzcd} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{B}),\underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "u_{[]},u_{[]}"] \arrow[d, "\mu^{\mathsf{Q}}"'] & \underline{\mathsf{Q}}(\mathcal{A};\mathcal{B})^{[]},\underline{\mathsf{Q}}(\mathcal{B};\mathcal{C})^{[]} \arrow[r, "\underline{S},\underline{S}"] \arrow[d, "(\mu^{\mathsf{Q}})^{[]}"'] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{B}^{[]}),\underline{\mathsf{Q}}(\mathcal{B}^{[]};\mathcal{C}^{[]}) \arrow[d, "\mu^{\mathsf{Q}}"] \\ \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \arrow[r, "u_{[]}"] & \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{[]} \arrow[r, "\underline{S}"] & \underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{C}^{[]}) \end{tikzcd}$$

The left square commutes by multinaturality of $u_{[]}$, commutativity of the right square is a consequence of (4.21.1). □

10.24 Proposition. *For an arbitrary morphism* $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ *in* Q *and a quiver* $\mathcal{C}$ *the following diagram commutes in* Q*:*

$$\begin{tikzcd} \underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "\_^{[]}"] \arrow[d, "\underline{\mathsf{Q}}(f;1)"'] & \underline{\mathsf{Q}}(\mathcal{B}^{[]};\mathcal{C}^{[]}) \arrow[d, "\underline{\mathsf{Q}}(f^{[]};1)"] \\ \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "\_^{[]}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{C}^{[]}) \end{tikzcd} \tag{10.24.1}$$

Proof. This is the exterior of the diagram

$$\begin{tikzcd} \underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "u_{[]}"] \arrow[d, "\underline{\mathsf{Q}}(f;1)"'] & \underline{\mathsf{Q}}(\mathcal{B};\mathcal{C})^{[]} \arrow[r, "\underline{S}"] \arrow[d, "\underline{\mathsf{Q}}(f;1)^{[]}"'] & \underline{\mathsf{Q}}(\mathcal{B}^{[]};\mathcal{C}^{[]}) \arrow[d, "\underline{\mathsf{Q}}(f^{[]};1)"] \\ \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "u_{[]}"] & \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})^{[]} \arrow[r, "\underline{S}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{C}^{[]}) \end{tikzcd}$$

The left square commutes due to naturality of $u_{[]}$, the right square is commutative by (4.23.1). □

10.25 Proposition. *For an arbitrary morphism* $f : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ *in* Q*, quivers* $\mathcal{A}_i$*,* $i \in I$*, and a map* $\phi : I \to J$ *in* $\operatorname{Mor}\mathcal{S}$ *the following diagram commutes in* Q*:*

$$\begin{tikzcd} (\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j))_{j\in J} \arrow[r, "(\_^{[]})_J"] \arrow[d, "\underline{\mathsf{Q}}(\phi;f)"'] & (\underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in\phi^{-1}j};\mathcal{B}_j^{[]}))_{j\in J} \arrow[d, "\underline{\mathsf{Q}}(\phi;f^{[]})"] \\ \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "\_^{[]}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{C}^{[]}) \end{tikzcd} \tag{10.25.1}$$

Proof. This is the exterior of the diagram

$$\begin{tikzcd}
(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j))_{j\in J} \arrow[r, "(u_{[]})_J"] \arrow[d, "\underline{\mathsf{Q}}(\phi;f)"'] & (\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)^{[]})_{j\in J} \arrow[r, "(\underline{S})_J"] \arrow[d, "\underline{\mathsf{Q}}(\phi;f)^{[]}"'] & (\underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in\phi^{-1}j};\mathcal{B}_j^{[]}))_{j\in J} \arrow[d, "\underline{\mathsf{Q}}(\phi;f^{[]})"] \\
\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "u_{[]}"] & \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C})^{[]} \arrow[r, "\underline{S}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{C}^{[]})
\end{tikzcd}$$

The left square commutes by multinaturality of $u_{[]}$, commutativity of the right square is a consequence of (4.22.1). □

10.26 Actions and the A_∞-multifunctor of shifts. In the remaining part of the chapter we extend the obtained monad of shifts to A_∞-categories. We employ the extension of the action used above to the action of the symmetric multicategory of differential graded categories on the symmetric multicategory of A_∞-categories. Let us compare the graded and the differential graded versions of the action.

There is an embedding of symmetric multicategories $e : \mathsf{A}^{\mathrm{u}}_\infty \hookrightarrow \mathsf{A}_\infty$ and the erasing symmetric multifunctor $E : \mathsf{A}_\infty \to \mathsf{Q}$ which forgets the differentials b. As explained in Sections C.5–C.13 these multifunctors agree with the following actions in $\mathcal{SMC}\mathrm{atm}$:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \arrow[r, "e\boxtimes\mathrm{Id}"] \arrow[d, "\boxdot"'] \arrow[dr, phantom, "="] & \mathsf{A}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \arrow[r, "E\boxtimes\widehat{E}"] \arrow[d, "\boxdot"'] \arrow[dr, phantom, "="] & \mathsf{Q} \boxtimes \widehat{\mathbf{gr}\text{-}\mathcal{C}at} \arrow[d, "\boxdot"] \\
\mathsf{A}^{\mathrm{u}}_\infty \arrow[r, "e"] & \mathsf{A}_\infty \arrow[r, "E"] & \mathsf{Q}
\end{tikzcd}$$

where the erasing functor $E : \mathbf{dg}\text{-}\mathcal{C}at \to \mathbf{gr}\text{-}\mathcal{C}at$ forgets the differential. Choose the algebra object $\mathcal{Z}$ in $\mathbf{dg}\text{-}\mathcal{C}at$ – the graded category with zero differentials described in Section 10.1. Its action gives the multifunctors of shifts $S = -^{[]}$ in the three considered multicategories which commute with e, E:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty \arrow[r, "e"] \arrow[d, "-^{[]}"', "= \ -\boxdot\mathcal{Z}"] \arrow[dr, phantom, "="] & \mathsf{A}_\infty \arrow[r, "E"] \arrow[d, "-^{[]}"', "= \ -\boxdot\mathcal{Z}"] \arrow[dr, phantom, "="] & \mathsf{Q} \arrow[d, "-^{[]}"', "= \ -\boxdot\mathcal{Z}"] \\
\mathsf{A}^{\mathrm{u}}_\infty \arrow[r, "e"] & \mathsf{A}_\infty \arrow[r, "E"] & \mathsf{Q}
\end{tikzcd}$$

According to Remark C.9 these multifunctors of shifts are augmented, where the augmentation multinatural transformation $u_{[]} : \mathrm{Id} \to -^{[]} = S$ comes from the unit $\eta_{\mathcal{Z}} : \mathbb{1}_p \to \mathcal{Z}$ of the algebra $\mathcal{Z}$. This gives equations between multinatural transformations – commutative cylinders:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{u}}_\infty \arrow[r, "e"] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, bend left, "-^{[]}"] \arrow[d, phantom, "\overset{u_{[]}}{\Longrightarrow}"] \arrow[dr, phantom, "="] & \mathsf{A}_\infty \arrow[r, "E"] \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, bend left, "-^{[]}"] \arrow[d, phantom, "\overset{u_{[]}}{\Longrightarrow}"] \arrow[dr, phantom, "="] & \mathsf{Q} \arrow[d, bend right, "\mathrm{Id}"'] \arrow[d, bend left, "-^{[]}"] \arrow[d, phantom, "\overset{u_{[]}}{\Longrightarrow}"] \\
\mathsf{A}^{\mathrm{u}}_\infty \arrow[r, "e"] & \mathsf{A}_\infty \arrow[r, "E"] & \mathsf{Q}
\end{tikzcd}$$

Therefore, assumptions of Proposition 4.36 are satisfied for both e and E and for $F = -^{[]}$, $G = -^{[]}$. Proposition 4.30 gives an $\mathsf{A}^{\mathrm{u}}_\infty$-multifunctor $S' = u_{[]} \cdot \underline{S} : \underline{\mathsf{A}^{\mathrm{u}}_\infty} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}$, an A_∞-multifunctor $S' = u_{[]} \cdot \underline{S} : \underline{\mathsf{A}_\infty} \to \underline{\mathsf{A}_\infty}$ and a Q-multifunctor $S' = u_{[]} \cdot \underline{S} : \underline{\mathsf{Q}} \to \underline{\mathsf{Q}}$. By abuse of notation all these will be also denoted $-^{[]}$. Proposition 4.36 implies that these multifunctors agree with $\underline{e}$ and $\underline{E}$, which are natural full embeddings:

$$
\begin{tikzcd}
\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \arrow[r, hook] \arrow[d, "-^{[]}"', "="{description}, "S'"] \arrow[rd, phantom, "="] & \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \arrow[r, hook] \arrow[d, "-^{[]}"', "="{description}, "S'"] \arrow[rd, phantom, "="] & \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}) \arrow[d, "-^{[]}"', "="{description}, "S'"] \\
\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{B}^{[]}) \arrow[r, hook] & \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{B}^{[]}) \arrow[r, hook] & \underline{\mathsf{Q}}((\mathcal{A}_i^{[]})_{i\in I};\mathcal{B}^{[]})
\end{tikzcd}
$$

Due to Proposition 10.16 and formula (10.19.3) the morphisms S' in Q are strict. Therefore, the other two multifunctors S' are strict as well, that is, all A_∞-functors $S'_{(\mathcal{A}_i);\mathcal{B}}$ are strict.

10.27 Differential. When $\mathcal{A}$ is an A_∞-category, it has the differential $b : \mathrm{id} \to \mathrm{id} : \mathcal{A} \to \mathcal{A}$ – an element of $\underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A})(\mathrm{id}, \mathrm{id})$ of degree 1. Equivalently, the differential is a quiver morphism $b : \mathbb{1}_p[-1] \to s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$, $* \mapsto \mathrm{id}_\mathcal{A}$, or a quiver morphism

$$\varphi^{\mathscr{Q}_p}(b) = \bigl[Ts\mathcal{A} \boxtimes \mathbb{1}_p[-1] \xrightarrow{1\boxtimes b} Ts\mathcal{A} \boxtimes \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\bigr].$$

We are going to analyze in detail the differential $b^{[]} = b^{\mathcal{A}\boxdot\mathcal{Z}}$ in $\mathcal{A}^{[]} = \mathcal{A} \boxdot \mathcal{Z}$. According to (C.10.1) this element of $\underline{\mathscr{Q}_p}(Ts(\mathcal{A}\boxdot\mathcal{Z}), s(\mathcal{A}\boxdot\mathcal{Z}))(\mathrm{id}, \mathrm{id})$ is given by the composition

$$b^{\mathcal{A}\boxdot\mathcal{Z}} = \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\varkappa} Ts\mathcal{A}\boxtimes T\mathcal{Z} \xrightarrow{b\boxtimes\mu_\mathcal{Z}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr].$$

Equivalently, it can be presented as the composition of quiver morphisms

$$
\begin{aligned}
b^{\mathcal{A}\boxdot\mathcal{Z}} = \bigl[\mathbb{1}_p[-1] &\xrightarrow{\lambda^{\mathrm{I}\cdot}} \mathbb{1}_p[-1]\boxtimes\mathbb{1}_p \xrightarrow{b\boxtimes\dot{\mu}_\mathcal{Z}} \underline{\mathscr{Q}_p}(Ts\mathcal{A}, s\mathcal{A}) \boxtimes \underline{\mathscr{Q}_p}(T\mathcal{Z},\mathcal{Z}) \\
&\xrightarrow{\boxtimes^{\mathbf{2}}} \underline{\mathscr{Q}_p}(Ts\mathcal{A}\boxtimes T\mathcal{Z}, s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\underline{\mathscr{Q}_p}(\varkappa;1)} \underline{\mathscr{Q}_p}(T(s\mathcal{A}\boxtimes\mathcal{Z}), s\mathcal{A}\boxtimes\mathcal{Z})\bigr].
\end{aligned}
$$

10.28 Proposition. *The differential $b^{\mathcal{A}\boxdot\mathcal{Z}}$ coincides with the compositions in $\mathscr{Q}$*

$$
\begin{aligned}
b^{[]} &= \bigl[\mathbb{1}_p[-1] \xrightarrow{\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow[S']{-^{[]}} s\underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{A}^{[]})\bigr] \\
&= \bigl[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow[S'_1]{(-^{[]})_1} s\underline{\mathsf{Q}}(\mathcal{A}^{[]};\mathcal{A}^{[]})\bigr].
\end{aligned}
$$

Proof. Let us show that $\varphi^{\mathscr{Q}_p}(b^{\mathcal{A}\boxdot\mathcal{Z}}) = \varphi^{\mathscr{Q}_p}(b^{[]})$. In fact,

$$\begin{aligned}
\varphi^{\mathscr{Q}_p}(b^{\mathcal{A}\boxdot\mathcal{Z}}) &= \bigl[Ts(\mathcal{A}^{[]})\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes b^{\mathcal{A}\boxdot\mathcal{Z}}} T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\underline{\mathscr{Q}_p}(T(s\mathcal{A}\boxtimes\mathcal{Z}),s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&= \bigl[Ts(\mathcal{A}^{[]})\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes\lambda^{\mathsf{I}\,\cdot}} Ts(\mathcal{A}^{[]})\boxtimes(\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p) \xrightarrow{1\boxtimes(b\boxtimes\dot\mu_{\mathcal{Z}})} \\
&\qquad Ts(\mathcal{A}^{[]})\boxtimes[\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})\boxtimes\underline{\mathscr{Q}_p}(T\mathcal{Z},\mathcal{Z})] \xrightarrow{1\boxtimes\boxtimes^{\mathbf{2}}} Ts(\mathcal{A}^{[]})\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A}\boxtimes T\mathcal{Z},s\mathcal{A}\boxtimes\mathcal{Z}) \\
&\qquad\qquad \xrightarrow{1\boxtimes\underline{\mathscr{Q}_p}(\varkappa;1)} T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\underline{\mathscr{Q}_p}(T(s\mathcal{A}\boxtimes\mathcal{Z}),s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&= \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathbb{1}_p[-1] \xrightarrow{\varkappa\boxtimes\lambda^{\mathsf{I}\,\cdot}} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes(\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p) \\
&\qquad \xrightarrow{1\boxtimes(b\boxtimes\dot\mu_{\mathcal{Z}})} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes[\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})\boxtimes\underline{\mathscr{Q}_p}(T\mathcal{Z},\mathcal{Z})] \\
&\qquad\qquad \xrightarrow{1\boxtimes\boxtimes^{\mathbf{2}}} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A}\boxtimes T\mathcal{Z},s\mathcal{A}\boxtimes\mathcal{Z}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&= \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathbb{1}_p[-1] \xrightarrow{\varkappa\boxtimes\lambda^{\mathsf{I}\,\cdot}} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes(\mathbb{1}_p[-1]\boxtimes\mathbb{1}_p) \\
&\qquad \xrightarrow{1\boxtimes(b\boxtimes\dot\mu_{\mathcal{Z}})} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes[\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})\boxtimes\underline{\mathscr{Q}_p}(T\mathcal{Z},\mathcal{Z})] \\
&\qquad \xrightarrow{\sigma_{(12)}} [Ts\mathcal{A}\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})]\boxtimes[T\mathcal{Z}\boxtimes\underline{\mathscr{Q}_p}(T\mathcal{Z},\mathcal{Z})] \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}\boxtimes\mathrm{ev}^{\mathscr{Q}_p}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&= \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes\mathbb{1}_p[-1] \xrightarrow{\varkappa\boxtimes b} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A}) \\
&\qquad \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}\cdot(23)^\sim\cdot\lambda^{\mathsf{VI}}} [Ts\mathcal{A}\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})]\boxtimes T\mathcal{Z} \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}\boxtimes\mu_{\mathcal{Z}}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr].
\end{aligned} \tag{10.28.1}$$

On the other hand, formula (10.19.4) gives for $k = 1$ the expression

$$\begin{aligned}
&\bigl[Ts(\mathcal{A}^{[]})\boxtimes s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{1\boxtimes S'_1} Ts(\mathcal{A}^{[]})\boxtimes\underline{\mathscr{Q}_p}(Ts(\mathcal{A}^{[]}),s(\mathcal{A}^{[]})) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s(\mathcal{A}^{[]})\bigr] \\
&\quad= \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\varkappa\boxtimes\lambda^{\mathsf{I}\,\cdot}} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes[\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})\boxtimes\mathbb{1}_p] \xrightarrow{(1\boxtimes\mu_{\mathcal{Z}})\boxtimes[1\boxtimes\eta_{\mathcal{Z}}]} \\
&(Ts\mathcal{A}\boxtimes\mathcal{Z})\boxtimes[\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})\boxtimes\mathcal{Z}] \xrightarrow{\sigma_{(12)}} [Ts\mathcal{A}\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})]\boxtimes(\mathcal{Z}\boxtimes\mathcal{Z}) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}\boxtimes\otimes_{\mathcal{Z}}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr] \\
&\quad= \bigl[T(s\mathcal{A}\boxtimes\mathcal{Z})\boxtimes s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\varkappa\boxtimes 1} (Ts\mathcal{A}\boxtimes T\mathcal{Z})\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A}) \\
&\qquad\qquad \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}\cdot(23)^\sim\cdot\lambda^{\mathsf{VI}}} [Ts\mathcal{A}\boxtimes\underline{\mathscr{Q}_p}(Ts\mathcal{A},s\mathcal{A})]\boxtimes T\mathcal{Z} \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}\boxtimes\mu_{\mathcal{Z}}} s\mathcal{A}\boxtimes\mathcal{Z}\bigr].
\end{aligned}$$

Therefore,

$$\varphi^{\mathscr{Q}_p}(b^{[]}) = \bigl[Ts(\mathcal{A}^{[]})\boxtimes\mathbb{1}_p[-1] \xrightarrow{1\boxtimes b^{[]}} Ts(\mathcal{A}^{[]})\boxtimes\underline{\mathscr{Q}_p}(Ts(\mathcal{A}^{[]}),s(\mathcal{A}^{[]})) \xrightarrow{\mathrm{ev}^{\mathscr{Q}_p}} s(\mathcal{A}^{[]})\bigr]$$

equals the last expression in (10.28.1). This implies that $b^{\mathcal{A}\boxdot\mathcal{Z}} = b^{[]}$. □

This allows to compute the components of the differential $b^{[]} \in \underline{\mathscr{Q}_p}(Ts\mathcal{A}^{[]}, s\mathcal{A}^{[]})(\mathrm{id},\mathrm{id})$,

or $b^{[]} : \mathrm{id} \to \mathrm{id} : \mathcal{A}^{[]} \to \mathcal{A}^{[]}$ via general formula (10.19.5):

$$
\begin{gathered}
b_p^{[]} = (s^{n_1-n_0} \otimes \cdots \otimes s^{n_p-n_{p-1}})^{-1} s^{n_p-n_0} b_p^{[n_p-n_0]} \\
= (-1)^{n_p-n_0} (s^{n_1-n_0} \otimes \cdots \otimes s^{n_p-n_{p-1}})^{-1} b_p s^{n_p-n_0} : \\
\otimes^{i\in\mathbf{p}} s\mathcal{A}^{[]}(X_{i-1}[n_{i-1}], X_i[n_i]) = \otimes^{i\in\mathbf{p}} (s\mathcal{A}(X_{i-1}, X_i)[n_i - n_{i-1}]) \xrightarrow{\sigma^+_{(n_i-n_{i-1})_i}} \\
(\otimes^{i\in\mathbf{p}} s\mathcal{A}(X_{i-1}, X_i))[n_p - n_0] \xrightarrow{b_p^{[n_p-n_0]}} s\mathcal{A}(X_0, X_p)[n_p - n_0] = s\mathcal{A}^{[]}(X_0[n_0], X_p[n_p]).
\end{gathered}
\tag{10.28.2}
$$

10.29 Remark. If A_∞-category $\mathcal{C}$ is unital, then $\mathcal{C}^{[]}$ is unital as well and its unit elements are

$$
{}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]}} = {}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X, X) = s\mathcal{C}^{[]}(X[n], X[n]) \tag{10.29.1}
$$

by Lemma C.14. Furthermore, if A_∞-category $\mathcal{C}$ is strictly unital, then $\mathcal{C}^{[]}$ is strictly unital as well, and the above unit elements are strict. Indeed, if $\mathbf{i}_0^{\mathcal{C}}$ is a strict unit, then $(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2 = 1$ implies

$$
(1 \otimes {}_{Y[m]}\mathbf{i}_0^{\mathcal{C}^{[]}})b_2^{[]} = (-s)^{n-m}(1 \otimes {}_Y\mathbf{i}_0^{\mathcal{C}})b_2(-s)^{m-n} = 1 : s\mathcal{C}^{[]}(X[n], Y[m]) \to s\mathcal{C}^{[]}(X[n], Y[m]).
$$

Similarly, $({}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]}} \otimes 1)b_2^{[]} = -1$ in the strict case. Since $(1^{\otimes\alpha} \otimes {}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]}} \otimes 1^{\otimes\beta})b_{\alpha+1+\beta}^{[]}$ has a factor $(1^{\otimes\alpha} \otimes {}_X\mathbf{i}_0^{\mathcal{C}} \otimes 1^{\otimes\beta})b_{\alpha+1+\beta}$, it vanishes for $\alpha + \beta > 1$.

10.30 The A_∞-2-monad of shifts. Proposition 10.20 is based only on the existence of action $\boxdot : \mathsf{Q} \boxtimes \widehat{\mathbf{gr}\text{-}\mathcal{C}at} \to \mathsf{Q}$ and the algebra property of differential graded category $\mathcal{Z}$. It gives the monad $(-^{[]}, u_{[]}, m_{[]})$, where $-^{[]} = S = -\boxdot\mathcal{Z} : \mathsf{Q} \to \mathsf{Q}$ is the multifunctor of shifts, the unit $u_{[]} = -\boxdot(\eta_{\mathcal{Z}} : \mathbb{1}_p \to \mathcal{Z})$ is a multinatural transformation, and the multiplication $m_{[]} = -\boxdot(\otimes_{\mathcal{Z}} : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z})$ is a natural transformation which satisfies condition (10.20.1). Since there is also action $\boxdot : \mathsf{A}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}_\infty$ by Section C.10 and action $\boxdot : \mathsf{A}_\infty^{\mathrm{u}} \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}_\infty^{\mathrm{u}}$ by Section C.13, the same algebra $\mathcal{Z}$ gives also monads $(-^{[]}, u_{[]}, m_{[]})$ in A_∞ and in $\mathsf{A}_\infty^{\mathrm{u}}$ with the same properties as in Q. As explained in Section 9.20 these monads give rise to an A_∞-monad (A_∞-2-monad) $(-^{[]}, u_{[]}, m_{[]}) : \underline{\mathsf{A}_\infty} \to \underline{\mathsf{A}_\infty}$ and respectively to an $\mathsf{A}_\infty^{\mathrm{u}}$-monad ($A_\infty^u$-2-monad) $(-^{[]}, u_{[]}, m_{[]}) : \underline{\mathsf{A}_\infty^{\mathrm{u}}} \to \underline{\mathsf{A}_\infty^{\mathrm{u}}}$. As explained in Section 10.26 these monads and the corresponding Q-monad agree with the natural full embeddings $\underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \overset{e}{\hookrightarrow} \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \overset{E}{\hookrightarrow} \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$. Let us describe the unit and the multiplication in detail.

10.30.1 Unit A_∞-2-transformation $u_{[]}$. For any A_∞-category $\mathcal{A}$ the unit A_∞-2-transformation $u_{[]} = u_{[]}^{\mathcal{A}} = \eta(\mathcal{A}) : \mathcal{A} \to \mathcal{A}^{[]}$ is a strict A_∞-functor given by the formulas $X \mapsto X[0]$, $\eta(\mathcal{A})_1 = \mathrm{id} : s\mathcal{A}(X_0, X_1) \to s\mathcal{A}^{[]}(X_0[0], X_1[0])$. If $\mathcal{A}$ is unital, then $u_{[]}^{\mathcal{A}}$ is unital,

because it takes a unit element ${}_X\mathbf{i}_0^{\mathcal{A}}$ to the unit element ${}_{X[0]}\mathbf{i}_0^{\mathcal{A}^{[]}}$. In the unital case $u_{[]}$ is a natural A_∞^u-2-transformation.

10.30.2 Multiplication A_∞-2-transformation $m_{[]}$. We describe the natural A_∞-2-transformation $m_{[]} : -^{[][]} \to -^{[]}$ in the A_∞-2-monad $(-^{[]}, m_{[]}, u_{[]})$, which is obtained via algebra $(\mathcal{Z}, \otimes_{\mathcal{Z}})$ of Section 10.1. The A_∞-2-transformation $m_{[]}$ is specified by the collection of A_∞-functors

$$m_{[]} = \bigl[\mathcal{A}^{[][]} = (\mathcal{A} \boxdot \mathcal{Z}) \boxdot \mathcal{Z} \xrightarrow{\alpha^{\mathbf{2}}} \mathcal{A} \boxdot (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxdot\otimes_{\mathcal{Z}}} \mathcal{A} \boxdot \mathcal{Z} = \mathcal{A}^{[]}\bigr]. \tag{10.30.1}$$

All the above A_∞-functors are strict. Indeed, $\alpha^{\mathbf{2}}$ given by (C.7.1) is simply an isomorphism of underlying quivers. The A_∞-functor $1 \boxdot \otimes_{\mathcal{Z}}$ is given by formula (C.5.1)

$$\overline{1 \boxdot \otimes_{\mathcal{Z}}} = \bigl[T(s\mathcal{A} \boxtimes (\mathcal{Z} \boxtimes \mathcal{Z})) \xrightarrow{\varkappa} Ts\mathcal{A} \boxtimes T(\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{\mathrm{pr}_1 \boxtimes \mu_{\mathcal{Z}\boxtimes\mathcal{Z}}} s\mathcal{A} \boxtimes (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} s\mathcal{A} \boxtimes \mathcal{Z}\bigr].$$

Clearly, it is strict with the first component equal to $1 \boxtimes \otimes_{\mathcal{Z}}$. Therefore, A_∞-functor $m_{[]}$ is also strict with the first component expressed by

$$(m_{[]})_1 = \bigl[(s\mathcal{A} \boxtimes \mathcal{Z}) \boxtimes \mathcal{Z} \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} s\mathcal{A} \boxtimes \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\lambda^{\mathsf{IV}}} s\mathcal{A} \boxtimes (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxtimes\otimes_{\mathcal{Z}}} s\mathcal{A} \boxtimes \mathcal{Z}\bigr]. \tag{10.30.2}$$

On objects this quiver map is given by the formula: $X[n][m] \mapsto X[n+m]$. On morphisms it equals

$$\begin{aligned}(m_{[]})_1 = \bigl[s\mathcal{A}^{[][]}(X[n][m], Y[k][l]) &= s\mathcal{A}(X,Y)[k-n][l-m] \\ &\xrightarrow{(-1)^{k(m-l)}} s\mathcal{A}(X,Y)[k+l-n-m] = s\mathcal{A}^{[]}(X[n+m], Y[k+l])\bigr].\end{aligned}$$

10.31 Remark. Let $\mathcal{B}$ be a unital A_∞-category. If $g : \mathcal{B} \to \mathcal{A}$ a contractible A_∞-functor, then $g^{[]} : \mathcal{B}^{[]} \to \mathcal{A}^{[]}$ is contractible as well. Indeed, if

$$g_1 = tb_1 + b_1 t : s\mathcal{B}(X,Y) \to s\mathcal{A}(Xg, Yg)$$

for some map $t : s\mathcal{B}(X,Y) \to s\mathcal{A}(Xg,Yg)$ of degree -1, then

$$g_1^{[]} = (s^{n-m}t(-s)^{m-n})b_1^{[]} + b_1^{[]}(s^{n-m}t(-s)^{m-n}) : s\mathcal{B}^{[]}(X[n], Y[m]) \to s\mathcal{A}^{[]}(Xg[n], Yg[m]).$$

10.32 Definition. We say that a unital A_∞-category $\mathcal{C}$ is *closed under shifts* if every object $X[n]$ of $\mathcal{C}^{[]}$ is isomorphic in $\mathcal{C}^{[]}$ to some object $Y[0]$, $Y = [X,n] \in \operatorname{Ob}\mathcal{C}$.

10.33 Proposition. *A unital A_∞-category $\mathcal{C}$ is closed under shifts if and only if the A_∞-functor $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an equivalence.*

Proof. Let us prove that for a closed under shifts unital A_∞-category $\mathcal{C}$ the A_∞-functor $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an equivalence. Choose a function $\operatorname{Ob}\psi : \operatorname{Ob}\mathcal{C}^{[]} \to \operatorname{Ob}\mathcal{C}$, $X[n] \mapsto [X,n]$, and cycles $p_0 \in s\mathcal{C}^{[]}(X[n],[X,n][0])$, $r_0 \in s\mathcal{C}^{[]}([X,n][0],X[n])$ such that

$$(p_0 \otimes r_0)b_2^{[]} - {}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]}} \in \operatorname{Im} b_1^{[]}, \qquad (r_0 \otimes p_0)b_2^{[]} - {}_{[X,n][0]}\mathbf{i}_0^{\mathcal{C}^{[]}} \in \operatorname{Im} b_1^{[]}.$$

As $u_{[]1} = \mathrm{id}$ is invertible, theorem 8.8 of [Lyu03] implies that $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an equivalence, and the map $\operatorname{Ob}\psi$ can be extended to an A_∞-functor $\psi : \mathcal{C}^{[]} \to \mathcal{C}$, quasi-inverse to $u_{[]}$.

On the other hand, if $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an equivalence, then it is essentially surjective on objects. This means precisely the existence of p_0, r_0 as above, that is, closedness under shifts. □

10.34 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$ be equivalent unital A_∞-categories. If one of them is closed under shifts, then so is the other.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-equivalence, then $f^{[]} : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$ is A_∞-equivalence as well, since the A_∞^u-2-functor $-^{[]}$ defines an ordinary strict 2-functor $\overline{-^{[]}} : \overline{A_\infty^u} \to \overline{A_\infty^u}$ [LM06a, Section 3.2]. In the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u_{[]}"] \arrow[d, "f"'] & \mathcal{A}^{[]} \arrow[d, "f^{[]}"] \\ \mathcal{B} \arrow[r, "u_{[]}"] & \mathcal{B}^{[]} \end{tikzcd}$$

three out of four A_∞-functors are equivalences. Hence, so is the fourth. □

10.35 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then $\mathcal{C}^{[]}$ is closed under shifts.*

Proof. Let $X[n][m]$ be an object of $\mathcal{C}^{[][]}$. We claim that it is isomorphic to $X[n+m][0]$. Indeed, there is a cocycle

$$q \in s\mathcal{C}^{[][]}(X[n][m], X[n+m][0]) = s\mathcal{C}^{[]}(X[n], X[n+m])[-m] = s\mathcal{C}(X,X),$$

which coincides with ${}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X,X)$, and a cocycle

$$t \in s\mathcal{C}^{[][]}(X[n+m][0], X[n][m]) = s\mathcal{C}^{[]}(X[n+m], X[n])[m] = s\mathcal{C}(X,X),$$

which coincides with $(-1)^m {}_X\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X,X)$.

We have

$$\begin{aligned}(q \otimes t)b_2^{[][]} - {}_{X[n][m]}\mathbf{i}_0^{\mathcal{C}^{[][]}} &= (q \otimes t)(s^m \otimes s^{-m})b_2^{[]} - {}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]}} \\ &= (q \otimes t)(s^m \otimes s^{-m})(s^{-m} \otimes s^m)b_2 - {}_X\mathbf{i}_0^{\mathcal{C}} = ({}_X\mathbf{i}_0^{\mathcal{C}} \otimes {}_X\mathbf{i}_0^{\mathcal{C}})b_2 - {}_X\mathbf{i}_0^{\mathcal{C}} \in \operatorname{Im} b_1^{[][]}.\end{aligned}$$

Similarly, $(t \otimes q)b_2^{[][]} - {}_{X[n+m][0]}\mathbf{i}_0^{\mathcal{C}^{[][]}} \in \operatorname{Im} b_1^{[][]}$. Hence, objects $X[n][m]$ and $X[n+m][0]$ are isomorphic. Therefore, $\mathcal{C}^{[]}$ is closed under shifts. □

10.36 Corollary. *If $\mathcal{C}$ is a unital A_∞-category, then A_∞-functors $u_{[]}, u_{[]}^{[]} : \mathcal{C}^{[]} \to \mathcal{C}^{[][]}$ and $m_{[]} : \mathcal{C}^{[][]} \to \mathcal{C}^{[]}$ are equivalences, quasi-inverse to each other.*

Indeed, such $u_{[]}$ is an equivalence by Proposition 10.33, and $u_{[]}m_{[]} = \mathrm{id}_{\mathcal{C}^{[]}} = u_{[]}^{[]}m_{[]}$.

10.37 Proposition. *For an arbitrary A_∞-category $\mathcal{C}$ closed under shifts there exists an A_∞-equivalence $U_{[]} = U_{[]}^{\mathcal{C}} : \mathcal{C}^{[]} \to \mathcal{C}$ such that $u_{[]} \cdot U_{[]} = \mathrm{id}_{\mathcal{C}}$. In particular, $U_{[]}$ is quasi-inverse to $u_{[]}$.*

Proof. Apply Proposition 9.22 to the following data: $\mathcal{B} = \mathcal{C}^{[]}$, $\mathcal{D} = \mathcal{C}$, the A_∞-equivalence $\phi = u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$, the embedding of a full A_∞-subcategory $\iota = u_{[]} : \mathcal{C} \hookrightarrow \mathcal{C}^{[]}$, the A_∞-functor $w = \mathrm{id} : \mathcal{C} \to \mathcal{C}$, the natural A_∞-transformation $q = u_{[]}\mathbf{i}^{\mathcal{C}^{[]}} : u_{[]} \to u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ which represents the identity 2-morphism of the 1-morphism $u_{[]}$ in $\overline{A_\infty^u}$. Choose a map $h : \operatorname{Ob}\mathcal{C}^{[]} \to \operatorname{Ob}\mathcal{C}$ in such a way that $(X[0])h = X = Xw$ for all objects X of $\mathcal{C}$ and that the objects Y and $Yh[0]$ were isomorphic in $H^0(\mathcal{C}^{[]})$ for all objects Y of $\mathcal{C}^{[]}$. Choose cycles ${}_Yr_0 \in \mathcal{C}^{[]}(Y, Yh[0])[1]^{-1}$, $Y \in \operatorname{Ob}\mathcal{C}^{[]}$ which represent these isomorphisms taking ${}_{X[0]}r_0 = {}_{X[0]}\mathbf{i}_0^{\mathcal{C}^{[]}} = {}_Xq_0$ for all objects X of $\mathcal{C}$. The hypotheses of Proposition 9.22 are satisfied. We deduce from it that there exists an A_∞-equivalence $\psi = U_{[]} : \mathcal{C}^{[]} \to \mathcal{C}$ such that $u_{[]} \cdot U_{[]} = \mathrm{id}_{\mathcal{C}}$. □

10.38 Remark. Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be unital A_∞-categories. Equation (4.17.1), which can be written for an arbitrary symmetric closed multicategory, takes for A_∞^u the form

$$\begin{aligned}
&\big[Ts\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}) \boxtimes Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y}) \xrightarrow{c} Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y}) \boxtimes Ts\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}) \xrightarrow{M} Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Z})\big] \\
&= \big[Ts\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}) \boxtimes Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y}) \xrightarrow{1\boxtimes \underline{\mathsf{A}_\infty^u}(_;\mathcal{Z})} Ts\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}) \boxtimes Ts\underline{\mathsf{A}_\infty^u}(\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}), \underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Z})) \\
&\qquad \xrightarrow{\mathrm{ev}^{\mathsf{A}_\infty^u}} Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Z})\big].
\end{aligned} \tag{10.38.1}$$

By the closedness of the multicategory A_∞^u, there exists a unique A_∞-functor

$$\underline{\mathsf{A}_\infty^u}(\_;\mathcal{Z}) : \underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y}) \to \underline{\mathsf{A}_\infty^u}(\underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}), \underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Z}))$$

which makes equation (10.38.1) hold true. A similar A_∞-functor $\underline{\mathsf{A}_\infty}(\_;\mathcal{Z})$ is derived in [LM08c, Appendix B.1]. The proof of Proposition 3.4 of [Lyu03] contains a recipe for finding the components of $\underline{\mathsf{A}_\infty^u}(\_;\mathcal{Z})$. Namely, the equation

$$(p \boxtimes 1)M = [p.\underline{\mathsf{A}_\infty^u}(\_;\mathcal{Z})]\theta \tag{10.38.2}$$

has to hold for all $p \in Ts\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y})$. In particular,

$$\begin{aligned}
f.\underline{\mathsf{A}_\infty^u}(_;\mathcal{Z}) &= (f \boxtimes 1)M = (f \boxtimes 1_{\mathcal{Z}})M : \underline{\mathsf{A}_\infty^u}(\mathcal{Y};\mathcal{Z}) \to \underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Z}) \quad \text{for } f \in \operatorname{Ob}\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y}), \\
r.\underline{\mathsf{A}_\infty^u}(_;\mathcal{Z})_1 &= (r \boxtimes 1)M = (r \boxtimes 1_{\mathcal{Z}})M : (f \boxtimes 1)M \to (g \boxtimes 1)M \text{ for } r \in s\underline{\mathsf{A}_\infty^u}(\mathcal{X};\mathcal{Y})(f,g).
\end{aligned}$$

Other components of $\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})$ are obtained from the recurrent relation, which is equation (10.38.2) written for $p = p^1 \otimes \cdots \otimes p^n$:

$$(p^1 \otimes \cdots \otimes p^n)\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_n = (p^1 \otimes \cdots \otimes p^n \boxtimes 1)M - \sum_{i_1+\cdots+i_l=n}^{l>1} [(p^1 \otimes \cdots \otimes p^n).(\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_{i_1} \otimes \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_{i_2} \otimes \cdots \otimes \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_{i_l})]\theta. \tag{10.38.3}$$

In particular, for $r \otimes t \in T^2 s\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{X};\mathcal{Y})$ we get

$$(r \otimes t)\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_2 = (r \otimes t \boxtimes 1)M - [(r \boxtimes 1)M \otimes (t \boxtimes 1)M]\theta.$$

Given $g^0 \xrightarrow{p^1} g^1 \xrightarrow{p^2} \dots g^{n-1} \xrightarrow{p^n} g^n$ we find from (10.38.3) the components of the A_∞-transformation

$$(p^1 \otimes \cdots \otimes p^n)\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_n \in s\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{Y};\mathcal{Z}), \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{X};\mathcal{Z}))((g^0 \boxtimes 1)M, (g^n \boxtimes 1)M)$$

in the form

$$[(p^1 \otimes \cdots \otimes p^n)\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_n]_m = (p^1 \otimes \cdots \otimes p^n \boxtimes 1)M_{nm}.$$

So they vanish for $m > 1$.

Since the A_∞-functor $\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})$ is unital, its first component takes the unit transformation $\mathbf{i}^{\mathcal{X}}$ of $\mathrm{id} : \mathcal{X} \to \mathcal{X}$ to the unit transformation

$$\mathbf{i}^{\mathcal{X}}.\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\_;\mathcal{Z})_1 = (\mathbf{i}^{\mathcal{X}} \boxtimes 1)M : \mathrm{id} \to \mathrm{id} : \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{X};\mathcal{Z}) \to \underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{X};\mathcal{Z}).$$

Notice that the A_∞-category $\underline{\mathsf{A}_\infty^{\mathrm{u}}}(\mathcal{X};\mathcal{Z})$ has another unit transformation $(1\boxtimes\mathbf{i}^{\mathcal{Z}})M$, thereby equivalent to $(\mathbf{i}^{\mathcal{X}} \boxtimes 1)M$.

10.39 Proposition. *Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories, and let $\mathcal{C}$ be closed under shifts. Let A_∞-equivalence $U_{[]} = U_{[]}^{\mathcal{C}} : \mathcal{C}^{[]} \to \mathcal{C}$ satisfy the equation $u_{[]} \cdot U_{[]} = \mathrm{id}_{\mathcal{C}}$ (it exists by Proposition 10.37). Then the strict A_∞-functor $A_\infty^u(u_{[]},\mathcal{C}) = (u_{[]} \boxtimes 1)M : A_\infty^u(\mathcal{A}^{[]},\mathcal{C}) \to A_\infty^u(\mathcal{A},\mathcal{C})$ is an A_∞-equivalence which admits a one-sided inverse*

$$F_{[]} = \bigl[A_\infty^u(\mathcal{A},\mathcal{C}) \xrightarrow{-^{[]}} A_\infty^u(\mathcal{A}^{[]},\mathcal{C}^{[]}) \xrightarrow{A_\infty^u(\mathcal{A}^{[]},U_{[]})} A_\infty^u(\mathcal{A}^{[]},\mathcal{C})\bigr]$$

(quasi-inverse to $A_\infty^u(u_{[]},\mathcal{C})$), namely, $F_{[]} \cdot A_\infty^u(u_{[]},\mathcal{C}) = \mathrm{id}_{A_\infty^u(\mathcal{A},\mathcal{C})}$.

Proof. The naturality of the A_∞^u-2-transformation $u_{[]}$ is expressed by the equation

$$\bigl[A_\infty^u(\mathcal{A},\mathcal{B}) \xrightarrow{-^{[]}} A_\infty^u(\mathcal{A}^{[]},\mathcal{B}^{[]}) \xrightarrow{(u_{[]}\boxtimes 1)M} A_\infty^u(\mathcal{A},\mathcal{B}^{[]})\bigr] = (1 \boxtimes u_{[]})M. \tag{10.39.1}$$

It implies that the A_∞-functor $F_{[]}$ is a one-sided inverse to $A^u_\infty(u_{[]},\mathcal{C})$. Indeed,

$$\begin{aligned} F_{[]}\cdot A^u_\infty(u_{[]},\mathcal{C}) &= \bigl[A^u_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{-^{[]}} A^u_\infty(\mathcal{A}^{[]},\mathcal{C}^{[]}) \xrightarrow{A^u_\infty(\mathcal{A}^{[]},U_{[]})} A^u_\infty(\mathcal{A}^{[]},\mathcal{C}) \xrightarrow{A^u_\infty(u_{[]},\mathcal{C})} A^u_\infty(\mathcal{A},\mathcal{C})\bigr] \\ &= \bigl[A^u_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{-^{[]}} A^u_\infty(\mathcal{A}^{[]},\mathcal{C}^{[]}) \xrightarrow{A^u_\infty(u_{[]},1)} A^u_\infty(\mathcal{A},\mathcal{C}^{[]}) \xrightarrow{A^u_\infty(1,U_{[]})} A^u_\infty(\mathcal{A},\mathcal{C})\bigr] \\ &= \bigl[A^u_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{A^u_\infty(1,u_{[]})} A^u_\infty(\mathcal{A},\mathcal{C}^{[]}) \xrightarrow{A^u_\infty(1,U_{[]})} A^u_\infty(\mathcal{A},\mathcal{C})\bigr] = A^u_\infty(1,u_{[]}U_{[]}) = \mathrm{id} \end{aligned}$$

due to Lemmata 4.13 and 4.15.

Composition of these A_∞-functors in the other order gives on objects (unital A_∞-functors $f:\mathcal{A}^{[]}\to\mathcal{C}$) the following:

$$f\mapsto u_{[]}f\mapsto (u_{[]}f)^{[]}U_{[]} = u^{[]}_{[]}f^{[]}U_{[]} \simeq u_{[]}f^{[]}U_{[]} = fu_{[]}U_{[]} = f.$$

Indeed, due to Corollary 10.36 there is an isomorphism of A_∞-functors $r : u^{[]}_{[]} \to u_{[]} : \mathcal{A}^{[]}\to\mathcal{A}^{[][]}$. Let us prove that this composition is isomorphic to identity A_∞-functor. It is given by the top–right–bottom exterior path in the following diagram, which describes a natural A_∞-transformation:

$$\begin{tikzcd}
A^u_\infty(\mathcal{A}^{[]},\mathcal{C}) \arrow[r, "A^u_\infty(u_{[]}{,}\mathcal{C})"] \arrow[d, equal] & A^u_\infty(\mathcal{A},\mathcal{C}) \arrow[r, "-^{[]}"] \arrow[d, phantom, "="] & A^u_\infty(\mathcal{A}^{[]},\mathcal{C}^{[]}) \arrow[d, equal] \\
A^u_\infty(\mathcal{A}^{[]},\mathcal{C}) \arrow[r, "-^{[]}"] \arrow[d, "\mathrm{id}"'] \arrow[drr, "A^u_\infty(\mathcal{A}^{[]}{,}u_{[]})"'] & A^u_\infty(\mathcal{A}^{[][]},\mathcal{C}^{[]}) \arrow[r, shift left=2, "A^u_\infty(u^{[]}_{[]}{,}\mathcal{C}^{[]})"] \arrow[r, shift right=2, "A^u_\infty(u_{[]}{,}\mathcal{C}^{[]})"'] \arrow[r, phantom, "(r\boxtimes1)M\Downarrow"] & A^u_\infty(\mathcal{A}^{[]},\mathcal{C}^{[]}) \arrow[d, equal] \\
A^u_\infty(\mathcal{A}^{[]},\mathcal{C}) & & A^u_\infty(\mathcal{A}^{[]},\mathcal{C}^{[]}) \arrow[ll, "A^u_\infty(\mathcal{A}^{[]}{,}U_{[]})"]
\end{tikzcd}$$

Since the A_∞-functor $A^u_\infty(\_,\mathcal{C}^{[]})$ is unital, its first component takes the isomorphism r to an isomorphism $r.A^u_\infty(\_,\mathcal{C}^{[]})_1 = (r\boxtimes 1)M : (u^{[]}_{[]}\boxtimes 1)M \to (u_{[]}\boxtimes 1)M$. Thus the above diagram gives an isomorphism of the top–right–bottom exterior path with the left column, which is the identity functor. The proposition is proven. $\square$

10.40 Corollary. *Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories, and let $\mathcal{C}$ be closed under shifts. Then the restriction map $\mathsf{A}^{\mathrm{u}}_\infty(u_{[]},\mathcal{C}) : \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{A}^{[]},\mathcal{C}) \to \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{A},\mathcal{C})$ is surjective.*

10.41 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. Then the A_∞-functor*

$$-^{[]} : A^u_\infty(\mathcal{A},\mathcal{B}) \to A^u_\infty(\mathcal{A}^{[]},\mathcal{B}^{[]})$$

is homotopy full and faithful, that is, its first component is homotopy invertible.

Proof. Consider equation (10.39.1). The first component of $A^u_\infty(\mathcal{A}, u_{[]}) = (1 \boxtimes u_{[]})M$ in the right hand side (composition with $u_{[]}$) is an isomorphism, since $u_{[]}$ is a strict full embedding. The first component of the second functor $A^u_\infty(u_{[]}, \mathcal{B}^{[]})$ in the left hand side is homotopy invertible by Proposition 10.39. Therefore, the first component of the first functor $-^{[]}$ in the left hand side is homotopy invertible. □

10.42 Lemma. *Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-equivalence. Let objects Xf, Yf of $\mathcal{B}$ be isomorphic via inverse to each other isomorphisms $r \in s\mathcal{B}(Xf, Yf)$, $p \in s\mathcal{B}(Yf, Xf)$ (in the sense of Definition 9.12). Then the objects X, Y of $\mathcal{A}$ are isomorphic via inverse to each other isomorphisms $q \in s\mathcal{A}(X, Y)$, $t \in s\mathcal{A}(Y, X)$ such that $qf_1 - r \in \operatorname{Im} b_1$, $tf_1 - p \in \operatorname{Im} b_1$.*

Proof. Let chain maps $g_{X,Y} : s\mathcal{B}(Xf, Yf) \to s\mathcal{A}(X, Y)$, $g_{Y,X} : s\mathcal{B}(Yf, Xf) \to s\mathcal{A}(Y, X)$ be homotopy inverse to maps $f_1 : s\mathcal{A}(X, Y) \to s\mathcal{B}(Xf, Yf)$, $f_1 : s\mathcal{A}(Y, X) \to s\mathcal{B}(Yf, Xf)$. Define $q = rg_{X,Y}$, $t = pg_{Y,X}$. Then

$$\begin{aligned}[(q \otimes t)b_2 - {}_X\mathbf{i}^{\mathcal{A}}_0]f_1 &= (rg \otimes pg)b_2 f_1 - {}_X\mathbf{i}^{\mathcal{A}}_0 f_1 \equiv (rg \otimes pg)(f_1 \otimes f_1)b_2 - {}_{Xf}\mathbf{i}^{\mathcal{B}}_0 \\ &= [(r + vb_1) \otimes (p + wb_1)]b_2 - {}_{Xf}\mathbf{i}^{\mathcal{B}}_0 \equiv (r \otimes p)b_2 - {}_{Xf}\mathbf{i}^{\mathcal{B}}_0 \equiv 0 \pmod{\operatorname{Im} b_1}.\end{aligned}$$

Hence,

$$(q \otimes t)b_2 - {}_X\mathbf{i}^{\mathcal{A}}_0 \equiv [(q \otimes t)b_2 - {}_X\mathbf{i}^{\mathcal{A}}_0]f_1 g_{X,X} \equiv 0 \pmod{\operatorname{Im} b_1}.$$

By symmetry, $(t \otimes q)b_2 - {}_Y\mathbf{i}^{\mathcal{A}}_0 \in \operatorname{Im} b_1$. Other properties are easy to verify. □

10.43 Quotients and shifts. Let $\mathcal{B}$ be a full subcategory of a unital A_∞-category $\mathcal{C}$. Denote by $i : \mathcal{B} \hookrightarrow \mathcal{C}$ the inclusion A_∞-functor, and by $e : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})$ the quotient functor. By construction e can be chosen so that $\operatorname{Ob} \mathsf{q}(\mathcal{C}|\mathcal{B}) = \operatorname{Ob} \mathcal{C}$, $\operatorname{Ob} e = \mathrm{id}_{\operatorname{Ob} \mathcal{C}}$ [LM08c]. Similarly, in the diagram below $i^{[]}$ is a full embedding and e' is the quotient functor:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, hook, "i"] \arrow[d, "u^{\mathcal{B}}_{[]}"'] \arrow[dr, phantom, "="] & \mathcal{C} \arrow[rr, "e"] \arrow[d, "u^{\mathcal{C}}_{[]}"] & & \mathsf{q}(\mathcal{C}|\mathcal{B}) \arrow[dl, "f"'] \arrow[dd, "u_{[]}"] \\
 & {} & \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \arrow[r, Rightarrow, "\chi"] \arrow[d, Rightarrow, "\beta"] \arrow[dr, "g"] & {} \\
\mathcal{B}^{[]} \arrow[r, hook, "i^{[]}"] & \mathcal{C}^{[]} \arrow[ur, "e'"] \arrow[rr, "e^{[]}"'] & {} & \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}
\end{tikzcd} \tag{10.43.1}$$

Here existence of A_∞-functors f, g and natural isomorphisms α, β follows from universality of quotients. Since $\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{u^{\mathcal{C}}_{[]}} \mathcal{C}^{[]} \xrightarrow{e'} \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})$ is contractible, there exists a unital A_∞-functor f and an isomorphism $\alpha : ef \xrightarrow{\sim} u^{\mathcal{C}}_{[]}e' : \mathcal{C} \to \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})$. Since $i^{[]}e^{[]}$ is

contractible by Remark 10.31, there exists a unital A_∞-functor g and an isomorphism $\beta : e'g \xrightarrow{\sim} e^{[]} : \mathcal{C}^{[]} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$. Since $\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{u^{\mathcal{C}}_{[]}} \mathcal{C}^{[]} \xrightarrow{e^{[]}} \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$ is contractible, there exists a unital A_∞-functor $\phi : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$ together with an isomorphism $e\phi \xrightarrow{\sim} u^{\mathcal{C}}_{[]} e^{[]} : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$. Actually, previous data allow to construct two such pairs: fg and $u_{[]}$ together with isomorphisms

$$e(fg) \xrightarrow{(\alpha g)\beta} u^{\mathcal{C}}_{[]} e^{[]} \overset{\mathrm{id}}{=\!=\!=} e u_{[]} : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}. \tag{10.43.2}$$

Theorem 1.3 of [LM08c] implies that

$$(e \boxtimes 1)M : A^u_\infty(\mathsf{q}(\mathcal{C}|\mathcal{B}), \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}) \to A^u_\infty(\mathcal{C}, \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]})_{\mathrm{mod}\,\mathcal{B}},$$

is an A_∞-equivalence. By Lemma 10.42 isomorphism (10.43.2) is equal to $e\chi$ for some isomorphism $\chi : fg \to u_{[]} : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$.

10.44 Lemma. *Let $i : \mathcal{B} \hookrightarrow \mathcal{C}$, $i' : \mathcal{B}' \hookrightarrow \mathcal{C}'$ be full subcategories of unital A_∞-categories. Denote $e : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})$, $e' : \mathcal{C}' \to \mathsf{q}(\mathcal{C}'|\mathcal{B}')$ the quotient functors. Let $\phi : \mathcal{B} \to \mathcal{B}'$, $\psi : \mathcal{C} \to \mathcal{C}'$ be unital A_∞-functors such that $i\psi = \phi i'$. There exists a unital A_∞-functor $f : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}'|\mathcal{B}')$ and an isomorphism $\alpha : ef \to \psi e'$. If ϕ, ψ are A_∞-equivalences, then f is an A_∞-equivalence as well.*

Proof. Choose a map $\mathrm{Ob}\,\xi : \mathrm{Ob}\,\mathcal{C}' \to \mathrm{Ob}\,\mathcal{C}$, $X' \mapsto X'\xi$ so that $X'\xi\psi$ were isomorphic to X' in $\mathcal{C}'$. Since $\phi : \mathcal{B} \to \mathcal{B}'$ is essentially surjective on objects we may assume that $(\mathrm{Ob}\,\mathcal{B}')\,\mathrm{Ob}\,\xi \subset \mathrm{Ob}\,\mathcal{B}$. Extend the map $\mathrm{Ob}\,\xi$ to an A_∞-equivalence $\xi : \mathcal{C}' \to \mathcal{C}$, quasi-inverse to ψ via [Lyu03, Theorem 8.8]. It restricts to the subcategory $\mathcal{B}'$ as an A_∞-equivalence $\gamma : \mathcal{B}' \to \mathcal{B}$, quasi-inverse to ϕ. There exists a unital A_∞-functor $h : \mathsf{q}(\mathcal{C}'|\mathcal{B}') \to \mathsf{q}(\mathcal{C}|\mathcal{B})$ and an isomorphism $\beta : e'h \to \xi e$ as shown in the diagram:

$$\begin{tikzcd}
\mathcal{B} \arrow[r, hook, "i"] \arrow[d, "\phi"'] \arrow[dr, phantom, "="] & \mathcal{C} \arrow[r, "e"] \arrow[d, "\psi"'] & \mathsf{q}(\mathcal{C}|\mathcal{B}) \arrow[d, "f"] \arrow[dl, Rightarrow, "\alpha"'] \\
\mathcal{B}' \arrow[r, hook, "i'"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "="] & \mathcal{C}' \arrow[r, "e'"] \arrow[d, "\xi"'] & \mathsf{q}(\mathcal{C}'|\mathcal{B}') \arrow[d, "h"] \arrow[dl, Rightarrow, "\beta"'] \\
\mathcal{B} \arrow[r, hook, "i"] & \mathcal{C} \arrow[r, "e"] & \mathsf{q}(\mathcal{C}|\mathcal{B})
\end{tikzcd}$$

Theorem 1.3 of [LM08c] implies that

$$(e \boxtimes 1)M : A^u_\infty(\mathsf{q}(\mathcal{C}|\mathcal{B}), \mathsf{q}(\mathcal{C}|\mathcal{B})) \to A^u_\infty(\mathcal{C}, \mathsf{q}(\mathcal{C}|\mathcal{B}))_{\mathrm{mod}\,\mathcal{B}},$$

is an A_∞-equivalence. The object e of the target A_∞-category comes from the object id of the source A_∞-category. The A_∞-functor e is isomorphic to the objects $\psi\xi e \simeq efh$ of the

target A_∞-category. By Lemma 10.42 the objects fh and id of the source A_∞-category are isomorphic. Similarly, $hf \simeq \mathrm{id}$, therefore, f and h are A_∞-equivalences, quasi-inverse to each other. $\square$

10.45 Proposition. *Assume that $\mathcal{B}$, $\mathcal{C}$ are closed under shifts. Then $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is closed under shifts as well, and $f : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})$, $g : \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$ from diagram* (10.43.1) *are A_∞-equivalences.*

Proof. We have $\operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B}) = \operatorname{Ob}\mathcal{C}$ and $\operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B})^{[]} = \operatorname{Ob}\mathcal{C} \times \mathbb{Z}$. There is a function $\psi : \operatorname{Ob}\mathcal{C} \times \mathbb{Z} \to \operatorname{Ob}\mathcal{C}$, $(X,n) \mapsto [X,n]$ together with inverse to each other isomorphisms

$$r_0 \in s\mathcal{C}^{[]}([X,n][0], X[n]), \qquad p_0 \in s\mathcal{C}^{[]}(X[n],[X,n][0]).$$

Take the same function $\psi : \operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B})^{[]} \to \operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B})$ together with inverse to each other isomorphisms

$$r_0 e_1^{[]} \in s\mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}([X,n][0], X[n]), \qquad p_0 e_1^{[]} \in s\mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}(X[n],[X,n][0]).$$

Notice that $\operatorname{Ob} e^{[]} = \mathrm{id}_{\mathcal{C}^{[]}}$. Their existence shows that $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is closed under shifts. By Proposition 10.33 the A_∞-functor $u_{[]} : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$ is an equivalence.

By Lemma 10.44 the A_∞-functor f is an equivalence. Since $fg \simeq u_{[]}$ is an A_∞-equivalence, so is g. $\square$

10.46 Proposition. *The functor $g : \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]}$ from* (10.43.1) *is an A_∞-equivalence.*

Proof. Let us describe the following diagram:

$$\begin{tikzcd}
\mathcal{B}^{[]} \arrow[r, hook, "i^{[]}"] \arrow[ddd, "(u^{\mathcal{B}}_{[]})^{[]}"'] \arrow[dddr, phantom, "="] & \mathcal{C}^{[]} \arrow[rr, "e^{[]}"] \arrow[dr, "e'"] \arrow[ddd, "(u^{\mathcal{C}}_{[]})^{[]}"] & & \mathsf{q}(\mathcal{C}|\mathcal{B})^{[]} \arrow[ddd, "f^{[]}"] \\
& & \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \arrow[u, phantom, "\Uparrow\beta"] \arrow[ur, "g"'] \arrow[d, "k"] \arrow[ddl, Rightarrow, "\xi"'] & \\
& & \mathsf{q}(\mathcal{C}^{[][]}|\mathcal{B}^{[][]}) \arrow[d, phantom, "\Downarrow\gamma"] \arrow[dr, "h"] & \\
\mathcal{B}^{[][]} \arrow[r, hook, "i^{[][]}"] & \mathcal{C}^{[][]} \arrow[rr, "e'^{[]}"'] \arrow[ur, "e''"] & & \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]}
\end{tikzcd}$$

Here $i^{[][]} : \mathcal{B}^{[][]} \to \mathcal{C}^{[][]}$ is a full embedding and $e'' : \mathcal{C}^{[][]} \to \mathsf{q}(\mathcal{C}^{[][]}|\mathcal{B}^{[][]})$ is a quotient map. Note that $(u^{\mathcal{A}}_{[]})^{[]}$ is an A_∞-equivalence by Corollary 10.36. By Lemma 10.44 there exist a unital A_∞-functor $k : \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \to \mathsf{q}(\mathcal{C}^{[][]}|\mathcal{B}^{[][]})$ and an isomorphism $\xi : e'k \to$

$(u^{\mathcal{C}}_{[]})^{[]}e'' : \mathcal{C}^{[]} \to \mathsf{q}(\mathcal{C}^{[][]}|\mathcal{B}^{[][]})$. By the same lemma k is an A_∞-equivalence. The functor $i^{[][]}e'^{[]}$ is contractible by Remark 10.31, therefore, there exist a unital A_∞-functor $h : \mathsf{q}(\mathcal{C}^{[][]}|\mathcal{B}^{[][]}) \to \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]}$ and an isomorphism $\gamma : e''h \to e'^{[]}$. Since $\mathcal{C}^{[]}$ and $\mathcal{B}^{[]}$ are closed under shifts, the functor h is an A_∞-equivalence by Proposition 10.45. We have the following isomorphism

$$e'gf^{[]} \xrightarrow[\sim]{\beta f^{[]}} e^{[]}f^{[]} \xrightarrow[\sim]{\alpha^{[]}} (u^{\mathcal{C}}_{[]})^{[]}e'^{[]} \xrightarrow[\sim]{(u^{\mathcal{C}}_{[]})^{[]}\gamma^{-1}} (u^{\mathcal{C}}_{[]})^{[]}e''h \xrightarrow[\sim]{\xi^{-1}h} e'kh : \mathcal{C}^{[]} \to \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]}. \tag{10.46.1}$$

Theorem 1.7 implies that

$$(e' \boxtimes 1)M : A^u_\infty(\mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}), \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]}) \to A^u_\infty(\mathcal{C}^{[]}, \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]})_{\operatorname{mod}\mathcal{B}^{[]}}$$

is an A_∞-equivalence. By Lemma 10.42 isomorphism (10.46.1) is equal to $e'\zeta$ for some isomorphism $\zeta : gf^{[]} \to kh : \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]}) \to \mathsf{q}(\mathcal{C}^{[]}|\mathcal{B}^{[]})^{[]}$. In particular, $gf^{[]}$ is an A_∞-equivalence. On the other hand, $f^{[]}g^{[]} \simeq (u^{\mathsf{q}(\mathcal{C}|\mathcal{B})}_{[]})^{[]}$ via the isomorphism $\chi^{[]}$. The functor $(u^{\mathsf{q}(\mathcal{C}|\mathcal{B})}_{[]})^{[]}$ is an A_∞-equivalence, hence so is $f^{[]}g^{[]}$. This shows that $f^{[]}$ has left and right quasi-inverses, hence it is an A_∞-equivalence. Since $gf^{[]}$ is an A_∞-equivalence, so is g. □

10.47 Embedding $A_\infty(\mathcal{A},\mathcal{C})^{[]} \hookrightarrow A_\infty(\mathcal{A},\mathcal{C}^{[]})$. As was suggested by Kontsevich in his letter to Beilinson [Kon99], the monad $-^{[]}$ (and similar monads introduced below) behaves like a completion. Thus it should, in a sense, commute with taking the A_∞-category of A_∞-functors. Let us provide the details.

Let $\mathcal{A}$, $\mathcal{C}$ be A_∞-categories. A strict A_∞-functor $F : A_\infty(\mathcal{A},\mathcal{C})^{[]} \to A_\infty(\mathcal{A},\mathcal{C}^{[]})$ is defined as the composition

$$F = \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C})^{[]} \xrightarrow{\underline{S}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{[]},\mathcal{C}^{[]}) \xrightarrow{\underline{\mathsf{A}_\infty}(u_{[]};1)} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}^{[]})\big].$$

It assigns to an object $(f : \mathcal{A} \to \mathcal{C}, n)$ of $A_\infty(\mathcal{A},\mathcal{C})^{[]}$ the A_∞-functor $f\cdot[n] : \mathcal{A} \to \mathcal{C}^{[]}$ that maps $X \in \operatorname{Ob}\mathcal{A}$ to $Xf[n] = (Xf, n) \in \mathcal{C}^{[]}$ and whose components are given by

$$f\cdot[n]_k : \otimes^{i\in\mathbf{k}} s\mathcal{A}(X_{i-1},X_i) \xrightarrow{f_k} s\mathcal{C}(X_0f, X_kf) = s\mathcal{C}^{[]}(X_0f[n], X_kf[n]).$$

The first component is found from (10.19.5) as

$$\begin{aligned} F_1 : sA_\infty(\mathcal{A},\mathcal{C})^{[]}((f,n),(g,m)) &= sA_\infty(\mathcal{A},\mathcal{C})(f,g)[m-n] \\ &= \prod_{k\geqslant 0}^{X_0,\dots,X_k\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}(\otimes^{i\in\mathbf{k}} s\mathcal{A}(X_{i-1},X_i), s\mathcal{C}(X_0f,X_kg))[m-n] \\ &\to \prod_{k\geqslant 0}^{X_0,\dots,X_k\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}(\otimes^{i\in\mathbf{k}} s\mathcal{A}(X_{i-1},X_i), s\mathcal{C}(X_0f,X_kg)[m-n]) \end{aligned}$$

$$= \prod_{k\geqslant 0}^{X_0,\dots,X_k\in \operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}(\otimes^{i\in\mathbf{k}} s\mathcal{A}(X_{i-1},X_i), s\mathcal{C}^{[]}(X_0f[n],X_kg[m])) = sA_\infty(\mathcal{A},\mathcal{C}^{[]})(f\cdot[n],g\cdot[m]).$$

It maps $rs^{m-n}\in sA_\infty(\mathcal{A},\mathcal{C})^{[]}((f,n),(g,m))$ to the transformation $(rs^{m-n})F_1 : f\cdot[n]\to g\cdot[m] : \mathcal{A}\to\mathcal{C}^{[]}$ given by the components

$$\otimes^{i\in\mathbf{k}} s\mathcal{A}(X_{i-1},X_i) \xrightarrow{r_k} s\mathcal{C}(X_0f,X_kg) \xrightarrow{s^{m-n}} s\mathcal{C}^{[]}(X_0f[n],X_kg[m]).$$

Clearly, F_1 is a bijection. Therefore we have a full embedding $F : A_\infty(\mathcal{A},\mathcal{C})^{[]} \hookrightarrow A_\infty(\mathcal{A},\mathcal{C}^{[]})$. If $f:\mathcal{A}\to\mathcal{C}$ is a unital A_∞-functor, then so is $f\cdot[n]:\mathcal{A}\to\mathcal{C}^{[]}$. Indeed, if ${}_X\mathbf{i}_0^{\mathcal{A}} f_1 = {}_{Xf}\mathbf{i}^{\mathcal{C}} + vb_1$ for some $v\in s\mathcal{C}(Xf,Xf) = s\mathcal{C}^{[]}(Xf[n],Xf[n])$, then also ${}_X\mathbf{i}_0^{\mathcal{A}} f\cdot[n]_1 = {}_{Xf[n]}\mathbf{i}^{\mathcal{C}^{[]}} + vb_1^{[]}$. Therefore F restricts to an embedding $A_\infty^u(\mathcal{A},\mathcal{C})^{[]} \hookrightarrow A_\infty^u(\mathcal{A},\mathcal{C}^{[]})$.

10.48 Corollary. *Let $\mathcal{C}$ be an A_∞-category closed under shifts. Then for an arbitrary A_∞-category $\mathcal{A}$ the category $A_\infty(\mathcal{A},\mathcal{C})$ is closed under shifts. If $\mathcal{A}$ is unital, then the category $A_\infty^u(\mathcal{A},\mathcal{C})$ is closed under shifts as well.*

Proof. Consider the composite functor

$$A_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{u_{[]}} A_\infty(\mathcal{A},\mathcal{C})^{[]} \xrightarrow{F} A_\infty(\mathcal{A},\mathcal{C}^{[]}).$$

It coincides with the functor $(1\boxtimes u_{[]})M$, which is an equivalence since so is $u_{[]}:\mathcal{C}\to\mathcal{C}^{[]}$. In particular, it is essentially surjective. Let (f,n) be an object of $A_\infty(\mathcal{A},\mathcal{C})^{[]}$, then the object $(f,n)F$ is isomorphic to an object $gu_{[]}F = (g,0)F$ for some $g\in\operatorname{Ob}A_\infty(\mathcal{A},\mathcal{C})$. Since F is a full embedding, it follows that $(g,0)$ is isomorphic to (f,n), which means closedness of $A_\infty(\mathcal{A},\mathcal{C})$ under shifts. The case of $A_\infty^u(\mathcal{A},\mathcal{C})$ is treated similarly. □

Chapter 11

The Maurer–Cartan A_∞-2-monad

We shall study a functor assigning to a given A_∞-category $\mathcal{C}$ another A_∞-category which, in a sense, consists of solutions to the Maurer-Cartan equation in $\mathcal{C}$. First we construct a precursor of this functor for which no Maurer-Cartan equation is written. Nevertheless there is a functor $-^{\mathsf{mC}}$ in the category of quivers which will have applications to the A_∞-category case. We construct also the multifunctor version $-^{\mathsf{MC}}$ of it.

Next, we consider the A_∞-category $\mathcal{A}^{\mathsf{Mc}}$ of bounded complexes in an arbitrary A_∞-category $\mathcal{A}$. Its underlying quiver is a full subquiver of $\mathcal{A}^{\mathsf{MC}}$. We may say that objects of $\mathcal{A}^{\mathsf{Mc}}$ are solutions to the Maurer-Cartan equation. Equivalently, they are A_∞-functors of the form $(J_k)_{k\in K} \to \mathcal{A}$, where J_k are A_n-type quivers viewed as A_∞-categories with zero operations. This gives an augmented multifunctor $-^{\mathsf{Mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$. A smaller version $\mathcal{A}^{\mathsf{mc}} \hookrightarrow \mathcal{A}^{\mathsf{Mc}}$ consists of A_∞-functors of the form $J \to \mathcal{A}$ with $K = \mathbf{1}$. This provides an A_∞-2-monad $-^{\mathsf{mc}} : \mathsf{A}^{\mathrm{u}}_\infty \to \mathsf{A}^{\mathrm{u}}_\infty$. When the unit of this monad $u_{\mathsf{mc}} : \mathcal{A} \to \mathcal{A}^{\mathsf{mc}}$ is an A_∞-equivalence, the unital A_∞-category $\mathcal{A}$ is said to be mc-*closed*. The multiplication $m_{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}$ of this monad resembles taking the total complex of a bicomplex. It is always an A_∞-equivalence, thus the monad $-^{\mathsf{mc}}$ is a kind of completion. In a certain sense it commutes with taking quotients and A_∞-categories of A_∞-functors.

11.1 Simple and multiple Maurer–Cartan quivers. Consider the category of partitions $\mathcal{P}$. Its objects are sets $\mathbf{m} = \{1, 2, \dots, m\}$, where $m \in \mathbb{Z}_{\geqslant 0}$. The morphisms $\mathbf{m} \to \mathbf{n}$ are embeddings $x \mapsto x + y$ for some $y \in \mathbb{Z}_{\geqslant 0}$. The subsets $I = \mathbb{Z} \cap [a, b] \subset \mathbb{Z}$ for some positive integers $a \leqslant b$ are called *intervals*.

Let $I \in \operatorname{Ob}\mathcal{P}$ be the interval $\mathbf{m}$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Turn I into a graded $\Bbbk$-linear quiver with the set of objects I. The only non-trivial graded modules of morphisms are $I(i, i+1) = \Bbbk[-1]$ concentrated in degree 1.

Let $\mathcal{C}$ be a graded $\Bbbk$-linear quiver. We associate with it its *simple Maurer–Cartan quiver* $\mathcal{C}^{\mathsf{mC}}$. Objects of $\mathcal{C}^{\mathsf{mC}}$ are elements X of

$$\operatorname{Ob}\mathcal{C}^{\mathsf{mC}} = \sqcup_{m\geqslant 0} \mathsf{Q}(\mathbf{m}; \mathcal{C}),$$

that is, morphisms $X : I \to \mathcal{C} \in \mathsf{Q}$, $\operatorname{Ob} X : i \mapsto X_i = (i)X$, where $i \in I \in \operatorname{Ob}\mathcal{P}$. Besides X_i the morphism X, or equivalently $X : T^{\geqslant 1}sI \to s\mathcal{C} \in \mathscr{Q}$, is determined by $\Bbbk$-linear maps of degree 0 for all $i < j$, $i, j \in I$,

$$X_{ij} : \Bbbk \simeq sI(i, i+1) \otimes \cdots \otimes sI(j-1, j) \to s\mathcal{C}(X_i, X_j),$$

that is, by elements $x_{ij} \in s\mathcal{C}(X_i, X_j)$ of degree 0, $i < j$, $i, j \in I$. The graded $\Bbbk$-module of morphisms between objects $X : I \to \mathcal{C}$ and $Y : J \to \mathcal{C}$ of $\mathcal{C}^{\mathsf{mC}}$ is defined as

$$\mathcal{C}^{\mathsf{mC}}(X, Y) \overset{\text{def}}{=} \prod_{i\in I,\, j\in J} \mathcal{C}(X_i, Y_j).$$

Given $I \in \operatorname{Ob} \mathcal{P}$ we define an embedding of graded $\Bbbk$-linear quivers $\iota : \underline{\mathsf{Q}}(I; \mathcal{C}) \hookrightarrow \mathcal{C}^{\mathsf{mC}}$ which takes an object $X : I \to \mathcal{C} \in \mathsf{Q}$, $\operatorname{Ob} X : i \mapsto X_i$ to itself. On morphisms it is the split embedding

$$\iota : \underline{\mathsf{Q}}(I; \mathcal{C})(X, Y) \simeq \prod_{i,j\in I,\, i\leqslant j} \mathcal{C}(X_i, Y_j) \hookrightarrow \prod_{i,j\in I} \mathcal{C}(X_i, Y_j) = \mathcal{C}^{\mathsf{mC}}(X, Y). \tag{11.1.1}$$

Iterations of the map $-^{\mathsf{mC}}$ are denoted $\mathcal{C}^{\mathsf{mC}^n} = (\mathcal{C}^{\mathsf{mC}^{n-1}})^{\mathsf{mC}}$. In particular, $\mathcal{C}^{\mathsf{mC}^0} = \mathcal{C}$.

Let us define inductively in $n \geqslant 0$ a map $\iota : \mathsf{Q}(I_1, \dots, I_n; \mathcal{C}) \hookrightarrow \operatorname{Ob} \mathcal{C}^{\mathsf{mC}^n}$, $X \mapsto \tilde{X}$, $I_j \in \operatorname{Ob} \mathcal{P}$. For $n = 0$ this is the identification $\mathsf{Q}(; \mathcal{C}) = \mathscr{Q}_u(\boxtimes^{\varnothing}(), s\mathcal{C}) = \mathscr{Q}(\mathbb{1}_u, s\mathcal{C}) \simeq \operatorname{Ob} \mathcal{C}$. For $n = 1$ this is the canonical embedding. Assuming the map ι already defined for $n - 1$ we define it for $n > 1$ as the composition

$$\begin{aligned} \mathsf{Q}(I_1, \dots, I_n; \mathcal{C}) \simeq \mathsf{Q}(I_2, \dots, I_n; \underline{\mathsf{Q}}(I_1; \mathcal{C})) &\xhookrightarrow{\mathsf{Q}((1);\iota)} \mathsf{Q}(I_2, \dots, I_n; \mathcal{C}^{\mathsf{mC}}) \\ &\xhookrightarrow{\iota} \operatorname{Ob}(\mathcal{C}^{\mathsf{mC}})^{\mathsf{mC}^{n-1}} = \operatorname{Ob} \mathcal{C}^{\mathsf{mC}^n}, \qquad X \mapsto \tilde{X}. \end{aligned} \tag{11.1.2}$$

Let $Y \in \mathsf{Q}(K_1, \dots, K_n; \mathcal{C})$ be one more multimorphism. Define a partially ordered set $\operatorname{supp} X = \prod_{j\in\mathbf{n}} I_j \ni i$, $i \leqslant i'$ iff $i_j \leqslant i'_j$ for all $j \in \mathbf{n}$. Naturally, $i < i'$ iff $i \leqslant i'$ and $i \neq i'$. For $n = 0$ the set $\operatorname{supp} X$ is a 1-element set. Similarly for $\operatorname{supp} Y = \prod_{j\in\mathbf{n}} K_j$. Let us prove by induction in $n \geqslant 0$ that

$$\mathcal{C}^{\mathsf{mC}^n}(\tilde{X}, \tilde{Y}) = \prod_{i\in\operatorname{supp} X,\, k\in\operatorname{supp} Y} \mathcal{C}(iX, kY) = \prod_{i_j\in I_j,\, k_j\in K_j,\, j\in\mathbf{n}} \mathcal{C}\bigl((i_1, \dots, i_n)X, (k_1, \dots, k_n)Y\bigr).$$

This holds for $n = 0, 1$. For $n > 1$ the first isomorphism in (11.1.2) takes X to X' such that $\operatorname{Ob} X' : (i_2, \dots, i_n) \mapsto (-, i_2, \dots, i_n)X \in \mathsf{Q}(I_1; \mathcal{C})$, where the element $(-, i_2, \dots, i_n)X$ is constructed by fixing certain indices in given data for X. By induction assumption we may write

$$\begin{aligned} \mathcal{C}^{\mathsf{mC}^n}(\tilde{X}, \tilde{Y}) &= \prod_{i_2\in I_2,\dots,i_n\in I_n,\, k_2\in K_2,\dots,k_n\in K_n} \mathcal{C}^{\mathsf{mC}}\bigl((-, i_2, \dots, i_n)X, (-, k_2, \dots, k_n)Y\bigr) \\ &= \prod_{i_1\in I_1,\dots,i_n\in I_n,\, k_1\in K_1,\dots,k_n\in K_n} \mathcal{C}\bigl((i_1, \dots, i_n)X, (k_1, \dots, k_n)Y\bigr). \end{aligned}$$

We extend by induction the embedding ι of graded $\Bbbk$-linear quivers to

$$\underline{\mathsf{Q}}(I_1, \dots, I_n; \mathcal{C}) \simeq \underline{\mathsf{Q}}(I_2, \dots, I_n; \underline{\mathsf{Q}}(I_1; \mathcal{C})) \xhookrightarrow{\underline{\mathsf{Q}}(\triangleright;\iota)} \underline{\mathsf{Q}}(I_2, \dots, I_n; \mathcal{C}^{\mathsf{mC}}) \xhookrightarrow{\iota} (\mathcal{C}^{\mathsf{mC}})^{\mathsf{mC}^{n-1}} = \mathcal{C}^{\mathsf{mC}^n}.$$

On objects it gives (11.1.2). One can prove by induction that on morphisms ι coincides with the split embedding

$$\iota : \underline{\mathsf{Q}}(I_1,\dots,I_n;\mathcal{C})(X,Y) \simeq \underline{\mathscr{Q}_p}(\boxtimes^{j\in\mathbf{n}}TsI_j, s\mathcal{C})(X,Y)[-1] \simeq \prod_{i_j\leqslant k_j\in I_j,\, j\in\mathbf{n}} \mathcal{C}\big((i_1,\dots,i_n)X,(k_1,\dots,k_n)Y\big) \hookrightarrow \prod_{i_j,k_j\in I_j,\, j\in\mathbf{n}} \mathcal{C}\big((i_1,\dots,i_n)X,(k_1,\dots,k_n)Y\big) = \mathcal{C}^{\mathsf{mC}^n}(\tilde{X},\tilde{Y}). \tag{11.1.3}$$

For $n = 0$ this is an isomorphism. The above formula gives another presentation of embedding ι:

$$\underline{\mathsf{Q}}(I_1,\dots,I_n;\mathcal{C}) \simeq \underline{\mathsf{Q}}(I_n;\underline{\mathsf{Q}}(I_1,\dots,I_{n-1};\mathcal{C})) \xhookrightarrow{\underline{\mathsf{Q}}(1;\iota)} \underline{\mathsf{Q}}(I_n;\mathcal{C}^{\mathsf{mC}^{n-1}}) \xhookrightarrow{\iota} (\mathcal{C}^{\mathsf{mC}^{n-1}})^{\mathsf{mC}} = \mathcal{C}^{\mathsf{mC}^n}. \tag{11.1.4}$$

We have a full embedding of graded $\Bbbk$-linear quivers $u_{\mathsf{mC}} : \mathcal{C} \hookrightarrow \mathcal{C}^{\mathsf{mC}}$, $X \mapsto (X' : \mathbf{1} \to \mathcal{C})$, $X'_1 = X$, with the obvious identification of morphisms. The full embeddings u_{mC} give an inductive system of quivers and its colimit

$$\mathcal{C}^{\mathsf{mC}^\infty} \overset{\text{def}}{=} \varinjlim\big(\mathcal{C} \xhookrightarrow{u_{\mathsf{mC}}} \mathcal{C}^{\mathsf{mC}} \xhookrightarrow{u_{\mathsf{mC}}} \mathcal{C}^{\mathsf{mC}^2} \xhookrightarrow{u_{\mathsf{mC}}} \mathcal{C}^{\mathsf{mC}^3} \dots\big).$$

The *multiple Maurer–Cartan quiver* $\mathcal{C}^{\mathsf{MC}}$ of $\mathcal{C}$ is defined as a full subquiver of $\mathcal{C}^{\mathsf{mC}^\infty}$. More precisely, objects of $\mathcal{C}^{\mathsf{MC}}$ are elements X of $\mathsf{Q}((I_j)_{j\in\mathbf{n}};\mathcal{C})$, $n \geqslant 0$, $I_j \in \operatorname{Ob}\mathcal{P}$. Besides $X_i = (i)X$, $i \in \operatorname{supp} X \overset{\text{def}}{=} \prod_{j\in\mathbf{n}} I_j$, the multimorphism $X : (I_j)_{j\in\mathbf{n}} \to \mathcal{C} \in \mathsf{Q}$, or equivalently $X : \boxtimes_u^{j\in\mathbf{n}} T^{\geqslant 1}sI_j \to s\mathcal{C} \in \mathscr{Q}$, or $\overline{X} : \boxtimes^{j\in\mathbf{n}}TsI_j \to s\mathcal{C} \in \mathscr{Q}$, is determined by $\Bbbk$-linear maps of degree 0 for $i,k \in \prod_{j\in\mathbf{n}} I_j$, $i \neq k$, $i_j \leqslant k_j$ for all $1 \leqslant j \leqslant n$,

$$X_{ik} : \Bbbk \to s\mathcal{C}(X_i,X_k),$$

that is, by elements $x_{ik} \in s\mathcal{C}(X_i,X_k)$ of degree 0 for $i,k \in \operatorname{supp} X$, such that $i \neq k$ and $i_j \leqslant k_j$ for all $1 \leqslant j \leqslant n$.

Let us define a map $\operatorname{Ob}\mathcal{C}^{\mathsf{MC}} \to \operatorname{Ob}\mathcal{C}^{\mathsf{mC}^\infty} = \cup_{n\in\mathbb{Z}_{\geqslant 0}} \operatorname{Ob}\mathcal{C}^{\mathsf{mC}^n}$, $X \mapsto \iota X = \tilde{X}$ by (11.1.2). We use it to define $\mathcal{C}^{\mathsf{MC}}(X,Y)$ as $\mathcal{C}^{\mathsf{mC}^\infty}(\tilde{X},\tilde{Y})$.

Let $X : (I_1,\dots,I_n) \to \mathcal{C}$ be a multimorphism in Q, specified by $X_i \in \operatorname{Ob}\mathcal{C}$, $i \in \operatorname{supp} X$, and by $x_{ik} \in s\mathcal{C}(X_i,X_k)$, $i,k \in \operatorname{supp} X$, such that $i \neq k$ and $i_j \leqslant k_j$ for all $1 \leqslant j \leqslant n$. Then it extends to the multimorphism $X' : (I_1,\dots,I_n,\mathbf{1}) \to \mathcal{C}$ in Q, specified by $X'_{(i,1)} = X_i$, $i \in \operatorname{supp} X$, and by $x'_{(i,1),(k,1)} = x_{ik}$, where $i,k \in \operatorname{supp} X$. Notice that $\operatorname{supp} X' = (\operatorname{supp} X) \times \mathbf{1}$. Then the particular case of (11.1.4):

$$\mathsf{Q}(I_1,\dots,I_n,\mathbf{1};\mathcal{C}) \simeq \mathsf{Q}(\mathbf{1};\underline{\mathsf{Q}}(I_1,\dots,I_n;\mathcal{C})) \xhookrightarrow{\mathsf{Q}(1;\iota)} \mathsf{Q}(\mathbf{1};\mathcal{C}^{\mathsf{mC}^n}) \xhookrightarrow{\iota} (\mathcal{C}^{\mathsf{mC}^n})^{\mathsf{mC}} = \mathcal{C}^{\mathsf{mC}^{n+1}}$$

shows that $\widetilde{X'} = u_{\mathsf{mC}}(\tilde{X})$.

Given two objects of $\mathcal{C}^{\mathsf{MC}}$, namely, $X : (I_j)_{j\in\mathbf{n}} \to \mathcal{C} \in \mathsf{Q}$ and $Y : (K_l)_{l\in\mathbf{m}} \to \mathcal{C} \in \mathsf{Q}$, we find the graded $\Bbbk$-module $\mathcal{C}^{\mathsf{MC}}(X,Y)$ as follows. Let p be an integer, $p \geqslant n, m$. Let $X' : (I_j)_{j\in\mathbf{n}}, (\mathbf{1})_{\mathbf{p-n}} \to \mathcal{C} \in \mathsf{Q}$ and $Y' : (K_l)_{l\in\mathbf{m}}, (\mathbf{1})_{\mathbf{p-m}} \to \mathcal{C} \in \mathsf{Q}$ extend X and Y by iterating the above procedure. Then

$$\begin{aligned}\mathcal{C}^{\mathsf{MC}}(X,Y) &= \mathcal{C}^{\mathsf{mC}^\infty}(\tilde{X},\tilde{Y}) \simeq \mathcal{C}^{\mathsf{mC}^p}(\widetilde{X'},\widetilde{Y'}) \\ &= \prod_{i'\in\operatorname{supp}X',\, k'\in\operatorname{supp}Y'} \mathcal{C}(i'X',k'Y') \simeq \prod_{i\in\operatorname{supp}X,\, k\in\operatorname{supp}Y} \mathcal{C}(iX,kY), \quad (11.1.5)\end{aligned}$$

due to the bijection $\operatorname{supp}X' = (\operatorname{supp}X) \times \mathbf{1} \times \dots \times \mathbf{1} \simeq \operatorname{supp}X$, $(i,1,\dots,1) \mapsto i$ and a similar bijection for Y. Moreover, this shows that $\mathcal{C}^{\mathsf{MC}}(X,Y) \simeq \mathcal{C}^{\mathsf{MC}}(X',Y')$, where X', Y' are obtained from X, Y by adding an arbitrary number of intervals $\mathbf{1}$, say, $p-n$ for X and $q-m$ for Y such that q may differ from p.

We have the full embedding $u_{\mathsf{mC}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mC}}$, $\operatorname{Ob}u_{\mathsf{mC}} : \operatorname{Ob}\mathcal{C} \simeq \mathsf{Q}(\mathbf{1};\mathcal{C}) \hookrightarrow \mathcal{C}^{\mathsf{mC}}$, $X \mapsto (1 \mapsto X)$ into the simple Maurer–Cartan quiver. There is also another embedding $u_{\mathsf{MC}} : \mathcal{C} \to \mathcal{C}^{\mathsf{MC}}$, $\operatorname{Ob}u_{\mathsf{MC}} : \operatorname{Ob}\mathcal{C} \simeq \mathsf{Q}(;\mathcal{C}) \hookrightarrow \mathcal{C}^{\mathsf{MC}}$, $X \mapsto X$ into the multiple Maurer–Cartan quiver.

11.2 Maurer–Cartan multifunctor.

Let $f : (\mathcal{C}_j)_{j\in J} \to \mathcal{B}$ be a multimorphism in Q, in the form $\overline{f} : \boxtimes^{j\in J} Ts\mathcal{C}_j \to s\mathcal{B}$. We would like to construct a multimorphism $f^{\mathsf{MC}} : (\mathcal{C}_j^{\mathsf{MC}})_{j\in J} \to \mathcal{B}^{\mathsf{MC}}$ out of it, in the form $\overline{f^{\mathsf{MC}}} : \boxtimes^{j\in J} Ts\mathcal{C}_j^{\mathsf{MC}} \to s\mathcal{B}^{\mathsf{MC}}$. It will take objects $X_j : (I_j^k)_{k\in\mathbf{n_j}} \to \mathcal{C}_j$ of $\mathcal{C}_j^{\mathsf{MC}}$ to the object

$$(X_j)_{j\in J} f^{\mathsf{MC}} = \big[((I_j^k)_{k\in\mathbf{n_j}})_{j\in J} \xrightarrow{(X_j)_{j\in J}} (\mathcal{C}_j)_{j\in J} \xrightarrow{f} \mathcal{B}\big]$$

of $\mathcal{B}^{\mathsf{MC}}$. Here the indexing set of the system of intervals is $\mathbf{n} \simeq \sqcup_{j\in J}\mathbf{n_j}$.

In order to describe it on morphisms we associate with an object $X : (I_1,\dots,I_n) \to \mathcal{C}$ of $\mathcal{C}^{\mathsf{MC}}$ some other objects of $\mathcal{C}^{\mathsf{MC}}$. Let $\mathcal{C}^*$ denote the graded $\Bbbk$-linear quiver with one object 0 added such that $\mathcal{C}^*(0,-) = \mathcal{C}^*(-,0) = 0$. Let $e_j = +N_j : I_j \hookrightarrow K_j$, $i_j \mapsto i_j + N_j$ be embeddings of intervals, $N_j \in \mathbb{Z}_{\geqslant 0}$, $K_j \in \operatorname{Ob}\mathcal{P}$, $1 \leqslant j \leqslant n$. Also $e_j : \mathbf{1} \hookrightarrow K_j$ are embeddings for $n < j \leqslant m$. Define an object ${}^eX = {}^eX^* = {}^*X = X^* : (K_1,\dots,K_m) \to \mathcal{C}^*$ of $\mathcal{C}^{*\mathsf{MC}}$ as follows:

$$(k_1,\dots,k_m)X^* = (i_1,\dots,i_n)X \quad \text{if} \quad k_j = i_j + N_j \quad \text{for all} \quad 1 \leqslant j \leqslant n$$

and all $k_j \in \operatorname{Im}e_j$ for $n < j \leqslant m$, and $(k_1,\dots,k_m)X^* = 0$ if $k_j \notin \operatorname{Im}(e_j : I_j \hookrightarrow K_j)$ for some $j \in \mathbf{m}$. Furthermore, for all $k,k' \in \prod_{j=1}^m K_j$ the element ${}^ex_{kk'}$ has to vanish if k_j or k'_j is not in $\operatorname{Im}e_j$ for some $j \in \mathbf{m}$, otherwise, ${}^ex_{kk'} = x_{ii'}$, where $k_j = i_j + N_j$, $k'_j = i'_j + N_j$, for $1 \leqslant j \leqslant n$, $i,i' \in \prod_{j=1}^n I_j$.

Let $X : (I_1,\dots,I_n) \to \mathcal{C}$, $Y : (L_1,\dots,L_p) \to \mathcal{C}$ be objects of $\mathcal{C}^{\mathsf{MC}}$. Let $m \geqslant n,p$, and let $I_j = \mathbf{1}$ for $n < j \leqslant m$, $L_j = \mathbf{1}$ for $p < j \leqslant m$. Let $e_j : I_j \hookrightarrow K_j$, $e'_j : L_j \hookrightarrow K_j$,

$j \leqslant m$, be embeddings in $\mathcal{P}$. Then there is the embedding

$$\iota : \underline{\mathsf{Q}}(K_1, \dots, K_m; \mathcal{C})({}^eX, {}^{e'}Y) \hookrightarrow \mathcal{C}^{\mathsf{MC}}({}^eX, {}^{e'}Y) \xrightarrow{\sim} \mathcal{C}^{\mathsf{MC}}(X, Y),$$

given by (11.1.3). The second arrow, an isomorphism due to (11.1.5), consists in dropping zeroes from the direct product. If $\operatorname{Im} e_j \leqslant \operatorname{Im} e'_j$ for all $1 \leqslant j \leqslant m$, then the embedding ι is an isomorphism.

Let $X^r_j : ({}^sI^r_j)^{n^r_j}_{s=1} \to \mathcal{C}_j$, $0 \leqslant r \leqslant m_j$, be objects of $\mathcal{C}^{\mathsf{MC}}_j$. Let $n_j \geqslant \max_r(n^r_j)$. Consider embeddings ${}^se^r_j : {}^sI^r_j \hookrightarrow K^s_j$, $1 \leqslant s \leqslant n^r_j$, ${}^se^r_j : \mathbf{1} \hookrightarrow K^s_j$, $n^r_j < s \leqslant n_j$ such that $\operatorname{Im} {}^se^{r-1}_j \leqslant \operatorname{Im} {}^se^r_j$ for all j, r, s. Then we have extended objects ${}^*X^r_j = {}^{e^r_j}X^r_j : (K^s_j)^{n_j}_{s=1} \to \mathcal{C}^*_j$. The multimorphism $f : (\mathcal{C}_j)_{j\in J} \to \mathcal{B}$ or $\overline{f} : \boxtimes^{j\in J} Ts\mathcal{C}_j \to s\mathcal{B}$ extends to $f^* : (\mathcal{C}^*_j)_{j\in J} \to \mathcal{B}^*$ or $\overline{f^*} : \boxtimes^{j\in J} Ts\mathcal{C}^*_j \to s\mathcal{B}^*$ in a unique way by $(\dots, 0, \dots)f^* = 0$ on objects and by zero on morphisms. Applying f^{MC} we get objects

$$\bigl({}^{e^r_j}X^r_j\bigr)_{j\in J} f^{\mathsf{MC}} = \bigl[(K^s_j)^{s\in\mathbf{n_j}}_{j\in J} \xrightarrow{({}^{e^r_j}X^r_j)_{j\in J}} (\mathcal{C}^*_j)_{j\in J} \xrightarrow{f^*} \mathcal{B}^*\bigr]$$

of $\mathcal{B}^{\mathsf{MC}}$.

Consider the following diagram:

$$\begin{tikzcd}
\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\underline{\mathsf{Q}}\bigl((K^s_j)_{s\in\mathbf{n_j}}; \mathcal{C}^*_j\bigr)({}^*X^{r-1}_j, {}^*X^r_j) \arrow[r, "s\underline{\mathsf{Q}}(1,\overline{f^*})"] \arrow[d, hook, "\otimes^{j\in J}\otimes^{\mathbf{m_j}}\iota"'] & s\underline{\mathsf{Q}}\bigl((K^s_j)^{s\in\mathbf{n_j}}_{j\in J}; \mathcal{B}^*\bigr)\bigl(({}^*X^0_j)f^*, ({}^*X^{m_j}_j)f^*\bigr) \arrow[d, hook, "\iota"] \\
\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\mathcal{C}^{\mathsf{MC}}_j({}^*X^{r-1}_j, {}^*X^r_j) \arrow[r, dashed, "g"'] \arrow[d, "\wr"'] & s\mathcal{B}^{\mathsf{MC}}\bigl(({}^*X^0_j)f^*, ({}^*X^{m_j}_j)f^*\bigr) \arrow[d, "\wr"] \\
\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\mathcal{C}^{\mathsf{MC}}_j(X^{r-1}_j, X^r_j) \arrow[r, "\overline{f^{\mathsf{MC}}}"'] & s\mathcal{B}^{\mathsf{MC}}\bigl((X^0_j)f, (X^{m_j}_j)f\bigr)
\end{tikzcd} \tag{11.2.1}$$

If conditions $\operatorname{Im} {}^se^{r-1}_j \leqslant \operatorname{Im} {}^se^r_j$ are satisfied, then all vertical arrows are isomorphisms and there are unique morphisms g and

$$\overline{f^{\mathsf{MC}}} : \boxtimes^{j\in J} Ts\mathcal{C}^{\mathsf{MC}}_j(X^0_j, X^{m_j}_j) \to s\mathcal{B}^{\mathsf{MC}}\bigl((X^0_j)_{j\in J} f^{\mathsf{MC}}, (X^{m_j}_j)_{j\in J} f^{\mathsf{MC}}\bigr)$$

which make this diagram commutative. Let us prove that $\overline{f^{\mathsf{MC}}}$ does not depend on the choice of embeddings ${}^se^r_j$.

Let $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} p^{jr}$ be an element of $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\underline{\mathsf{Q}}((K^s_j)_{s\in\mathbf{n_j}}; \mathcal{C}^*_j)({}^*X^{r-1}_j, {}^*X^r_j)$. It is mapped by $s\underline{\mathsf{Q}}(1, \overline{f^*}) = (1 \boxtimes \overline{f^*})\mu^{\underline{\mathsf{Q}}}$ to

$$[\boxtimes^{j\in J}(\otimes^{r\in\mathbf{m_j}} p^{jr})\theta]f^* = \sum \bigl[\boxtimes^{j\in J}\bigl(({}^*X^0_j)^{\otimes t^0_j} \otimes p^{j1} \otimes ({}^*X^1_j)^{\otimes t^1_j} \otimes \dots \otimes p^{jm_j} \otimes ({}^*X^{m_j}_j)^{\otimes t^{m_j}_j}\bigr)\bigr] f^*_\ell, \tag{11.2.2}$$

where $\ell = (\ell^j)_{j\in J}$, $\ell^j = t_j^0 + \dots + t_j^{m_j} + m_j$, $j \in J$, and summation is taken over all $t_j^r \geqslant 0$, $j \in J$, $0 \leqslant r \leqslant m_j$ by (7.17.1), (7.16.3) and (7.6.4). Let us define the dashed arrow g by imitating (11.2.2). An element $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} {}^*q^{jr}$ of $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\mathcal{C}_j^{\mathsf{MC}}({}^*X_j^{r-1}, {}^*X_j^r)$ is mapped by g to

$$\sum \bigl[\boxtimes^{j\in J}\bigl(({}^*X_j^0)^{\otimes t_j^0} \otimes {}^*q^{j1} \otimes ({}^*X_j^1)^{\otimes t_j^1} \otimes \dots \otimes {}^*q^{jm_j} \otimes ({}^*X_j^{m_j})^{\otimes t_j^{m_j}}\bigr)\bigr] f_\ell^*. \tag{11.2.3}$$

Then the top square always commutes independently of conditionsIm ${}^se_j^{r-1} \leqslant \operatorname{Im} {}^se_j^r$.

Let us find explicitly the lower arrow. Let $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} q^{jr}$ be an element of $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\mathcal{C}_j^{\mathsf{MC}}(X_j^{r-1}, X_j^r)$. It induces the corresponding element $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} {}^*q^{jr}$ of $\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} s\mathcal{C}_j^{\mathsf{MC}}({}^*X_j^{r-1}, {}^*X_j^r)$. Dropping zeroes in (11.2.3) we get the following formula for $\overline{f^{\mathsf{MC}}}$:

$$(\otimes^{j\in J} \otimes^{r\in\mathbf{m_j}} q^{jr})\overline{f^{\mathsf{MC}}} = \sum \bigl[\boxtimes^{j\in J}\bigl((X_j^0)^{\otimes t_j^0} \otimes q^{j1} \otimes (X_j^1)^{\otimes t_j^1} \otimes \dots \otimes q^{jm_j} \otimes (X_j^{m_j})^{\otimes t_j^{m_j}}\bigr)\bigr] f_\ell, \tag{11.2.4}$$

which does not depend on embeddings ${}^se_j^r$. The bottom square of diagram (11.2.1) always commutes independently of conditions $\operatorname{Im} {}^se_j^{r-1} \leqslant \operatorname{Im} {}^se_j^r$. Thus, the diagram is commutative.

Associativity of composition in $\underline{\mathsf{Q}}$ implies that $\overline{f^{\mathsf{MC}}}$ is a multimorphism. For similar reasons, $-^{\mathsf{MC}} : \mathsf{Q} \to \mathsf{Q}$ is a multifunctor and the embedding $u_{\mathsf{MC}} : \mathrm{Id} \to -^{\mathsf{MC}} : \mathsf{Q} \to \mathsf{Q}$ is a multinatural transformation. Thus, $(-^{\mathsf{MC}}, u_{\mathsf{MC}})$ is an augmented multifunctor.

11.3 Closing transformation $-^{\mathsf{MC}}$. According to Section 4.18, there is a natural morphism in Q

$$\underline{\mathsf{MC}}_{(\mathcal{A}_i);\mathcal{C}} : \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I}; \mathcal{C})^{\mathsf{MC}} \to \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in I}; \mathcal{C}^{\mathsf{MC}}),$$

which is the closing transformation for the multifunctor $-^{\mathsf{MC}}$. We are going to compute this morphism explicitly in the case $I = \mathbf{1}$. It is determined as a unique solution of the following equation in Q:

$$\bigl[\mathcal{A}^{\mathsf{MC}}, \underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C})^{\mathsf{MC}} \xrightarrow{\mathrm{id},\underline{\mathsf{MC}}} \mathcal{A}^{\mathsf{MC}}, \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}}; \mathcal{C}^{\mathsf{MC}}) \xrightarrow{\mathrm{ev}^{\mathsf{Q}}} \mathcal{C}^{\mathsf{MC}}\bigr] = (\mathrm{ev}^{\mathsf{Q}})^{\mathsf{MC}}. \tag{11.3.1}$$

An object $X : (I_p)_{p\in\mathbf{n}} \to \mathcal{A}$ of $\mathcal{A}^{\mathsf{MC}}$ is determined by a map $\operatorname{Ob} X : \prod_{p\in\mathbf{n}} I_p \to \operatorname{Ob}\mathcal{A}$, $i \mapsto X_i$, and a family of $\Bbbk$-linear maps $X_{ij} : \Bbbk \to s\mathcal{C}(X_i, X_j)$ of degree 0, where $i, j \in \prod_{p\in\mathbf{n}} I_p$, $i \neq j$, $i_p \leqslant j_p$ for all $1 \leqslant p \leqslant n$. Similarly, an object $f : (K_s)_{s\in\mathbf{m}} \to \underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C})$ of $\underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C})^{\mathsf{MC}}$ is determined by a map $\operatorname{Ob} f : \prod_{s\in\mathbf{m}} K_s \to \operatorname{Ob}\underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C})$, $k \mapsto f^k$, and a family of $\Bbbk$-linear maps $f^{kl} : \Bbbk \to s\underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C})(f^k, f^l)$ of degree 0, where $k, l \in \prod_{s\in\mathbf{m}} K_s$, $k \neq l$, $k_s \leqslant l_s$ for all $1 \leqslant s \leqslant m$. The morphism $(\mathrm{ev}^{\mathsf{Q}})^{\mathsf{MC}}$ assigns to the objects X, f the object

$$\bigl[(I_p)_{p\in\mathbf{n}}, (K_s)_{s\in\mathbf{m}} \xrightarrow{X,f} \mathcal{A}, \underline{\mathsf{Q}}(\mathcal{A}; \mathcal{C}) \xrightarrow{\mathrm{ev}^{\mathsf{Q}}} \mathcal{C}\bigr] \tag{11.3.2}$$

of $\mathcal{C}^{\mathsf{MC}}$. It is given by the map $\prod_{p\in\mathbf{n}}\times\prod_{s\in\mathbf{m}}\to\operatorname{Ob}\mathcal{C}$, $(i,k)\mapsto X_if^k$, and a family of elements for $(i,k),(j,l)\in\prod_{p\in\mathbf{n}}I_p\times\prod_{s\in\mathbf{m}}K_s$, $(i,k)\neq(j,l)$, $i_p\leqslant j_p$, $k_s\leqslant l_s$ for all $1\leqslant p\leqslant n$, $1\leqslant s\leqslant m$,

$$\sum(X_{i_0i_1}\otimes\dots\otimes X_{i_{r-1}i_r})f_r^{kl}\in s\mathcal{C}(X_if^k,X_jf^l), \tag{11.3.3}$$

where the summation is taken over all sequences $i_0,i_1,\dots,i_r\in\prod_{p\in\mathbf{n}}I_p$ with $i_{t-1}\neq i_t$, $i_{t-1,p}\leqslant i_{tp}$ for all $1\leqslant p\leqslant n$, $1\leqslant t\leqslant r$, $i_0=i$, $i_r=j$. The latter expression can be abbreviated to $\sum_{r\geqslant0}(X^{\otimes r})f$. The components

$$\operatorname{ev}^{\mathsf{Q}}_{a,c}:T^as\mathcal{A}\boxtimes T^cs\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})\to s\mathcal{C}$$

of the evaluation morphism vanish unless $c=0$ or $c=1$. It follows from equation (11.2.4) that the same is true for the components

$$(\operatorname{ev}^{\mathsf{Q}})^{\mathsf{MC}}_{a,c}:T^as\mathcal{A}^{\mathsf{MC}}\boxtimes T^cs\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}\to s\mathcal{C}^{\mathsf{MC}}.$$

By the same equation

$$\begin{aligned}
&(p^1\otimes\dots\otimes p^r\boxtimes f)(\operatorname{ev}^{\mathsf{Q}})^{\mathsf{MC}}_{r,0}\\
&\quad=\sum_{t_0,\dots,t_r\geqslant0}((X^0)^{\otimes t_0}\otimes p_1\otimes(X^1)^{\otimes t_1}\otimes\dots\otimes p_r\otimes(X^r)^{\otimes t_r}\boxtimes f)\operatorname{ev}^{\mathsf{Q}}_{t_0+\dots+t_r+r,0}\\
&\quad=\sum_{t_0,\dots,t_r\geqslant0}((X^0)^{\otimes t_0}\otimes p_1\otimes(X^1)^{\otimes t_1}\otimes\dots\otimes p_r\otimes(X^r)^{\otimes t_r})f_{t_0+\dots+t_r+r}, &(11.3.4)\\
&(p^1\otimes\dots\otimes p^r\boxtimes q)(\operatorname{ev}^{\mathsf{Q}})^{\mathsf{MC}}_{r,1}\\
&\quad=\sum_{t_0,\dots,t_r\geqslant0}((X^0)^{\otimes t_0}\otimes p_1\otimes(X^1)^{\otimes t_1}\otimes\dots\otimes p_r\otimes(X^r)^{\otimes t_r}\boxtimes q)\operatorname{ev}^{\mathsf{Q}}_{t_0+\dots+t_r+r,1}\\
&\quad=\sum_{t_0,\dots,t_r\geqslant0}((X^0)^{\otimes t_0}\otimes p_1\otimes(X^1)^{\otimes t_1}\otimes\dots\otimes p_r\otimes(X^r)^{\otimes t_r})q_{t_0+\dots+t_r+r}, &(11.3.5)
\end{aligned}$$

where $X^0,\dots,X^r\in\operatorname{Ob}\mathcal{A}^{\mathsf{MC}}$, $p^1\otimes\dots\otimes p^r\in\otimes^{t\in\mathbf{r}}s\mathcal{A}^{\mathsf{MC}}(X^{t-1},X^t)$, $f,g\in\operatorname{Ob}\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}$, $q\in s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,g)$.

From equation (11.3.1) we obtain

$$\begin{aligned}
&(1^{\otimes a}\otimes\operatorname{Ob}\underline{\mathsf{MC}})\operatorname{ev}^{\mathsf{Q}}_{a,0}=(\operatorname{ev}^{\mathsf{Q}})^{\mathsf{MC}}_{a,0}:\\
&\qquad\otimes^{\alpha\in\mathbf{a}}s\mathcal{A}^{\mathsf{MC}}(X^{\alpha-1},X^\alpha)\otimes T^0s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,f)\to s\mathcal{C}^{\mathsf{MC}}(X^0f^{\underline{\mathsf{MC}}},X^af^{\underline{\mathsf{MC}}})\\
&(1^{\otimes a}\otimes\underline{\mathsf{MC}}_c)\operatorname{ev}^{\mathsf{Q}}_{a,1}=(\operatorname{ev}^{\mathsf{Q}})^{\mathsf{MC}}_{a,c}:\\
&\qquad\otimes^{\alpha\in\mathbf{a}}s\mathcal{A}^{\mathsf{MC}}(X^{\alpha-1},X^\alpha)\otimes\otimes^{\beta\in\mathbf{c}}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f^{\beta-1},f^\beta)\to s\mathcal{C}^{\mathsf{MC}}(X^0(f^0)^{\underline{\mathsf{MC}}},X^a(f^c)^{\underline{\mathsf{MC}}}).
\end{aligned}$$

We conclude that $\underline{\mathsf{MC}}_c=0$ if $c>1$. An element $f:(K_s)_{s\in\mathbf{m}}\to\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})$ of $\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})$ is mapped by $\operatorname{Ob}\underline{\mathsf{MC}}$ to the morphism $f^{\underline{\mathsf{MC}}}:\mathcal{A}^{\mathsf{MC}}\to\mathcal{C}^{\mathsf{MC}}$ specified below. It maps an object

$X : (I_p)_{p\in\mathbf{n}} \to \mathcal{A}$ of $\mathcal{A}^{\mathsf{MC}}$ to the object $Xf^{\underline{\mathsf{MC}}} : (I_p)_{p\in\mathbf{n}}, (K_s)_{s\in\mathbf{m}} \to \mathcal{C}$ of $\mathcal{C}^{\mathsf{MC}}$, determined by the objects $X_i f^k \in \operatorname{Ob}\mathcal{C}$, $(i,k) \in \prod_{p\in\mathbf{n}} I_p \times \prod_{s\in\mathbf{m}} K_s$, and elements (11.3.3). Components of the morphism $f^{\underline{\mathsf{MC}}}$ are given by

$$(p^1\otimes\cdots\otimes p^r)f_r^{\underline{\mathsf{MC}}} = \sum_{t_0,\dots,t_r\geqslant 0} ((X^0)^{\otimes t_0}\otimes p^1\otimes(X^1)^{\otimes t_1}\otimes\cdots\otimes p^r\otimes(X^r)^{\otimes t_r})f_{t_0+\dots+t_r+r} \qquad (11.3.6)$$

for $p^1\otimes\cdots\otimes p^r \in \otimes^{t\in\mathbf{r}} s\mathcal{A}^{\mathsf{MC}}(X^{t-1},X^t)$. An element $q \in s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,g)$ is mapped by $\underline{\mathsf{MC}}_1$ to the element $q^{\underline{\mathsf{MC}}} \in s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}},\mathcal{C}^{\mathsf{MC}})(f^{\underline{\mathsf{MC}}},g^{\underline{\mathsf{MC}}})$ given by its components:

$$(p^1\otimes\cdots\otimes p^r)q_r^{\underline{\mathsf{MC}}} = \sum_{t_0,\dots,t_r\geqslant 0} ((X^0)^{\otimes t_0}\otimes p^1\otimes(X^1)^{\otimes t_1}\otimes\cdots\otimes p^r\otimes(X^r)^{\otimes t_r})q_{t_0+\dots+t_r+r} \qquad (11.3.7)$$

for $p^1\otimes\cdots\otimes p^r \in \otimes^{t\in\mathbf{r}} s\mathcal{A}^{\mathsf{MC}}(X^{t-1},X^t)$.

Abusing the notation we define a strict morphism in Q (cf. Section 4.29)

$$-^{\mathsf{MC}} \overset{\text{def}}{=} \bigl(\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B}) \xrightarrow{u_{\mathsf{MC}}} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{B})^{\mathsf{MC}} \xrightarrow{\underline{\mathsf{MC}}} \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{B}^{\mathsf{MC}})\bigr), \qquad f\mapsto f^{\mathsf{MC}}, \quad r\mapsto r^{\mathsf{MC}}.$$

The proofs of Propositions 11.4, 11.5, 11.6 repeat those of Propositions 10.23, 10.24, 10.25 with $-^{\mathsf{MC}}$ in place of $-^{[]}$ and $\underline{\mathsf{MC}}$ in place of $\underline{F}$.

11.4 Proposition. *The following diagram commutes in Q for arbitrary quivers $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$:*

$$\begin{tikzcd}
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B}),\underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "-^{\mathsf{MC}},-^{\mathsf{MC}}"] \arrow[d, "\mu^{\mathsf{Q}}"'] & \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{B}^{\mathsf{MC}}),\underline{\mathsf{Q}}(\mathcal{B}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \arrow[d, "\mu^{\mathsf{Q}}"] \\
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \arrow[r, "-^{\mathsf{MC}}"] & \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}})
\end{tikzcd} \qquad (11.4.1)$$

11.5 Proposition. *For an arbitrary morphism $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ in Q and a quiver $\mathcal{C}$ the following diagram commutes:*

$$\begin{tikzcd}
\underline{\mathsf{Q}}(\mathcal{B};\mathcal{C}) \arrow[r, "-^{\mathsf{MC}}"] \arrow[d, "\underline{\mathsf{Q}}(f;1)"'] & \underline{\mathsf{Q}}(\mathcal{B}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \arrow[d, "\underline{\mathsf{Q}}(f^{\mathsf{MC}};1)"] \\
\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "-^{\mathsf{MC}}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in I};\mathcal{C}^{\mathsf{MC}})
\end{tikzcd} \qquad (11.5.1)$$

11.6 Proposition. *For an arbitrary morphism $f : (\mathcal{B}_j)_{j\in J} \to \mathcal{C}$ in Q, quivers $\mathcal{A}_i$, $i\in I$, and a map $\phi : I \to J$ in* $\operatorname{Mor}\mathcal{S}$ *the following diagram commutes:*

$$\begin{tikzcd}
(\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j))_{j\in J} \arrow[r, "(-^{\mathsf{MC}})_J"] \arrow[d, "\underline{\mathsf{Q}}(\phi;f)"'] & (\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\phi^{-1}j};\mathcal{B}_j^{\mathsf{MC}}))_{j\in J} \arrow[d, "\underline{\mathsf{Q}}(\phi;f^{\mathsf{MC}})"] \\
\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{C}) \arrow[r, "-^{\mathsf{MC}}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in I};\mathcal{C}^{\mathsf{MC}})
\end{tikzcd} \qquad (11.6.1)$$

11.7 The case of $\mathcal{A}^{\mathsf{mC}}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in Q. It produces a morphism $f^{\mathsf{MC}} : \mathcal{A}^{\mathsf{MC}} \to \mathcal{B}^{\mathsf{MC}}$. There is a full embedding $\mathcal{C}^{\mathsf{mC}} \hookrightarrow \mathcal{C}^{\mathsf{MC}}$. Clearly, f^{MC} maps the full subquiver $\mathcal{A}^{\mathsf{mC}}$ into the full subquiver $\mathcal{C}^{\mathsf{mC}}$. Indeed, an object $X : I \to \mathcal{A}$ of $\mathcal{A}^{\mathsf{mC}}$ is mapped by f^{MC} to the object $Xf : I \to \mathcal{B}$ of $\mathcal{B}^{\mathsf{mC}}$. It is represented by the matrix $Xf^{\mathsf{MC}} = \sum_{n\geqslant 1}(X^{\otimes n})f_n = \sum_{1\leqslant n<|I|}(X^{\otimes n})f_n$. The morphism f is extended to $f^* : \mathcal{A}^* \to \mathcal{B}^*$ by the requirement $0f^* = 0$.

Define $f^{\mathsf{mC}} : \mathcal{A}^{\mathsf{mC}} \to \mathcal{B}^{\mathsf{mC}}$ as the restriction of f^{MC}. The components of f^{mC} are computed via

$$f_n^{\mathsf{mC}} = \bigl[s\mathcal{A}^{\mathsf{mC}}(X^0,X^1) \otimes \dots \otimes s\mathcal{A}^{\mathsf{mC}}(X^{n-1},X^n) \\ \xrightarrow[\sim]{\otimes\iota_1^{-1}} sA_\infty(K,\mathcal{A}^*)(X^{0*},X^{1*}) \otimes \dots \otimes sA_\infty(K,\mathcal{A}^*)(X^{n-1*},X^{n*}) \\ \xrightarrow{(1\boxtimes f^*)M_{n0}} sA_\infty(K,\mathcal{B}^*)(X^{0*}f^*,X^{n*}f^*) \xrightarrow[\sim]{\iota_1} s\mathcal{B}^{\mathsf{mC}}(X^0f,X^nf)\bigr]$$

for an arbitrary embedding $e : I^0 \sqcup I^1 \sqcup \dots \sqcup I^n \hookrightarrow K$ and $X^p : I^p \to \mathcal{A}$. Let $r^p \in s\mathcal{A}^{\mathsf{mC}}(X^{p-1},X^p)$. The corresponding r^{p*} are mapped by $(1\boxtimes f^*)M_{n0}$ to

$$(r^{1*}\otimes\dots\otimes r^{n*}\boxtimes f^*)M_{n0} = \sum_{l>0}(r^{1*}\otimes\dots\otimes r^{n*})\theta_{\bullet l}f_l^* \\ = \sum_{t_0,\dots,t_n\geqslant 0}\bigl[(X^{0*})^{\otimes t_0}\otimes r^{1*}\otimes(X^{1*})^{\otimes t_1}\otimes\dots\otimes r^{n*}\otimes(X^{n*})^{\otimes t_n}\bigr]f^*_{t_0+\dots+t_n+n}$$

by [Lyu03, (4.1.3)]. Dropping zeroes we get a map of degree 0

$$(r^1\otimes\dots\otimes r^n)f_n^{\mathsf{mC}} = \sum_{t_0,\dots,t_n\geqslant 0}\bigl[(X^0)^{\otimes t_0}\otimes r^1\otimes(X^1)^{\otimes t_1}\otimes\dots\otimes r^n\otimes(X^n)^{\otimes t_n}\bigr]f_{t_0+\dots+t_n+n},$$

which does not depend on the embedding e.

Suppose that $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are morphisms in Q. Then $(fg)^* = \bigl(\mathcal{A}^* \xrightarrow{f^*} \mathcal{B}^* \xrightarrow{g^*} \mathcal{C}^*\bigr)$ and $(1\boxtimes(fg)^*)M = (1\boxtimes f^*g^*)M = (1\boxtimes f^*)M(1\boxtimes g^*)M$. Arbitrariness of the embeddings e implies that $(fg)^{\mathsf{mC}} = f^{\mathsf{mC}}g^{\mathsf{mC}}$. Therefore, $-^{\mathsf{mC}} : \mathsf{Q} \to \mathsf{Q}$ is a functor (of ordinary categories).

11.8 The closed version of $-^{\mathsf{mC}}$. Let $p : f \to g : \mathcal{A} \to \mathcal{B}$ be an element of $\underline{\mathsf{Q}}(\mathcal{A};\mathcal{B})(f,g)$. It extends in a unique way (by zero) to an element $p^* : f^* \to g^* : \mathcal{A}^* \to \mathcal{B}^*$. Define the element $p^{\mathsf{mC}} : f^{\mathsf{mC}} \to g^{\mathsf{mC}} : \mathcal{A}^{\mathsf{mC}} \to \mathcal{B}^{\mathsf{mC}}$ as follows. The components of p^{mC} are defined via

$$p_n^{\mathsf{mC}} = \bigl[s\mathcal{A}^{\mathsf{mC}}(X^0,X^1) \otimes \dots \otimes s\mathcal{A}^{\mathsf{mC}}(X^{n-1},X^n) \\ \xrightarrow[\sim]{\otimes\iota^{-1}} sA_\infty(K,\mathcal{A}^*)(X^{0*},X^{1*}) \otimes \dots \otimes sA_\infty(K,\mathcal{A}^*)(X^{n-1*},X^{n*}) \\ \xrightarrow{(1\boxtimes p^*)M_{n1}} sA_\infty(K,\mathcal{B}^*)(X^{0*}f^*,X^{n*}g^*) \xhookrightarrow{\iota} s\mathcal{B}^{\mathsf{mC}}(X^0f,X^ng)\bigr]$$

for an arbitrary embedding $e : I^0 \sqcup I^1 \sqcup \cdots \sqcup I^n \hookrightarrow K$. Let $r^q \in s\mathcal{A}^{\mathsf{mC}}(X^{q-1}, X^q)$. The last map ι_1 is given by (11.1.1). It is an isomorphism if $n > 0$, and it is the embedding of upper-triangular matrices into all matrices if $n = 0$. The corresponding r^{q*} are mapped by $(1 \boxtimes p^*)M_{n1}$ to

$$\begin{aligned}(r^{1*} \otimes \cdots \otimes r^{n*} \boxtimes p^*)M_{n1} &= \sum_{l\geqslant 0}(r^{1*} \otimes \cdots \otimes r^{n*})\theta_{\bullet l}p_l^* \\ &= \sum_{t_0,\dots,t_n\geqslant 0} \big[(x^{0*})^{\otimes t_0} \otimes r^{1*} \otimes (x^{1*})^{\otimes t_1} \otimes \cdots \otimes r^{n*} \otimes (x^{n*})^{\otimes t_n}\big]p^*_{t_0+\cdots+t_n+n}\end{aligned}$$

by [Lyu03, (4.1.4)]. This formula holds for $n = 0$ as well. Dropping zeroes we get a map of degree 0

$$(r^1 \otimes \cdots \otimes r^n)p_n^{\mathsf{mC}} = \sum_{t_0,\dots,t_n\geqslant 0} \big[(x^0)^{\otimes t_0} \otimes r^1 \otimes (x^1)^{\otimes t_1} \otimes \cdots \otimes r^n \otimes (x^n)^{\otimes t_n}\big]p_{t_0+\cdots+t_n+n},$$

which does not depend on the embedding e. In particular, for $n = 0$ we get

$$_Xp_0^{\mathsf{mC}} = \sum_{t\geqslant 0}(x^{\otimes t})p_t.$$

11.9 Consequences of the closing transformation for multifunctor MC. The closing transformation for the multifunctor $\mathsf{MC} : \mathsf{Q} \to \mathsf{Q}$ allows to define a morphism in Q

$$\varpi_{\mathsf{MC}} = \big[\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \xrightarrow{\underline{\mathsf{MC}}} \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \xrightarrow{\underline{\mathsf{Q}}(u_{\mathsf{MC}};1)} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})\big].$$

It makes commutative the following diagram in the multicategory Q:

$$\begin{tikzcd}
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \arrow[rrr, "u_{\mathsf{MC}},1"] \arrow[dr, "1{,}\underline{\mathsf{MC}}"] \arrow[drr, "u_{\mathsf{MC}}{,}\underline{\mathsf{MC}}"] \arrow[dd, "1{,}\varpi_{\mathsf{MC}}"'] & & & \mathcal{A}^{\mathsf{MC}},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \arrow[dl, "1{,}\underline{\mathsf{MC}}"'] \arrow[dd, "\mathrm{ev}^{\mathsf{MC}}"] \\
& \mathcal{A},\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \arrow[r, "u_{\mathsf{MC}},1"'] \arrow[dl, "1{,}\underline{\mathsf{Q}}(u_{\mathsf{MC}}{,}1)"] & \mathcal{A}^{\mathsf{MC}},\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \arrow[dr, "\mathrm{ev}"] & \\
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}}) \arrow[rrr, "\mathrm{ev}"] & & & \mathcal{C}^{\mathsf{MC}}
\end{tikzcd} \tag{11.9.1}$$

11.10 Proposition. *The morphism ϖ_{MC} is a strict full embedding, that is, all its components vanish except $(\varpi_{\mathsf{MC}})_1 : \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,g) \to \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})(f\varpi_{\mathsf{MC}}, g\varpi_{\mathsf{MC}})$, which is an isomorphism.*

Proof. The morphism $\varpi_{\mathsf{MC}} = \underline{\mathsf{MC}} \cdot (u_{\mathsf{MC}} \boxtimes 1)M$ in Q is strict as a composition of two strict morphisms. Let us describe it on objects. Let $\mathcal{A}$, $\mathcal{C}$ be graded quivers. An object f of

$\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}$ can be given by the following data: a family of functors $\operatorname{supp} f = \prod_{j\in\mathbf{n}} I_j \ni i \mapsto f^i \in \mathsf{Q}(\mathcal{A};\mathcal{C})$, a family of transformations (of degree 0) $f^{ij} \in s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})(f^i,f^j)$, i, $j \in \operatorname{supp} f$, $i<j$ such that the Maurer-Cartan equation holds: for all i, $j\in \operatorname{supp} f$, $i<j$

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (f^{ik_1}\otimes f^{k_1k_2}\otimes\dots\otimes f^{k_{m-1}j})B_m = 0.$$

The morphism $f\varpi_{\mathsf{MC}} : \mathcal{A} \to \mathcal{C}^{\mathsf{MC}}$ assigned to f is due to diagram (11.9.1) and formula (11.3.3). An object $X\in \operatorname{Ob}\mathcal{A}$ is mapped to the object $Xf : I_1,\dots,I_n \to \mathcal{C}$, $i\mapsto Xf^i$ of $\mathcal{C}^{\mathsf{MC}}$. It is determined by the family of elements of degree 0 $x^{ij} = {}_Xf_0^{ij} \in s\mathcal{C}(Xf^i,Xf^j)$, i, $j\in\operatorname{supp} f$, $i<j$. Components of $f\varpi_{\mathsf{MC}}$ are obtained as follows: the composition

$$\otimes^{p\in\mathbf{n}} s\mathcal{A}(X_{p-1},X_p) \xrightarrow{(f\varpi_{\mathsf{MC}})_n} s\mathcal{C}^{\mathsf{mC}}(X_0f,X_nf) = \prod_{i,j\in\operatorname{supp} f} s\mathcal{C}(X_0f^i,X_nf^j) \xrightarrow{\mathrm{pr}} s\mathcal{C}(X_0f^i,X_nf^j)$$

equals f_n^i for $i=j$, f_n^{ij} for $i<j$, and vanishes otherwise.

Let us describe the first component $(\varpi_{\mathsf{MC}})_1$. An element $q\in s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,g)$ is mapped to element $q(\varpi_{\mathsf{MC}})_1 \in s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})(f\varpi_{\mathsf{MC}},g\varpi_{\mathsf{MC}})$, given by the formula

$$\bigl[(p^1\otimes\dots\otimes p^r)\bigl(q(\varpi_{\mathsf{MC}})_1\bigr)_r\bigr]^{ij} = (p^1\otimes\dots\otimes p^r)(q^{ij})_r \in s\mathcal{C}(Xf^i,Yg^j)$$

for $p^1\otimes\dots\otimes p^r \in T^rs\mathcal{A}(X,Y)$, $i\in\operatorname{supp} f$, $j\in\operatorname{supp} g$. This follows from formula (11.3.5), since the sum there reduces to a single summand with $t_0=\dots=t_r=0$. The same mapping can be presented as follows:

$$\begin{aligned}
(\varpi_{\mathsf{MC}})_1 : s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}}(f,g) &= \prod_{i\in\operatorname{supp} f,\, j\in\operatorname{supp} g} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})(f^i,g^j)\\
&= \prod_{i\in\operatorname{supp} f,\, j\in\operatorname{supp} g}\ \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}\bigl(Ts\mathcal{A}(X,Y), s\mathcal{C}(Xf^i,Yg^j)\bigr)\\
&\xrightarrow{\sim} \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}\Bigl(Ts\mathcal{A}(X,Y), \prod_{i\in\operatorname{supp} f,\, j\in\operatorname{supp} g} s\mathcal{C}(Xf^i,Yg^j)\Bigr)\\
&= \prod_{X,Y\in\operatorname{Ob}\mathcal{A}} \underline{\mathsf{C}}_{\Bbbk}\bigl(Ts\mathcal{A}(X,Y), s\mathcal{C}^{\mathsf{MC}}(X(f\varpi_{\mathsf{MC}}),Y(g\varpi_{\mathsf{MC}}))\bigr)\\
&= s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})(f\varpi_{\mathsf{MC}},g\varpi_{\mathsf{MC}}). \qquad (11.10.1)
\end{aligned}$$

Clearly, this is an isomorphism. □

11.11 Proposition. *There exists a unique morphism* $\varpi_{\mathsf{mC}} : \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{mC}} \to \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{mC}})$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{mC}} \arrow[r, hook] \arrow[d, "\varpi_{\mathsf{mC}}"'] & \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \arrow[r, "\mathsf{MC}"] \arrow[dr, "\varpi_{\mathsf{MC}}"'] & \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{C}^{\mathsf{MC}}) \arrow[d, "\underline{\mathsf{Q}}(u_{\mathsf{MC}};1)"] \\
\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{mC}}) \arrow[rr, hook] & & \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})
\end{tikzcd}$$

Proof. It suffices to show that the image of

$$\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{mC}} \xhookrightarrow{1,\mathrm{in}} \mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \xrightarrow{1,\varpi_{\mathsf{MC}}} \mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}}) \xrightarrow{\mathrm{ev}} \mathcal{C}^{\mathsf{MC}}$$

is contained in the subcategory $\mathcal{C}^{\mathsf{mC}} \subset \mathcal{C}^{\mathsf{MC}}$. In fact, the above composition coincides with

$$\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{mC}} \xhookrightarrow{u_{\mathsf{MC}},\mathrm{in}} \mathcal{A}^{\mathsf{MC}},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \xrightarrow{\mathrm{ev}^{\mathsf{MC}}} \mathcal{C}^{\mathsf{MC}}$$

due to diagram (11.9.1). The image Z in $\mathcal{C}^{\mathsf{MC}}$ of an object (X,f) of the source quiver, $X \in \operatorname{Ob}\mathcal{A}$, $f : I \to \underline{\mathsf{Q}}(\mathcal{A},\mathcal{C})$, $f : i \mapsto f^i$, $f^{ij} \in \underline{\mathsf{Q}}(\mathcal{A},\mathcal{C})[1]^0$, is obtained as

$$Z = (X,f)\,\mathrm{ev}^{\mathsf{MC}} : I \to \mathcal{C}, \quad i \mapsto Xf^i, \quad z_{ij} = {}_X f_0^{ij} \in s\mathcal{C}(Xf^i, Xf^j)$$

via (11.3.3) (r must be 0 there). This proves the claim. □

11.12 Lemma. *For arbitrary graded quivers $\mathcal{A}$, $\mathcal{C}$ the following equation holds:*

$$\big[\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \xrightarrow{u_{\mathsf{MC}}} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \xrightarrow{\varpi_{\mathsf{MC}}} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})\big] = \underline{\mathsf{Q}}(\mathcal{A};u_{\mathsf{MC}}) = (1 \boxtimes u_{\mathsf{MC}})M.$$

Proof. The left diagram in multicategory Q:

$$\begin{tikzcd}
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \arrow[rr, "\mathrm{ev}"] \arrow[d, "{1,u_{\mathsf{MC}}}"'] \arrow[dr, "{u_{\mathsf{MC}},u_{\mathsf{MC}}}"] & & \mathcal{C} \arrow[dd, "u_{\mathsf{MC}}"] \\
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \arrow[r, "{u_{\mathsf{MC}},1}"] \arrow[d, "{1,\varpi_{\mathsf{MC}}}"'] & \mathcal{A}^{\mathsf{MC}},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{MC}} \arrow[dr, "\mathrm{ev}^{\mathsf{MC}}"] & \\
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}}) \arrow[rr, "\mathrm{ev}"] & & \mathcal{C}^{\mathsf{MC}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \arrow[r, "\mathrm{ev}"] \arrow[d, "{1,\underline{\mathsf{Q}}(\mathcal{A};u_{\mathsf{MC}})}"'] & \mathcal{C} \arrow[d, "u_{\mathsf{MC}}"] \\
\mathcal{A},\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}}) \arrow[r, "\mathrm{ev}"] & \mathcal{C}^{\mathsf{MC}}
\end{tikzcd}$$

commutes due to diagram (11.9.1) and multinaturality of u_{MC}. Compare it with the right diagram, where the arrow $\underline{\mathsf{Q}}(\mathcal{A};u_{\mathsf{MC}}) : \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \to \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{MC}})$ exists and is unique due to universality of evaluation. It is computed via recipe of [Lyu03, Section 4] as $(1 \boxtimes u_{\mathsf{MC}})M$. Uniqueness of such arrow implies the claim. □

11.13 Corollary. *For arbitrary graded quivers $\mathcal{A}$, $\mathcal{C}$ the following equation holds:*

$$\big[\underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}) \xrightarrow{u_{\mathsf{mC}}} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C})^{\mathsf{mC}} \xrightarrow{\varpi_{\mathsf{mC}}} \underline{\mathsf{Q}}(\mathcal{A};\mathcal{C}^{\mathsf{mC}})\big] = \underline{\mathsf{Q}}(\mathcal{A};u_{\mathsf{mC}}) = (1 \boxtimes u_{\mathsf{mC}})M.$$

11.14 Multifunctor Mc. Now we shall study versions of Maurer–Cartan functors for A_∞-categories.

Let $I \in \operatorname{Ob}\mathcal{P}$ be the set $\mathbf{m}$ for some $m \in \mathbb{Z}_{\geqslant 0}$. Turn I into a non-unital graded $\Bbbk$-linear category with the set of objects I. The only non-trivial graded modules of morphisms are $I(i,i+1) = \Bbbk[-1]$ concentrated in degree 1. Thus, the composition in this category vanishes. We shall view I as a non-unital A_∞-category with $b_n = 0$ for all $n \geqslant 1$. The empty category $I = \varnothing$ is also allowed.

11.15 Lemma. *Let $\mathcal{A}$ be an A_∞-category, $b \in \underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})(\mathrm{id}_\mathcal{A}, \mathrm{id}_\mathcal{A})^1$ its differential. The element $b^{\mathsf{MC}} \in \underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})(\mathrm{id}_{\mathcal{A}^{\mathsf{MC}}}, \mathrm{id}_{\mathcal{A}^{\mathsf{MC}}})^1$ satisfies condition* (8.2.1), *that is,* $(\widehat{b^{\mathsf{MC}}\theta}) \cdot \mathrm{in}_1 \cdot b^{\mathsf{MC}} = 0$.

Proof. Equivalently, the differential in $\mathcal{A}$ is a quiver morphism $b : \mathbb{1}_p[-1] \to s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A})$, $* \mapsto \mathrm{id}_\mathcal{A}$, or a $T^{\geqslant 1}$-coalgebra morphism $\hat{b} = b \cdot \mathrm{in}_1 : (\mathbb{1}_p[-1], \mathrm{in}_1) \to (T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}), \Delta)$. Composing the latter morphism with the $T^{\geqslant 1}$-coalgebra morphism $\widehat{-^{\mathsf{MC}}} : T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \to T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})$ we get compositions in $\mathscr{Q}$:

$$
\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\widehat{-^{\mathsf{MC}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\mathrm{in}_1} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{\Delta} T^{\geqslant 1}T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{T^{\geqslant 1}(-^{\mathsf{MC}})} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{-^{\mathsf{MC}}} s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}}) \xrightarrow{\mathrm{in}_1} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big] \\
&= \big[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{(-^{\mathsf{MC}})_1} s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}}) \xrightarrow{\mathrm{in}_1} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big] = \widehat{b^{\mathsf{MC}}},
\end{aligned}
$$

where $b^{\mathsf{MC}} = \big[\mathbb{1}_p[-1] \xrightarrow{b} s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \xrightarrow{(-^{\mathsf{MC}})_1} s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big]$ satisfies condition (8.3.1). Indeed, the differential $\hat{b}$ satisfies equation (8.3.1) between quiver morphisms. Composing it with the morphism $\widehat{-^{\mathsf{MC}}} : T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \to T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})$ and using (11.4.1) we get an equation in $\mathscr{Q}$:

$$
\begin{aligned}
\big[\mathbb{1}_p[-2] &\xrightarrow{\sim} \mathbb{1}_p[-1] \boxtimes \mathbb{1}_p[-1] \hookrightarrow \mathbb{1}_p[-1] \boxtimes_u \mathbb{1}_p[-1] \xrightarrow{\hat{b}\boxtimes_u\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A};\mathcal{A}) \\
&\xrightarrow{\widehat{-^{\mathsf{MC}}}\boxtimes_u\widehat{-^{\mathsf{MC}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}}) \boxtimes_u T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}}) \xrightarrow{\widehat{\mu^{\mathsf{Q}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}^{\mathsf{MC}};\mathcal{A}^{\mathsf{MC}})\big] = 0.
\end{aligned}
$$

This is precisely equation (8.3.1) for b^{MC}, equivalent to (8.2.1). □

Note however that the 0-th component ${}_Xb_0^{\mathsf{MC}} = \sum_{t>0}(X^{\otimes t})b_n \in s\mathcal{A}^{\mathsf{MC}}(X,X)$, where $X : (J_k)_{k\in K} \to \mathcal{A} \in \operatorname{Ob}\mathcal{A}^{\mathsf{MC}}$, does not necessarily vanish. It does if and only if $X : (J_k)_{k\in K} \to \mathcal{A}$ is an A_∞-functor! Let $\mathcal{A}^{\mathsf{Mc}}$ denote the full subquiver of $\mathcal{A}^{\mathsf{MC}}$, whose objects X are A_∞-functors. The restriction b^{Mc} of b^{MC} to the quiver $\mathcal{A}^{\mathsf{Mc}}$ satisfies the condition $b_0^{\mathsf{Mc}} = 0$, thus defines a structure of an A_∞-category on $\mathcal{A}^{\mathsf{Mc}}$.

11.16 Proposition. *Let $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be an A_∞-functor. Then $f^{\mathsf{MC}} : (\mathcal{A}_i^{\mathsf{MC}})_{i\in I} \to \mathcal{B}^{\mathsf{MC}}$ commutes with elements b^{MC}. It restricts to an A_∞-functor $f^{\mathsf{Mc}} : (\mathcal{A}_i^{\mathsf{Mc}})_{i\in I} \to \mathcal{B}^{\mathsf{Mc}}$. The assignment $\mathcal{A} \mapsto \mathcal{A}^{\mathsf{Mc}}$, $f \mapsto f^{\mathsf{Mc}}$ defines a multifunctor $\mathsf{A}_\infty \to \mathsf{A}_\infty$.*

Proof. Let $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ be an A_∞-functor, that is, a morphism in Q that satisfies equation (8.9.4). Compose in $\widehat{\mathscr{Q}_{uT^{\geqslant 1}}}$ this equation with the $T^{\geqslant 1}$-coalgebra morphism $\widehat{-^{\mathsf{MC}}} : T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in\mathbf{n}};\mathcal{B}) \to T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\mathbf{n}};\mathcal{B}^{\mathsf{MC}})$. It follows from (11.5.1) and (11.6.1)

that

$$\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{\hat{b}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B};\mathcal{B}) \xrightarrow{\widehat{-^{\mathsf{MC}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B}^{\mathsf{MC}};\mathcal{B}^{\mathsf{MC}}) \xrightarrow{\underline{\mathsf{Q}}(f^{\mathsf{MC}};1)\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\mathbf{n}};\mathcal{B}^{\mathsf{MC}})\big] \\
&= \sum_{j=1}^n \big[\mathbb{1}_p[-1] \xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i<j},\hat{b},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{})_{i>j}} (T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i))_{i\in\mathbf{n}} \xrightarrow{(\widehat{-^{\mathsf{MC}}})_{\mathbf{n}}} (T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i^{\mathsf{MC}};\mathcal{A}_i^{\mathsf{MC}}))_{i\in\mathbf{n}} \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{Q}}(1;f^{\mathsf{MC}})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\mathbf{n}};\mathcal{B}^{\mathsf{MC}})\big].
\end{aligned}$$

Notice that in $\widehat{\mathscr{Q}_{uT^{\geqslant 1}}}$

$$\big[() \xrightarrow{(\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i;\mathcal{A}_i) \xrightarrow{\widehat{-^{\mathsf{MC}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i^{\mathsf{MC}};\mathcal{A}_i^{\mathsf{MC}})\big] = (\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i^{\mathsf{MC}}})\hat{}.$$

Thus, the above equation is equivalent to

$$\begin{aligned}
&\big[\mathbb{1}_p[-1] \xrightarrow{\widehat{b^{\mathsf{MC}}}} T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{B}^{\mathsf{MC}};\mathcal{B}^{\mathsf{MC}}) \xrightarrow{\underline{\mathsf{Q}}(f^{\mathsf{MC}};1)\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\mathbf{n}};\mathcal{B}^{\mathsf{MC}})\big] \\
&= \sum_{j=1}^n \big[\mathbb{1}_p[-1] \xrightarrow{((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i^{\mathsf{MC}}})\hat{})_{i<j},\widehat{b^{\mathsf{MC}}},((\dot{\mathrm{id}}^{\mathsf{Q}}_{\mathcal{A}_i^{\mathsf{MC}}})\hat{})_{i>j}} (T^{\geqslant 1}s\underline{\mathsf{Q}}(\mathcal{A}_i^{\mathsf{MC}};\mathcal{A}_i^{\mathsf{MC}}))_{i\in\mathbf{n}} \\
&\qquad\qquad \xrightarrow{\underline{\mathsf{Q}}(1;f^{\mathsf{MC}})\hat{}} T^{\geqslant 1}s\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in\mathbf{n}};\mathcal{B}^{\mathsf{MC}})\big].
\end{aligned}$$

The above equation states that f^{MC} commutes with elements b^{MC}.

The morphism $f^{\mathsf{MC}} : (\mathcal{A}_i^{\mathsf{MC}})_{i\in I} \to \mathcal{B}^{\mathsf{MC}}$ restricts to a morphism $f^{\mathsf{Mc}} : (\mathcal{A}_i^{\mathsf{Mc}})_{i\in I} \to \mathcal{B}^{\mathsf{Mc}}$. Indeed, it maps a family of A_∞-functors $X_i : (J_k^i)_{k\in K_i} \to \mathcal{A}_i$, $i \in I$ to the composition

$$\big[((J_k^i)_{k\in K_i})_{i\in I} \xrightarrow{(X_i)_{i\in I}} (\mathcal{A}_i)_{i\in I} \xrightarrow{f} \mathcal{B}\big],$$

which is again an A_∞-functor. The above equation implies that f^{Mc} commutes with differentials b^{Mc}, that is, it is an A_∞-functor. The assignment $\mathcal{A} \mapsto \mathcal{A}^{\mathsf{Mc}}$, $f \mapsto f^{\mathsf{Mc}}$ defines a multifunctor $\mathsf{A}_\infty \to \mathsf{A}_\infty$. Compatibility with multiplications follows from the fact that $-^{\mathsf{MC}}$ is a multifunctor. □

Likewise $-^{\mathsf{MC}}$ the multifunctor $-^{\mathsf{Mc}}$ is equipped with the multinatural transformation $u_{\mathsf{Mc}} : \mathrm{Id} \to -^{\mathsf{Mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$. Thus, $(-^{\mathsf{Mc}}, u_{\mathsf{Mc}})$ is an augmented multifunctor. It has the corresponding A_∞-multifunctor $\mathsf{Mc}' = u_{\mathsf{Mc}} \cdot \underline{\mathsf{Mc}} : \underline{\mathsf{A}_\infty} \to \underline{\mathsf{A}_\infty}$, which we denote also $-^{\mathsf{Mc}} : p \mapsto p^{\mathsf{Mc}}$ by abuse of notation.

11.17 Proposition. *The morphisms* $\underline{\mathsf{Mc}}$ *and* $\underline{\mathsf{MC}}$ *agree in the sense of the following*

diagram:

$$\begin{tikzcd}
\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{Mc}} \arrow[rr, "\underline{\mathsf{Mc}}"] \arrow[d, hook, "\iota"'] & & \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[d, hook, "\underline{E}"] \\
\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{MC}} \arrow[d, hook, "\underline{E}^{\mathsf{MC}}"'] & = & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[d, hook, "\underline{\mathsf{Q}}(\triangleright;\iota)"] \\
\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{MC}} \arrow[r, "\underline{\mathsf{MC}}"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in I};\mathcal{B}^{\mathsf{MC}}) \arrow[r, "\underline{\mathsf{Q}}((\iota);1)"] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{MC}})
\end{tikzcd}$$

Proof. The multifunctors $-^{\mathsf{Mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$ and $-^{\mathsf{MC}} : \mathsf{Q} \to \mathsf{Q}$ are related by the multinatural transformation $\iota : \mathcal{C}^{\mathsf{Mc}} \hookrightarrow \mathcal{C}^{\mathsf{MC}}$, that is,

$$\begin{tikzcd}
\mathsf{A}_\infty \arrow[r, "_^{\mathsf{Mc}}"] \arrow[d, "E"'] & \mathsf{A}_\infty \arrow[d, "E"] \arrow[dl, Rightarrow, "\iota"'] \\
\mathsf{Q} \arrow[r, "_^{\mathsf{MC}}"] & \mathsf{Q}
\end{tikzcd}$$

It remains to apply Lemma 4.25 twice and Lemma 4.24. □

11.18 Corollary. *For an arbitrary sequence of composable A_∞-transformations*

$$f^0 \xrightarrow{p^1} f^1 \xrightarrow{p^2} \dots f^{m-1} \xrightarrow{p^m} f^m : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$$

we have

$$(p^{1\mathsf{Mc}}\otimes\dots\otimes p^{m\mathsf{Mc}})B_m = [(p^1\otimes\dots\otimes p^m)B_m]^{\mathsf{Mc}} : f^{0\mathsf{Mc}} \to f^{m\mathsf{Mc}} : (\mathcal{A}_i^{\mathsf{Mc}})_{i\in I} \to \mathcal{B}^{\mathsf{Mc}}. \quad (11.18.1)$$

Proof. The morphisms $\underline{\mathsf{MC}}$ of Q are strict by Section 11.3. It follows from the above proposition that the A_∞-functors $\underline{\mathsf{Mc}}$ are strict. □

The first component $\underline{\mathsf{Mc}}_1$ is given by formula (11.3.7) similarly to $\underline{\mathsf{MC}}_1$.

11.19 A_∞-category of bounded complexes.

Let $\mathcal{C}$ be an A_∞-category. The A_∞-category $\mathcal{C}^{\mathsf{mc}}$ of bounded complexes in $\mathcal{C}$ is a full subquiver of $\mathcal{C}^{\mathsf{mC}}$. It is also a full A_∞-subcategory of $\mathcal{C}^{\mathsf{Mc}}$. It can be viewed as a pull-back in the following diagram of full embeddings of quivers:

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{mc}} \arrow[r, hook, "\text{full}"] \arrow[d, hook, "\text{full}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}^{\mathsf{mC}} \arrow[d, hook, "\text{full}"] \\
\mathcal{C}^{\mathsf{Mc}} \arrow[r, hook, "\text{full}"] & \mathcal{C}^{\mathsf{MC}}
\end{tikzcd}$$

The quivers on the left are A_∞-categories. Thus objects of $\mathcal{C}^{\mathsf{mc}}$ are A_∞-functors $X : I \to \mathcal{C}$, $\operatorname{Ob} X : i \mapsto X_i$, where I is some set $\mathbf{m}$ for $m \in \mathbb{Z}_{\geqslant 0}$. If $I = \varnothing$, the A_∞-functor $\varnothing \to \mathcal{C}$ gives the zero object of $\mathcal{C}^{\mathsf{mc}}$. Besides X_i the A_∞-functor X is determined by $\Bbbk$-linear maps of degree 0 for all $i < j$, $i, j \in I$,

$$X_{ij} : \Bbbk = sI(i, i+1) \otimes \cdots \otimes sI(j-1, j) \to s\mathcal{C}(X_i, X_j),$$

that is, by elements $x_{ij} \in s\mathcal{C}(X_i, X_j)$ of degree 0. They give an A_∞-functor if and only if the Maurer–Cartan equation holds for all $i < j$, $i, j \in I$:

$$\sum_{i<k_1<\cdots<k_{m-1}<j}^{m>0} (X_{ik_1} \otimes X_{k_1k_2} \otimes \cdots \otimes X_{k_{m-1}j}) b_m^{\mathcal{C}} = 0 : \Bbbk = \bigotimes_{k=i}^{j-1} sI(k, k+1) \to s\mathcal{C}(X_i, X_j).$$

In this equation we may replace X_{ac} with elements $x_{ac} \in (s\mathcal{C})^0(X_a, X_c)$.

The general procedure of defining $\mathcal{C}^{\mathsf{Mc}}(X, Y)$ applies also to $\mathcal{C}^{\mathsf{mc}}(X, Y)$ as follows. Given $X : I \to \mathcal{C}$, $Y : J \to \mathcal{C}$ we consider isotonic embeddings $e_1 : I \hookrightarrow K$, $e_2 : J \hookrightarrow K$ into a finite totally ordered set K such that $\operatorname{Im} e_1$, $\operatorname{Im} e_2$ are subintervals of K and $\operatorname{Im} e_1 < \operatorname{Im} e_2$. Let $\mathcal{C}^*$ denote the A_∞-category $\mathcal{C}$ with one object 0 added such that $\mathcal{C}^*(0, -) = \mathcal{C}^*(-, 0) = 0$. The resulting A_∞-category $\mathcal{C}^*$ is unital with ${}_0\mathbf{i}_0^{\mathcal{C}^*} = 0$ if $\mathcal{C}$ is unital. Let us extend the A_∞-functor $X : I \to \mathcal{C}$ to an A_∞-functor $X^* = X^*_{I\subset K} : K \to \mathcal{C}^*$ by zero, that is, $X^* : e_1(i) \mapsto X_i$ if $i \in I$ and $X^* : k \mapsto 0$ if $k \in K \smallsetminus e_1(I)$. The components of X^* for $k, m \in K$, $k < m$

$$X^*_{km} : \Bbbk = \bigotimes_{p=k}^{m-1} sK(p, p+1) \to s\mathcal{C}^*(X^*_k, X^*_m)$$

are set equal to X_{il} if $k = e_1(i)$, $m = e_1(l)$, $i, l \in I$, and they have to vanish if k or m is not in $e_1(I)$. If the extension procedure is repeated for $I \hookrightarrow K \hookrightarrow L$, we do not add 0 twice. Therefore, the double extension $(X^*_{I\subset K})^*_{K\subset L} : L \to \mathcal{C}^*$ coincides with the single one $X^*_{I\subset L} : L \to \mathcal{C}^*$. Similarly, $Y : J \to \mathcal{C}$ is extended to $Y^* = Y^*_{J\subset K} : K \to \mathcal{C}^*$.

The graded $\Bbbk$-module of morphisms between X and Y is defined as

$$s\mathcal{C}^{\mathsf{mc}}(X, Y) = \prod_{i\in I, j\in J} s\mathcal{C}(X_i, Y_j) \xleftarrow[\sim]{\iota_1} \prod_{k,m\in K, k<m} \underline{\mathsf{C}}_\Bbbk(\Bbbk, s\mathcal{C}^*(X^*_k, Y^*_m)))$$
$$= sA_\infty(K, \mathcal{C}^*)(X^*, Y^*). \quad (11.19.1)$$

The isomorphism consists of inserting (dropping) the zero factors. The dependence on the embeddings $e_1 : I \hookrightarrow K$, $e_2 : J \hookrightarrow K$ does not show up in the graded $\Bbbk$-module structure of $\mathcal{C}^{\mathsf{mc}}(X, Y)$. Notice, nevertheless, that the factor $s\mathcal{C}(X_i, Y_j)$ belongs to the $\Bbbk$-module of $e_2(j) - e_1(i)$ components of (X^*, Y^*)-coderivations.

Let $X^0 : I^0 \to \mathcal{C}$, …, $X^n : I^n \to \mathcal{C}$ be objects of $\mathcal{C}^{\mathsf{mc}}$, $n \geqslant 1$. Choose embeddings $e^p : I^p \hookrightarrow K$, $0 \leqslant p \leqslant n$ so that $\operatorname{Im}(e^p)$ are subintervals of a finite totally ordered set

K and $\operatorname{Im}(e^p) < \operatorname{Im}(e^{p+1})$ in K. Consider extensions $X^{p*} = X^{p*}_{I^p \subset K} : K \to \mathcal{C}^*$. Define the differential b^{mc} in $\mathcal{C}^{\mathsf{mc}}$ as $b^{\mathsf{MC}} = b^{\mathsf{Mc}}$. In components it gives

$$b_n^{\mathsf{mc}} = \bigl[s\mathcal{C}^{\mathsf{mc}}(X^0,X^1) \otimes \cdots \otimes s\mathcal{C}^{\mathsf{mc}}(X^{n-1},X^n) \\ \xrightarrow[\sim]{\otimes\iota_1^{-1}} sA_\infty(K,\mathcal{C}^*)(X^{0*},X^{1*}) \otimes \cdots \otimes sA_\infty(K,\mathcal{C}^*)(X^{n-1*},X^{n*}) \\ \xrightarrow{B_n} sA_\infty(K,\mathcal{C}^*)(X^{0*},X^{n*}) \xrightarrow[\sim]{\iota_1} s\mathcal{C}^{\mathsf{mc}}(X^0,X^n)\bigr].$$

As in general case the operation b_n^{mc} does not depend on the choice of embeddings.

Let $r^{p*} \in sA_\infty(K,\mathcal{C}^*)(X^{p-1*},X^{p*})$ for $1 \leqslant p \leqslant n$ represent $r^p \in s\mathcal{C}^{\mathsf{mc}}(X^{p-1},X^p)$. Then for $n > 1$

$$[(r^{1*} \otimes \cdots \otimes r^{n*})B_n]_q = \sum_{l>0} (r^{1*} \otimes \cdots \otimes r^{n*})\theta_{ql} b_l : \Bbbk = \bigotimes_{t=k}^{m-1} sK(t,t+1) \to s\mathcal{C}^*(X^{0*}_k, X^{n*}_m) \tag{11.19.2}$$

by [Lyu03, (5.1.3)]. The same formula holds for $n = 1$ as well since in our case $b^K = 0$ and $rB_1 = rb^{\mathcal{C}^*} - (-)^r b^K r = (r)\theta b^{\mathcal{C}^*}$. The coderivation component number $q = m - k$ in (11.19.2) is determined uniquely by objects $k,m \in \operatorname{Ob} K = K$. So we may drop it from notation and write $r^p = (r^p_{ij})^{j \in I^p}_{i \in I^{p-1}}$, $r^{p*} = (r^{p*}_{km})_{k,m \in K}$ in a matrix form. Similarly, $X^p = (X^p_{ij})_{i,j \in I^p}$, $X^{p*} = (X^{p*}_{km})_{k,m \in K}$ are matrices of degree 0 elements. The tensor product $A \otimes B$ of two matrices $A = (A_{kl})^{k \in K}_{l \in L}$, $B = (B_{lm})^{l \in L}_{m \in M}$ means $(A \otimes B)_{km} = \sum_{l \in L} A_{kl} \otimes B_{lm}$. In these matrix notations formula (11.19.2) reads

$$[(r^{1*} \otimes \cdots \otimes r^{n*})B_n]_{km} \\ = \sum_{t_0,\dots,t_n \geqslant 0} \bigl[(X^{0*})^{\otimes t_0} \otimes r^{1*} \otimes (X^{1*})^{\otimes t_1} \otimes \cdots \otimes r^{n*} \otimes (X^{n*})^{\otimes t_n}\bigr]_{km} b^{\mathcal{C}^*}_{t_0+\dots+t_n+n} : \\ \Bbbk^{\otimes m-k} \to s\mathcal{C}^*(X^{0*}_k, X^{n*}_m). \tag{11.19.3}$$

Clearly, the sum is finite. Zero matrix entries are irrelevant in this formula, so we may write it as

$$(r^1 \otimes \cdots \otimes r^n) b_n^{\mathsf{mc}} \\ = \sum_{t_0,\dots,t_n \geqslant 0} \bigl[(X^0)^{\otimes t_0} \otimes r^1 \otimes (X^1)^{\otimes t_1} \otimes \cdots \otimes r^n \otimes (X^n)^{\otimes t_n}\bigr] b^{\mathcal{C}}_{t_0+\dots+t_n+n} : \\ \Bbbk \to s\mathcal{C}^{\mathsf{mc}}(X^0,X^n), \tag{11.19.4}$$

which does not depend on embeddings e^p.

11.20 The Maurer–Cartan functor on A_∞-functors. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-functor. The A_∞-functor $f^{\mathsf{Mc}} : \mathcal{A}^{\mathsf{Mc}} \to \mathcal{B}^{\mathsf{Mc}}$ restricts to an A_∞-functor $f^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to$

$\mathcal{B}^{\mathsf{mc}}$. An object $X : I \to \mathcal{A}$ of $\mathcal{A}^{\mathsf{mc}}$ is mapped by f^{mc} to the object $Xf : I \to \mathcal{B}$ of $\mathcal{B}^{\mathsf{mc}}$. It is represented by the matrix $Xf^{\mathsf{mc}} = \sum_{n\geqslant 1}(X^{\otimes n})f_n = \sum_{1\leqslant n<|I|}(X^{\otimes n})f_n$. Extend f to $f^* : \mathcal{A}^* \to \mathcal{B}^*$ by the requirement $0f^* = 0$. The components of f^{mc} can be found from

$$\begin{aligned} f_n^{\mathsf{mc}} = \bigl[& s\mathcal{A}^{\mathsf{mc}}(X^0, X^1) \otimes \cdots \otimes s\mathcal{A}^{\mathsf{mc}}(X^{n-1}, X^n) \\ & \xrightarrow[\sim]{\otimes \iota_1^{-1}} sA_\infty(K, \mathcal{A}^*)(X^{0*}, X^{1*}) \otimes \cdots \otimes sA_\infty(K, \mathcal{A}^*)(X^{n-1*}, X^{n*}) \\ & \xrightarrow{(1\boxtimes f^*)M_{n0}} sA_\infty(K, \mathcal{B}^*)(X^{0*}f^*, X^{n*}f^*) \xrightarrow[\sim]{\iota_1} s\mathcal{B}^{\mathsf{mc}}(X^0 f, X^n f)\bigr] \end{aligned} \tag{11.20.1}$$

for an arbitrary embedding $e : I^0 \sqcup I^1 \sqcup \cdots \sqcup I^n \hookrightarrow K$ and $X^p : I^p \to \mathcal{A}$. Let $r^p \in s\mathcal{A}^{\mathsf{mc}}(X^{p-1}, X^p)$. The corresponding r^{p*} are mapped by $(1 \boxtimes f^*)M_{n0}$ to

$$\begin{aligned} (r^{1*} \otimes \cdots \otimes r^{n*} \boxtimes f^*)M_{n0} &= \sum_{l>0} (r^{1*} \otimes \cdots \otimes r^{n*})\theta_{\bullet l} f_l^* \\ &= \sum_{t_0,\dots,t_n \geqslant 0} \bigl[(X^{0*})^{\otimes t_0} \otimes r^{1*} \otimes (X^{1*})^{\otimes t_1} \otimes \cdots \otimes r^{n*} \otimes (X^{n*})^{\otimes t_n}\bigr] f^*_{t_0+\dots+t_n+n} \end{aligned} \tag{11.20.2}$$

by [Lyu03, (4.1.3)]. Dropping zeroes we get a map of degree 0

$$(r^1 \otimes \cdots \otimes r^n) f_n^{\mathsf{mc}} = \sum_{t_0,\dots,t_n \geqslant 0} \bigl[(X^0)^{\otimes t_0} \otimes r^1 \otimes (X^1)^{\otimes t_1} \otimes \cdots \otimes r^n \otimes (X^n)^{\otimes t_n}\bigr] f_{t_0+\dots+t_n+n}, \tag{11.20.3}$$

which does not depend on the embedding e.

Suppose that $\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$ are A_∞-functors. Since $(fg)^{\mathsf{Mc}} = f^{\mathsf{Mc}} g^{\mathsf{Mc}}$, we have also $(fg)^{\mathsf{mc}} = f^{\mathsf{mc}} g^{\mathsf{mc}}$. Therefore, $-^{\mathsf{mc}} : A_\infty \to A_\infty$ is a functor (of ordinary categories). We are going to equip it with a monad structure.

11.21 The Maurer–Cartan functor on A_∞-transformations. Let $p : f \to g : \mathcal{A} \to \mathcal{B}$ be an A_∞-transformation. It extends in a unique way (by zero) to an A_∞-transformation $p^* : f^* \to g^* : \mathcal{A}^* \to \mathcal{B}^*$. Part of the data defining the A_∞-transformation p^{Mc} gives the A_∞-transformation $p^{\mathsf{mc}} : f^{\mathsf{mc}} \to g^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{B}^{\mathsf{mc}}$. The components of p^{mc} are found from

$$\begin{aligned} p_n^{\mathsf{mc}} = \bigl[& s\mathcal{A}^{\mathsf{mc}}(X^0, X^1) \otimes \cdots \otimes s\mathcal{A}^{\mathsf{mc}}(X^{n-1}, X^n) \\ & \xrightarrow[\sim]{\otimes \iota_1^{-1}} sA_\infty(K, \mathcal{A}^*)(X^{0*}, X^{1*}) \otimes \cdots \otimes sA_\infty(K, \mathcal{A}^*)(X^{n-1*}, X^{n*}) \\ & \xrightarrow{(1\boxtimes p^*)M_{n1}} sA_\infty(K, \mathcal{B}^*)(X^{0*}f^*, X^{n*}g^*) \xhookrightarrow{\iota_1} s\mathcal{B}^{\mathsf{mc}}(X^0 f, X^n g)\bigr] \end{aligned} \tag{11.21.1}$$

for an arbitrary embedding $e : I^0 \sqcup I^1 \sqcup \cdots \sqcup I^n \hookrightarrow K$. Let $r^q \in s\mathcal{A}^{\mathsf{mc}}(X^{q-1}, X^q)$. The last map ι_1 is given by (11.19.1). It is an isomorphism if $n > 0$, and it is the embedding of

upper-triangular matrices into all matrices if $n = 0$. The corresponding r^{q*} are mapped by $(1 \boxtimes p^*)M_{n1}$ to

$$
\begin{aligned}
(r^{1*} \otimes \cdots \otimes r^{n*} \boxtimes p^*)M_{n1} &= \sum_{l\geqslant 0} (r^{1*} \otimes \cdots \otimes r^{n*})\theta_{\bullet l} p_l^* \\
&= \sum_{t_0,\dots,t_n\geqslant 0} \bigl[(x^{0*})^{\otimes t_0} \otimes r^{1*} \otimes (x^{1*})^{\otimes t_1} \otimes \cdots \otimes r^{n*} \otimes (x^{n*})^{\otimes t_n}\bigr] p^*_{t_0+\dots+t_n+n} \qquad (11.21.2)
\end{aligned}
$$

by [Lyu03, (4.1.4)]. This formula holds for $n = 0$ as well. Dropping zeroes we get a map of degree 0

$$
(r^1 \otimes \cdots \otimes r^n) p_n^{\mathsf{mc}} = \sum_{t_0,\dots,t_n\geqslant 0} \bigl[(x^0)^{\otimes t_0} \otimes r^1 \otimes (x^1)^{\otimes t_1} \otimes \cdots \otimes r^n \otimes (x^n)^{\otimes t_n}\bigr] p_{t_0+\dots+t_n+n}, \qquad (11.21.3)
$$

which does not depend on the embedding e. In particular, for $n = 0$ we get

$$
{}_X p_0^{\mathsf{mc}} = \sum_{t\geqslant 0} (x^{\otimes t}) p_t.
$$

11.22 Corollary (to Corollary 11.18)**.** *Let $\mathcal{A}$, $\mathcal{B}$ be A_∞-categories. Then the correspondences $f \mapsto f^{\mathsf{mc}}$, $p \mapsto p^{\mathsf{mc}}$ given by* (11.20.3) *and* (11.21.3) *define a strict A_∞-functor $-^{\mathsf{mc}} : A_\infty(\mathcal{A}, \mathcal{B}) \to A_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$.*

11.23 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then $\mathcal{C}^{\mathsf{mc}}$ is unital with the unit transformation $\mathbf{i}^{(\mathcal{C}^{\mathsf{mc}})} = (\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}}$. For a unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ the A_∞-functor $f^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{B}^{\mathsf{mc}}$ is unital as well. If $\mathcal{A}$, $\mathcal{B}$ are unital A_∞-categories, then the A_∞-functor $-^{\mathsf{mc}} : A^u_\infty(\mathcal{A}, \mathcal{B}) \to A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$ is unital.*

Proof. The A_∞-category $\mathcal{C}$ has a unit transformation $\mathbf{i}^{\mathcal{C}} : \mathrm{id} \to \mathrm{id} : \mathcal{C} \to \mathcal{C}$ which satisfies

$$
\mathbf{i}^{\mathcal{C}} B_1 = 0, \qquad (\mathbf{i}^{\mathcal{C}} \otimes \mathbf{i}^{\mathcal{C}}) B_2 = \mathbf{i}^{\mathcal{C}} + v B_1
$$

for some $v : \mathrm{id} \to \mathrm{id} : \mathcal{C} \to \mathcal{C}$. Applying $-^{\mathsf{mc}}$ to these equations we get

$$
(\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}} B_1 = 0, \qquad ((\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}} \otimes (\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}}) B_2 = (\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}} + v^{\mathsf{mc}} B_1 : \mathrm{id} \to \mathrm{id} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}^{\mathsf{mc}}.
$$

The A_∞-transformation $(\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}}$ has degree -1. Let us write down its 0-th component. If $(X : I \to \mathcal{C}, i \mapsto X_i, x = (x_{ij})_{i,j\in I})$ is an object of $\mathcal{C}^{\mathsf{mc}}$, then

$$
{}_X(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} = \sum_{t\geqslant 0} (x^{\otimes t}) \mathbf{i}_t^{\mathcal{C}} \in s\mathcal{C}^{\mathsf{mc}}(X, X). \qquad (11.23.1)
$$

The summand corresponding to $t = 0$ is $\mathrm{diag}({}_{X_i}\mathbf{i}_0^{\mathcal{C}})$.

Let $(X : I \to \mathcal{C}, x)$, $(Y : J \to \mathcal{C}, y)$ be objects of $\mathcal{C}^{\mathsf{mc}}$ and let $r \in s\mathcal{C}^{\mathsf{mc}}(X, Y)$. Consider the map

$$r \mapsto (r \otimes_Y (\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}})b_2^{\mathsf{mc}} = \sum_{t_0,t_1,t_2 \geqslant 0} \big(x^{\otimes t_0} \otimes r \otimes y^{\otimes t_1} \otimes_Y (\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \otimes y^{\otimes t_2}\big) b_{t_0+t_1+t_2+2} : \\ \prod_{i\in I, j\in J} s\mathcal{C}(X_i, Y_j) \to \prod_{i\in I, j\in J} s\mathcal{C}(X_i, Y_j). \quad (11.23.2)$$

Filter this $\Bbbk$-module by graded submodules $\Phi_{kl} = \prod_{i\leqslant k, j\geqslant l} s\mathcal{C}(X_i, Y_j)$, $k \in I$, $l \in J$. They are subcomplexes with respect to the differential b_1^{mc}. Let

$$(1 \otimes_{Y_j} \mathbf{i}_0^{\mathcal{C}})b_2^{\mathcal{C}} = 1 + h_{ij}b_1 + b_1 h_{ij} : s\mathcal{C}(X_i, Y_j) \to s\mathcal{C}(X_i, Y_j).$$

Combine the maps h_{ij} into a $\Bbbk$-linear map $h = \prod_{i\in I, j\in J} h_{ij} : s\mathcal{C}^{\mathsf{mc}}(X, Y) \to s\mathcal{C}^{\mathsf{mc}}(X, Y)$ of degree -1. The chain map

$$a = (1 \otimes_Y (\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}})b_2^{\mathsf{mc}} - 1 - hb_1^{\mathsf{mc}} - b_1^{\mathsf{mc}}h : s\mathcal{C}^{\mathsf{mc}}(X, Y) \to s\mathcal{C}^{\mathsf{mc}}(X, Y)$$

maps the subcomplex Φ_{kl} into the sum of subcomplexes $\Phi_{k,l+1} + \Phi_{k-1,l}$. Therefore, a is nilpotent, $a^{|I|+|J|} = 0$. Hence, $1+a$ is invertible, and $(1 \otimes_Y (\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}})b_2^{\mathsf{mc}} \sim 1 + a$ is homotopy invertible. Similarly, $({}_X(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \otimes 1)b_2^{\mathsf{mc}}$ is homotopy invertible, hence, $\mathcal{C}^{\mathsf{mc}}$ is unital with the unit transformation $\mathbf{i}^{(\mathcal{C}^{\mathsf{mc}})} = (\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}}$.

For an arbitrary diagram $\mathcal{A} \xrightarrow{h} \mathcal{B} \underset{g}{\overset{f}{\underset{\Downarrow p}{\rightrightarrows}}} \mathcal{C} \xrightarrow{k} \mathcal{D}$ of A_∞-categories, A_∞-functors, an A_∞-transformation p and an A_∞-category $\mathcal{X}$ the equations

$$(1 \boxtimes p^*)M(1 \boxtimes k^*)M = (1 \boxtimes p^*k^*)M = (1 \boxtimes (pk)^*)M : \\ (1 \boxtimes (fk)^*)M \to (1 \boxtimes (gk)^*)M : A_\infty(\mathcal{X}, \mathcal{B}) \to A_\infty(\mathcal{X}, \mathcal{D}),$$

$$(1 \boxtimes h^*)M(1 \boxtimes p^*)M = (1 \boxtimes h^*p^*)M = (1 \boxtimes (hp)^*)M : \\ (1 \boxtimes (hf)^*)M \to (1 \boxtimes (hg)^*)M : A_\infty(\mathcal{X}, \mathcal{A}) \to A_\infty(\mathcal{X}, \mathcal{C})$$

imply that

$$p^{\mathsf{mc}}k^{\mathsf{mc}} = (pk)^{\mathsf{mc}}, \qquad h^{\mathsf{mc}}p^{\mathsf{mc}} = (hp)^{\mathsf{mc}}. \quad (11.23.3)$$

Let $f : \mathcal{A} \to \mathcal{B}$ be a unital A_∞-functor. Then there is an A_∞-transformation $v : \mathbf{i}^{\mathcal{A}}f \to f\mathbf{i}^{\mathcal{B}} : f \to f : \mathcal{A} \to \mathcal{B}$, that is, equation $vB_1 = \mathbf{i}^{\mathcal{A}}f - f\mathbf{i}^{\mathcal{B}}$ holds. Applying $-^{\mathsf{mc}}$ we get an equation

$$v^{\mathsf{mc}}B_1 = (\mathbf{i}^{\mathcal{A}})^{\mathsf{mc}}f^{\mathsf{mc}} - f^{\mathsf{mc}}(\mathbf{i}^{\mathcal{B}})^{\mathsf{mc}},$$

which implies that f^{mc} is unital. The first component $(-^{\mathsf{mc}})_1 : A_\infty^u(\mathcal{A}, \mathcal{B})(f, f) \to A_\infty^u(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})(f^{\mathsf{mc}}, f^{\mathsf{mc}})$ takes the unit element $f\mathbf{i}^{\mathcal{B}}$ of $\mathcal{B}$ to the unit element $(f\mathbf{i}^{\mathcal{B}})^{\mathsf{mc}} = f^{\mathsf{mc}}(\mathbf{i}^{\mathcal{B}})^{\mathsf{mc}} = f^{\mathsf{mc}}\mathbf{i}^{(\mathcal{B}^{\mathsf{mc}})}$ of $\mathcal{B}^{\mathsf{mc}}$. Therefore, $-^{\mathsf{mc}} : A_\infty^u(\mathcal{A}, \mathcal{B}) \to A_\infty^u(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$ is unital. $\square$

11.24 Lemma. *If $\mathcal{C}$ is strictly unital, then $\mathcal{C}^{\mathsf{mc}}$ is strictly unital as well.*

Proof. Let $(X : I \to \mathcal{C}, i \mapsto X_i, x_{ij} \in s\mathcal{C}(X_i, X_j))$ be an object of $\mathcal{C}^{\mathsf{mc}}$. Consider the element $a = {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \in s\mathcal{C}^{\mathsf{mc}}(X, X) = \prod_{i,j\in I} s\mathcal{C}(X_i, X_j)$ of degree -1 with the matrix elements $a_{ii} = {}_{X_i}\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(X_i, X_i)$, $a_{ij} = 0 \in s\mathcal{C}(X_i, X_j)$ if $i \neq j$. In other terms, ${}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} = \operatorname{diag}({}_{X_i}\mathbf{i}_0^{\mathcal{C}})$. We have

$$({}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}})b_1^{\mathsf{mc}} = \sum_{t,p\geqslant 0} (x^{\otimes t} \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes x^{\otimes p})b^{\mathcal{C}}_{t+1+p} = (x \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}})b_2^{\mathcal{C}} + ({}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes x)b_2^{\mathcal{C}} = x - x = 0.$$

For all $r \in s\mathcal{C}^{\mathsf{mc}}(Y, X)$ we have

$$(r \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}})b_2^{\mathsf{mc}} = \sum_{t,p,q\geqslant 0} (y^{\otimes t} \otimes r \otimes x^{\otimes p} \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes x^{\otimes q})b^{\mathcal{C}}_{t+p+q+2} = (r \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}})b_2^{\mathcal{C}} = r.$$

Similarly, for all $r \in s\mathcal{C}^{\mathsf{mc}}(X, Y)$ we have

$$r({}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes 1)b_2^{\mathsf{mc}} = (-)^r({}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes r)b_2^{\mathsf{mc}} = (-)^r({}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes r)b_2^{\mathcal{C}} = -r : \Bbbk \to s\mathcal{C}^{\mathsf{mc}}(Y, X).$$

If $n > 2$, the expression $(r^1 \otimes \cdots \otimes r^k \otimes {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} \otimes r^{k+2} \otimes \cdots \otimes r^n)b_n^{\mathsf{mc}}$ vanishes. □

11.25 A^u_∞-2-functor $-^{\mathsf{mc}}$. Let us show that the Maurer–Cartan construction gives an example of an A_∞-2-functor $-^{\mathsf{mc}} : A_\infty \to A_\infty$ which restricts to an A^u_∞-2-functor $-^{\mathsf{mc}} : A^u_\infty \to A^u_\infty$. Indeed, we have

1. maps $-^{\mathsf{mc}} : \operatorname{Ob} A_\infty \to \operatorname{Ob} A_\infty$ and $-^{\mathsf{mc}} : \operatorname{Ob} A^u_\infty \to \operatorname{Ob} A^u_\infty$, $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{mc}}$;

2. a strict A_∞-functor $-^{\mathsf{mc}} : A_\infty(\mathcal{A}, \mathcal{B}) \to A_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$, $f \mapsto f^{\mathsf{mc}}$, $r \mapsto r^{\mathsf{mc}}$, as Corollary 11.22 shows, whose restriction is a unital A_∞-functor $-^{\mathsf{mc}} : A^u_\infty(\mathcal{A}, \mathcal{B}) \to A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$, if A_∞-categories $\mathcal{A}$, $\mathcal{B}$ are unital (by Proposition 11.23 and by equations (11.23.3)).

11.26 Proposition. *$-^{\mathsf{mc}} : A_\infty \to A_\infty$ is an A_∞-2-functor (equivalently, an A_∞-functor) which restricts to an A^u_∞-2-functor $-^{\mathsf{mc}} : A^u_\infty \to A^u_\infty$ (equivalently, to an $\mathsf{A}^{\mathsf{u}}_\infty$-functor).*

Proof. As required, $\mathrm{id}_{\mathcal{C}^{\mathsf{mc}}} = (\mathrm{id}_{\mathcal{C}})^{\mathsf{mc}}$. Let us prove that the following equation holds:

$$\begin{array}{ccc}
\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}), \underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) & \xrightarrow{\mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}}} & \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}) \\
{\scriptstyle -^{\mathsf{mc}},-^{\mathsf{mc}}}\big\downarrow & = & \big\downarrow{\scriptstyle -^{\mathsf{mc}}} \\
\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}), \underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}}) & \xrightarrow{\mu^{\mathsf{A}_\infty}_{\mathbf{1}\to\mathbf{1}}} & \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}})
\end{array} \tag{11.26.1}$$

By Section 9.20 the multifunctor $-^{\mathsf{Mc}} : \mathsf{A}_\infty \to \mathsf{A}_\infty$ together with the natural transformation $u_{\mathsf{Mc}} : \mathrm{Id} \to -^{\mathsf{Mc}}$ produces an A_∞-functor $\mathsf{Mc}' : \underline{\mathsf{A}_\infty} \to \underline{\mathsf{A}_\infty}$. It is related to $-^{\mathsf{mc}}$ by the following equation

$$\begin{tikzcd} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}) \arrow[r, "\mathsf{Mc}'"] \arrow[d, "\_^{\mathsf{mc}}"'] \arrow[dr, phantom, "="] & \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{Mc}};\mathcal{B}^{\mathsf{Mc}}) \arrow[d, "\underline{\mathsf{A}_\infty}(\iota;1)"] \\ \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}) \arrow[r, hook, "\underline{\mathsf{A}_\infty}(1;\iota)"] & \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{Mc}}) \end{tikzcd}$$

where ι denotes the embedding $\mathcal{C}^{\mathsf{mc}} \hookrightarrow \mathcal{C}^{\mathsf{Mc}}$. This implies that

$$\begin{aligned}
&\big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{_^{\mathsf{mc}},_^{\mathsf{mc}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}}) \\
&\qquad \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}}) \xhookrightarrow{\underline{\mathsf{A}_\infty}(1;\iota)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{_^{\mathsf{mc}},_^{\mathsf{mc}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}}) \\
&\qquad \xrightarrow{1,\underline{\mathsf{A}_\infty}(1;\iota)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{_^{\mathsf{mc}},\mathsf{Mc}'} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \\
&\qquad \xrightarrow{1,\underline{\mathsf{A}_\infty}(\iota;1)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{_^{\mathsf{mc}},\mathsf{Mc}'} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \\
&\qquad \xrightarrow{\underline{\mathsf{A}_\infty}(1;\iota),1} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{Mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{\mathsf{Mc}',\mathsf{Mc}'} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{Mc}};\mathcal{B}^{\mathsf{Mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \\
&\qquad \xrightarrow{\underline{\mathsf{A}_\infty}(\iota;1),1} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{B}^{\mathsf{Mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{\mathsf{Mc}',\mathsf{Mc}'} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{Mc}};\mathcal{B}^{\mathsf{Mc}}),\underline{\mathsf{A}_\infty}(\mathcal{B}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \\
&\qquad \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\underline{\mathsf{A}_\infty}(\iota;1)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}) \xrightarrow{\mathsf{Mc}'} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{Mc}};\mathcal{C}^{\mathsf{Mc}}) \xrightarrow{\underline{\mathsf{A}_\infty}(\iota;1)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big] \\
&= \big[\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B}),\underline{\mathsf{A}_\infty}(\mathcal{B};\mathcal{C}) \xrightarrow{\mu^{\underline{\mathsf{A}_\infty}}_{\mathbf{1}\to\mathbf{1}}} \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}) \xrightarrow{_^{\mathsf{mc}}} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{mc}}) \xhookrightarrow{\underline{\mathsf{A}_\infty}(1;\iota)} \underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathsf{mc}};\mathcal{C}^{\mathsf{Mc}})\big].
\end{aligned}$$

Since $\underline{\mathsf{A}_\infty}(1;\iota)$ is an embedding, equation (11.26.1) follows. □

11.27 Unit for the Maurer–Cartan monad. Denote by $u_{\mathsf{mc}} = u^{\mathcal{A}}_{\mathsf{mc}} : \mathcal{A} \to \mathcal{A}^{\mathsf{mc}}$ the strict A_∞-functor

$$\operatorname{Ob}\mathcal{A} \xrightarrow{\sim} \operatorname{Ob} A_\infty(\mathbf{1},\mathcal{A}) \hookrightarrow \operatorname{Ob}\mathcal{A}^{\mathsf{mc}},$$

$X \mapsto (\mathbf{1} \ni 1 \mapsto X, 0)$, with the first component given by $(u_{\mathsf{mc}})_1 = \mathrm{id} : s\mathcal{A}(X,Y) \to s\mathcal{A}^{\mathsf{mc}}((1 \mapsto X, 0), (1 \mapsto Y, 0)) = s\mathcal{A}(X,Y)$. We claim that the collection of $u^{\mathcal{A}}_{\mathsf{mc}}$ defines a strict A_∞-2-transformation $u_{\mathsf{mc}} : \mathrm{Id} \to (-)^{\mathsf{mc}}$.

1) If $\mathcal{A}$ is unital, then A_∞-functor $u^{\mathcal{A}}_{\mathsf{mc}}$ is unital. Indeed, (11.21.3) implies $(r^1 \otimes \cdots \otimes r^n)(\mathbf{i}^{\mathcal{A}})^{\mathsf{mc}}_n = (r^1 \otimes \cdots \otimes r^n)\mathbf{i}^{\mathcal{A}}_n$, that is, $(\mathbf{i}^{\mathcal{A}})^{\mathsf{mc}}_n = \mathbf{i}^{\mathcal{A}}_n : (s\mathcal{A})^{\otimes n}(X,Y) \to s\mathcal{A}(X,Y)$. In other terms, $u^{\mathcal{A}}_{\mathsf{mc}}\mathbf{i}^{(\mathcal{A}^{\mathsf{mc}})} = \mathbf{i}^{\mathcal{A}}u^{\mathcal{A}}_{\mathsf{mc}} : u^{\mathcal{A}}_{\mathsf{mc}} \to u^{\mathcal{A}}_{\mathsf{mc}} : \mathcal{A} \to \mathcal{A}^{\mathsf{mc}}$. Therefore, $u^{\mathcal{A}}_{\mathsf{mc}}$ is unital.

2) We claim that the following equation holds:

$$(1 \boxtimes u^{\mathcal{B}}_{\mathsf{mc}})M = \bigl(A_\infty(\mathcal{A},\mathcal{B}) \xrightarrow{-^{\mathsf{mc}}} A_\infty(\mathcal{A}^{\mathsf{mc}},\mathcal{B}^{\mathsf{mc}}) \xrightarrow{(u^{\mathcal{A}}_{\mathsf{mc}}\boxtimes 1)M} A_\infty(\mathcal{A},\mathcal{B}^{\mathsf{mc}})\bigr). \tag{11.27.1}$$

Indeed, on objects we have the equation $fu^{\mathcal{B}}_{\mathsf{mc}} = u^{\mathcal{A}}_{\mathsf{mc}}f^{\mathsf{mc}}$, since both sides map an object $X \in \mathrm{Ob}\,\mathcal{A}$ to the object $(1 \mapsto Xf, 0)$ of $\mathcal{B}^{\mathsf{mc}}$, and $f^{\mathsf{mc}}_n = f_n : s\mathcal{A}^{\otimes n} \to s\mathcal{B}^{\mathsf{mc}}$ due to formula (11.20.3). Furthermore, both sides of equation (11.27.1) consist of strict A_∞-functors. It remains to verify for each A_∞-transformation $p : f \to g : \mathcal{A} \to \mathcal{B}$ the equation

$$pu^{\mathcal{B}}_{\mathsf{mc}} = u^{\mathcal{A}}_{\mathsf{mc}}p^{\mathsf{mc}} : fu^{\mathcal{B}}_{\mathsf{mc}} \to gu^{\mathcal{B}}_{\mathsf{mc}} : \mathcal{A} \to \mathcal{B}^{\mathsf{mc}}.$$

Its equivalent form $p_n = p^{\mathsf{mc}}_n : T^n s\mathcal{A} \to s\mathcal{B}^{\mathsf{mc}}$ holds true due to equation (11.21.3). Therefore, $u_{\mathsf{mc}} : \mathrm{Id} \to (-)^{\mathsf{mc}} : A_\infty \to A_\infty$ is a strict A_∞-2-transformation, whose restriction $u_{\mathsf{mc}} : \mathrm{Id} \to (-)^{\mathsf{mc}} : A^u_\infty \to A^u_\infty$ is a strict A^u_∞-2-transformation.

11.28 The A_∞-2-transformation Tot**.** Let $\mathcal{A}$ be an A_∞-category. We want to define a strict A_∞-functor $m^{\mathcal{A}}_{\mathsf{mc}} = \mathrm{Tot}_{\mathcal{A}} : (\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}$. Objects of $(\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}}$ are A_∞-functors $X : I \to \mathcal{A}^{\mathsf{mc}}$, $i \mapsto (X^i : J^i \to \mathcal{A})$, $x^{ii'} \in [s\mathcal{A}^{\mathsf{mc}}(X^i, X^{i'})]^0 \simeq [sA_\infty(K,\mathcal{A}^*)(X^{i*}, X^{i'*})]^0$ for $i, i' \in I$, $i < i'$. Here we use the partition $K = \sqcup_{i\in I} J^i = J^0 \sqcup \cdots \sqcup J^n$. We claim that A_∞-functors X with fixed sets I, $(J^i)_{i\in I}$ are in bijection with A_∞-functors $\tilde{X} : K \to \mathcal{A}$. This defines a map $\mathrm{Ob}\,\mathrm{Tot}_{\mathcal{A}} : \mathrm{Ob}(\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}} \to \mathrm{Ob}\,\mathcal{A}^{\mathsf{mc}}$.

To prove this claim we fix $(I, (J^i)_{i\in I})$ and consider two sets:

$$S = \{(I \ni i \mapsto X^i : J^i \to \mathcal{A}, x^{ii'} \in [s\mathcal{A}^{\mathsf{mc}}(X^i, X^{i'})]^0)_{i,i'\in I} \mid i \geqslant i' \implies x^{ii'} = 0, \\ \text{and } \sum_{t>0} x^{\otimes t} b^{\mathsf{mc}}_t = 0 \text{ for } x = (x^{ii'})_{i,i'\in I}\},$$

$$\tilde{S} = \{\tilde{X} : K \to \mathcal{A} - A_\infty\text{-functor }\}.$$

Let us transform the equation which determines S. The A_∞-functor $X^i : J^i \to \mathcal{A}$ consists of $\mathrm{Ob}\,X^i : J^i \ni j \mapsto X^i_j \in \mathrm{Ob}\,\mathcal{A}$ and of $x^{ii} \overset{\text{def}}{=} x^i = (x^i_{jj'})_{j,j'\in J^i}$. We have an equation between matrices of size $|J^i| \times |J^{i'}|$:

$$\sum_{t>0} (x^{\otimes t} b^{\mathsf{mc}}_t)^{ii'} = \sum_{i=i_0<i_1<\cdots<i_p=i'}^{t_0,\dots,t_p\geqslant 0} ((x^{i_0})^{\otimes t_0} \otimes x^{i_0i_1} \otimes (x^{i_1})^{\otimes t_1} \otimes \cdots \otimes x^{i_{p-1}i_p} \otimes (x^{i_p})^{\otimes t_p}) b_{t_0+\cdots+t_p+p}$$

$$= \sum_{i=l_0\leqslant l_1\leqslant\cdots\leqslant l_{q-1}\leqslant l_q=i'} (x^{l_0l_1} \otimes x^{l_1l_2} \otimes \cdots \otimes x^{l_{q-1}l_q}) b_q.$$

A map $S \to \tilde{S}$, $X \mapsto \tilde{X}$ is given by the assignment $J^i \ni j \mapsto X^i_j$, $\tilde{x}^{ii'}_{jj'} = x^{ii'}_{jj'}$, where $j \in J^i$, $j' \in J^{i'}$, $i \leqslant i'$. For this choice of $\tilde{x}$

$$\sum_{t>0} (x^{\otimes t} b^{\mathsf{mc}}_t)^{ii'}_{jj'} = \sum_{q>0} (\tilde{x}^{\otimes q} b_q)^{ii'}_{jj'},$$

hence, both sides vanish simultaneously. The inverse map $\tilde{S} \to S$, $\tilde{X} \mapsto X$ is given by restriction: $\operatorname{Ob} X^i = \operatorname{Ob} \tilde{X}\big|_{J^i}$, $x^i_{jj'} = \tilde{x}^{ii}_{jj'}$, $j, j' \in J^i$, $x^{ii'} = (\tilde{x}^{ii'}_{jj'})^{j\in J^i}_{j'\in J^{i'}}$ for $i < i'$. This is the bijection between the discussed sets of objects, which defines $\operatorname{Ob} \operatorname{Tot}_{\mathcal{A}}$.

Given two objects of $(\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}}$, $X : I \to \mathcal{A}^{\mathsf{mc}}$, $i \mapsto (X^i : J^i \to \mathcal{A}, j \mapsto X^i_j)$, and $Y : L \to \mathcal{A}^{\mathsf{mc}}$, $l \mapsto (Y^l : M^l \to \mathcal{A}, m \mapsto Y^l_m)$, we describe the $\Bbbk$-module of morphisms between them:

$$\begin{aligned}(\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}}(X, Y) = \prod_{i\in I,\, l\in L} \mathcal{A}^{\mathsf{mc}}(X^i, Y^l) = \prod_{i\in I,\, l\in L}\ \prod_{j\in J^i,\, m\in M^l} \mathcal{A}(X^i_j, Y^l_m) \\ \simeq \prod_{k\in K,\, n\in N} \mathcal{A}(\tilde{X}^k, \tilde{Y}^n) = \mathcal{A}^{\mathsf{mc}}(\tilde{X}, \tilde{Y}),\end{aligned}$$

where $K = \sqcup_{i\in I} J^i$, $N = \sqcup_{l\in L} M^l$ and $\tilde{X} = X \operatorname{Tot}$, $\tilde{Y} = Y \operatorname{Tot}$.

We make $\operatorname{Tot}_{\mathcal{A}}$ into a strict A_∞-functor setting $(\operatorname{Tot}_{\mathcal{A}})_1$ to be the above isomorphism and $(\operatorname{Tot}_{\mathcal{A}})_k = 0$, $k > 1$. We must verify the equations

$$((\operatorname{Tot}_{\mathcal{A}})_1 \otimes \dots \otimes (\operatorname{Tot}_{\mathcal{A}})_1) b^{\mathsf{mc}}_k = b^{\mathsf{mc}\,\mathsf{mc}}_k (\operatorname{Tot}_{\mathcal{A}})_1. \tag{11.28.1}$$

Since $(\operatorname{Tot}_{\mathcal{A}})_1$ is an isomorphism it is sufficient to check that $b^{\mathsf{mc}\,\mathsf{mc}}_k$ and b^{mc}_k coincide under the natural identification.

Let ${}^pX : {}^pI \to \mathcal{A}^{\mathsf{mc}}$, $0 \leqslant p \leqslant k$, be A_∞-functors, that is, objects of $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}$. They can be described as functions ${}^pI \ni i \mapsto {}^pX^i : {}^pJ^i \to \mathcal{A}$, $j \mapsto {}^pX^i_j$, together with a matrix ${}^px = ({}^px^{ii'}_{jj'})^{i,i'\in {}^pI}_{j\in {}^pJ^i, j'\in {}^pJ^{i'}}$. Let ${}^pr = ({}^pr^{ii'}_{jj'})^{i\in {}^{p-1}I, i'\in {}^pI}_{j\in {}^{p-1}J^i, j'\in {}^pJ^{i'}}$. Then, in obvious notation

$$\begin{aligned}({}^1r \otimes \dots \otimes {}^kr) b^{\mathsf{mc}\,\mathsf{mc}}_k = \sum ({}^0x^{\bullet<\bullet\otimes l_0} \otimes {}^1r^{\bullet\bullet} \otimes {}^1x^{\bullet<\bullet\otimes l_1} \otimes \dots \otimes {}^kr^{\bullet\bullet} \otimes {}^kx^{\bullet<\bullet\otimes l_k}) b^{\mathsf{mc}}_{l_0+\dots+l_k+k} \\ = \sum ({}^0x^{\bullet=\bullet\otimes n^0_0} \otimes {}^0x^{\bullet<\bullet} \otimes {}^0x^{\bullet=\bullet\otimes n^1_0} \otimes \dots \otimes {}^0x^{\bullet<\bullet} \otimes {}^0x^{\bullet=\bullet\otimes n^{l_0}_0} \otimes {}^1r^{\bullet\bullet} \otimes {}^1x^{\bullet=\bullet\otimes n^0_1} \otimes \dots) b_{\sum n^q_p + \sum l_p + k} \\ = \sum ({}^0x^{\bullet\leqslant\bullet\otimes l_0} \otimes {}^1r^{\bullet\bullet} \otimes {}^1x^{\bullet\leqslant\bullet\otimes l_1} \otimes \dots \otimes {}^kr^{\bullet\bullet} \otimes {}^kx^{\bullet\leqslant\bullet\otimes l_k}) b^{\mathsf{mc}}_{l_0+\dots+l_k+k} \\ = \sum ({}^0x^{\bullet\bullet\otimes l_0} \otimes {}^1r^{\bullet\bullet} \otimes {}^1x^{\bullet\bullet\otimes l_1} \otimes \dots \otimes {}^kr^{\bullet\bullet} \otimes {}^kx^{\bullet\bullet\otimes l_k}) b^{\mathsf{mc}}_{l_0+\dots+l_k+k},\end{aligned} \tag{11.28.2}$$

where $\otimes$ means the matrix tensor product $(A \otimes B)_{jj''} = \sum_{j'} A_{jj'} \otimes B_{j'j''}$. Denoting by $\odot$ the matrix tensor product $(A \odot B)^{ii''}_{jj''} = \sum_{i',j'} A^{ii'}_{jj'} \otimes B^{i'i''}_{j'j''}$, we get

$$({}^1r \otimes \dots \otimes {}^kr) b^{\mathsf{mc}}_k = \sum ({}^0x^{\odot l_0} \odot {}^1r \odot {}^1x^{\odot l_1} \odot \dots \odot {}^kr \odot {}^kx^{\odot l_k}) b^{\mathsf{mc}}_{l_0+\dots+l_k+k}. \tag{11.28.3}$$

This coincides with the above expression, hence, (11.28.1) is proven. Hence $\mathrm{Tot}_{\mathcal{A}}$ is a strict A_∞-functor.

When $\mathcal{A}$ runs over all A_∞-categories, the collection of functors $\mathrm{Tot}_{\mathcal{A}}$ determines a natural transformation $\mathrm{Tot} : ((-)^{\mathsf{mc}})^{\mathsf{mc}} \to (-)^{\mathsf{mc}}$. We must show that for any A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ the diagram commutes:

$$\begin{tikzcd}
(\mathcal{A}^{\mathsf{mc}})^{\mathsf{mc}} \arrow[r, "\mathrm{Tot}_{\mathcal{A}}"] \arrow[d, "(f^{\mathsf{mc}})^{\mathsf{mc}}"'] & \mathcal{A}^{\mathsf{mc}} \arrow[d, "f^{\mathsf{mc}}"] \\
(\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}} \arrow[r, "\mathrm{Tot}_{\mathcal{B}}"] & \mathcal{B}^{\mathsf{mc}}
\end{tikzcd}$$

Since $\mathrm{Tot}_{\mathcal{A}}$ and $\mathrm{Tot}_{\mathcal{B}}$ are strict functors, it is sufficient to establish the equation

$$((\mathrm{Tot}_{\mathcal{A}})_1 \otimes \cdots \otimes (\mathrm{Tot}_{\mathcal{A}})_1) f_k^{\mathsf{mc}} = f_k^{\mathsf{mc}\,\mathsf{mc}} (\mathrm{Tot}_{\mathcal{A}})_1.$$

The verification is quite similar to that made for b (replace b with f in equations (11.28.2), (11.28.3)).

Let us show that Tot is an A_∞-2-transformation. We have already verified equation

$$\begin{tikzcd}
A_\infty(\mathcal{A}, \mathcal{B}) \arrow[r, "_^{\mathsf{mc}\,\mathsf{mc}}"] \arrow[d, "_^{\mathsf{mc}}"'] \arrow[dr, phantom, "="] & A_\infty(\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}\,\mathsf{mc}}) \arrow[d, "(1\boxtimes \mathrm{Tot}_{\mathcal{B}})M"] \\
A_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}}) \arrow[r, "(\mathrm{Tot}_{\mathcal{A}} \boxtimes 1)M"] & A_\infty(\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})
\end{tikzcd}$$

on objects. Let us do it on morphisms. All arrows in the above diagram are strict A_∞-functors, so we have to consider only the first components. Let $p : f \to g : \mathcal{A} \to \mathcal{B}$ be an A_∞-transformation. We have to check that

$$\mathrm{Tot}_{\mathcal{A}}\, p^{\mathsf{mc}} = p^{\mathsf{mc}\,\mathsf{mc}}\, \mathrm{Tot}_{\mathcal{B}} : \mathrm{Tot}_{\mathcal{B}}\, f^{\mathsf{mc}} \to \mathrm{Tot}_{\mathcal{B}}\, g^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{B}^{\mathsf{mc}}, \tag{11.28.4}$$

that is,

$$(\mathrm{Tot}_{\mathcal{A}\,1})^{\otimes k} p_k^{\mathsf{mc}} = p_k^{\mathsf{mc}\,\mathsf{mc}}\, \mathrm{Tot}_{\mathcal{B}\,1} : T^k s(\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}) \to s\mathcal{B}^{\mathsf{mc}}.$$

Considering isomorphisms $\mathrm{Tot}_{\mathcal{A}\,1}$ and $\mathrm{Tot}_{\mathcal{B}\,1}$ as natural identifications we have to verify that p_k^{mc} and $p_k^{\mathsf{mc}\,\mathsf{mc}}$ coincide. Using definition (11.21.3) of p_k^{mc} we prove the general case of p similarly to the particular case $p = b : \mathrm{id} \to \mathrm{id} : \mathcal{A} \to \mathcal{A}$ (replace b with p in equations (11.28.2), (11.28.3)). Therefore, Tot is a strict A_∞-2-transformation.

If $\mathcal{A}$ is unital, then the A_∞-functor $\mathrm{Tot}_{\mathcal{A}} : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}$ is unital. Indeed, (11.28.4) implies that

$$\mathrm{Tot}_{\mathcal{A}}\, \mathbf{i}^{(\mathcal{A}^{\mathsf{mc}})} = \mathrm{Tot}_{\mathcal{A}}\, \mathbf{i}^{\mathsf{mc}} = \mathbf{i}^{\mathsf{mc}\,\mathsf{mc}}\, \mathrm{Tot}_{\mathcal{A}} = \mathbf{i}^{(\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}})}\, \mathrm{Tot}_{\mathcal{A}} : \mathrm{Tot}_{\mathcal{A}} \to \mathrm{Tot}_{\mathcal{A}} : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}. \tag{11.28.5}$$

Therefore, restriction of Tot to unital A_∞-categories and functors is a strict A_∞^u-2-transformation.

11.29 The A_∞-2-monad $(-^{\mathsf{mc}}, \mathrm{Tot}, u_{\mathsf{mc}})$. Now we claim that $(-^{\mathsf{mc}}, \mathrm{Tot}, u_{\mathsf{mc}})$ is an A_∞-2-monad. Indeed, equations

$$\bigl(\mathcal{A}^{\mathsf{mc}} \xrightarrow{u_{\mathsf{mc}}^{\mathcal{A}^{\mathsf{mc}}}} \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \xrightarrow{\mathrm{Tot}_{\mathcal{A}}} \mathcal{A}^{\mathsf{mc}}\bigr) = \mathrm{id},$$
$$\bigl(\mathcal{A}^{\mathsf{mc}} \xrightarrow{(u_{\mathsf{mc}}^{\mathcal{A}})^{\mathsf{mc}}} \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \xrightarrow{\mathrm{Tot}_{\mathcal{A}}} \mathcal{A}^{\mathsf{mc}}\bigr) = \mathrm{id},$$

$$\begin{tikzcd}
\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}\,\mathsf{mc}} \arrow[r, "\mathrm{Tot}_{\mathcal{A}^{\mathsf{mc}}}"] \arrow[d, "(\mathrm{Tot}_{\mathcal{A}})^{\mathsf{mc}}"'] \arrow[dr, phantom, "="] & \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \arrow[d, "\mathrm{Tot}_{\mathcal{A}}"] \\
\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \arrow[r, "\mathrm{Tot}_{\mathcal{A}}"] & \mathcal{A}^{\mathsf{mc}}
\end{tikzcd} \tag{11.29.1}$$

are equations between strict A_∞-functors. The left hand sides of the first two equations give on objects the maps

$$(X : J \to \mathcal{A}) \xmapsto{u_{\mathsf{mc}}^{\mathcal{A}^{\mathsf{mc}}}} (1 \mapsto (X : J \to \mathcal{A}), 0) \xmapsto{\mathrm{Tot}_{\mathcal{A}}} (X : J \to \mathcal{A})$$

due to $\sqcup_{i\in\mathbf{1}} J = J$, and

$$(X : I \to \mathcal{A}) \xmapsto{(u_{\mathsf{mc}}^{\mathcal{A}})^{\mathsf{mc}}} (I \to \mathcal{A}^{\mathsf{mc}}, i \mapsto (1 \mapsto X^i, 0)) \xmapsto{\mathrm{Tot}_{\mathcal{A}}} (X : I \to \mathcal{A})$$

due to $\sqcup_{i\in I}\mathbf{1} = I$. Thus, both maps are identity maps.

Equation (11.29.1) on objects follows from the fact that in the category of partitions $\mathcal{P}$ the ordered sets $\sqcup_{i\in I} \sqcup_{j\in J^i} K^j$ and $\sqcup_{j\in\sqcup_{i\in I} J^i} K^j$ are not only isomorphic, but equal. Equations between first components on morphisms follow from the fact that $(u_{\mathsf{mc}})_1 = \mathrm{id}$ and Tot_1 are natural identifications. Thus an A_∞-2-monad $(-^{\mathsf{mc}}, \mathrm{Tot}, u_{\mathsf{mc}})$ is constructed. Its restriction to unital A_∞-categories and unital A_∞-functors is an A_∞^u-2-monad.

11.30 Remark. Let $g : \mathcal{B} \to \mathcal{A}$ be a contractible unital A_∞-functor. Then $g^{\mathsf{mc}} : \mathcal{B}^{\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}$ is contractible as well. Indeed, by criterion of [LO06] (propositions 6.1(C1), 6.3) the complexes $s\mathcal{A}(Ug, Vg)$ are contractible for all $U, V \in \operatorname{Ob}\mathcal{B}$. Hence, for all $X, Y \in \operatorname{Ob}\mathcal{B}^{\mathsf{mc}}$ the complex $s\mathcal{A}^{\mathsf{mc}}(Xg^{\mathsf{mc}}, Yg^{\mathsf{mc}}) = \prod_{i\in I, j\in J} s\mathcal{A}(X_i g, Y_j g)$ is contractible. By [LO06, Proposition 6.1(C3)] the functor g^{mc} is contractible.

11.31 Definition. We say that a unital A_∞-category $\mathcal{C}$ is mc-*closed* if every object X of $\mathcal{C}^{\mathsf{mc}}$ is isomorphic in $\mathcal{C}^{\mathsf{mc}}$ to Yu_{mc} for some object $Y \in \operatorname{Ob}\mathcal{C}$.

If $\mathcal{C}$ is mc-closed, then the zero object $\mathbf{0} \to \mathcal{C}$ of $\mathcal{C}^{\mathsf{mc}}$ is isomorphic in $\mathcal{C}^{\mathsf{mc}}$ to $\mathcal{O}u_{\mathsf{mc}}$ for some object $\mathcal{O} \in \operatorname{Ob}\mathcal{C}$. Therefore, the only morphisms $0 \in s\mathcal{C}^{\mathsf{mc}}(\mathbf{0} \to \mathcal{C}, \mathcal{O}) = 0$ and $0 \in s\mathcal{C}^{\mathsf{mc}}(\mathcal{O}, \mathbf{0} \to \mathcal{C}) = 0$ are inverse to each other. This means precisely that

(c1) the unit element ${}_{\mathcal{O}}\mathbf{i}_0^{\mathcal{C}} \in s\mathcal{C}(\mathcal{O}, \mathcal{O})$ is the boundary vb_1 for some $v \in \mathcal{C}(\mathcal{O}, \mathcal{O})[1]^{-2}$.

This condition is equivalent to any of the following three:

(c2) the complex $(s\mathcal{C}(\mathcal{O},\mathcal{O}), b_1)$ is acyclic;

(c3) the complex $(s\mathcal{C}(\mathcal{O},\mathcal{O}), b_1)$ is contractible;

(c4) for any $X \in \operatorname{Ob}\mathcal{C}$ the complexes $(s\mathcal{C}(\mathcal{O},X), b_1)$ and $(s\mathcal{C}(X,\mathcal{O}), b_1)$ are contractible.

Their equivalence is shown in [LO06, Proposition 6.1]. An object $\mathcal{O}$ of a unital A_∞-category $\mathcal{C}$ which satisfies equivalent conditions (c1)–(c4) is called *contractible*. Contractible objects play the rôle similar to that of zero (initial and final) objects in ordinary categories. A unital A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ is called *contractible* if equivalent conditions (C1)–(C11) of [LO06, Section 6] hold. One of them says that f is a contractible object of $A_\infty(\mathcal{A},\mathcal{B})$ in the sense of (c1).

11.32 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ contains a contractible object, and each object $\bigl(W : \mathbf{2} \to \mathcal{C}, \left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right)\bigr)$ of $\mathcal{C}^{\mathsf{mc}}$ is isomorphic in $\mathcal{C}^{\mathsf{mc}}$ to Cu_{mc} for some object $C \in \operatorname{Ob}\mathcal{C}$;

(ii) $\mathcal{C}$ is mc-closed;

(iii) the A_∞-functor $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ is an equivalence.

Proof. (ii) $\Longrightarrow$ (i) is obvious.

(i) $\Longrightarrow$ (ii): If $|K| = 0,1,2$, then an object $Z : K \to \mathcal{C}$ of $\mathcal{C}^{\mathsf{mc}}$ is isomorphic in $\mathcal{C}^{\mathsf{mc}}$ to Cu_{mc} for some object $C \in \operatorname{Ob}\mathcal{C}$ by condition (i). We proceed by induction on $n = |K|$. Let $Z : K \to \mathcal{C}$ be an object of $\mathcal{C}^{\mathsf{mc}}$ with $|K| > 1$. Then K can be presented as a disjoint union $K = I \sqcup J$ with $|I|, |J| < |K|$. Define the objects $X = \bigl(I \hookrightarrow K \xrightarrow{Z} \mathcal{C}\bigr)$, $Y = \bigl(J \hookrightarrow K \xrightarrow{Z} \mathcal{C}\bigr)$ of $\mathcal{C}^{\mathsf{mc}}$ as compositions of full embeddings with the A_∞-functor Z. Then $X_i = Z_i$, $x_{ii'} = z_{ii'}$ for $i, i' \in I$, and $Y_j = Z_j$, $y_{jj'} = z_{jj'}$ for $j, j' \in J$. Denote by f the matrix $(z_{ij})_{i\in I}^{j\in J}$, $z_{ij} \in \mathcal{C}(X_i, Y_j)[1]^0$. The object Z is determined by X, Y and f, since $z = (z_{kk'})_{k,k'\in K} = \left(\begin{smallmatrix} x & f \\ 0 & y \end{smallmatrix}\right)$. The matrix z has to satisfy the Maurer–Cartan equation:

$$
\begin{aligned}
0 = \sum_{t>0} (z^{\otimes t}) b_t^{\mathcal{C}} = \sum_{t>0} \begin{pmatrix} x & f \\ 0 & y \end{pmatrix}^{\otimes t} b_t \\
= \begin{pmatrix} \sum_{t>0} x^{\otimes t} b_t & \sum_{a,c\geqslant 0} (x^{\otimes a} \otimes f \otimes y^{\otimes c}) b_{a+1+c} \\ 0 & \sum_{t>0} y^{\otimes t} b_t \end{pmatrix} = \begin{pmatrix} 0 & f b_1^{\mathsf{mc}} \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

We see that given objects X, Y of $\mathcal{C}^{\mathsf{mc}}$ and an element $f \in \mathcal{C}^{\mathsf{mc}}(X,Y)[1]^0$ determine such object Z if and only if f is a cycle.

By induction assumption there are objects $\tilde{U}$, $\tilde{V}$ of $\mathcal{C}$ and the corresponding objects $U = \tilde{U}u_{\mathsf{mc}} = (\mathbf{1} \to \mathcal{C}, 1 \mapsto \tilde{U}, 0)$, $V = \tilde{V}u_{\mathsf{mc}} = (\mathbf{1} \to \mathcal{C}, 1 \mapsto \tilde{V}, 0)$, there are cycles $\alpha \in \mathcal{C}^{\mathsf{mc}}(X,U)[1]^{-1}$, $\alpha' \in \mathcal{C}^{\mathsf{mc}}(U,X)[1]^{-1}$ and $\beta \in \mathcal{C}^{\mathsf{mc}}(Y,V)[1]^{-1}$, $\beta' \in \mathcal{C}^{\mathsf{mc}}(V,Y)[1]^{-1}$, pairwise mutually inverse to each other. This means that morphisms

$$\begin{aligned} \alpha s^{-1} &: \Bbbk \to \mathcal{C}^{\mathsf{mc}}(X,U)^0, & \alpha' s^{-1} &: \Bbbk \to \mathcal{C}^{\mathsf{mc}}(U,X)^0, \\ \beta s^{-1} &: \Bbbk \to \mathcal{C}^{\mathsf{mc}}(Y,V)^0, & \beta' s^{-1} &: \Bbbk \to \mathcal{C}^{\mathsf{mc}}(V,Y)^0 \end{aligned}$$

define mutually inverse elements in $\mathcal{K}$, that is, $(\alpha' s^{-1} \otimes \alpha s^{-1})m_2^{\mathsf{mc}} - 1_U \in \operatorname{Im} m_1^{\mathsf{mc}}$ etc. Since αs^{-1} is invertible, there is a unique element $gs^{-1} : \Bbbk \to \mathcal{C}^{\mathsf{mc}}(U,V)^1$ in $\mathcal{K}$ such that equation

$$\big(X \xrightarrow{fs^{-1}} Y \xrightarrow{\beta s^{-1}} V\big) = \big(X \xrightarrow{\alpha s^{-1}} U \xrightarrow{gs^{-1}} V\big) \tag{11.32.1}$$

holds in $\mathsf{k}\mathcal{C}^{\mathsf{mc}}$, namely, $gs^{-1} = (\alpha' s^{-1})\cdot(fs^{-1})\cdot(\beta s^{-1})$. Lift gs^{-1} to a chain map $gs^{-1} : \Bbbk \to \mathcal{C}^{\mathsf{mc}}(U,V)^1$, and consider the corresponding cycle $g \in \mathcal{C}^{\mathsf{mc}}(U,V)[1]^0$. Equation (11.32.1) means that

$$(fs^{-1} \otimes \beta s^{-1})m_2^{\mathsf{mc}} - (\alpha s^{-1} \otimes gs^{-1})m_2^{\mathsf{mc}} \in \operatorname{Im} m_1^{\mathsf{mc}}.$$

Therefore, there exists an element $\eta \in \mathcal{C}^{\mathsf{mc}}(X,V)[1]^{-1}$ such that

$$(f \otimes \beta)b_2^{\mathsf{mc}} + (\alpha \otimes g)b_2^{\mathsf{mc}} + \eta b_1^{\mathsf{mc}} = 0.$$

Due to g being a cycle, we may define an object $\big(W : \mathbf{2} \to \mathcal{C}, 1 \mapsto \tilde{U}, 2 \mapsto \tilde{V}, \left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right)\big)$ of $\mathcal{C}^{\mathsf{mc}}$. Consider an element $\gamma \in \mathcal{C}^{\mathsf{mc}}(Z,W)[1]^{-1}$, represented by the matrix $\gamma = \left(\begin{smallmatrix} \alpha & \eta \\ 0 & \beta \end{smallmatrix}\right)$. We claim that it is a cycle. We shall prove it in more general assumptions: $(W : I' \sqcup J' \to \mathcal{C}, w)$ is determined by $(U : I' \to \mathcal{C}, u)$, $(V : J' \to \mathcal{C}, v)$ and a cycle $g \in \mathcal{C}^{\mathsf{mc}}(U,V)[1]^0$. Thus, $w = \left(\begin{smallmatrix} u & g \\ 0 & v \end{smallmatrix}\right)$ instead of $\left(\begin{smallmatrix} 0 & g \\ 0 & 0 \end{smallmatrix}\right)$. We have

$$\begin{aligned} \gamma b_1^{\mathsf{mc}} &= \sum_{k,m\geqslant 0} \left[\begin{pmatrix} x & f \\ 0 & y \end{pmatrix}^{\otimes k} \otimes \begin{pmatrix} \alpha & \eta \\ 0 & \beta \end{pmatrix} \otimes \begin{pmatrix} u & g \\ 0 & v \end{pmatrix}^{\otimes m} \right] b_{k+1+m} \\ &= \sum_{k,m\geqslant 0} \left[\begin{pmatrix} x^{\otimes k} & \sum_{n+1+l=k} x^{\otimes n} \otimes f \otimes y^{\otimes l} \\ 0 & y^{\otimes k} \end{pmatrix} \otimes \begin{pmatrix} \alpha & \eta \\ 0 & \beta \end{pmatrix} \otimes \begin{pmatrix} u^{\otimes m} & \sum_{p+1+q=m} u^{\otimes p} \otimes g \otimes v^{\otimes q} \\ 0 & v^{\otimes m} \end{pmatrix} \right] b_{k+1+m} \\ &= \sum_{k,m\geqslant 0} \begin{pmatrix} x^{\otimes k} \otimes \alpha \otimes u^{\otimes m} & t_{k,m} \\ 0 & y^{\otimes k} \otimes \beta \otimes v^{\otimes m} \end{pmatrix} b_{k+1+m} = \begin{pmatrix} \alpha b_1^{\mathsf{mc}} & (f \otimes \beta)b_2^{\mathsf{mc}} + \eta b_1^{\mathsf{mc}} + (\alpha \otimes g)b_2^{\mathsf{mc}} \\ 0 & \beta b_1^{\mathsf{mc}} \end{pmatrix} \\ &= 0, \end{aligned} \tag{11.32.2}$$

where the matrix $t_{k,m}$ denotes the sum

$$\sum_{n+1+l=k} x^{\otimes n} \otimes f \otimes y^{\otimes l} \otimes \beta \otimes v^{\otimes m} + x^{\otimes k} \otimes \eta \otimes v^{\otimes m} + \sum_{p+1+q=m} x^{\otimes k} \otimes \alpha \otimes u^{\otimes p} \otimes g \otimes v^{\otimes q}.$$

We also have the equation

$$\bigl(U \xrightarrow{gs^{-1}} V \xrightarrow{\beta' s^{-1}} Y\bigr) = \bigl(U \xrightarrow{\alpha' s^{-1}} X \xrightarrow{fs^{-1}} Y\bigr)$$

in $\mathsf{k}\mathcal{C}^{\mathsf{mc}}$, which means that there is an element $\eta' \in \mathcal{C}^{\mathsf{mc}}(U,Y)[1]^{-1}$ such that

$$(g \otimes \beta')b_2^{\mathsf{mc}} + (\alpha' \otimes f)b_2^{\mathsf{mc}} + \eta' b_1^{\mathsf{mc}} = 0.$$

Exchanging the rôles of Z and W, f and g, α and α', β and β' in the above formulae we obtain that $\gamma' = \bigl(\begin{smallmatrix} \alpha' & \eta' \\ 0 & \beta' \end{smallmatrix}\bigr) \in \mathcal{C}^{\mathsf{mc}}(W,Z)[1]^{-1}$ is a cycle. Consider the element

$$\begin{aligned}
\omega = (\gamma \otimes \gamma')b_2^{\mathsf{mc}} &= \sum_{k,l,m \geqslant 0} \left[\begin{pmatrix} x & f \\ 0 & y \end{pmatrix}^{\otimes k} \otimes \begin{pmatrix} \alpha & \eta \\ 0 & \beta \end{pmatrix} \otimes \begin{pmatrix} u & g \\ 0 & v \end{pmatrix}^{\otimes l} \otimes \begin{pmatrix} \alpha' & \eta' \\ 0 & \beta' \end{pmatrix} \otimes \begin{pmatrix} x & f \\ 0 & y \end{pmatrix}^{\otimes m} \right] b_{k+l+m+2} \\
&= \begin{pmatrix} (\alpha \otimes \alpha')b_2^{\mathsf{mc}} & p \\ 0 & (\beta \otimes \beta')b_2^{\mathsf{mc}} \end{pmatrix} = \begin{pmatrix} {}_X(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} + r b_1^{\mathsf{mc}} & p \\ 0 & {}_Y(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} + q b_1^{\mathsf{mc}} \end{pmatrix} \\
&= {}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} + \begin{pmatrix} r & 0 \\ 0 & q \end{pmatrix} b_1^{\mathsf{mc}} + \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix} \in \mathcal{C}^{\mathsf{mc}}(Z,Z)[1]^{-1}
\end{aligned}$$

for some $n \in \mathcal{C}^{\mathsf{mc}}(X,Y)[1]^{-1}$, because formula (11.23.1) shows that ${}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} = \bigl(\begin{smallmatrix} {}_X(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} & * \\ 0 & {}_Y(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \end{smallmatrix}\bigr)$. Computation (11.32.2) proves that the necessary property $\bigl(\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\bigr) b_1^{\mathsf{mc}} = 0$ is equivalent to the condition $n b_1^{\mathsf{mc}} = 0$. Denote by a the chain map

$$a = \bigl[\bigl(\begin{smallmatrix} 0 & n \\ 0 & 0 \end{smallmatrix}\bigr) \otimes 1\bigr] b_2^{\mathsf{mc}} : s\mathcal{C}^{\mathsf{mc}}(Z,Z) \to s\mathcal{C}^{\mathsf{mc}}(Z,Z).$$

Matrix form of this expression shows that $\operatorname{Im} a \subset s\mathcal{C}^{\mathsf{mc}}(X,Z)$ and $a^2 = 0$. The chain map

$$-\bigl[\bigl({}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} + \bigl(\begin{smallmatrix} r & 0 \\ 0 & q \end{smallmatrix}\bigr) b_1^{\mathsf{mc}}\bigr) \otimes 1\bigr] b_2^{\mathsf{mc}} : s\mathcal{C}^{\mathsf{mc}}(Z,Z) \to s\mathcal{C}^{\mathsf{mc}}(Z,Z)$$

is homotopic to the identity map since $(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}}$ are unit elements of $\mathcal{C}^{\mathsf{mc}}$ by Proposition 11.23. Therefore, the chain map $-(\omega \otimes 1)b_2^{\mathsf{mc}} \sim 1 - a$ is homotopy invertible, since $(1-a)^{-1} = 1 + a$ exists. Thus, it is a homology isomorphism, and

$$-H^{-1}\bigl[(\omega \otimes 1)b_2^{\mathsf{mc}}\bigr] : H^{-1}\bigl(s\mathcal{C}^{\mathsf{mc}}(Z,Z)\bigr) \to H^{-1}\bigl(s\mathcal{C}^{\mathsf{mc}}(Z,Z)\bigr)$$

is invertible. In particular, there is a cycle $\sigma \in \mathcal{C}^{\mathsf{mc}}(Z,Z)[1]^{-1}$ such that $-\sigma(\omega \otimes 1)b_2^{\mathsf{mc}} - {}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \in \operatorname{Im} b_1^{\mathsf{mc}}$. That is, $[(\gamma \otimes \gamma')b_2^{\mathsf{mc}} \otimes \sigma]b_2^{\mathsf{mc}} - {}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \in \operatorname{Im} b_1^{\mathsf{mc}}$. Equivalently, $[\gamma \otimes (\gamma' \otimes \sigma)b_2^{\mathsf{mc}}]b_2^{\mathsf{mc}} - {}_Z(\mathbf{i}^{\mathcal{C}})_0^{\mathsf{mc}} \in \operatorname{Im} b_1^{\mathsf{mc}}$, hence, γ is invertible on the right. Similarly, γ is invertible on the left. We have established that $\gamma : Z \to W$ is an isomorphism in $\mathcal{C}^{\mathsf{mc}}$. The object $W : \mathbf{2} \to \mathcal{C}$ is isomorphic in $\mathcal{C}^{\mathsf{mc}}$ to Cu_{mc} for some object $C \in \operatorname{Ob} \mathcal{C}$, thus, $Z \simeq Cu_{\mathsf{mc}}$.

(ii) $\iff$ (iii): Since $u_{\mathsf{mc}1} = \mathrm{id}$ is invertible, the A_∞-functor $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ is an equivalence if and only if it is essentially surjective on objects [Lyu03, Theorem 8.8]. □

11.33 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$ be equivalent unital A_∞-categories. If one of them is* mc*-closed, then so is the other.*

Proof. The A^u_∞-2-functor $-^{\mathsf{mc}}$ projects to an ordinary strict 2-functor $\overline{-^{\mathsf{mc}}} : \overline{A^u_\infty} \to \overline{A^u_\infty}$ [LM06a, Section 3.2]. In particular, it takes A_∞-equivalences to A_∞-equivalences. Take an A_∞-equivalence $f : \mathcal{A} \to \mathcal{B}$, then $f^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{B}^{\mathsf{mc}}$ is A_∞-equivalence as well. In the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "u_{\mathsf{mc}}"] \arrow[d, "f"'] & \mathcal{A}^{\mathsf{mc}} \arrow[d, "f^{\mathsf{mc}}"] \\ \mathcal{B} \arrow[r, "u_{\mathsf{mc}}"] & \mathcal{B}^{\mathsf{mc}} \end{tikzcd}$$

three out of four A_∞-functors are equivalences. Hence, so is the fourth. □

11.34 Proposition. *Let $\mathcal{A}$ be a unital A_∞-category. Then $\mathcal{A}^{\mathsf{mc}}$ is* mc*-closed.*

Proof. Let X be an object of $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}$. We claim that it is isomorphic to $X\operatorname{Tot} u_{\mathsf{mc}} \in \operatorname{Ob}\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}$. Such X is specified by a family

$$(I \ni i \mapsto (X^i : J^i \to \mathcal{A}), x^{ii'} \in [s\mathcal{A}^{\mathsf{mc}}(X^i, X^{i'})]^0)^{i<i'}_{i,i' \in I}.$$

The corresponding $\tilde{X} = X\operatorname{Tot}_{\mathcal{A}}$ is determined by the partition $K = \sqcup_{i\in I} J^i = J^0 \sqcup \cdots \sqcup J^n$ and the A_∞-functor $\tilde{X} : K \to \mathcal{A}$ with $\tilde{x}^{ii'}_{jj'} = x^{ii'}_{jj'}$ if $i < i'$, and with $\tilde{x}^{ii}_{jj'} = x^i_{jj'}$, coming from X^i. Then $Y = X\operatorname{Tot} u_{\mathsf{mc}} = (\mathbf{1} \ni 1 \mapsto \tilde{X}, 0)$. Notice that $\tilde{Y} = \tilde{X}$, hence, the isomorphism Tot_1 identifies $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(X,Y)$, $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(Y,X)$, $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(X,X)$, $\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(Y,Y)$ with $\mathcal{A}^{\mathsf{mc}}(\tilde{X},\tilde{X})$.

Consider the elements $q \in s\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(X,Y)$, $t \in s\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(Y,X)$, which are identified with ${}_{\tilde{X}}\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}}}$. The map

$$\operatorname{Tot}_1 : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}(X,Y) \xrightarrow{\sim} \mathcal{A}^{\mathsf{mc}}(\tilde{X},\tilde{Y}) = \mathcal{A}^{\mathsf{mc}}(\tilde{X},\tilde{X})$$

and three other similar isomorphisms strictly commute with the operations b_k in the sense of (11.28.1). The equation $qb_1^{\mathsf{mc}\,\mathsf{mc}}\operatorname{Tot}_1 = \mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}}} b_1^{\mathsf{mc}} = 0$ implies that $qb_1^{\mathsf{mc}\,\mathsf{mc}} = 0$. Similarly, $tb_1^{\mathsf{mc}\,\mathsf{mc}} = 0$. The property

$$((q \otimes t)b_2^{\mathsf{mc}\,\mathsf{mc}} - {}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}})\operatorname{Tot}_1 = ({}_{\tilde{X}}\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}}} \otimes {}_{\tilde{X}}\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}}})b_2^{\mathsf{mc}} - {}_{\tilde{X}}\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}}} \in \operatorname{Im} b_1^{\mathsf{mc}} = (\operatorname{Im} b_1^{\mathsf{mc}\,\mathsf{mc}})\operatorname{Tot}_1$$

holds true, as 0-th component of (11.28.5) shows. It implies that $(q \otimes t)b_2^{\mathsf{mc}\,\mathsf{mc}} - {}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}} \in \operatorname{Im} b_1^{\mathsf{mc}\,\mathsf{mc}}$. Similarly, $(t \otimes q)b_2^{\mathsf{mc}\,\mathsf{mc}} - {}_X\mathbf{i}_0^{\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}} \in \operatorname{Im} b_1^{\mathsf{mc}\,\mathsf{mc}}$. Thus, X and Y are isomorphic. Therefore, $\mathcal{A}^{\mathsf{mc}}$ is mc-closed. □

11.35 Corollary. *If $\mathcal{A}$ is a unital A_∞-category, then A_∞-functors $u_{\mathsf{mc}}, u_{\mathsf{mc}}^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}$ and $m_{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}}$ are equivalences, quasi-inverse to each other.*

Indeed, such u_{mc} is an equivalence by Proposition 11.32, and $u_{\mathsf{mc}} m_{\mathsf{mc}} = \mathrm{id}_{\mathcal{A}^{\mathsf{mc}}} = u_{\mathsf{mc}}^{\mathsf{mc}} m_{\mathsf{mc}}$.

11.36 Proposition. *For an arbitrary* mc*-closed A_∞-category $\mathcal{C}$ there exists an A_∞-equivalence $U_{\mathsf{mc}} = U_{\mathsf{mc}}^{\mathcal{C}} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}$ such that $u_{\mathsf{mc}} \cdot U_{\mathsf{mc}} = \mathrm{id}_{\mathcal{C}}$. In particular, U_{mc} is quasi-inverse to u_{mc}.*

Proof. Apply Proposition 9.22 to the following data: $\mathcal{B} = \mathcal{C}^{\mathsf{mc}}$, $\mathcal{D} = \mathcal{C}$, the A_∞-equivalence $\phi = u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$, the embedding of a full A_∞-subcategory $\iota = u_{\mathsf{mc}} : \mathcal{C} \hookrightarrow \mathcal{C}^{\mathsf{mc}}$, the A_∞-functor $w = \mathrm{id} : \mathcal{C} \to \mathcal{C}$, the natural A_∞-transformation $q = u_{\mathsf{mc}} \mathbf{i}^{\mathcal{C}^{\mathsf{mc}}} : u_{\mathsf{mc}} \to u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ which represents the identity 2-morphism of the 1-morphism u_{mc} in $\overline{A^u_\infty}$. Choose a map $h : \mathrm{Ob}\, \mathcal{C}^{\mathsf{mc}} \to \mathrm{Ob}\, \mathcal{C}$ in such a way that $X u_{\mathsf{mc}} h = X = Xw$ for all objects X of $\mathcal{C}$ and that the objects Y and $Y h u_{\mathsf{mc}}$ were isomorphic in $H^0(\mathcal{C}^{\mathsf{mc}})$ for all objects Y of $\mathcal{C}^{\mathsf{mc}}$. Choose cycles ${}_Y r_0 \in \mathcal{C}^{\mathsf{mc}}(Y, Y h u_{\mathsf{mc}})[1]^{-1}$, $Y \in \mathrm{Ob}\, \mathcal{C}^{\mathsf{mc}}$ which represent these isomorphisms taking ${}_{X u_{\mathsf{mc}}} r_0 = {}_{X u_{\mathsf{mc}}} \mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} = {}_X q_0$ for all objects X of $\mathcal{C}$. The hypotheses of Proposition 9.22 are satisfied. We deduce from it that there exists an A_∞-equivalence $\psi = U_{\mathsf{mc}} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}$ such that $u_{\mathsf{mc}} \cdot U_{\mathsf{mc}} = \mathrm{id}_{\mathcal{C}}$. $\square$

11.37 Proposition. *Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories, and let $\mathcal{C}$ be* mc*-closed. Let A_∞-equivalence $U_{\mathsf{mc}} = U_{\mathsf{mc}}^{\mathcal{C}} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}$ satisfy the equation $u_{\mathsf{mc}} \cdot U_{\mathsf{mc}} = \mathrm{id}_{\mathcal{C}}$ (it exists by Proposition 11.36). Then the strict A_∞-functor $A^u_\infty(u_{\mathsf{mc}}, \mathcal{C}) = (u_{\mathsf{mc}} \boxtimes 1) M : A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) \to A^u_\infty(\mathcal{A}, \mathcal{C})$ is an A_∞-equivalence which admits a one-sided inverse*

$$F_{\mathsf{mc}} = \big[A^u_\infty(\mathcal{A}, \mathcal{C}) \xrightarrow{-^{\mathsf{mc}}} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \xrightarrow{A^u_\infty(\mathcal{A}^{\mathsf{mc}}, U_{\mathsf{mc}})} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) \big]$$

(quasi-inverse to $A^u_\infty(u_{\mathsf{mc}}, \mathcal{C})$), namely, $F_{\mathsf{mc}} \cdot A^u_\infty(u_{\mathsf{mc}}, \mathcal{C}) = \mathrm{id}_{A^u_\infty(\mathcal{A}, \mathcal{C})}$.

Proof. Naturality of the A^u_∞-2-transformation u_{mc} is expressed by the equation

$$\big[A^u_\infty(\mathcal{A}, \mathcal{B}) \xrightarrow{-^{\mathsf{mc}}} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}}) \xrightarrow{(u_{\mathsf{mc}} \boxtimes 1) M} A^u_\infty(\mathcal{A}, \mathcal{B}^{\mathsf{mc}}) \big] = (1 \boxtimes u_{\mathsf{mc}}) M. \tag{11.37.1}$$

It implies that the A_∞-functor F_{mc} is a one-sided inverse to $A^u_\infty(u_{\mathsf{mc}}, \mathcal{C})$. Indeed,

$$\begin{aligned}
& F_{\mathsf{mc}} \cdot A^u_\infty(u_{\mathsf{mc}}, \mathcal{C}) \\
&= \big[A^u_\infty(\mathcal{A}, \mathcal{C}) \xrightarrow{-^{\mathsf{mc}}} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \xrightarrow{A^u_\infty(\mathcal{A}^{\mathsf{mc}}, U_{\mathsf{mc}})} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) \xrightarrow{A^u_\infty(u_{\mathsf{mc}}, \mathcal{C})} A^u_\infty(\mathcal{A}, \mathcal{C}) \big] \\
&= \big[A^u_\infty(\mathcal{A}, \mathcal{C}) \xrightarrow{-^{\mathsf{mc}}} A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \xrightarrow{A^u_\infty(u_{\mathsf{mc}}, 1)} A^u_\infty(\mathcal{A}, \mathcal{C}^{\mathsf{mc}}) \xrightarrow{A^u_\infty(1, U_{\mathsf{mc}})} A^u_\infty(\mathcal{A}, \mathcal{C}) \big] \\
&= \big[A^u_\infty(\mathcal{A}, \mathcal{C}) \xrightarrow{A^u_\infty(1, u_{\mathsf{mc}})} A^u_\infty(\mathcal{A}, \mathcal{C}^{\mathsf{mc}}) \xrightarrow{A^u_\infty(1, U_{\mathsf{mc}})} A^u_\infty(\mathcal{A}, \mathcal{C}) \big] = A^u_\infty(1, u_{\mathsf{mc}} U_{\mathsf{mc}}) = \mathrm{id}
\end{aligned}$$

due to Lemmata 4.13 and 4.15.

Composition of these A_∞-functors in the other order gives on objects (unital A_∞-functors $f : \mathcal{A}^{\mathsf{mc}} \to \mathcal{C}$) the following:

$$f \mapsto u_{\mathsf{mc}} f \mapsto (u_{\mathsf{mc}} f)^{\mathsf{mc}} U_{\mathsf{mc}} = u_{\mathsf{mc}}^{\mathsf{mc}} f^{\mathsf{mc}} U_{\mathsf{mc}} \simeq u_{\mathsf{mc}} f^{\mathsf{mc}} U_{\mathsf{mc}} = f u_{\mathsf{mc}} U_{\mathsf{mc}} = f.$$

Indeed, due to Corollary 11.35 there is an isomorphism of A_∞-functors $r : u^{\mathsf{mc}}_{\mathsf{mc}} \to u_{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}$. Let us prove that this composition is isomorphic to identity A_∞-functor. It is given by the top–right–bottom exterior path in the following diagram, which describes a natural A_∞-transformation:

$$
\begin{tikzcd}[column sep=large]
A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) \arrow[r, "A^u_\infty(u_{\mathsf{mc}}, \mathcal{C})"] \arrow[d, equal] & A^u_\infty(\mathcal{A}, \mathcal{C}) \arrow[r, "\_^{\mathsf{mc}}"] \arrow[d, phantom, "="] & A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \arrow[d, equal] \\
A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) \arrow[r, "\_^{\mathsf{mc}}"] \arrow[d, "\mathrm{id}"'] \arrow[drr, "A^u_\infty(\mathcal{A}^{\mathsf{mc}}, u_{\mathsf{mc}})"'] \arrow[dr, phantom, "="] & A^u_\infty(\mathcal{A}^{\mathsf{mc}\,\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \arrow[r, shift left=2, "A^u_\infty(u^{\mathsf{mc}}_{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}})"] \arrow[r, shift right=2, "A^u_\infty(u_{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}})"'] \arrow[r, phantom, "(r\boxtimes 1)M\Downarrow"] \arrow[dr, phantom, "="] & A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \arrow[d, equal] \\
A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}) & & A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{C}^{\mathsf{mc}}) \arrow[ll, "A^u_\infty(\mathcal{A}^{\mathsf{mc}}, U_{\mathsf{mc}})"]
\end{tikzcd}
$$

Since the A_∞-functor $A^u_\infty(\_, \mathcal{C}^{\mathsf{mc}})$ is unital, its first component takes the isomorphism r to an isomorphism $r.A^u_\infty(\_, \mathcal{C}^{\mathsf{mc}})_1 = (r \boxtimes 1)M : (u^{\mathsf{mc}}_{\mathsf{mc}} \boxtimes 1)M \to (u_{\mathsf{mc}} \boxtimes 1)M$. Thus the above diagram gives an isomorphism of the top–right–bottom exterior path with the left column, which is the identity functor. The proposition is proven. $\square$

11.38 Corollary. *Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories, and let $\mathcal{C}$ be* mc*-closed. Then the restriction map $\mathsf{A}^{\mathrm{u}}_\infty(u_{\mathsf{mc}}; 1) : \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{A}^{\mathsf{mc}}; \mathcal{C}) \to \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{A}; \mathcal{C})$ is surjective.*

11.39 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. Then the A_∞-functor*

$$-^{\mathsf{mc}} : A^u_\infty(\mathcal{A}, \mathcal{B}) \to A^u_\infty(\mathcal{A}^{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$$

is homotopy full and faithful, that is, its first component is homotopy invertible.

Proof. Consider equation (11.37.1). The first component of $A^u_\infty(\mathcal{A}, u_{\mathsf{mc}}) = (1 \boxtimes u_{\mathsf{mc}})M$ in the right hand side (composition with u_{mc}) is an isomorphism, since u_{mc} is a strict full embedding. The first component of the second functor $A^u_\infty(u_{\mathsf{mc}}, \mathcal{B}^{\mathsf{mc}})$ in the left hand side is homotopy invertible by Proposition 11.37. Therefore, the first component of the first functor $-^{\mathsf{mc}}$ in the left hand side is homotopy invertible. $\square$

11.40 Quotients and solutions to Maurer–Cartan equation. Let $\mathcal{B}$ be a full subcategory of a unital A_∞-category $\mathcal{C}$. Denote by $i : \mathcal{B} \hookrightarrow \mathcal{C}$ the inclusion strict A_∞-functor, and by $e : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})$ the quotient functor. By construction e can be chosen so that $\operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B}) = \operatorname{Ob}\mathcal{C}$, $\operatorname{Ob} e = \mathrm{id}_{\operatorname{Ob}\mathcal{C}}$ [LM08c]. Similarly, in the diagram below

i^{mc} is a full embedding and e' is the quotient functor:

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, hook, "i"] \arrow[dd, "u^{\mathcal{B}}_{\mathsf{mc}}"'] \arrow[rdd, phantom, "="] & \mathcal{C} \arrow[rr, "e"] \arrow[dd, "u^{\mathcal{C}}_{\mathsf{mc}}"'] \arrow[rdd, phantom, "\Swarrow\alpha"] & & \mathsf{q}(\mathcal{C}|\mathcal{B}) \arrow[dl, "f"'] \arrow[dd, "u_{\mathsf{mc}}"] \\
& & \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \arrow[dr, "g"] \arrow[r, phantom, "\overset{\chi}{\Longrightarrow}"] \arrow[d, phantom, "\Downarrow\beta"] & {} \\
\mathcal{B}^{\mathsf{mc}} \arrow[r, hook, "i^{\mathsf{mc}}"] & \mathcal{C}^{\mathsf{mc}} \arrow[ur, "e'"] \arrow[rr, "e^{\mathsf{mc}}"'] & {} & \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}
\end{tikzcd}
\tag{11.40.1}
$$

Here existence of A_∞-functors f, g and natural isomorphisms α, β follows from universality of quotients. Since $\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{u^{\mathcal{C}}_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}} \xrightarrow{e'} \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})$ is contractible, there exists a unital A_∞-functor f and an isomorphism $\alpha : ef \xrightarrow{\sim} u^{\mathcal{C}}_{\mathsf{mc}}e' : \mathcal{C} \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})$. Since $i^{\mathsf{mc}}e^{\mathsf{mc}}$ is contractible by Remark 11.30, there exists a unital A_∞-functor g and an isomorphism $\beta : e'g \xrightarrow{\sim} e^{\mathsf{mc}} : \mathcal{C}^{\mathsf{mc}} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$. Since $\mathcal{B} \overset{i}{\hookrightarrow} \mathcal{C} \xrightarrow{u^{\mathcal{C}}_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}} \xrightarrow{e^{\mathsf{mc}}} \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$ is contractible, there exists a unital A_∞-functor $\phi : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$ together with an isomorphism $e\phi \xrightarrow{\sim} u^{\mathcal{C}}_{\mathsf{mc}}e^{\mathsf{mc}} : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$. Actually, previous data allow to construct two such pairs: fg and u_{mc} together with isomorphisms

$$
e(fg) \xrightarrow{(\alpha g)\beta} u^{\mathcal{C}}_{\mathsf{mc}}e^{\mathsf{mc}} \overset{\mathrm{id}}{=\!=\!=} eu_{\mathsf{mc}} : \mathcal{C} \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}. \tag{11.40.2}
$$

Theorem 1.3 of [LM08c] implies that

$$
(e \boxtimes 1)M : A^u_\infty(\mathsf{q}(\mathcal{C}|\mathcal{B}), \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}) \to A^u_\infty(\mathcal{C}, \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}})_{\mathrm{mod}\,\mathcal{B}},
$$

is an A_∞-equivalence. By Lemma 10.42 isomorphism (11.40.2) is equal to $e\chi$ for some isomorphism $\chi : fg \to u_{\mathsf{mc}} : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$.

11.41 Proposition. *Assume that $\mathcal{B}$, $\mathcal{C}$ are mc-closed. Then $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is mc-closed as well, and $f : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})$, $g : \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$ from diagram* (11.40.1) *are A_∞-equivalences.*

Proof. We have $\operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B}) = \operatorname{Ob}\mathcal{C}$. There is a function $\psi : \operatorname{Ob}\mathcal{C}^{\mathsf{mc}} \to \operatorname{Ob}\mathcal{C}$, $X \mapsto \overline{X}$ together with inverse to each other isomorphisms

$$
r_0 \in s\mathcal{C}^{\mathsf{mc}}(\overline{X}u_{\mathsf{mc}}, X), \qquad p_0 \in s\mathcal{C}^{\mathsf{mc}}(X, \overline{X}u_{\mathsf{mc}}).
$$

Take the same function $\psi : \operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}} \to \operatorname{Ob}\mathsf{q}(\mathcal{C}|\mathcal{B})$, $X \mapsto \overline{X}$ together with inverse to each other isomorphisms

$$
r_0e^{\mathsf{mc}}_1 \in s\mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}(\overline{X}u_{\mathsf{mc}}, X), \qquad p_0e^{\mathsf{mc}}_1 \in s\mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}(X, \overline{X}u_{\mathsf{mc}}).
$$

Notice that $\operatorname{Ob} e^{\mathsf{mc}} = \mathrm{id}_{\mathcal{C}^{\mathsf{mc}}}$. Their existence shows that $\mathsf{q}(\mathcal{C}|\mathcal{B})$ is mc-closed. By Proposition 11.32 the A_∞-functor $u_{\mathsf{mc}} : \mathsf{q}(\mathcal{C}|\mathcal{B}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$ is an equivalence.

By Lemma 10.44 the A_∞-functor f is an equivalence. Since $fg \simeq u_{\mathsf{mc}}$ is an A_∞-equivalence, so is g. $\square$

11.42 Proposition. *The functor* $g : \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}}$ *from* (11.40.1) *is an* A_∞*-equivalence.*

Proof. Let us describe the following diagram:

$$\begin{tikzcd}
\mathcal{B}^{\mathsf{mc}} \arrow[r, hook, "i^{\mathsf{mc}}"] \arrow[ddd, "(u^{\mathcal{B}}_{\mathsf{mc}})^{\mathsf{mc}}"'] \arrow[dddr, phantom, "="] & \mathcal{C}^{\mathsf{mc}} \arrow[rr, "e^{\mathsf{mc}}"] \arrow[dr, "e'"'] \arrow[ddd, "(u^{\mathcal{C}}_{\mathsf{mc}})^{\mathsf{mc}}"'] & & \mathsf{q}(\mathcal{C}|\mathcal{B})^{\mathsf{mc}} \arrow[ddd, "f^{\mathsf{mc}}"] \\
& & \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \arrow[ur, "g"'] \arrow[u, Rightarrow, "\beta"'] \arrow[d, "k"] \arrow[ddl, Rightarrow, "\xi"'] & \\
& & \mathsf{q}(\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}}) \arrow[dr, "h"] \arrow[d, Rightarrow, "\gamma"'] & \\
\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}} \arrow[r, hook, "i^{\mathsf{mc}\,\mathsf{mc}}"] & \mathcal{C}^{\mathsf{mc}\,\mathsf{mc}} \arrow[rr, "e'^{\mathsf{mc}}"'] \arrow[ur, "e''"'] & & \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}}
\end{tikzcd}$$

Here $i^{\mathsf{mc}\,\mathsf{mc}} : \mathcal{B}^{\mathsf{mc}\,\mathsf{mc}} \to \mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}$ is a full embedding and $e'' : \mathcal{C}^{\mathsf{mc}\,\mathsf{mc}} \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}})$ is a quotient map. Note that $(u^{\mathcal{A}}_{\mathsf{mc}})^{\mathsf{mc}}$ is an A_∞-equivalence by Corollary 11.35. By Lemma 10.44 there exist a unital A_∞-functor $k : \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}})$ and an isomorphism $\xi : e'k \to (u^{\mathcal{C}}_{\mathsf{mc}})^{\mathsf{mc}} e'' : \mathcal{C}^{\mathsf{mc}} \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}})$. By the same lemma k is an A_∞-equivalence. The functor $i^{\mathsf{mc}\,\mathsf{mc}} e'^{\mathsf{mc}}$ is contractible by Remark 11.30, therefore, there exist a unital A_∞-functor $h : \mathsf{q}(\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}\,\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}}$ and an isomorphism $\gamma : e''h \to e'^{\mathsf{mc}}$. Since $\mathcal{C}^{\mathsf{mc}}$ and $\mathcal{B}^{\mathsf{mc}}$ are mc-closed, the functor h is an A_∞-equivalence by Proposition 11.41. We have the following isomorphism

$$e'gf^{\mathsf{mc}} \xrightarrow[\sim]{\beta f^{\mathsf{mc}}} e^{\mathsf{mc}} f^{\mathsf{mc}} \xrightarrow[\sim]{\alpha^{\mathsf{mc}}} (u^{\mathcal{C}}_{\mathsf{mc}})^{\mathsf{mc}} e'^{\mathsf{mc}} \xrightarrow[\sim]{(u^{\mathcal{C}}_{\mathsf{mc}})^{\mathsf{mc}}\gamma^{-1}} (u^{\mathcal{C}}_{\mathsf{mc}})^{\mathsf{mc}} e''h \xrightarrow[\sim]{\xi^{-1}h} e'kh : \mathcal{C}^{\mathsf{mc}} \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}}. \tag{11.42.1}$$

Theorem 1.7 implies that

$$(e' \boxtimes 1)M : A^u_\infty(\mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}), \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}}) \to A^u_\infty(\mathcal{C}^{\mathsf{mc}}, \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}})_{\operatorname{mod}\mathcal{B}^{\mathsf{mc}}}$$

is an A_∞-equivalence. By Lemma 10.42 isomorphism (11.42.1) is equal to $e'\zeta$ for some isomorphism $\zeta : gf^{\mathsf{mc}} \to kh : \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}}) \to \mathsf{q}(\mathcal{C}^{\mathsf{mc}}|\mathcal{B}^{\mathsf{mc}})^{\mathsf{mc}}$. In particular, gf^{mc} is an A_∞-equivalence. On the other hand, $f^{\mathsf{mc}}g^{\mathsf{mc}} \simeq (u^{\mathsf{q}(\mathcal{C}|\mathcal{B})}_{\mathsf{mc}})^{\mathsf{mc}}$ via the isomorphism χ^{mc}. The functor $(u^{\mathsf{q}(\mathcal{C}|\mathcal{B})}_{\mathsf{mc}})^{\mathsf{mc}}$ is an A_∞-equivalence, hence so is $f^{\mathsf{mc}}g^{\mathsf{mc}}$. This shows that f^{mc} has left and right quasi-inverses, hence it is an A_∞-equivalence. Since gf^{mc} is an A_∞-equivalence, so is g. $\square$

11.43 Embedding $A_\infty(\mathcal{A},\mathcal{B})^{\mathsf{mc}} \hookrightarrow A_\infty(\mathcal{A},\mathcal{B}^{\mathsf{mc}})$. Let $\mathcal{A}$, $\mathcal{B}$ be A_∞-categories. Let $\varpi_{\mathsf{Mc}} : \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{Mc}} \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{Mc}})$ be an A_∞-functor defined as follows:

$$\varpi_{\mathsf{Mc}} = \big[\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{Mc}} \xrightarrow{\underline{\mathsf{Mc}}} \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \xrightarrow{\underline{\mathsf{A}_\infty}((u_{\mathsf{Mc}});1)} \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{Mc}})\big].$$

11.44 Proposition. *The A_∞-functor ϖ_{Mc} is a strict full embedding, that is, all its components vanish except*

$$(\varpi_{\mathsf{Mc}})_1 : \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{Mc}}(f,g) \to \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{Mc}})(f\varpi_{\mathsf{Mc}}, g\varpi_{\mathsf{Mc}})$$

which is an isomorphism.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{Mc}} \arrow[r, "\underline{\mathsf{Mc}}"] \arrow[d, hook, "\iota"'] & \underline{\mathsf{A}_\infty}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[r, "\underline{\mathsf{A}_\infty}((u_{\mathsf{Mc}});1)"] \arrow[d, hook, "\underline{E}"] & \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[d, hook, "\underline{E}"] \\
\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{MC}} \arrow[d, hook, "\underline{E}^{\mathsf{MC}}"'] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[r, "\underline{\mathsf{Q}}((u_{\mathsf{Mc}});1)"] \arrow[d, hook, "\underline{\mathsf{Q}}(\triangleright;\iota)"] & \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{Mc}}) \arrow[dd, hook, "\underline{\mathsf{Q}}(\triangleright;\iota)"] \\
\underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B})^{\mathsf{MC}} \arrow[d, "\underline{\mathsf{MC}}"'] & \underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{Mc}})_{i\in I};\mathcal{B}^{\mathsf{MC}}) \arrow[dr, "\underline{\mathsf{Q}}((u_{\mathsf{Mc}});1)"] & \\
\underline{\mathsf{Q}}((\mathcal{A}_i^{\mathsf{MC}})_{i\in I};\mathcal{B}^{\mathsf{MC}}) \arrow[ur, "\underline{\mathsf{Q}}((\iota);1)"] \arrow[rr, "\underline{\mathsf{Q}}((u_{\mathsf{MC}});1)"] & & \underline{\mathsf{Q}}((\mathcal{A}_i)_{i\in I};\mathcal{B}^{\mathsf{MC}})
\end{tikzcd}$$

The left heptagon is commutative due to Proposition 11.17. The upper and the lower quadrilaterals commute by Lemmata 4.23 and 4.13 respectively. Commutativity of the bottom triangle is a particular case of Lemma 4.14. The morphism ϖ_{MC} is a strict morphism with the bijective first component by Proposition 11.10. It is related to ϖ_{Mc} by full embeddings, therefore, the latter shares the same properties. □

As a corollary the same statement holds for $\varpi_{\mathsf{mc}} : \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C})^{\mathsf{mc}} \to \underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{C}^{\mathsf{mc}})$, which is a restriction of ϖ_{Mc}. Let us describe this A_∞-functor on objects. An object of $A_\infty(\mathcal{A},\mathcal{C})^{\mathsf{mc}}$ is given by the following data: a family of functors $I \ni i \mapsto f^i \in \operatorname{Ob} A_\infty(\mathcal{A},\mathcal{C})$, a family of transformations (of degree 0) $r^{ij} \in sA_\infty(\mathcal{A},\mathcal{C})(f^i,f^j)$, i, $j \in I$, $i < j$ such that the Maurer-Cartan equation holds: for all i, $j \in I$, $i < j$

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (r^{ik_1} \otimes r^{k_1k_2} \otimes \dots \otimes r^{k_{m-1}j})B_m = 0.$$

Note that the first component of this equation reads as follows:

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (r_0^{ik_1} \otimes r_0^{k_1k_2} \otimes \dots \otimes r_0^{k_{m-1}j})b_m = 0. \tag{11.44.1}$$

To this object ϖ_{mc} assigns the A_∞-functor $f:\mathcal{A}\to\mathcal{C}^{\mathsf{mc}}$ defined by the following prescriptions. An object $X\in\operatorname{Ob}\mathcal{A}$ is mapped to the object $Xf:I\to\mathcal{C}$, $i\mapsto Xf^i$ of $\mathcal{C}^{\mathsf{mc}}$. It is determined by the family of elements of degree 0 $x^{ij}={}_Xr_0^{ij}\in s\mathcal{C}(Xf^i,Xf^j)$, $i,j\in I$, $i<j$. The Maurer-Cartan equation is fulfilled due to (11.44.1). Components of f are determined as follows: the composition

$$\otimes^{p\in\mathbf{n}}s\mathcal{A}(X_{p-1},X_p)\xrightarrow{f_n}s\mathcal{C}^{\mathsf{mc}}(X_0f,X_nf)=\prod_{i,j\in I}s\mathcal{C}(X_0f^i,X_nf^j)\xrightarrow{\mathrm{pr}}s\mathcal{C}(X_0f^i,X_nf^j)$$

equals f_n^i for $i=j$, r_n^{ij} for $i<j$, and vanishes otherwise. We write briefly

$$f_n=\begin{bmatrix} f_n^1 & r_n^{12} & \dots & r_n^{1n}\\ 0 & f_n^2 & \dots & r_n^{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & f_n^{|I|}\end{bmatrix}\qquad\text{or}\qquad f_n^{ij}=\begin{cases} f_n^i, & \text{if } i=j,\\ r_n^{ij}, & \text{if } i<j,\\ 0, & \text{otherwise.}\end{cases}\tag{11.44.2}$$

The components of ϖ_{mc} are determined by (11.10.1).

11.45 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category. Let $(X:I\ni i\mapsto X_i,x=(x_{ij}))$, $x_{ij}\in s\mathcal{B}(X_i,X_j)$ be an object of $\mathcal{B}^{\mathsf{mc}}$, let $p=(p_{ij})\in s\mathcal{B}^{\mathsf{mc}}(X,X)$ be an element of degree -1 such that $pb_1^{\mathsf{mc}}=0$, $p_{ii}={}_{X_i}\mathbf{i}_0^{\mathcal{B}}$ for all $i\in I$, and $p_{ij}=0$ for $i>j$. Then p is invertible modulo boundaries.*

Proof. By (11.23.1) we can write p as ${}_X\mathbf{i}_0^{\mathcal{B}^{\mathsf{mc}}}+n$, where $n=(n_{ij})\in s\mathcal{B}^{\mathsf{mc}}(X,X)$, $nb_1^{\mathsf{mc}}=0$, $n_{ij}=0$ for $i\geqslant j$. Consider the chain map $a=(1\otimes n)b_2^{\mathsf{mc}}:s\mathcal{B}^{\mathsf{mc}}(X,X)\to s\mathcal{B}^{\mathsf{mc}}(X,X)$. Then $a^k=(1\otimes n^k)b_2^{\mathsf{mc}}$, where by definition $n^1=n$, $n^k=(n^{k-1}\otimes n)b_2^{\mathsf{mc}}$. Since n is represented by a strictly upper triangular matrix, we have $n^k=0$ for all sufficiently large k, hence a is a nilpotent chain map. Therefore $1+a$ is invertible, as $(1+a)^{-1}=\sum_{k=0}^\infty(-a)^k$ exists. This implies that the chain map $(1\otimes p)b_2=(1\otimes{}_X\mathbf{i}_0^{\mathcal{B}^{\mathsf{mc}}})b_2^{\mathsf{mc}}+a\sim 1+a$ is homotopy invertible. In particular the induced map $H^{-1}(s\mathcal{B}^{\mathsf{mc}}(X,X))\to H^{-1}(s\mathcal{B}^{\mathsf{mc}}(X,X))$ is an isomorphism. Therefore there exists a cycle $q\in s\mathcal{B}^{\mathsf{mc}}(X,X)^{-1}$ such that $(q\otimes p)b_2^{\mathsf{mc}}-{}_X\mathbf{i}_0^{\mathcal{B}^{\mathsf{mc}}}\in\operatorname{Im}b_1^{\mathsf{mc}}$, that is, p is invertible on the left. Similarly, it is invertible on the right. $\square$

Suppose that $\mathcal{A}$ and $\mathcal{C}$ are unital A_∞-categories. Let us show that ϖ_{mc} restricts to an A_∞-functor $A_\infty^u(\mathcal{A},\mathcal{C})^{\mathsf{mc}}\to A_\infty^u(\mathcal{A},\mathcal{C}^{\mathsf{mc}})$. We preserve the above notations. It suffices to show that the A_∞-functor $f:\mathcal{A}\to\mathcal{C}^{\mathsf{mc}}$ is unital if each $f^i:\mathcal{A}\to\mathcal{C}$, $i\in I$ is unital. Assume $\mathbf{i}^{\mathcal{A}}f^i=f^i\mathbf{i}^{\mathcal{C}}+v^iB_1$, $i\in I$, where $v^i:f^i\to f^i:\mathcal{A}\to\mathcal{C}$ is an A_∞-transformation of degree -2. Since f is an A_∞-functor, it follows that ${}_X\mathbf{i}_0^{\mathcal{A}}f_1\in s\mathcal{C}^{\mathsf{mc}}(Xf,Xf)$ is a cycle, idempotent modulo boundary. According to (11.44.2)

$${}_X\mathbf{i}_0^{\mathcal{A}}f_1=\begin{bmatrix} {}_X\mathbf{i}_0^{\mathcal{A}}f_1^1 & {}_X\mathbf{i}_0^{\mathcal{A}}r_1^{12} & \dots & {}_X\mathbf{i}_0^{\mathcal{A}}r_1^{1n}\\ 0 & {}_X\mathbf{i}_0^{\mathcal{A}}f_1^2 & \dots & {}_X\mathbf{i}_0^{\mathcal{A}}r_1^{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & {}_X\mathbf{i}_0^{\mathcal{A}}f_1^{|I|}\end{bmatrix}=p+\operatorname{diag}({}_Xv_0^i)b_1^{\mathsf{mc}},$$

where $p = (p_{ij}) \in s\mathcal{C}^{\mathsf{mc}}(Xf, Xf)$ is an element of degree -1 such that $pb_1^{\mathsf{mc}} = 0$, $p_{ii} = {}_{Xf^i}\mathbf{i}_0^{\mathcal{C}}$ for all $i \in I$, and $p_{ij} = 0$ for $i > j$. By Lemma 11.45 p is invertible modulo boundaries. Therefore, the same is true for ${}_X\mathbf{i}_0^{\mathcal{A}} f_1$. Being also idempotent, ${}_X\mathbf{i}_0^{\mathcal{A}} f_1$ equals the unit ${}_{Xf}\mathbf{i}_0^{\mathcal{C}}$ modulo boundaries.

11.46 Corollary. *Let $\mathcal{C}$ be an* mc*-closed A_∞-category. Then for an arbitrary A_∞-category $\mathcal{A}$ the A_∞-category $A_\infty(\mathcal{A},\mathcal{C})$ is* mc*-closed. If $\mathcal{A}$ is unital, then the A_∞-category $A_\infty^u(\mathcal{A},\mathcal{C})$ is* mc*-closed.*

Proof. Due to Corollary 11.13 for arbitrary A_∞-categories $\mathcal{A}$, $\mathcal{C}$ the following equation holds:

$$\big[A_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{u_{\mathsf{mc}}} A_\infty(\mathcal{A},\mathcal{C})^{\mathsf{mc}} \xrightarrow{\varpi_{\mathsf{mc}}} A_\infty(\mathcal{A},\mathcal{C}^{\mathsf{mc}})\big] = A_\infty(\mathcal{A},u_{\mathsf{mc}}).$$

When $\mathcal{C}$ is mc-closed, the above A_∞-functor is an equivalence since so is $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$. In particular, it is essentially surjective. Let $f : I \to A_\infty(\mathcal{A},\mathcal{C})$ be an object of $A_\infty(\mathcal{A},\mathcal{C})^{\mathsf{mc}}$, then the object $f\varpi_{\mathsf{mc}}$ is isomorphic to an object $gu_{\mathsf{mc}}\varpi_{\mathsf{mc}}$ for some $g \in \operatorname{Ob} A_\infty(\mathcal{A},\mathcal{C})$. Since ϖ_{mc} is a full embedding, it follows that gu_{mc} is isomorphic to f, which means that $A_\infty(\mathcal{A},\mathcal{C})$ is mc-closed. The case of $A_\infty^u(\mathcal{A},\mathcal{C})$ is treated similarly. □

Chapter 12

The monad of pretriangulated A_∞-categories

In this chapter we construct a commutation morphism between the two $\mathsf{A}^{\mathrm{u}}_\infty$-monads considered above, the monad of shifts and the Maurer–Cartan monad. Since it is coherent with the units and the multiplications in these monads, a new monad $-^{\mathsf{tr}}$, their composition, is obtained as a corollary. It is called the $\mathsf{A}^{\mathrm{u}}_\infty$-monad of pretriangulated A_∞-categories. Using it we define pretriangulated A_∞-categories and study some of their properties. In particular, an A_∞-category is pretriangulated if and only if it is closed under shifts and mc-closed. By Proposition 11.32 this is equivalent to closedness under shifts and existence of cones of closed morphisms. The monad $-^{\mathsf{tr}}$ behaves like a completion. In a certain sense it commutes with taking quotients and the A_∞-category of A_∞-functors.

12.1 The composition of two monads. Define $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$. Its objects are A_∞-functors $X : I \to \mathcal{C}^{[]}$, $\operatorname{Ob} X : i \mapsto X_i[n_i]$, specified by the elements

$$x_{ij} \in \{s\mathcal{C}^{[]}(X_i[n_i], X_j[n_j])\}^0 = \{s\mathcal{C}(X_i, X_j)[n_j - n_i]\}^0 = \mathcal{C}^{n_j-n_i+1}(X_i, X_j).$$

These elements have to satisfy the Maurer–Cartan equation

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (x_{ik_1} \otimes x_{k_1k_2} \otimes \dots \otimes x_{k_{m-1}j}) b_m^{[]} = 0. \tag{12.1.1}$$

where the operation $b_m^{[]}$ is described by (10.28.2). This equation can be written as

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (x_{ik_1} \otimes x_{k_1k_2} \otimes \dots \otimes x_{k_{m-1}j})(s^{n_{k_1}-n_i} \otimes s^{n_{k_2}-n_{k_1}} \otimes \dots \otimes s^{n_j-n_{k_{m-1}}})^{-1} b_m^{\mathcal{C}} = 0.$$

12.2 Commutation morphism between two monads. To equip $-^{\mathsf{tr}}$ with a monad structure, we construct a commutation morphism between the A_∞-2-monads mc and $[]$,

$$\mathfrak{c} = \mathfrak{c}_{\mathcal{C}} : \mathcal{C}^{\mathsf{mc}[]} \to \mathcal{C}^{[]\mathsf{mc}}.$$

The strict A_∞-functor $\mathfrak{c}_{\mathcal{C}}$ takes an object $X[n]$ of $\mathcal{C}^{\mathsf{mc}[]}$, where $X : I \to \mathcal{C}$ is an A_∞-functor, to the object $X' = X[n]\mathfrak{c} : I \to \mathcal{C}^{[]}$, $i \mapsto X_i[n]$, $x'_{ij} = x_{ij} \in s\mathcal{C}^{[]}(X_i[n], X_j[n]) = s\mathcal{C}(X_i, X_j)$.

Equation (12.1.1) for x' takes (correctly) the form

$$\sum_{i<k_1<\dots<k_{m-1}<j}^{m>0} (x_{ik_1} \otimes x_{k_1k_2} \otimes \dots \otimes x_{k_{m-1}j}) b_m^{\mathcal{C}} = 0.$$

and thus holds.

The first component of $\mathfrak{c}$ consists of the identity morphisms

$$\begin{aligned}\mathfrak{c}_1 : s\mathcal{C}^{\mathsf{mc}[]}(X[n], Y[m]) &= s\mathcal{C}^{\mathsf{mc}}(X, Y)[m-n] \\ &= \Bigl\{ \prod_{i\in I, j\in J} s\mathcal{C}(X_i, Y_j) \Bigr\}[m-n] = \prod_{i\in I,\, j\in J} \bigl\{ s\mathcal{C}(X_i, Y_j)[m-n] \bigr\} \\ &= \prod_{i\in I,\, j\in J} s\mathcal{C}^{[]}(X_i[n], Y_j[m]) = s\mathcal{C}^{[]\mathsf{mc}}(X[n]\mathfrak{c}, Y[m]\mathfrak{c}).\end{aligned}$$

To be a strict A_∞-functor, $\mathfrak{c}$ has to satisfy the equation

$$(\mathfrak{c}_1 \otimes \dots \otimes \mathfrak{c}_1) b_k^{[]\mathsf{mc}} = b_k^{\mathsf{mc}[]} \mathfrak{c}_1 : \otimes^{p\in\mathbf{k}} s\mathcal{C}^{\mathsf{mc}[]}(X^{p-1}[n_{p-1}], X^p[n_p]) \to s\mathcal{C}^{[]\mathsf{mc}}(X^0[n_0]\mathfrak{c}, X^k[n_k]\mathfrak{c}). \tag{12.2.1}$$

Let objects $X^p[n_p]$, $0 \leqslant p \leqslant k$ of $\mathcal{C}^{\mathsf{mc}[]}$ be specified by elements

$$x^p = (x^p_{ij}) \in \prod_{i\in I_p,\, j\in I_p} s\mathcal{C}^{[]}(X^p_i[n_p], X^p_j[n_p]) = s\mathcal{C}^{\mathsf{mc}[]}(X^p[n_p], X^p[n_p]). \tag{12.2.2}$$

Consider morphisms

$$r^p = (r^p_{ij}) \in \prod_{i\in I_{p-1},\, j\in I_p} s\mathcal{C}^{[]}(X^{p-1}_i[n_{p-1}], X^p_j[n_p]) = s\mathcal{C}^{\mathsf{mc}[]}(X^{p-1}[n_{p-1}], X^p[n_p]). \tag{12.2.3}$$

Applying the left hand side of (12.2.1) to them we obtain

$$\begin{aligned}(r^1 \otimes \dots \otimes r^k) b_k^{[]\mathsf{mc}} &= \sum_{t_0,\dots,t_k\geqslant 0} \bigl[(x^0)^{\otimes t_0} \otimes r^1 \otimes (x^1)^{\otimes t_1} \otimes \dots \otimes r^k \otimes (x^k)^{\otimes t_k}\bigr] b^{[]}_{t_0+\dots+t_k+k} \\ = \sum_{t_0,\dots,t_k\geqslant 0} &\bigl[(x^0)^{\otimes t_0} \otimes r^1 \otimes (x^1)^{\otimes t_1} \otimes \dots \otimes r^k \otimes (x^k)^{\otimes t_k}\bigr] \\ &(1^{\otimes t_0} \otimes s^{n_0-n_1} \otimes 1^{\otimes t_1} \otimes \dots \otimes s^{n_{k-1}-n_k} \otimes 1^{\otimes t_k}) b_{t_0+\dots+t_k+k} (-s)^{n_k-n_0} \\ = (-)^\sigma \sum_{t_0,\dots,t_k\geqslant 0} &\bigl[(x^0)^{\otimes t_0} \otimes r^1 s^{n_0-n_1} \otimes (x^1)^{\otimes t_1} \otimes \dots \otimes r^k s^{n_{k-1}-n_k} \otimes (x^k)^{\otimes t_k}\bigr] b_{t_0+\dots+t_k+k} (-s)^{n_k-n_0}.\end{aligned}$$

Here the sign is determined by

$$\sigma = \sum_{1\leqslant q<p\leqslant k} (n_{q-1} - n_q) \deg r_p \tag{12.2.4}$$

due to $\deg x^p_{ij}$ being 0. Applying the right hand side of (12.2.1) to the same $r^p \in s\mathcal{C}^{\mathsf{mc}}(X^{p-1}, X^p)[n_p - n_{p-1}]$ we get

$$
\begin{aligned}
(r^1 \otimes \dots \otimes r^k) b_k^{\mathsf{mc}[]} &= (r^1 \otimes \dots \otimes r^k)(s^{n_0-n_1} \otimes \dots \otimes s^{n_{k-1}-n_k}) b_k^{\mathsf{mc}} (-s)^{n_k-n_0} \\
&= (-)^\sigma (r^1 s^{n_0-n_1} \otimes \dots \otimes r^k s^{n_{k-1}-n_k}) b_k^{\mathsf{mc}} (-s)^{n_k-n_0} \\
= (-)^\sigma \sum_{t_0,\dots,t_k \geqslant 0} \bigl[(x^0)^{\otimes t_0} &\otimes r^1 s^{n_0-n_1} \otimes (x^1)^{\otimes t_1} \otimes \dots \otimes r^k s^{n_{k-1}-n_k} \otimes (x^k)^{\otimes t_k}\bigr] b_{t_0+\dots+t_k+k} (-s)^{n_k-n_0},
\end{aligned}
$$

which coincides with the left hand side. Thus $\mathfrak{c}_{\mathcal{C}}$ is a strict A_∞-functor.

12.2.1 Unitality of the commutation morphism.

Let us prove that $\mathfrak{c}$ is a natural A_∞-2-transformation. First of all, we shall prove that $\mathfrak{c}_{\mathcal{C}} : \mathcal{C}^{\mathsf{mc}[]} \to \mathcal{C}^{[]\mathsf{mc}}$ is unital if $\mathcal{C}$ is. Indeed, the unit elements of $\mathcal{C}^{[]\mathsf{mc}}$ and $\mathcal{C}^{\mathsf{mc}[]}$ belonging to the same $\Bbbk$-module

$$
\begin{aligned}
s\mathcal{C}^{[]\mathsf{mc}}(X[n]\mathfrak{c}, X[n]\mathfrak{c}) = \prod_{i,j\in I} s\mathcal{C}^{[]}(X_i[n], X_j[n]) &= \prod_{i,j\in I} s\mathcal{C}(X_i, X_j) \\
&= s\mathcal{C}^{\mathsf{mc}}(X, X) = s\mathcal{C}^{\mathsf{mc}[]}(X[n], X[n])
\end{aligned}
$$

are

$$
{}_{X[n]\mathfrak{c}}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}} = \sum_{t\geqslant 0} (x^{\otimes t}) \mathbf{i}_t^{\mathcal{C}^{[]}} = \sum_{t\geqslant 0} (x^{\otimes t}) (\mathbf{i}^{\mathcal{C}})_t^{[]} = \sum_{t\geqslant 0} (x^{\otimes t}) \mathbf{i}_t^{\mathcal{C}} = {}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}[]}},
$$

due to already proven equations $(\mathbf{i}^{\mathcal{C}})^{\mathsf{mc}} = \mathbf{i}^{\mathcal{C}^{\mathsf{mc}}}$ and $(\mathbf{i}^{\mathcal{C}})^{[]} = \mathbf{i}^{\mathcal{C}^{[]}}$, and due to formula (10.19.5) for $(\mathbf{i}^{\mathcal{C}})^{[]}$. Notice that here all objects $X_i \in \operatorname{Ob}\mathcal{C}$ are shifted by the same integer n. Therefore, the identity map $\mathfrak{c}_1$ maps the unit element ${}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}[]}}$ to the unit element ${}_{X[n]}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}}$, thus, $\mathfrak{c}$ is unital.

12.2.2 The natural transformation $\mathfrak{c}$.

Let us prove that $\mathfrak{c}$ is a natural transformation, that is, for an arbitrary A_∞-functor $f : \mathcal{A} \to \mathcal{B}$ the following equation holds:

$$
\begin{array}{ccc}
\mathcal{A}^{\mathsf{mc}[]} & \xrightarrow{\mathfrak{c}_{\mathcal{A}}} & \mathcal{A}^{[]\mathsf{mc}} \\
{\scriptstyle f^{\mathsf{mc}[]}}\downarrow & = & \downarrow{\scriptstyle f^{[]\mathsf{mc}}} \\
\mathcal{B}^{\mathsf{mc}[]} & \xrightarrow{\mathfrak{c}_{\mathcal{B}}} & \mathcal{B}^{[]\mathsf{mc}}
\end{array}
$$

Since $\mathfrak{c}$ is strict this equation reduces to:

$$
\begin{aligned}
(\mathfrak{c}_1 \otimes \dots \otimes \mathfrak{c}_1) f_k^{[]\mathsf{mc}} = f_k^{\mathsf{mc}[]} \mathfrak{c}_1 &: \\
\otimes^{p\in\mathbf{k}} s\mathcal{A}^{\mathsf{mc}[]}(X^{p-1}[n_{p-1}], X^p[n_p]) &\to s\mathcal{B}^{[]\mathsf{mc}}(X^0 f[n_0]\mathfrak{c}, X^k f[n_k]\mathfrak{c}). \quad (12.2.5)
\end{aligned}
$$

Let X^p, x^p, r^p be given by (12.2.2), (12.2.3). In the left hand side of (12.2.5) we obtain

$$(r^1\otimes\cdots\otimes r^k)f_k^{[]\mathsf{mc}} = \sum_{t_0,\dots,t_k\geqslant 0}\bigl[(x^0)^{\otimes t_0}\otimes r^1\otimes(x^1)^{\otimes t_1}\otimes\cdots\otimes r^k\otimes(x^k)^{\otimes t_k}\bigr]f^{[]}_{t_0+\dots+t_k+k}$$

$$= \sum_{t_0,\dots,t_k\geqslant 0}\bigl[(x^0)^{\otimes t_0}\otimes r^1\otimes(x^1)^{\otimes t_1}\otimes\cdots\otimes r^k\otimes(x^k)^{\otimes t_k}\bigr]$$
$$(1^{\otimes t_0}\otimes s^{n_0-n_1}\otimes 1^{\otimes t_1}\otimes\cdots\otimes s^{n_{k-1}-n_k}\otimes 1^{\otimes t_k})f_{t_0+\dots+t_k+k}s^{n_k-n_0}$$
$$= (-)^\sigma\sum_{t_0,\dots,t_k\geqslant 0}\bigl[(x^0)^{\otimes t_0}\otimes r^1s^{n_0-n_1}\otimes(x^1)^{\otimes t_1}\otimes\cdots\otimes r^ks^{n_{k-1}-n_k}\otimes(x^k)^{\otimes t_k}\bigr]f_{t_0+\dots+t_k+k}s^{n_k-n_0}.$$

The sign is determined by σ from (12.2.4). The right hand side of (12.2.5) gives

$$(r^1\otimes\cdots\otimes r^k)f_k^{\mathsf{mc}[]} = (r^1\otimes\cdots\otimes r^k)(s^{n_0-n_1}\otimes\cdots\otimes s^{n_{k-1}-n_k})f_k^{\mathsf{mc}}s^{n_k-n_0}$$
$$= (-)^\sigma(r^1s^{n_0-n_1}\otimes\cdots\otimes r^ks^{n_{k-1}-n_k})f_k^{\mathsf{mc}}s^{n_k-n_0}$$
$$= (-)^\sigma\sum_{t_0,\dots,t_k\geqslant 0}\bigl[(x^0)^{\otimes t_0}\otimes r^1s^{n_0-n_1}\otimes(x^1)^{\otimes t_1}\otimes\cdots\otimes r^ks^{n_{k-1}-n_k}\otimes(x^k)^{\otimes t_k}\bigr]f_{t_0+\dots+t_k+k}s^{n_k-n_0},$$

which coincides with the left hand side. Thus $\mathfrak{c} : -^{\mathsf{mc}[]}\to -^{[]\mathsf{mc}}$ is a natural transformation.

12.2.3 The natural A_∞-2-transformation $\mathfrak{c}$. Let us show that $\mathfrak{c}$ is an A_∞-2-transformation. We have already verified equation

$$\begin{tikzcd}
A_\infty(\mathcal{A},\mathcal{B}) \arrow[r, "_^{\mathsf{mc}[]}"] \arrow[d, "_^{[]\mathsf{mc}}"'] \arrow[dr, phantom, "="] & A_\infty(\mathcal{A}^{\mathsf{mc}[]},\mathcal{B}^{\mathsf{mc}[]}) \arrow[d, "(1\boxtimes\mathfrak{c}_{\mathcal{B}})M"] \\
A_\infty(\mathcal{A}^{[]\mathsf{mc}},\mathcal{B}^{[]\mathsf{mc}}) \arrow[r, "(\mathfrak{c}_{\mathcal{A}}\boxtimes 1)M"] & A_\infty(\mathcal{A}^{\mathsf{mc}[]},\mathcal{B}^{[]\mathsf{mc}})
\end{tikzcd}$$

on objects in Section 12.2.2. Let us verify this equation on morphisms. All A_∞-functors being strict in this diagram, we have to compare the first components:

$$\begin{tikzcd}
A_\infty(\mathcal{A},\mathcal{B})(f,g) \arrow[r, "(_^{\mathsf{mc}[]})_1"] \arrow[d, "(_^{[]\mathsf{mc}})_1"'] \arrow[dr, phantom, "="] & A_\infty(\mathcal{A}^{\mathsf{mc}[]},\mathcal{B}^{\mathsf{mc}[]})(f^{\mathsf{mc}[]},g^{\mathsf{mc}[]}) \arrow[d, "-\cdot\mathfrak{c}_{\mathcal{B}}"] \\
A_\infty(\mathcal{A}^{[]\mathsf{mc}},\mathcal{B}^{[]\mathsf{mc}})(f^{[]\mathsf{mc}},g^{[]\mathsf{mc}}) \arrow[r, "\mathfrak{c}_{\mathcal{A}}\cdot -"] & A_\infty(\mathcal{A}^{\mathsf{mc}[]},\mathcal{B}^{[]\mathsf{mc}})(f^{\mathsf{mc}[]}\mathfrak{c}_{\mathcal{B}},\mathfrak{c}_{\mathcal{A}}g^{[]\mathsf{mc}})
\end{tikzcd}$$

Consider an A_∞-transformation $q : f\to g : \mathcal{A}\to\mathcal{B}$. We have to verify that $q^{\mathsf{mc}[]}\cdot\mathfrak{c}_{\mathcal{B}} = \mathfrak{c}_{\mathcal{A}}\cdot q^{[]\mathsf{mc}}$. Since A_∞-functors $\mathfrak{c}$ are strict, this equation reduces to:

$$(\mathfrak{c}_{\mathcal{A}1}\otimes\cdots\otimes\mathfrak{c}_{\mathcal{A}1})q_k^{[]\mathsf{mc}} = q_k^{\mathsf{mc}[]}\mathfrak{c}_{\mathcal{B}1} :$$
$$\otimes^{p\in\mathbf{k}}s\mathcal{A}^{\mathsf{mc}[]}(X^{p-1}[n_{p-1}],X^p[n_p])\to s\mathcal{B}^{[]\mathsf{mc}}(X^0f[n_0]\mathfrak{c}_{\mathcal{B}},X^kg[n_k]\mathfrak{c}_{\mathcal{B}}).$$

It resembles equation (12.2.5) and is proved in the same way as in Section 12.2.2 (replace f_k with q_k, and add the sign $(-)^{q(n_k-n_0)}$ in appropriate lines). Therefore, $\mathfrak{c}$ is an A_∞-2-transformation which restricts to an A^u_∞-2-transformation in the unital case.

12.3 Proposition. *The A_∞-2-functor $\mathsf{tr} : A_\infty \to A_\infty$, (resp. A^u_∞-2-functor $\mathsf{tr} : A^u_\infty \to A^u_\infty$) $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$, equipped with the unit*

$$u_{\mathsf{tr}} = \bigl(\mathrm{Id} \xrightarrow{u_{[]}} \_^{[]} \xrightarrow{u_{\mathsf{mc}}} \_^{[]\mathsf{mc}}\bigr)$$

and with the multiplication

$$m_{\mathsf{tr}} = \bigl(\_^{[]\mathsf{mc}[]\mathsf{mc}} \xrightarrow{\mathfrak{c}^{\mathsf{mc}}} \_^{[][]\mathsf{mc}\,\mathsf{mc}} \xrightarrow{m_{\mathsf{mc}}} \_^{[][]\mathsf{mc}} \xrightarrow{m_{[]}^{\mathsf{mc}}} \_^{[]\mathsf{mc}}\bigr) \tag{12.3.1}$$

is an A_∞-2-monad (resp. A^u_∞-2-monad).

Proof. The conclusion of the proposition follows from commutation relations between $\mathfrak{c}$ and $u_{[]}$, u_{mc}, m_{mc}, $m_{[]}$ – distributivity laws of Beck [Bec69]. They are proven in four lemmata below.

12.4 Lemma. *For an arbitrary A_∞-category $\mathcal{C}$ the following equation holds*

$$u_{[]}^{\mathsf{mc}} = \bigl(\mathcal{C}^{\mathsf{mc}} \xrightarrow{u_{[]}} \mathcal{C}^{\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{C}^{[]\mathsf{mc}}\bigr). \tag{12.4.1}$$

Proof. Let $(X : I \ni i \mapsto X_i, x = (x_{ij}))$, $x_{ij} \in s\mathcal{C}(X_i, X_j)$ be an object of $\mathcal{C}^{\mathsf{mc}}$. Then $Xu_{[]}\mathfrak{c} = X[0]\mathfrak{c} : i \mapsto X_i[0]$ is equipped with the matrix (x_{ij}), $x_{ij} \in s\mathcal{C}^{[]}(X_i[0], X_j[0])$. The same data determine the object $Xu_{[]}^{\mathsf{mc}} : I \to \mathcal{C}^{[]}$, since $u_{[]}$ is strict. Therefore both sides of (12.4.1) give the same map on objects. The both sides are strict A_∞-functors. The first components of both sides give the same identity map on morphisms: $\mathrm{id} : \prod_{i\in I, j\in J} s\mathcal{C}(X_i, Y_j) \Longrightarrow \prod_{i\in I, j\in J} s\mathcal{C}^{[]}(X_i[0], Y_j[0])$. □

12.5 Lemma. *For an arbitrary A_∞-category $\mathcal{C}$ the following equation holds*

$$u_{\mathsf{mc}} = \bigl(\mathcal{C}^{[]} \xrightarrow{u_{\mathsf{mc}}^{[]}} \mathcal{C}^{\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{C}^{[]\mathsf{mc}}\bigr). \tag{12.5.1}$$

Proof. An object $X[n]$ of $\mathcal{C}^{[]}$ is mapped by both sides of (12.5.1) to $(\mathbf{1} \ni 1 \mapsto X[n], 0)$. The both sides are strict A_∞-functors. The first components of both sides give the same identity map on morphisms $\mathrm{id} : s\mathcal{C}^{[]}(X[n], Y[m]) \to s\mathcal{C}^{[]\mathsf{mc}}((1 \mapsto X[n], 0), (1 \mapsto Y[m], 0))$. □

12.6 Lemma. *For an arbitrary A_∞-category $\mathcal{C}$ the following equation holds*

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}[]} \arrow[r, "\mathfrak{c}"] \arrow[d, "m_{\mathsf{mc}}^{[]}"'] & \mathcal{C}^{\mathsf{mc}[]\mathsf{mc}} \arrow[r, "\mathfrak{c}^{\mathsf{mc}}"] \arrow[d, phantom, "="] & \mathcal{C}^{[]\mathsf{mc}\,\mathsf{mc}} \arrow[d, "m_{\mathsf{mc}}"] \\
\mathcal{C}^{\mathsf{mc}[]} \arrow[rr, "\mathfrak{c}"] & & \mathcal{C}^{[]\mathsf{mc}}
\end{tikzcd} \tag{12.6.1}$$

Proof. An object X of $\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}[]}$ is described by the following data:

$$(X : I \to \mathcal{C}^{\mathsf{mc}}, i \mapsto X^i, x)[n],\ x = (x_{ii'})_{i,i' \in I},\ (X^i : J^i \to \mathcal{C}, j \mapsto X^i_j, x^i),\ x^i = (x^i_{jj'})_{j,j' \in J^i}.$$

Denote $K = \sqcup_{i\in I} J^i$. Then

$$\begin{aligned}
X\mathfrak{c} &= (i \mapsto (X^i : J^i \to \mathcal{C}, j \mapsto X^i_j, x^i)[n], x), \\
X\mathfrak{c}\mathfrak{c}^{\mathsf{mc}} &= (i \mapsto (X^i : J^i \to \mathcal{C}, j \mapsto X^i_j[n], x^i), x), \\
X\mathfrak{c}\mathfrak{c}^{\mathsf{mc}} m_{\mathsf{mc}} &= (\tilde{X} : K \ni k = (i, j \in J^i) \mapsto X^i_j[n], x + \operatorname{diag}(x^i)), \\
X m^{[]}_{\mathsf{mc}} &= (K \ni k = (i, j \in J^i) \mapsto X^i_j, x + \operatorname{diag}(x^i))[n], \\
X m^{[]}_{\mathsf{mc}}\mathfrak{c} &= (\tilde{X} : K \ni k = (i, j \in J^i) \mapsto X^i_j[n], x + \operatorname{diag}(x^i)).
\end{aligned}$$

Therefore, the objects $X\mathfrak{c}\mathfrak{c}^{\mathsf{mc}} m_{\mathsf{mc}}$ and $X m^{[]}_{\mathsf{mc}}\mathfrak{c}$ coincide.

All A_∞-functors in diagram (12.6.1) are strict. All their first components are identity maps. Indeed, the top–right path gives

$$\begin{aligned}
&s\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}[]}\big((X : i \mapsto (X^i : J^i \ni j \mapsto X^i_j, x^i), x)[n], (Y : l \mapsto (Y^l : P^l \ni p \mapsto Y^l_p, y^l), y)[m]\big) \\
&\xrightarrow{\mathfrak{c}_1} \prod_{i\in I, l \in L} s\mathcal{C}^{\mathsf{mc}[]}\big((X^i : J^i \ni j \mapsto X^i_j, x^i)[n], (Y^l : P^l \ni p \mapsto Y^l_p, y^l)[m]\big) \\
&\xrightarrow{\mathfrak{c}^{\mathsf{mc}}_1} \prod_{i\in I, l\in L} \prod_{j \in J^i, p \in P^l} s\mathcal{C}^{[]}(X^i_j[n], Y^l_p[m]) \xrightarrow{m_{\mathsf{mc}1}} \prod_{k\in K, q \in Q} s\mathcal{C}^{[]}(\tilde{X}^k[n], \tilde{Y}^q[m]),
\end{aligned}$$

where $Q = \sqcup_{l\in L} P^l$. The left–bottom path gives

$$\begin{aligned}
&s\mathcal{C}^{\mathsf{mc}\,\mathsf{mc}}\big((X : i \mapsto (X^i : J^i \ni j \mapsto X^i_j, x^i), x), (Y : l \mapsto (Y^l : P^l \ni p \mapsto Y^l_p, y^l), y)\big)[m-n] \\
&\xrightarrow{s^{n-m} m_{\mathsf{mc}1} s^{m-n}} s\mathcal{C}^{\mathsf{mc}}\big((\tilde{X} : K \ni k \mapsto \tilde{X}^k, x + \operatorname{diag}(x^i)), (\tilde{Y} : Q \ni q \mapsto \tilde{Y}^q, y + \operatorname{diag}(y^l))\big)[m-n] \\
&\xrightarrow{\mathfrak{c}_1} \prod_{k\in K, q\in Q} s\mathcal{C}^{[]}(\tilde{X}^k[n], \tilde{Y}^q[m]).
\end{aligned}$$

Therefore the both compositions are equal to the identity map, hence, coincide. □

12.7 Lemma. *For an arbitrary A_∞-category $\mathcal{C}$ the following equation holds*

$$\begin{tikzcd}
\mathcal{C}^{\mathsf{mc}[][]} \arrow[r, "\mathfrak{c}^{[]}"] \arrow[d, "m_{[]}"'] & \mathcal{C}^{[]\mathsf{mc}[]} \arrow[r, "\mathfrak{c}"] & \mathcal{C}^{[][]\mathsf{mc}} \arrow[d, "m^{\mathsf{mc}}_{[]}"] \\
\mathcal{C}^{\mathsf{mc}[]} \arrow[rr, "\mathfrak{c}"] & & \mathcal{C}^{[]\mathsf{mc}}
\end{tikzcd} \tag{12.7.1}$$

Proof. On the top–right path the objects are mapped as follows:

$$
\begin{aligned}
(X : I \ni i \mapsto X^i, x)[n][m] &\xmapsto{\mathfrak{c}^{[]}} (X : I \ni i \mapsto X^i[n], x)[m] \\
&\xmapsto{\mathfrak{c}} (X : I \ni i \mapsto X^i[n][m], x) \xmapsto{m^{\mathsf{mc}}_{[]}} (X : I \ni i \mapsto X^i[n+m], x).
\end{aligned}
$$

The left–bottom path gives

$$
(X : I \ni i \mapsto X^i, x)[n][m] \xmapsto{m_{[]}} (X : I \ni i \mapsto X^i, x)[n+m] \xmapsto{\mathfrak{c}} (X : I \ni i \mapsto X^i[n+m], x),
$$

which coincides with the above result.

All A_∞-functors in diagram (12.7.1) are strict. Their first components are acting as follows. On the top–right path:

$$
\begin{aligned}
\prod_{i\in I,\, j\in J} s\mathcal{C}(X_i, Y_j)[k-n][l-m] &\xRightarrow{s^{m-l}\mathfrak{c}_1 s^{l-m}} \prod_{i\in I,\, j\in J} s\mathcal{C}^{[]}(X_i[n], Y_j[k])[l-m] \\
\xRightarrow{\mathfrak{c}_1} \prod_{i\in I,\, j\in J} s\mathcal{C}^{[][]}(X_i[n][m], Y_j[k][l]) &\xrightarrow[m_{[]1}]{(-1)^{k(m-l)}} \prod_{i\in I,\, j\in J} s\mathcal{C}^{[]}(X_i[n+m], Y_j[k+l]).
\end{aligned}
$$

On the left–bottom path:

$$
\begin{aligned}
s\mathcal{C}^{\mathsf{mc}[][]}(X[n][m], Y[k][l]) &\xrightarrow[m_{[]1}]{(-1)^{k(m-l)}} s\mathcal{C}^{\mathsf{mc}[]}(X[n+m], Y[k+l]) \\
&\xRightarrow{\mathfrak{c}_1} \prod_{i\in I,\, j\in J} s\mathcal{C}^{[]}(X_i[n+m], Y_j[k+l]).
\end{aligned}
$$

These compositions are both equal to multiplication by $(-1)^{k(m-l)}$, hence, coincide. □

The proof of Proposition 12.3 is complete. □

12.8 Remark. If $\mathcal{C}$ is strictly unital, then $\mathcal{C}^{\mathsf{tr}}$ is strictly unital as well. This follows from Remark 10.29 and Lemma 11.24.

Let us describe the strict A_∞-functor $m_{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}}$ explicitly. By definition (12.3.1) this is a composition of three strict A_∞-functors, two of which act as identity on morphisms and the third acts via a cocycle. First we consider the strict A_∞-functor m_{tr} on objects.

An object X of $\mathcal{C}^{\mathsf{tr}\,\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}[]\mathsf{mc}}$ is specified by the following data:

— finite totally ordered sets $I \in \operatorname{Ob}\mathcal{P}$, $J^i \in \operatorname{Ob}\mathcal{P}$ for every $i \in I$;
— objects X^i_j of $\mathcal{C}$ for all $i \in I$, $j \in J^i$;
— integers m^i, n^i_j for all $i \in I$, $j \in J^i$;

— and the following matrices x, x^{ii}, $i \in I$ of morphisms:

$$\begin{gathered}\bigl(X : I \ni i \mapsto (X^i : J^i \ni j \mapsto X^i_j[n^i_j], x^{ii} = (x^{ii}_{jj'})_{j,j' \in J^i})[m^i], x = (x^{ii'})_{i,i' \in I}\bigr),\\ x^{ii}_{jj'} \in \mathcal{C}^{[]}(X^i_j[n^i_j], X^i_{j'}[n^i_{j'}])[1]^0 = \mathcal{C}(X^i_j, X^i_{j'})[n^i_{j'} - n^i_j + 1]^0,\\ x^{ii'} \in \mathcal{C}^{[]\mathsf{mc}[]}(X^i[m^i], X^{i'}[m^{i'}])[1]^0 = \mathcal{C}(X^i, X^{i'})[m^{i'} - m^i + 1]^0,\end{gathered}$$

where X^i and X specify A_∞-functors. It means that $j \geqslant j'$ implies $x^{ii}_{jj'} = 0$, $i \geqslant i'$ implies $x^{ii'} = 0$, the Maurer–Cartan equations $\sum_{t \geqslant 0} x^{\otimes t} b_t^{[]\mathsf{mc}[]} = 0$ and $\sum_{t \geqslant 0} (x^{ii})^{\otimes t} b_t^{[]} = 0$ hold for all $i \in I$.

Applying $\mathfrak{c}^{\mathsf{mc}}$ to X gives

$$\begin{gathered}\bigl(X\mathfrak{c}^{\mathsf{mc}} : I \ni i \mapsto (\tilde{X}^i : J^i \ni j \mapsto X^i_j[n^i_j][m^i], x^{ii} = (x^{ii}_{jj'})_{j,j' \in J^i}), x = (x^{ii'})_{i,i' \in I}\bigr),\\ x^{ii}_{jj'} \in \mathcal{C}^{[][]}(X^i_j[n^i_j][m^i], X^i_{j'}[n^i_{j'}][m^i])[1]^0 = \mathcal{C}(X^i_j, X^i_{j'})[n^i_{j'} - n^i_j + 1]^0,\\ x^{ii'} \in \mathcal{C}^{[][]\mathsf{mc}}(\tilde{X}^i, \tilde{X}^{i'})[1]^0 = \prod_{j \in J^i, j' \in J^{i'}} \mathcal{C}^{[][]}(X^i_j[n^i_j][m^i], X^{i'}_{j'}[n^{i'}_{j'}][m^{i'}])[1]^0\\ = \prod_{j \in J^i, j' \in J^{i'}} \mathcal{C}(X^i_j, X^{i'}_{j'})[n^{i'}_{j'} + m^{i'} - n^i_j - m^i + 1]^0 = \mathcal{C}^{[]\mathsf{mc}[]}(X^i[m^i], X^{i'}[m^{i'}])[1]^0.\end{gathered}$$

The elements $x^{ii'}$ are identified with matrices $(x^{ii'}_{jj'})^{j' \in J^{i'}}_{j \in J^i}$, where

$$x^{ii'}_{jj'} \in \mathcal{C}^{[]}(X^i_j[n^i_j], X^{i'}_{j'}[n^{i'}_{j'}])[m^{i'} - m^i + 1]^0 = \mathcal{C}(X^i_j, X^{i'}_{j'})[n^{i'}_{j'} + m^{i'} - n^i_j - m^i + 1]^0.$$

Denote $IJ = \bigsqcup_{i \in I} J^i = \bigl\{(i,j) \in I \times \bigcup_{i \in I} J^i \mid j \in J^i\bigr\}$, and consider the $IJ \times IJ$-matrix $\tilde{x} = x + \operatorname{diag}(x^{ii})$, thus,

$$\tilde{x}^{ii'}_{jj'} = x^{ii'}_{jj'} \in \mathcal{C}^{[][]}(X^i_j[n^i_j][m^i], X^{i'}_{j'}[n^{i'}_{j'}][m^{i'}])[1]^0 = \mathcal{C}(X^i_j, X^{i'}_{j'})[m^{i'} + n^{i'}_{j'} - m^i - n^i_j + 1]^0$$

if $i < i'$, or if $i = i'$ and $j < j'$, otherwise $\tilde{x}^{ii'}_{jj'} = 0$.

Applying m_{mc} to $X\mathfrak{c}^{\mathsf{mc}}$ gives

$$\bigl(X\mathfrak{c}^{\mathsf{mc}} m_{\mathsf{mc}} : IJ \ni (i,j) \mapsto X^i_j[n^i_j][m^i], \tilde{x} = (\tilde{x}^{ii'}_{jj'})_{(i,j),(i',j') \in IJ}\bigr).$$

due to Section 11.28. Finally, $m^{\mathsf{mc}}_{[]}$ takes $X\mathfrak{c}^{\mathsf{mc}} m_{\mathsf{mc}}$ to $X m_{\mathsf{tr}}$:

$$\begin{gathered}\bigl(X m_{\mathsf{tr}} : IJ \ni (i,j) \mapsto X^i_j[n^i_j + m^i], \bar{x} = (\bar{x}^{ii'}_{jj'})_{(i,j),(i',j') \in IJ}\bigr),\\ \bar{x}^{ii'}_{jj'} = (-1)^{n^{i'}_{j'}(m^i - m^{i'})} x^{ii'}_{jj'} \in \mathcal{C}^{[]}(X^i_j[n^i_j + m^i], X^{i'}_{j'}[n^{i'}_{j'} + m^{i'}])[1]^0\end{gathered} \tag{12.8.1}$$

due to Section 10.30.2. In particular, $\bar{x}^{ii}_{jj'} = x^{ii}_{jj'}$, so diagonal blocks are not changed.

Now we consider the first component of the strict A_∞-functor m_{tr}. Let Y be another object of $\mathcal{C}^{\mathsf{tr}\,\mathsf{tr}}$,

$$\big(Y : K \ni k \mapsto (Y^k : L^k \ni l \mapsto Y^k_l[p^k_l], y^{kk} = (y^{kk}_{ll'})_{l,l'\in L^k})[q^k], y = (y^{kk'})_{k,k'\in K}\big).$$

A morphism $f \in \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}}(X,Y)[1]$ identifies with the matrix $(f^{ik}_{jl})^{i\in I,k\in K}_{j\in J^i,l\in L^k}$

$$f^{ik}_{jl} \in \mathcal{C}(X^i_j, Y^k_l)[p^k_l + q^k - n^i_j - m^i + 1] \simeq \mathcal{C}^{\mathsf{tr}}(X^i, Y^k)[q^k - m^i + 1].$$

It is mapped by $\mathfrak{c}^{\mathsf{mc}}_1$ to $f\mathfrak{c}^{\mathsf{mc}}_1 \in \mathcal{C}^{[][]\mathsf{mc}\,\mathsf{mc}}(X\mathfrak{c}^{\mathsf{mc}}, Y\mathfrak{c}^{\mathsf{mc}})[1]$ identified with the same matrix,

$$f^{ik}_{jl} \in \mathcal{C}^{[][]}(X^i_j[n^i_j][m^i], Y^k_l[p^k_l][q^k])[1].$$

Denote $KL = \bigsqcup_{k\in K} L^k$. The previous morphism is mapped by $(m_{\mathsf{mc}})_1$ to $f\mathfrak{c}^{\mathsf{mc}}_1(m_{\mathsf{mc}})_1 \in \mathcal{C}^{[][]\mathsf{mc}}(X\mathfrak{c}^{\mathsf{mc}}m_{\mathsf{mc}}, Y\mathfrak{c}^{\mathsf{mc}}m_{\mathsf{mc}})[1]$ identified with the same matrix $(f^{ik}_{jl})_{(i,j)\in IJ,(k,l)\in KL}$ due to Section 11.28. Finally, $(m^{\mathsf{mc}}_{[]})_1$ takes this morphism to $f(m_{\mathsf{tr}})_1 \in \mathcal{C}^{\mathsf{tr}}(Xm_{\mathsf{tr}}, Ym_{\mathsf{tr}})[1]$, determined by the matrix

$$f(m_{\mathsf{tr}})_1 = \big((-1)^{p^k_l(m^i-q^k)} f^{ik}_{jl}\big)_{(i,j)\in IJ,(k,l)\in KL}, \quad f^{ik}_{jl} \in \mathcal{C}^{[]}(X^i_j[n^i_j + m^i], Y^k_l[p^k_l + q^k])[1]$$

due to Section 10.30.2.

By Proposition 12.3 the A_∞-2-monad $\mathcal{C} \mapsto \mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ has the unit

$$u_{\mathsf{tr}} = \big(\mathcal{C} \xrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}} \xrightarrow{u^{\mathsf{mc}}_{[]}} \mathcal{C}^{[]\mathsf{mc}}\big) = \big(\mathcal{C} \xrightarrow{u_{[]}} \mathcal{C}^{[]} \xrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{[]\mathsf{mc}}\big).$$

12.9 Definition. We say that a unital A_∞-category $\mathcal{C}$ is *pretriangulated* if every object X of $\mathcal{C}^{\mathsf{tr}}$ is isomorphic in $\mathcal{C}^{\mathsf{tr}}$ to Yu_{tr} for some object Y of $\mathcal{C}$.

12.10 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ is pretriangulated;

(ii) the A_∞-functor $u_{\mathsf{tr}} : \mathcal{C} \to \mathcal{C}^{\mathsf{tr}}$ is an equivalence;

(iii) $\mathcal{C}$ is closed under shifts and mc*-closed;*

(iv) the A_∞-functors $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ and $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ are equivalences.

Proof. (i) $\Longleftrightarrow$ (ii): Both A_∞-functors $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ and $u^{\mathsf{mc}}_{[]} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}^{[]\mathsf{mc}}$ are strict, and their first components are isomorphisms. Their composition $u_{\mathsf{tr}} : \mathcal{C} \to \mathcal{C}^{\mathsf{tr}}$ has the same properties. So it is an equivalence if and only if it is essentially surjective on objects [Lyu03, Theorem 8.8].

(iii) $\Longleftrightarrow$ (iv): Proved in Propositions 10.33 and 11.32.

(iv) $\Longrightarrow$ (ii): Since $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an A_∞-equivalence, so is $u_{[]}^{\mathsf{mc}} : \mathcal{C}^{\mathsf{mc}} \to \mathcal{C}^{[]\mathsf{mc}}$. Therefore, its composition u_{tr} with the A_∞-equivalence $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ is also an A_∞-equivalence.

(ii) $\Longrightarrow$ (iv): The composition u_{tr} of two full strict embeddings $u_{\mathsf{mc}} : \mathcal{C} \hookrightarrow \mathcal{C}^{\mathsf{mc}}$ and $u_{[]}^{\mathsf{mc}} : \mathcal{C}^{\mathsf{mc}} \hookrightarrow \mathcal{C}^{[]\mathsf{mc}}$ is essentially surjective on objects. Therefore, both $u_{[]}^{\mathsf{mc}}$ and u_{mc} are essentially surjective on objects (the latter by Lemma 10.42). From the presentation of u_{tr} as composition of full embeddings $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ and $u_{\mathsf{mc}} : \mathcal{C}^{[]} \to \mathcal{C}^{[]\mathsf{mc}}$ we find that both are essentially surjective on objects (the former by Lemma 10.42). Therefore, all mentioned A_∞-functors are equivalences. □

12.11 Proposition. *Let $\mathcal{C}$ be a unital A_∞-category. Then the A_∞-category $\mathcal{C}^{\mathsf{tr}}$ is pretriangulated, closed under shifts and* mc*-closed. The A_∞-functors $u_{\mathsf{tr}}, u_{\mathsf{tr}}^{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}}$ and $m_{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}}$ are equivalences, quasi-inverse to each other.*

Proof. The A_∞-category $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ is mc-closed by Proposition 11.34, hence, $u_{\mathsf{mc}} : \mathcal{C}^{\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}\,\mathsf{mc}}$ is an A_∞-equivalence by Corollary 11.35. Similarly to Proposition 10.35 we are going to prove that $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ is closed under shifts. This amounts to proving that $u_{[]} : \mathcal{C}^{[]\mathsf{mc}} \to \mathcal{C}^{[]\mathsf{mc}[]}$ is an A_∞-equivalence. Recall that $\otimes_{\mathbb{Z}}$ and the multiplication A_∞-functor $m_{[]} : \mathcal{C}^{[][]} \to \mathcal{C}^{[]}$ depend on a sign, determined by the function $\psi_1(a,b,c,d) = (-1)^{c(b-d)}$. Consider the A_∞-functor

$$g = \big(\mathcal{C}^{[]\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{C}^{[][]\mathsf{mc}} \xrightarrow{m_{[]}^{\mathsf{mc}}} \mathcal{C}^{[]\mathsf{mc}}\big), \tag{12.11.1}$$

whose source and target are the same as target and source of the studied functor $u_{[]}$. Let $X : I \to \mathcal{C}^{[]}$, $i \mapsto X_i[n_i]$, $x = (x_{ij})_{i,j\in I}$, $x_{ij} \in \big[s\mathcal{C}^{[]}(X_i[n_i], X_j[n_j])\big]^0$ be an object of $\mathcal{C}^{\mathsf{tr}}$. The first functor from (12.11.1) takes an object $X[m]$ of $\mathcal{C}^{\mathsf{tr}[]}$ to the object $X' = X[m]\mathfrak{c} : I \to \mathcal{C}^{[][]}$, $i \mapsto X_i[n_i][m]$, $x' = x = (x_{ij})$, $x'_{ij} \in s\mathcal{C}^{[][]}(X_i[n_i][m], X_j[n_j][m])$ of $\mathcal{C}^{[][]\mathsf{mc}}$. The second takes it further to the object $Y : I \to \mathcal{C}^{[]}$, $i \mapsto X_i[n_i+m]$, $y = (\psi_1(n_i,m,n_j,m)x_{ij})_{i,j\in I} = (x_{ij})_{i,j\in I}$, $y_{ij} = x_{ij} \in s\mathcal{C}^{[]}(X_i[n_i+m], X_j[n_j+m])$ of $\mathcal{C}^{\mathsf{tr}}$, due to our choice of $\psi_1(n_i,m,n_j,m) = 1$. Let us prove that $X[m]$ is isomorphic to $Y[0]$ in $\mathcal{C}^{\mathsf{tr}[]}$.

Applying the same A_∞-functor g to the object $Y[0]$ of $\mathcal{C}^{\mathsf{tr}[]}$ we get the object $Y' = Y[0]\mathfrak{c} : I \to \mathcal{C}^{[][]}$, $i \mapsto X_i[n_i+m][0]$, $y' = y = x$, $y'_{ij} \in s\mathcal{C}^{[][]}(X_i[n_i+m][0], X_j[n_j+m][0])$ of $\mathcal{C}^{[][]\mathsf{mc}}$, and $Y[0]g = Y$. The first component

$$g_1 = \big(s\mathcal{C}^{[]\mathsf{mc}[]}(X[m], Y[0]) \xrightarrow{\mathfrak{c}_1} s\mathcal{C}^{[][]\mathsf{mc}}(X', Y') \xrightarrow{(m_{[]})_1^{\mathsf{mc}}} s\mathcal{C}^{[]\mathsf{mc}}(Y,Y)\big)$$

is an isomorphism. It takes some element $q \in s\mathcal{C}^{[]\mathsf{mc}[]}(X[m], Y[0])$ to the unit element ${}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}}$. Similarly, the isomorphism

$$g_1 = \big(s\mathcal{C}^{[]\mathsf{mc}[]}(Y[0], X[m]) \xrightarrow{\mathfrak{c}_1} s\mathcal{C}^{[][]\mathsf{mc}}(Y', X') \xrightarrow{(m_{[]})_1^{\mathsf{mc}}} s\mathcal{C}^{[]\mathsf{mc}}(Y,Y)\big)$$

takes some element $t \in s\mathcal{C}^{[]\mathsf{mc}[]}(Y[0],X[m])$ to the unit element ${}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}}$. Since $\mathfrak{c}$, $m_{[]}$ are A_∞-functors, the elements q, t are cocycles.

The equations $m_{[]}b^{[]} = b^{[]}m_{[]}$, $(m_{[]})^{\mathsf{mc}}(b^{[]})^{\mathsf{mc}} = (b^{[]})^{\mathsf{mc}}(m_{[]})^{\mathsf{mc}}$ and property (12.2.1) imply commutativity of the following diagram

$$\begin{tikzcd} s\mathcal{C}^{[]\mathsf{mc}[]}(X[m],Y[0]) \otimes s\mathcal{C}^{[]\mathsf{mc}[]}(Y[0],X[m]) \arrow[r, "b_2^{[]\mathsf{mc}[]}"] \arrow[d, "\mathfrak{c}_1\otimes\mathfrak{c}_1"'] & s\mathcal{C}^{[]\mathsf{mc}[]}(X[m],X[m]) \arrow[d, "\mathfrak{c}_1"] \\ s\mathcal{C}^{[][]\mathsf{mc}}(X',Y') \otimes s\mathcal{C}^{[][]\mathsf{mc}}(Y',X') \arrow[r, "b_2^{[][]\mathsf{mc}}", "(b^{[][]})_2^{\mathsf{mc}}"'] \arrow[d, "(m_{[]})_1^{\mathsf{mc}}\otimes(m_{[]})_1^{\mathsf{mc}}"'] & s\mathcal{C}^{[][]\mathsf{mc}}(X',X') \arrow[d, "(m_{[]})_1^{\mathsf{mc}}"] \\ s\mathcal{C}^{[]\mathsf{mc}}(Y,Y) \otimes s\mathcal{C}^{[]\mathsf{mc}}(Y,Y) \arrow[r, "(b^{[]})_2^{\mathsf{mc}}", "b_2^{[]\mathsf{mc}}"'] & s\mathcal{C}^{[]\mathsf{mc}}(Y,Y) \end{tikzcd}$$

The result of Section 12.2.1 implies that ${}_{X[m]}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}[]}}\mathfrak{c}_1 = {}_{X'}\mathbf{i}_0^{\mathcal{C}^{[][]\mathsf{mc}}}$. Since $\psi_1(n_i,m,n_j,m) = 1$, we have ${}_{X'}\mathbf{i}_0^{\mathcal{C}^{[][]\mathsf{mc}}}(m_{[]})_1^{\mathsf{mc}} = {}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}}$. The image of the element $q \otimes t$ from the left top corner of this diagram is $({}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}} \otimes {}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}})b_2^{[]\mathsf{mc}}$. It differs by a boundary from ${}_Y\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}}} = {}_{X[m]}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}[]}}\mathfrak{c}_1(m_{[]})_1^{\mathsf{mc}}$. Hence, the difference $(q\otimes t)b_2^{[]\mathsf{mc}[]} - {}_{X[m]}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}[]}}$ is mapped by the isomorphism g_1 to a boundary. Therefore, this difference is a boundary itself.

Similarly, $(t\otimes q)b_2^{[]\mathsf{mc}[]} - {}_{Y[0]}\mathbf{i}_0^{\mathcal{C}^{[]\mathsf{mc}[]}} \in \operatorname{Im} b_1^{[]\mathsf{mc}[]}$. We conclude that $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$ is closed under shifts, and $u_{[]} : \mathcal{C}^{[]\mathsf{mc}} \to \mathcal{C}^{[]\mathsf{mc}[]}$ is an A_∞-equivalence. Since $-^{\mathsf{mc}}$ is an A_∞^u-2-functor, it projects to an ordinary strict 2-functor $\overline{-^{\mathsf{mc}}} : \overline{A_\infty^u} \to \overline{A_\infty^u}$ [LM06a, Section 3.2]. In particular, it takes A_∞-equivalences to A_∞-equivalences. Hence, $u_{[]}^{\mathsf{mc}} : \mathcal{C}^{[]\mathsf{mc}\,\mathsf{mc}} \to \mathcal{C}^{[]\mathsf{mc}[]\mathsf{mc}}$ is an A_∞-equivalence. Therefore, the composition

$$u_{\mathsf{tr}} = \bigl(\mathcal{C}^{\mathsf{tr}} \xrightarrow{u_{\mathsf{mc}}} \mathcal{C}^{\mathsf{tr}\,\mathsf{mc}} \xrightarrow{u_{[]}^{\mathsf{mc}}} \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}}\bigr)$$

is an A_∞-equivalence. Equations $u_{\mathsf{tr}}m_{\mathsf{tr}} = \mathrm{id}_{\mathcal{C}^{\mathsf{tr}}} = u_{\mathsf{tr}}^{\mathsf{tr}}m_{\mathsf{tr}}$ imply that $m_{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}}$ and $u_{\mathsf{tr}}^{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}\,\mathsf{tr}}$ are A_∞-equivalences as well. □

12.12 Corollary. *Let $\mathcal{A}$, $\mathcal{B}$ be equivalent unital A_∞-categories. If one of them is pretriangulated, then so is the other.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be an A_∞-equivalence, then $f^{\mathsf{tr}} : \mathcal{A}^{\mathsf{tr}} \to \mathcal{B}^{\mathsf{tr}}$ is A_∞-equivalence as well. In the equation

$$u_{\mathsf{tr}}f^{\mathsf{tr}} = fu_{\mathsf{tr}} : \mathcal{A} \to \mathcal{B}^{\mathsf{tr}}$$

three out of four A_∞-functors are equivalences. Hence, so is the fourth. □

We say that a unital A_∞-functor $f : \mathcal{A} \to \mathcal{C}$ is an *A_∞-equivalence with its image* if f factorizes as follows: $f = (\mathcal{A} \xrightarrow{g} \mathcal{B} \xhookrightarrow{e} \mathcal{C})$, where $e : \mathcal{B} \hookrightarrow \mathcal{C}$ is a strict full embedding with $\operatorname{Ob}\mathcal{B} = \operatorname{Im}\operatorname{Ob} f$, and the uniquely defined g is an A_∞-equivalence. This

is equivalent to homotopy invertibility of the first component $f_1 : s\mathcal{A}(X,Y) \to s\mathcal{C}(Xf,Yf)$ for all objects X, Y of $\mathcal{A}$. Such A_∞-functor f is also called *homotopy fully faithful.*

Let $\mathcal{A}$ be a unital A_∞-category. Recall that the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ is an A_∞-equivalence with its image – full differential graded subcategory [LM08c, Appendix A].

12.13 Proposition. *If a unital A_∞-functor $f : \mathcal{A} \to \mathcal{C}$ is an equivalence with its image, then so are $f^{[]} : \mathcal{A}^{[]} \to \mathcal{C}^{[]}$, $f^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{C}^{\mathsf{mc}}$ and $f^{\mathsf{tr}} : \mathcal{A}^{\mathsf{tr}} \to \mathcal{C}^{\mathsf{tr}}$.*

Proof. Let $g : \mathcal{A} \to \mathcal{B}$ be an A_∞-equivalence. Then $g^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to \mathcal{B}^{\mathsf{mc}}$ is an A_∞-equivalence as well. This follows from the fact that $-^{\mathsf{mc}} : A^u_\infty \to A^u_\infty$ is an A^u_∞-2-functor (it produces a strict 2-functor $\overline{-^{\mathsf{mc}}} : \overline{A^u_\infty} \to \overline{A^u_\infty}$, which maps equivalences to equivalences).

Suppose that the functor $f : \mathcal{A} \to \mathcal{C}$ is an A_∞-equivalence with its image. That is, f factorizes as follows: $f = (\mathcal{A} \xrightarrow{g} \mathcal{B} \xhookrightarrow{e} \mathcal{C})$, where $\mathcal{B}$ denotes the image of f, g is an A_∞-equivalence, and $e : \mathcal{B} \hookrightarrow \mathcal{C}$ is a strict full embedding. Applying $-^{\mathsf{mc}}$ we obtain $f^{\mathsf{mc}} = (\mathcal{A}^{\mathsf{mc}} \xrightarrow{g^{\mathsf{mc}}} \mathcal{B}^{\mathsf{mc}} \xrightarrow{e^{\mathsf{mc}}} \mathcal{C}^{\mathsf{mc}})$. By the above observation g^{mc} is an A_∞-equivalence. Since e is a strict A_∞-functor with $e_1 = \mathrm{id}$, the same holds for e^{mc}. This shows that the functor f^{mc} is also an A_∞-equivalence with its essential image.

Similar statement holds for $-^{[]}$: we can look at the first component

$$
\begin{aligned}
f_1^{[]} : s\mathcal{A}^{[]}(X[n],Y[m]) = s\mathcal{A}(X,Y)[m-n]& \\
\xrightarrow{f_1^{[m-n]}} s\mathcal{C}^{[]}(X[n]f^{[]},Y[m]f^{[]}) &= s\mathcal{C}(Xf,Yf)[m-n],
\end{aligned}
$$

it is homotopy invertible if so is f_1.

The first two claims imply the third one. □

12.14 Corollary. *The A_∞-functors $\mathscr{Y}^{[]} : \mathcal{A}^{[]} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)^{[]}$, $\mathscr{Y}^{\mathsf{mc}} : \mathcal{A}^{\mathsf{mc}} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)^{\mathsf{mc}}$ and $\mathscr{Y}^{\mathsf{tr}} : \mathcal{A}^{\mathsf{tr}} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)^{\mathsf{tr}}$ are A_∞-equivalences with its image.*

12.15 Proposition. *1) For an arbitrary pretriangulated A_∞-category $\mathcal{C}$ there exists an A_∞-equivalence $U_{\mathsf{tr}} = U^{\mathcal{C}}_{\mathsf{tr}} = \bigl(\mathcal{C}^{\mathsf{tr}} \xrightarrow{U^{\mathsf{mc}}_{[]}} \mathcal{C}^{\mathsf{mc}} \xrightarrow{U_{\mathsf{mc}}} \mathcal{C}\bigr)$ such that $u_{\mathsf{tr}} \cdot U_{\mathsf{tr}} = \mathrm{id}_{\mathcal{C}}$. In particular, U_{tr} is quasi-inverse to u_{tr}.*

2) Let $\mathcal{A}$, $\mathcal{C}$ be unital A_∞-categories, and let $\mathcal{C}$ be pretriangulated. Then the strict A_∞-functor $A^u_\infty(u_{\mathsf{tr}}, \mathcal{C}) = (u_{\mathsf{tr}} \boxtimes 1)M : A^u_\infty(\mathcal{A}^{\mathsf{tr}}, \mathcal{C}) \to A^u_\infty(\mathcal{A}, \mathcal{C})$ is an A_∞-equivalence which admits a one-sided inverse

$$
F_{\mathsf{tr}} \overset{def}{=} \bigl[A^u_\infty(\mathcal{A},\mathcal{C}) \xrightarrow{-^{\mathsf{tr}}} A^u_\infty(\mathcal{A}^{\mathsf{tr}},\mathcal{C}^{\mathsf{tr}}) \xrightarrow{A^u_\infty(\mathcal{A}^{\mathsf{tr}},U_{\mathsf{tr}})} A^u_\infty(\mathcal{A}^{\mathsf{tr}},\mathcal{C})\bigr] = F_{[]} \cdot F_{\mathsf{mc}},
$$

namely, $F_{\mathsf{tr}} \cdot A^u_\infty(u_{\mathsf{tr}}, \mathcal{C}) = \mathrm{id}_{A^u_\infty(\mathcal{A},\mathcal{C})}$.

The restriction map $\mathsf{A}^{\mathsf{u}}_\infty(u_{\mathsf{tr}}; 1) : \mathsf{A}^{\mathsf{u}}_\infty(\mathcal{A}^{\mathsf{tr}}; \mathcal{C}) \to \mathsf{A}^{\mathsf{u}}_\infty(\mathcal{A}; \mathcal{C})$ is surjective.

3) Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. Then the A_∞-functor

$$
-^{\mathsf{tr}} : A^u_\infty(\mathcal{A},\mathcal{B}) \to A^u_\infty(\mathcal{A}^{\mathsf{tr}},\mathcal{B}^{\mathsf{tr}})
$$

is homotopy full and faithful, that is, its first component is homotopy invertible.

Proof. 1) Since u_{mc} is a natural transformation we have $u_{\mathsf{tr}} \cdot U_{\mathsf{tr}} = u_{[]}u_{\mathsf{mc}}U_{[]}^{\mathsf{mc}}U_{\mathsf{mc}} = u_{[]}U_{[]}u_{\mathsf{mc}}U_{\mathsf{mc}} = \mathrm{id}\cdot\mathrm{id} = \mathrm{id}$.

2) Since $u_{\mathsf{tr}} = u_{[]} \cdot u_{\mathsf{mc}}$, the considered A_∞-functor is a composition of two A_∞-equivalences

$$A_\infty^u(u_{\mathsf{tr}}, \mathcal{C}) = \bigl[A_\infty^u(\mathcal{A}^{\mathsf{tr}}, \mathcal{C}) \xrightarrow{A_\infty^u(u_{\mathsf{mc}},\mathcal{C})} A_\infty^u(\mathcal{A}^{[]}, \mathcal{C}) \xrightarrow{A_\infty^u(u_{[]},\mathcal{C})} A_\infty^u(\mathcal{A}, \mathcal{C})\bigr]$$

from Propositions 10.39 and 11.37. The same propositions imply that $F_{[]} \cdot F_{\mathsf{mc}}$ is a one-sided inverse of $A_\infty^u(u_{\mathsf{tr}}, \mathcal{C})$. Let us prove that $F_{\mathsf{tr}} = F_{[]} \cdot F_{\mathsf{mc}}$. Indeed, these A_∞-functors are compositions of the two paths from the left top corner to the right bottom corner of the following diagram

$$\begin{tikzcd}
A_\infty^u(\mathcal{A},\mathcal{C}) \arrow[r, "-^{[]}"] & A_\infty^u(\mathcal{A}^{[]},\mathcal{C}^{[]}) \arrow[r, "-^{\mathsf{mc}}"] \arrow[d, "A_\infty^u(1,U_{[]})"] & A_\infty^u(\mathcal{A}^{\mathsf{tr}},\mathcal{C}^{\mathsf{tr}}) \arrow[d, "A_\infty^u(1,U_{[]}^{\mathsf{mc}})"'] & \\
F_{[]}\cdot F_{\mathsf{mc}} & A_\infty^u(\mathcal{A}^{[]},\mathcal{C}) \arrow[r, "-^{\mathsf{mc}}"] & A_\infty^u(\mathcal{A}^{\mathsf{tr}},\mathcal{C}^{\mathsf{mc}}) \arrow[r, "A_\infty^u(1,U_{\mathsf{mc}})"] & A_\infty^u(\mathcal{A}^{\mathsf{tr}},\mathcal{C})
\end{tikzcd}
\qquad F_{\mathsf{tr}}$$

It commutes since $-^{\mathsf{mc}} : A_\infty^u \to A_\infty^u$ is an A_∞^u-2-functor.

The restriction map $\mathsf{A}_\infty^{\mathsf{u}}(u_{\mathsf{tr}}; \mathcal{C}) = \mathsf{A}_\infty^{\mathsf{u}}(u_{\mathsf{mc}}; \mathcal{C}) \cdot \mathsf{A}_\infty^{\mathsf{u}}(u_{[]}; \mathcal{C})$ is a composition of two surjective mappings.

3) The considered A_∞-functor is a composition of two homotopy full and faithful A_∞-functors

$$-^{\mathsf{tr}} = \bigl[A_\infty^u(\mathcal{A}, \mathcal{B}) \xrightarrow{-^{[]}} A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]}) \xrightarrow{-^{\mathsf{mc}}} A_\infty^u(\mathcal{A}^{\mathsf{tr}}, \mathcal{B}^{\mathsf{tr}})\bigr]$$

from Corollaries 10.41 and 11.39. □

12.16 Remark. Let $\mathcal{A}$ be a unital A_∞-category. It follows from Section 10.47 and Proposition 11.44 that there is a strict full embedding

$$A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk})^{\mathsf{tr}} \xrightarrow{\underline{S}^{\mathsf{mc}}\cdot \mathsf{A}_\infty(u_{[]};1)^{\mathsf{mc}}} A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk}{}^{[]})^{\mathsf{mc}} \xrightarrow{\varpi_{\mathsf{mc}}} A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk}{}^{\mathsf{tr}}),$$

whose first component is bijective on morphisms. Since the differential graded category $\underline{\mathsf{C}}_{\Bbbk}$ is pretriangulated, the above functor extends to an A_∞-equivalence of $A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk})^{\mathsf{tr}}$ with $A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk})$. It follows from Corollary 12.14 that $\mathcal{A}^{\mathsf{tr}}$ is A_∞-equivalent to a full differential graded subcategory of $A_\infty^u(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_{\Bbbk})$.

12.17 Closed multicategory of pretriangulated A_∞-categories.

Let $\mathsf{A}_\infty^{\mathrm{tr}}$ denote the full submulticategory of $\mathsf{A}_\infty^{\mathrm{u}}$, whose objects are pretriangulated A_∞-categories.

12.18 Proposition. *The multicategory $\mathsf{A}_\infty^{\mathrm{tr}}$ is closed with the inner object of morphism $\underline{\mathsf{A}_\infty^{\mathrm{tr}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) = \underline{\mathsf{A}_\infty^{\mathrm{u}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B})$ and the evaluation*

$$\mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{tr}}} = \mathrm{ev}^{\mathsf{A}_\infty^{\mathrm{u}}} : (\mathcal{A}_i)_{i\in I}, \underline{\mathsf{A}_\infty^{\mathrm{tr}}}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \to \mathcal{B}.$$

The proof follows from the following lemma.

12.19 Lemma. *Let $\mathcal{B}$ be a pretriangulated A_∞-category. Then for arbitrary A_∞-categories $\mathcal{A}_i$, $i \in I$ the A_∞-category $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is pretriangulated. If $\mathcal{A}_i$, $i \in I$ are unital, then the A_∞-category $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is pretriangulated.*

Proof. Let us prove that $\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is pretriangulated, the case of $\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$ is treated similarly. Let $\mathcal{A}$ be an A_∞-category. By Corollaries 10.48 and 11.46 the A_∞-category $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ is closed under shifts and mc-closed. Proposition 12.10 implies that $\underline{\mathsf{A}_\infty}(\mathcal{A};\mathcal{B})$ is pretriangulated. Therefore, the claim holds for 1-element sets I. For $I = \varnothing$ we have an isomorphism $\underline{\mathsf{A}_\infty}(;\mathcal{B}) \simeq \mathcal{B}$. For other I the claim follows by induction from the isomorphism

$$\underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in\mathbf{1}\sqcup I};\mathcal{B}) \simeq \underline{\mathsf{A}_\infty}((\mathcal{A}_i)_{i\in I};\underline{\mathsf{A}_\infty}(\mathcal{A}_1;\mathcal{B})).$$

The lemma is proven. $\square$

12.20 Remark. The A_∞-category $A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk)$ is pretriangulated by Lemma 12.19, for any unital A_∞-category $\mathcal{A}$. Therefore, by Proposition 12.15.1 there exists an A_∞-equivalence U_{tr} such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathscr{Y}"] \arrow[d, "u_{\mathsf{tr}}"'] & A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk) \arrow[r, Rightarrow, no head] \arrow[d, "u_{\mathsf{tr}}"'] & A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk) \\
\mathcal{A}^{\mathsf{tr}} \arrow[r, "\mathscr{Y}^{\mathsf{tr}}"] & A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk)^{\mathsf{tr}} \arrow[ur, "U_{\mathsf{tr}}"'] &
\end{tikzcd}$$

Thus, the homotopy fully faithful Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk)$ factors into u_{tr} and the homotopy fully faithful A_∞-functor $\mathscr{Y}^{\mathsf{tr}} \cdot U_{\mathsf{tr}} : \mathcal{A}^{\mathsf{tr}} \to A^u_\infty(\mathcal{A}^{\mathrm{op}},\underline{\mathsf{C}}_\Bbbk)$.

12.21 Pretriangulated differential graded categories. Suppose $\mathcal{C}$ is a differential graded category. Then $\mathcal{C}^{\mathsf{tr}}$ is a differential graded category as well. Explicit description of that is given below.

An object X of $\mathcal{C}^{\mathsf{tr}}$ consists of the following data: a set $I \in \mathcal{P}$, a function that assigns to every $i \in I$ an object X_i of $\mathcal{C}$ and an integer n_i, and a family of elements $x_{ij} \in s\mathcal{C}(X_i,X_j)[n_j - n_i]$, $i,j \in I$, $i < j$ of degree 0 which satisfy Maurer-Cartan equation (12.1.1): for every $i,j \in I$, $i < j$

$$(-)^{n_j-n_i}x_{ij}s^{n_i-n_j}b_1s^{n_j-n_i} + \sum_{i<k<j}(-)^{n_j-n_i}(x_{ik}\otimes x_{kj})(s^{n_k-n_i}\otimes s^{n_j-n_k})^{-1}b_2s^{n_j-n_i} = 0.$$

Let us introduce elements $q_{ij} = x_{ij}s^{-1} \in \mathcal{C}(X_i,X_j)[n_j - n_i]$, $i,j \in I$, $i < j$ of degree 1. Composing the above equation with s^{-1} we write it as follows:

$$(-)^{n_j-n_i}q_{ij}s^{n_i-n_j}m_1s^{n_j-n_i} - \sum_{i<k<j}(q_{ik}\otimes q_{kj})(s^{n_k-n_i}\otimes s^{n_j-n_k})^{-1}\mu s^{n_j-n_i} = 0, \quad i,j \in J, \quad i < j,$$

where $m_1 = sb_1s^{-1}$ and $\mu = m_2 = (s \otimes s)b_2s^{-1}$ are respectively the differential and the multiplication in the original category $\mathcal{C}$. Putting $q_{ij} = 0$ for $i \geqslant j$ we may extend this equation to all pairs $i, j \in I$. Similarly to [Dri04] we write briefly $X = (\bigoplus_{i\in I} X_i[n_i], x)$. When the strictly upper-triangular matrix x has to be specified explicitly we use the following compact notation:

$$X = \begin{pmatrix} X_1[n_1] & x_{12} & x_{13} & \dots \\ & X_2[n_2] & x_{23} & \dots \\ & & X_3[n_3] & \dots \\ & & & \ddots \end{pmatrix}.$$

Let $X = (\bigoplus_{i\in I} X_i[n_i], x)$, $Y = (\bigoplus_{j\in J} Y_j[m_j], y)$ be objects of $\mathcal{C}^{\mathsf{tr}}$, $q = xs^{-1}$, $r = ys^{-1}$. The graded $\Bbbk$-module of morphisms between X and Y is defined as

$$\mathcal{C}^{\mathsf{tr}}(X, Y) = \prod_{i\in I, j\in J} \mathcal{C}(X_i, Y_j)[m_j - n_i].$$

An element f of $\mathcal{C}^{\mathsf{tr}}(X, Y)$ is thought as a matrix with entries $f_{ij} \in \mathcal{C}(X_i, Y_j)[m_j - n_i]$, $i \in I$, $j \in J$. The composition map is matrix multiplication. More precisely, let $Z = (\bigoplus_{k\in K} Z_k[\ell_k], z)$ be another object of $\mathcal{C}^{\mathsf{tr}}$, $p = zs^{-1}$ and let g be a morphism from Y to Z. Then $fg \overset{\text{def}}{=} (f \otimes g)m_2^{\mathsf{tr}}$ has the entries

$$\big[(f \otimes g)m_2^{\mathsf{tr}}\big]_{ik} = \sum_{j\in J} (f_{ij} \otimes g_{jk})(s^{m_j-n_i} \otimes s^{\ell_k-m_j})^{-1}\mu s^{\ell_k-n_i}, \; i \in I, \; k \in K. \tag{12.21.1}$$

The differential $m_1^{\mathsf{tr}} : \mathcal{C}^{\mathsf{tr}}(X, Y) \to \mathcal{C}^{\mathsf{tr}}(X, Y)$ is given by

$$\begin{aligned} \big[fm_1^{\mathsf{tr}}\big]_{ij} = (-)^{m_j-n_i} f_{ij} s^{n_i-m_j} m_1 s^{m_j-n_i} - \sum_{t\in J} (f_{it} \otimes r_{tj})(s^{m_t-n_i} \otimes s^{m_j-m_t})^{-1}\mu s^{m_j-n_i} \\ + (-)^f \sum_{u\in I} (q_{iu} \otimes f_{uj})(s^{n_u-n_i} \otimes s^{m_j-n_u})^{-1}\mu s^{m_j-n_i} \end{aligned}$$

for every $i \in I$, $j \in J$. Let us denote by $d = m_1^{[]}$ the naive differential in $\mathcal{C}^{\mathsf{tr}}(X, Y)$ defined by

$$\big[fd\big]_{ij} = f_{ij}m_1^{[]} = (-)^{m_j-n_i} f_{ij} s^{n_i-m_j} m_1 s^{m_j-n_i}, \quad i \in I, \; j \in J.$$

Since the composition m_2^{tr} in $\mathcal{C}^{\mathsf{tr}}$ consists of matrix composition combined with $m_2^{[]}$, the differential d is a derivation of it: $m_2^{\mathsf{tr}}d = (1 \otimes d + d \otimes 1)m_2^{\mathsf{tr}}$. Denoting the composition of $f \in \mathcal{C}^{\mathsf{tr}}(X, Y)$ and $g \in \mathcal{C}^{\mathsf{tr}}(Y, Z)$ simply fg we may write the following expressions for the Maurer-Cartan equation and for the differential in $\mathcal{C}^{\mathsf{tr}}(X, Y)$:

$$\begin{aligned} qd &= q^2, \\ fm_1^{\mathsf{tr}} &= fd - fr + (-)^f qf. \end{aligned} \tag{12.21.2}$$

These expressions agree with those in [Dri04, Section 2.4] up to the fact that we use right operators.

The category $\mathcal{C}$ is embedded into $\mathcal{C}^{\mathsf{tr}}$ as a full differential graded subcategory via $X \mapsto (X[0], 0)$ and we identify $\mathcal{C}$ with its image. If $X, Y \in \operatorname{Ob}\mathcal{C}$ and $f : X \to Y$ is a closed morphism of degree 0 one defines $\operatorname{Cone}(f)$ to be the object $\bigl(X[1] \oplus Y, \bigl(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\bigr)\bigr)$ of $\mathcal{C}^{\mathsf{tr}}$ with $f \in \mathcal{C}^{[]}(X[1], Y)[1]^0 = \mathcal{C}(X, Y)^0$.

The differential graded category $\mathcal{C}$ is said to be pretriangulated (originally Bondal and Kapranov called them enhancements of triangulated categories [BK90]) in the sense of [Dri04, 2.4] if for every $X \in \operatorname{Ob}\mathcal{C}$, $k \in \mathbb{Z}$ the object $X[k]$ of $H^0(\mathcal{C}^{\mathsf{tr}})$ is isomorphic[1] to an object of $H^0(\mathcal{C})$ and for every closed morphism f in $\mathcal{C}$ of degree 0 the object $\operatorname{Cone}(f) \in \operatorname{Ob}\mathcal{C}^{\mathsf{tr}}$ is isomorphic in $H^0(\mathcal{C}^{\mathsf{tr}})$ to an object of $H^0(\mathcal{C})$. The first condition implies that $\mathcal{C}$ is closed under shifts. This combined with the second condition gives nothing else but condition (i) of Proposition 11.32. Therefore if $\mathcal{C}$ is pretriangulated in the sense of [Dri04, Section 2.4], then it is pretriangulated in the sense of this book. The converse is obvious.

[1]In [Dri04, 2.4] isomorphic objects are called homotopy equivalent.

Chapter 13

Strongly triangulated categories

In this chapter we prove that zeroth homology of a pretriangulated A_∞-category is strongly triangulated. This implies, for instance, that the derived category of an abelian category is strongly triangulated. Besides, this corollary is already known by work of Neeman [Nee05] and Maltsiniotis [Mal06]. The definition of strongly triangulated categories is given by Maltsiniotis [Mal06]. In particular, such category is triangulated in the ordinary (although relaxed) sense: we do not assume that the action of the group $\mathbb{Z}$ on the category by translations $[n]$ is strict. That is why we describe firstly translation structures on a category or more generally, on an object of a 2-category. For instance, an A_∞-category closed under shifts has a translation structure as an object of the 2-category $\overline{A^u_\infty}$. The essence of strongly triangulated categories is that they have a hierarchy of distinguished triangles, distinguished octahedra and so on, while ordinary triangulated categories need only the class of distinguished triangles. We define higher triangles using a particular translation structure on certain subcategories of $\mathbb{Z} \times \mathbb{Z}$. Thus the new axioms realize the suggestion expressed by Beilinson, Bernstein and Deligne in [BBD82, Remark 1.1.14] to work with finer structures than those of triangulated categories. The new structure is more attractive than the old one, although it does not resolve the usual problems of triangulated categories: non-functoriality of the cone, non-existence of direct product of such categories.

From our point of view, the new axioms just express a better truncation of the underlying thing, which is a pretriangulated **dg**-category or a pretriangulated A_∞-category. We describe in detail distinguished n-triangles in its zeroth homology and prove the main result of this chapter.

13.1 Translation structures.

Let $\mathfrak{C}$ be a 2-category, $\mathcal{C}$, $\mathcal{D}$ objects of $\mathfrak{C}$. An *adjunction from* $\mathcal{C}$ to $\mathcal{D}$ is a quadruple $(F, U, \eta, \varepsilon)$, where $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ are 1-morphisms, $\eta : \mathrm{Id}_\mathcal{C} \to FU$ and $\varepsilon : UF \to \mathrm{Id}_\mathcal{D}$ are 2-morphisms such that the following equations hold:

$$\big(F \xrightarrow{\eta F} FUF \xrightarrow{F\varepsilon} F\big) = \mathrm{id}_F, \tag{13.1.1}$$

$$\big(U \xrightarrow{U\eta} UFU \xrightarrow{\varepsilon U} U\big) = \mathrm{id}_U\,. \tag{13.1.2}$$

Let $\mathcal{C}$ be an object of $\mathfrak{C}$. The category $\mathfrak{C}(\mathcal{C}, \mathcal{C}) = \mathrm{End}(\mathcal{C})$ is a strict monoidal category with the tensor product given by composition of 1-morphisms. The set of integers $\mathbb{Z}$ can

be viewed as a discrete category. It is a strict monoidal category with the tensor product given by addition.

13.2 Definition. A *translation structure* on $\mathcal{C}$ is a monoidal functor $(\Sigma, \varsigma) : \mathbb{Z} \to \mathrm{End}(\mathcal{C})$. More specifically, a translation structure on $\mathcal{C}$ consists of the following data: for every $n \in \mathbb{Z}$ a 1-morphism $\Sigma^n = \Sigma(n) : \mathcal{C} \to \mathcal{C}$; for each pair $m, n \in \mathbb{Z}$ a 2-isomorphism $\varsigma_{m,n} : \Sigma^m \Sigma^n \xrightarrow{\sim} \Sigma^{m+n}$; a 2-isomorphism $\varsigma_0 : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} \Sigma^0$. These data must satisfy the following coherence relations:

(i) cocycle condition: for $k, m, n \in \mathbb{Z}$

$$\big(\Sigma^k \Sigma^m \Sigma^n \xrightarrow{\Sigma^k \varsigma_{m,n}} \Sigma^k \Sigma^{m+n} \xrightarrow{\varsigma_{k,m+n}} \Sigma^{k+m+n}\big) = \big(\Sigma^k \Sigma^m \Sigma^n \xrightarrow{\varsigma_{k,m}\Sigma^n} \Sigma^{k+m} \Sigma^n \xrightarrow{\varsigma_{k+m,n}} \Sigma^{k+m+n}\big); \tag{13.2.1}$$

(ii) for every $n \in \mathbb{Z}$:

$$\big(\Sigma^n \xrightarrow{\varsigma_0 \Sigma^n} \Sigma^0 \Sigma^n \xrightarrow{\varsigma_{0,n}} \Sigma^n\big) = \mathrm{id}_{\Sigma^n}, \tag{13.2.2}$$

$$\big(\Sigma^n \xrightarrow{\Sigma^n \varsigma_0} \Sigma^n \Sigma^0 \xrightarrow{\varsigma_{n,0}} \Sigma^n\big) = \mathrm{id}_{\Sigma^n}. \tag{13.2.3}$$

This definition is similar to definition of weak action of the group $\mathbb{Z}$ on a category $\mathcal{C}$ [Ver96, Définition 1.2.2], except that we do not require $\Sigma^0 = \mathrm{Id}_{\mathcal{C}}$. In our applications we might have restricted the family (Σ^p, ς) to the only self-equivalence Σ^1. To recover the rest one chooses a quasi-inverse Σ^{-1} to Σ^1 together with adjunction isomorphisms. Then one defines $\Sigma^p = (\Sigma^1)^p$, $\Sigma^0 = \mathrm{id}$, $\Sigma^{-p} = (\Sigma^{-1})^p$ for $p > 0$. The isomorphisms ς are constructed from the adjunction isomorphisms. This particular family (Σ^p, ς) does not actually have any advantage over the general case. That is why we have chosen the general approach. We stress, however, that the general structure is isomorphic to this particular one in the sense described below.

The notion of a $\mathbb{Z}$-functor (a functor commuting with the action of $\mathbb{Z}$ up to isomorphism) due to Verdier [Ver96, Définition 1.2.4] suggests the following generalization. Let us define a new 2-category $\mathrm{Trans}\,\mathfrak{C}$. An object of this category is an object of $\mathfrak{C}$ with a translation structure on it. A 1-morphism from $(\mathcal{C}, \Sigma, \varsigma)$ to $(\mathcal{D}, \Sigma, \varsigma)$ (a *translation preserving 1-morphism*) consists of the following data: a 1-morphism $F : \mathcal{C} \to \mathcal{D}$; for every $n \in \mathbb{Z}$ a 2-isomorphism $\phi_n : \Sigma^n F \xrightarrow{\sim} F \Sigma^n$ such that the following diagram commutes for each pair $m, n \in \mathbb{Z}$

$$\begin{tikzcd}
\Sigma^m \Sigma^n F \arrow[r, "\Sigma^m \phi_n"] \arrow[d, "\varsigma_{m,n} F"'] & \Sigma^m F \Sigma^n \arrow[r, "\phi_m \Sigma^n"] & F \Sigma^m \Sigma^n \arrow[d, "F \varsigma_{m,n}"] \\
\Sigma^{m+n} F \arrow[rr, "\phi_{m+n}"] & & F \Sigma^{m+n}
\end{tikzcd} \tag{13.2.4}$$

and the following equation holds:

$$\bigl(F \xrightarrow{\varsigma_0 F} \Sigma^0 F \xrightarrow{\phi_0} F\Sigma^0\bigr) = \bigl(F \xrightarrow{F\varsigma_0} F\Sigma^0\bigr). \tag{13.2.5}$$

A 2-morphism $\nu : (F, \phi_n) \to (G, \psi_n) : (\mathcal{C}, \Sigma, \varsigma) \to (\mathcal{D}, \Sigma, \varsigma)$ (a *translation preserving 2-morphism*) is a 2-morphism $\nu : F \to G$ such that the following diagram commutes for every $n \in \mathbb{Z}$:

$$\begin{tikzcd} \Sigma^n F \arrow[r, "\Sigma^n \nu"] \arrow[d, "\phi_n"'] & \Sigma^n G \arrow[d, "\psi_n"] \\ F\Sigma^n \arrow[r, "\nu\Sigma^n"] & G\Sigma^n \end{tikzcd} \tag{13.2.6}$$

13.3 Proposition. *Let $\mathcal{C}$, $\mathcal{D}$ be objects of $\mathfrak{C}$, $(F, U, \eta, \varepsilon)$ an adjunction from $\mathcal{C}$ to $\mathcal{D}$. The correspondence $T \mapsto FTU$ extends to a lax monoidal functor $\Gamma : \mathrm{End}(\mathcal{D}) \to \mathrm{End}(\mathcal{C})$.*

Proof. For each pair $T, S \in \mathrm{End}(\mathcal{D})$ put

$$\gamma_{T,S} = \bigl(T\Gamma \cdot S\Gamma = FTUFSU \xrightarrow{FT\varepsilon SU} FTSU = (TS)\Gamma\bigr).$$

We have to check that for $T, S, R \in \mathrm{End}(\mathcal{D})$ the following equation holds:

$$\begin{aligned}\bigl(FTUFSUFRU &\xrightarrow{FT\varepsilon SUFRU} FTSUFRU \xrightarrow{FTS\varepsilon RU} FTSRU\bigr) \\ &= \bigl(FTUFSUFRU \xrightarrow{FTUFS\varepsilon RU} FTUFSRU \xrightarrow{FT\varepsilon SRU} FTSRU\bigr).\end{aligned}$$

It follows from the equation

$$\bigl(UFSUF \xrightarrow{\varepsilon SUF} SUF \xrightarrow{S\varepsilon} S\bigr) = \bigl(UFSUF \xrightarrow{UFS\varepsilon} UFS \xrightarrow{\varepsilon S} S\bigr),$$

which is a consequence of distributivity of horizontal product of 2-morphisms with respect to vertical product. A 2-morphism $\mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{D}} \Gamma = FU$ coincides with η. We have to check that for every $T \in \mathrm{End}(\mathcal{D})$

$$\bigl(FTU \xrightarrow{FTU\eta} FTUFU \xrightarrow{FT\varepsilon U} FTU\bigr) = \mathrm{id},$$
$$\bigl(FTU \xrightarrow{\eta FTU} FUFTU \xrightarrow{F\varepsilon TU} FTU\bigr) = \mathrm{id}\,.$$

These equations follow from (13.1.1) and (13.1.2). □

13.4 Remark. If $(F, U, \eta, \varepsilon)$ is an adjunction–equivalence between $\mathcal{C}$ and $\mathcal{D}$ (an adjunction with invertible η, ε), the constructed above functor is monoidal. Assume $(F, U, \eta, \varepsilon)$ is an adjunction–equivalence from $\mathcal{C}$ to $\mathcal{D}$ and $\mathcal{D}$ is equipped with a translation structure. Then the composite functor $\mathbb{Z} \xrightarrow{\Sigma_{\mathcal{D}}} \mathrm{End}(\mathcal{D}) \xrightarrow{\Gamma} \mathrm{End}(\mathcal{C})$ is monoidal and defines

a translation structure on $\mathcal{C}$. Specifically, the translation structure on $\mathcal{C}$ is given by the following data: for each $n \in \mathbb{Z}$ a 1-morphism $\Sigma^n_{\mathcal{C}} = F\Sigma^n_{\mathcal{D}}U : \mathcal{C} \to \mathcal{C}$; for each pair $m, n \in \mathbb{Z}$ a 2-isomorphism

$$\varsigma^{\mathcal{C}}_{m,n} = \bigl(\Sigma^m_{\mathcal{C}}\Sigma^n_{\mathcal{C}} = F\Sigma^n_{\mathcal{D}}UF\Sigma^m_{\mathcal{D}}U \xrightarrow{F\Sigma^m_{\mathcal{D}}\varepsilon\Sigma^n_{\mathcal{D}}U} F\Sigma^m_{\mathcal{D}}\Sigma^n_{\mathcal{D}}U \xrightarrow{F\varsigma^{\mathcal{D}}_{m,n}U} F\Sigma^{m+n}_{\mathcal{D}}U = \Sigma^{m+n}_{\mathcal{C}}\bigr);$$

a 2-isomorphism $\varsigma^{\mathcal{C}}_0 = \eta(F\varsigma^{\mathcal{D}}_0 U) : \mathrm{Id}_{\mathcal{C}} \to F\Sigma^0_{\mathcal{D}}U = \Sigma^0_{\mathcal{C}}$. We may turn F into a morphism of objects with translation structure putting for every $n \in \mathbb{Z}$

$$\phi_n = \bigl(\Sigma^n_{\mathcal{C}}F = F\Sigma^n_{\mathcal{D}}UF \xrightarrow{F\Sigma^n_{\mathcal{D}}\varepsilon} F\Sigma^n_{\mathcal{D}}\bigr).$$

Let us check equations (13.2.4) and (13.2.5). Equation (13.2.4) reads explicitly as follows:

$$\begin{aligned}&\bigl(F\Sigma^m_{\mathcal{D}}UF\Sigma^n_{\mathcal{D}}UF \xrightarrow{F\Sigma^m_{\mathcal{D}}UF\Sigma^n_{\mathcal{D}}\varepsilon} F\Sigma^m_{\mathcal{D}}UF\Sigma^m_{\mathcal{D}} \xrightarrow{F\Sigma^m_{\mathcal{D}}\varepsilon\Sigma^n_{\mathcal{D}}} F\Sigma^m_{\mathcal{D}}\Sigma^n_{\mathcal{D}} \xrightarrow{F\varsigma^{\mathcal{D}}_{m,n}} F\Sigma^{m+n}_{\mathcal{D}}\bigr)\\ &\quad = \bigl(F\Sigma^m_{\mathcal{D}}UF\Sigma^n_{\mathcal{D}}UF \xrightarrow{F\Sigma^m_{\mathcal{D}}\varepsilon\Sigma^n_{\mathcal{D}}UF} F\Sigma^m_{\mathcal{D}}\Sigma^m_{\mathcal{D}}UF \xrightarrow{F\varsigma^{\mathcal{D}}_{m,n}UF} F\Sigma^{m+n}_{\mathcal{D}}UF \xrightarrow{F\Sigma^{m+n}_{\mathcal{D}}\varepsilon} F\Sigma^{m+n}_{\mathcal{D}}\bigr).\end{aligned}$$

It is a consequence of distributivity of products in 2-categories. Equation (13.2.5) reads as follows:

$$\bigl(F \xrightarrow{\eta F} FUF \xrightarrow{F\varsigma^{\mathcal{D}}_0 UF} F\Sigma^0_{\mathcal{D}}UF \xrightarrow{F\Sigma^0_{\mathcal{D}}\varepsilon} F\Sigma^0_{\mathcal{D}}\bigr) = \bigl(F \xrightarrow{F\varsigma^{\mathcal{D}}_0} F\Sigma^0_{\mathcal{D}}\bigr).$$

It follows from distributivity of products and equation (13.1.1).

A given equivalence $F : \mathcal{C} \to \mathcal{D}$ can be completed to an adjunction in a non-unique way. Let $(F, U, \eta, \varepsilon)$ and $(F, \tilde{U}, \tilde{\eta}, \tilde{\varepsilon})$ be adjunction–equivalences between $\mathcal{C}$ and $\mathcal{D}$. Being quasi–inverse to F the 1-morphisms U, $\tilde{U}$ are related by some 2-isomorphism $\alpha : U \to \tilde{U}$ such that $\tilde{\eta} = \eta \cdot F\alpha$ and $\tilde{\varepsilon} = \alpha^{-1}F \cdot \varepsilon$. The two obtained translation structures $(\Sigma_{\mathcal{C}}, \varsigma)$ and $(\tilde{\Sigma}_{\mathcal{C}}, \tilde{\varsigma})$ on $\mathcal{C}$ are isomorphic via the invertible 1-morphism $(\mathrm{Id}, \psi_n) : (\mathcal{C}, \Sigma_{\mathcal{C}}, \varsigma) \to (\mathcal{C}, \tilde{\Sigma}_{\mathcal{C}}, \tilde{\varsigma})$ of $\mathrm{Trans}\,\mathfrak{C}$, where the 2-isomorphism ψ_n is given by

$$\psi_n = F\Sigma^n_{\mathcal{D}}\alpha : \Sigma^n_{\mathcal{C}} = F\Sigma^n_{\mathcal{D}}U \to F\Sigma^n_{\mathcal{D}}\tilde{U} = \tilde{\Sigma}^n_{\mathcal{C}}.$$

Indeed,

$$\begin{aligned}\bigl(\mathrm{Id}_{\mathcal{C}} \xrightarrow{\varsigma^{\mathcal{C}}_0} \Sigma^0_{\mathcal{C}} \xrightarrow{\psi_0} \tilde{\Sigma}^0_{\mathcal{C}}\bigr) &= \bigl(\mathrm{Id}_{\mathcal{C}} \xrightarrow{\eta} FU \xrightarrow{F\varsigma^{\mathcal{D}}_0 U} F\Sigma^0_{\mathcal{D}}U \xrightarrow{F\Sigma^0_{\mathcal{D}}\alpha} F\Sigma^0_{\mathcal{D}}\tilde{U}\bigr)\\ &= \bigl(\mathrm{Id}_{\mathcal{C}} \xrightarrow[\tilde{\eta}]{\eta\cdot F\alpha} F\tilde{U} \xrightarrow{F\varsigma^{\mathcal{D}}_0\tilde{U}} F\Sigma^0_{\mathcal{D}}\tilde{U}\bigr) = \tilde{\varsigma}^{\mathcal{C}}_0,\end{aligned}$$

hence, (13.2.5) holds. Moreover,

$$
\begin{aligned}
&\bigl(\varSigma^m_{\mathcal{C}}\varSigma^n_{\mathcal{C}} \xrightarrow{\varSigma^m_{\mathcal{C}}\psi_n} \varSigma^m_{\mathcal{C}}\tilde{\varSigma}^n_{\mathcal{C}} \xrightarrow{\psi_m\tilde{\varSigma}^n_{\mathcal{C}}} \tilde{\varSigma}^m_{\mathcal{C}}\tilde{\varSigma}^n_{\mathcal{C}} \xrightarrow{\tilde{\varsigma}_{m,n}} \tilde{\varSigma}^{m+n}_{\mathcal{C}}\bigr) \\
&= \bigl(F\varSigma^m_{\mathcal{D}}UF\varSigma^n_{\mathcal{D}}U \xrightarrow{F\varSigma^m_{\mathcal{D}}UF\varSigma^n_{\mathcal{D}}\alpha} F\varSigma^m_{\mathcal{D}}UF\varSigma^n_{\mathcal{D}}\tilde{U} \xrightarrow{F\varSigma^m_{\mathcal{D}}\alpha F\varSigma^n_{\mathcal{D}}\tilde{U}} \\
&\qquad\qquad F\varSigma^m_{\mathcal{D}}\tilde{U}F\varSigma^n_{\mathcal{D}}\tilde{U} \xrightarrow{F\varSigma^m_{\mathcal{D}}\tilde{\varepsilon}\varSigma^n_{\mathcal{D}}\tilde{U}} F\varSigma^m_{\mathcal{D}}\varSigma^n_{\mathcal{D}}\tilde{U} \xrightarrow{F\varsigma^{\mathcal{D}}_{m,n}\tilde{U}} F\varSigma^{m+n}_{\mathcal{D}}\tilde{U}\bigr) \\
&= \bigl(F\varSigma^m_{\mathcal{D}}UF\varSigma^n_{\mathcal{D}}U \xrightarrow{F\varSigma^m_{\mathcal{D}}\varepsilon\varSigma^n_{\mathcal{D}}U} F\varSigma^m_{\mathcal{D}}\varSigma^n_{\mathcal{D}}U \xrightarrow{F\varsigma^{\mathcal{D}}_{m,n}U} F\varSigma^{m+n}_{\mathcal{D}}U \\
&\qquad\qquad \xrightarrow{F\varSigma^{m+n}_{\mathcal{D}}\alpha} F\varSigma^{m+n}_{\mathcal{D}}\tilde{U}\bigr) = \bigl(\varSigma^m_{\mathcal{C}}\varSigma^n_{\mathcal{C}} \xrightarrow{\varsigma_{m,n}} \varSigma^{m+n}_{\mathcal{C}} \xrightarrow{\psi_{m+n}} \tilde{\varSigma}^{m+n}_{\mathcal{C}}\bigr).
\end{aligned}
$$

implies that (13.2.4) holds. This shows that $\varSigma_{\mathcal{C}}$ depends functorially on the choice of adjunction data.

13.5 Translation structures determined by an automorphism.

Let category $\mathcal{C}$ have a translation structure. Let us introduce a new category $\mathcal{D}$. Its class of objects is $\operatorname{Ob}\mathcal{C} \times \mathbb{Z}$, that is, objects of $\mathcal{D}$ are pairs (X, n) with $X \in \operatorname{Ob}\mathcal{C}$, $n \in \mathbb{Z}$. The $\Bbbk$-module of morphisms from (X, n) to (Y, m) is defined as

$$\mathcal{D}((X, n), (Y, m)) = \mathcal{C}(X\varSigma^n, Y\varSigma^m).$$

Composition in $\mathcal{D}$ is induced by that in $\mathcal{C}$. There is a natural functor $P : \mathcal{D} \to \mathcal{C}$, $(X, n) \mapsto X\varSigma^n$, identity on morphisms. It is obviously an equivalence of categories.

13.6 Proposition. *The category $\mathcal{D}$ has a translation structure, which is a strict monoidal functor $\Sigma : \mathbb{Z} \to \operatorname{End}(\mathcal{D})$, such that $\Sigma^k = \Sigma(k) : \mathcal{D} \to \mathcal{D}$ maps an object (X, n) to $(X, n+k)$. A morphism $(f : X\varSigma^n \to Y\varSigma^m) \in \mathcal{D}((X, n), (Y, m))$ is mapped by Σ^k to the composition*

$$\bigl(X\varSigma^{n+k} \xrightarrow{\varsigma^{-1}_{n,k}} X\varSigma^n\varSigma^k \xrightarrow{f\varSigma^k} Y\varSigma^m\varSigma^k \xrightarrow{\varsigma_{m,k}} Y\varSigma^{m+k}\bigr) \in \mathcal{D}((X, n+k), (Y, m+k)).$$

In particular, $\Sigma^k\Sigma^l = \Sigma^{k+l}$, $\Sigma^0 = \mathrm{Id}_{\mathcal{D}}$, and Σ^k are automorphisms of $\mathcal{D}$.

The functor P together with the collection of natural transformations

$$\phi_k = \bigl((X, n)\Sigma^k P = X\varSigma^{n+k} \xrightarrow{\varsigma^{-1}_{n,k}} X\varSigma^n\varSigma^k = (X, n)P\varSigma^k\bigr)$$

is a morphism of categories with translation structure.

Proof. Clearly, the functors $\Sigma^k\Sigma^l$ and Σ^{k+l} coincide on objects. Let $f \in \mathcal{D}((X, n), (Y, m)) = \mathcal{C}(X\varSigma^n, Y\varSigma^m)$. The equation $f\Sigma^k\Sigma^l = f\Sigma^{k+l}$ is given by the exterior of the following diagram:

$$
\begin{tikzcd}
X\varSigma^{n+k}\varSigma^l \arrow[r, "\varsigma^{-1}_{n,k}\varSigma^l"] & X\varSigma^n\varSigma^k\varSigma^l \arrow[r, "f\varSigma^k\varSigma^l"] & Y\varSigma^m\varSigma^k\varSigma^l \arrow[r, "\varsigma_{m,k}\varSigma^l"] & X\varSigma^{m+k}\varSigma^l \arrow[d, "\varsigma_{m+k,l}"] \\
X\varSigma^{n+k+l} \arrow[u, "\varsigma^{-1}_{n+k,l}"] \arrow[r, "\varsigma^{-1}_{n,k+l}"] & X\varSigma^n\varSigma^{k+l} \arrow[u, "\varSigma^n\varsigma^{-1}_{k,l}"'] \arrow[r, "f\varSigma^{k+l}"] & Y\varSigma^m\varSigma^{k+l} \arrow[u, "\varSigma^m\varsigma^{-1}_{k,l}"] \arrow[r, "\varsigma_{m,k+l}"] & Y\varSigma^{m+k+l}
\end{tikzcd}
$$

The middle square is commutative due to naturality of $\varsigma_{k,l}$. Two other squares commute by (13.2.1).

Similarly, the functors Σ^0 and $\mathrm{Id}_{\mathcal{D}}$ coincide on objects. The equation $f\Sigma^0 = f$ is proven in the diagram below:

$$\begin{tikzcd}
 & X\varSigma^n\varSigma^0 \arrow[r, "f\varSigma^0"] & Y\varSigma^m\varSigma^0 \arrow[dr, "\varsigma_{m,0}"] & \\
X\varSigma^n \arrow[ur, "\varsigma_{n,0}^{-1}"] \arrow[r, equal] & X\varSigma^n \arrow[u, "\varSigma^n\varsigma_0"'] \arrow[r, "f"] & Y\varSigma^m \arrow[u, "\varSigma^m\varsigma_0"] \arrow[r, equal] & Y\varSigma^m
\end{tikzcd}$$

The middle square is commutative due to naturality of ς_0. The triangles commute due to equations (13.2.2) and (13.2.3).

The transformations $\phi_k = \varsigma_{n,k}^{-1}$ are commutation isomorphisms of P with translation functors due to equation (13.2.1). □

13.7 Proposition. *Let unital A_∞-category $\mathcal{C}$ be closed under shifts. Then the object $\mathcal{C}$ of 2-category $\overline{A_\infty^u}$, the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$ and the $\Bbbk$-linear category $H^0(\mathcal{C})$ have a natural translation structure, determined in a unique way up to an isomorphism.*

Proof. First we construct a translation structure on $\mathcal{C}^{[]}$ for an arbitrary unital A_∞-category $\mathcal{C}$. Denote by $\boldsymbol{\sigma}^n$ the differential graded functor

$$\boldsymbol{\sigma}^n = \bigl(\mathcal{Z} \xrightarrow{\lambda^{\mathsf{I}\cdot}} \mathcal{Z} \boxtimes \mathbb{1}_p \xrightarrow{1\boxtimes\dot{n}} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_{\mathcal{Z}}} \mathcal{Z}\bigr),$$

where the quiver $\mathbb{1}_p$ with one object $*$ is made into a differential graded category in the obvious way, and the differential graded functor $\dot{n} : \mathbb{1}_p \to \mathcal{Z}$, $* \mapsto n$ gives $\mathrm{id}_\Bbbk : \Bbbk \to \mathcal{Z}(n,n)$ on morphisms. On objects $\boldsymbol{\sigma}^n(m) = m + n$. Explicit formula for $\otimes_{\mathcal{Z}}$ based on $\psi_1(l,n,k,n) = 1$ shows that

$$\boldsymbol{\sigma}^n = \mathrm{id} : \mathcal{Z}(l,k) = \Bbbk[k-l] \to \Bbbk[k+n-l-n] = \mathcal{Z}(l+n,k+n)$$

on morphisms. Therefore, $\boldsymbol{\sigma}^m\boldsymbol{\sigma}^n = \boldsymbol{\sigma}^{m+n}$ and $\boldsymbol{\sigma}^0 = \mathrm{Id}_{\mathcal{Z}}$. This gives a strict monoidal functor $\boldsymbol{\sigma} : \mathbb{Z} \to \mathrm{End}\,\mathcal{Z}$, which is a translation structure on $\mathcal{Z}$.

Denote by $\varSigma^n = [n] : \mathcal{C}^{[]} \to \mathcal{C}^{[]}$ the unital strict A_∞-functor $\mathrm{id}_{\mathcal{C}} \boxdot \boldsymbol{\sigma}^n : \mathcal{C} \boxdot \mathcal{Z} \to \mathcal{C} \boxdot \mathcal{Z}$, see Proposition C.15. On objects it gives the shift $\varSigma^n : X[m] \mapsto X[m+n]$. The first component gives on morphisms the identity map

$$\begin{aligned}\varSigma_1^n = \mathrm{id} : \mathcal{C}^{[]}(X[k],Y[m])[1] &= \mathcal{C}(X,Y)[1] \otimes \Bbbk[m-k] \\ &\to \mathcal{C}(X,Y)[1] \otimes \Bbbk[m+n-k-n] = \mathcal{C}^{[]}(X[k+n],Y[m+n])[1].\end{aligned}$$

Since the action $\boxdot$ is a symmetric multifunctor, we have $\varSigma^m\varSigma^n = \varSigma^{m+n}$ and $\varSigma^0 = \mathrm{id}_{\mathcal{C}^{[]}}$. Equipping this family with identity 2-isomorphisms $\varsigma_{m,n} = \mathrm{id}$ and $\varsigma_0 = \mathrm{id}$, we get a strict monoidal functor $(\varSigma, \mathrm{id}) : \mathbb{Z} \to \overline{A_\infty^u}(\mathcal{C}^{[]}, \mathcal{C}^{[]})$, which is a translation structure on $\mathcal{C}^{[]}$.

For an arbitrary A_∞-category $\mathcal{C}$ closed under shifts the A_∞-functor $u_{[]} : \mathcal{C} \to \mathcal{C}^{[]}$ is an A_∞-equivalence by Proposition 10.33. Choose an A_∞-equivalence $U : \mathcal{C}^{[]} \to \mathcal{C}$ quasi-inverse to $u_{[]}$ and 2-isomorphisms $\eta : \mathrm{id}_\mathcal{C} \to u_{[]}U$, $\varepsilon : Uu_{[]} \to \mathrm{id}_{\mathcal{C}^{[]}}$ in $\overline{A^u_\infty}$ so that the quadruple $(u_{[]}, U, \eta, \varepsilon)$ were an adjunction–equivalence. According to Remark 13.4 the translation structure on $\mathcal{C}^{[]}$ constructed above transfers along this adjunction–equivalence to a translation structure $(\Sigma, \varsigma) : \mathbb{Z} \to \overline{A^u_\infty}(\mathcal{C}, \mathcal{C})$ on $\mathcal{C}$. It is proven in the same remark that choosing other adjunction–equivalence data will lead to an isomorphic translation structure on $\mathcal{C}$.

A strict 2-functor $\mathsf{k} : \overline{A^u_\infty} \to \mathcal{K}\text{-}\mathcal{C}at$ is constructed in [Lyu03, Proposition 8.6]. It takes a unital A_∞-category $\mathcal{C}$ to the $\mathcal{K}$-category $\mathsf{k}\mathcal{C}$, a unital A_∞-functor f to the $\mathcal{K}$-functor $\mathsf{k}f = sf_1s^{-1}$, the cohomology class of a natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ to $\mathsf{k}r = r_0s^{-1} : \mathsf{k}f \to \mathsf{k}g : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{B}$, that is, for each object X of $\mathcal{A}$ the component $(\mathsf{k}r)_X$ is the homotopy equivalence class of chain map ${}_Xr_0s^{-1} : \Bbbk \to \mathcal{B}(Xf, Xg)$. This 2-functor induces a translation structure on $\mathsf{k}\mathcal{C}$ as the composition of monoidal functors

$$(\mathsf{k}\Sigma, \mathsf{k}\varsigma) = \bigl[\mathbb{Z} \xrightarrow{(\Sigma,\varsigma)} \overline{A^u_\infty}(\mathcal{C}, \mathcal{C}) \xrightarrow{\mathsf{k}} \mathcal{K}\text{-}\mathcal{C}at(\mathsf{k}\mathcal{C}, \mathsf{k}\mathcal{C})\bigr].$$

Following [Lyu03, Section 8.13] we compose the 2-functor k with the strict 2-functor $H^0 : \mathcal{K}\text{-}\mathcal{C}at \to \Bbbk\text{-}\mathcal{C}at$. The composition $\overline{A^u_\infty} \xrightarrow{\mathsf{k}} \mathcal{K}\text{-}\mathcal{C}at \xrightarrow{H^0} \Bbbk\text{-}\mathcal{C}at$ is again denoted H^0. It takes a unital A_∞-category $\mathcal{C}$ to $H^0(\mathcal{C})$, a unital A_∞-functor f to $H^0(f)$ induced by $\mathsf{k}f$, a natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ to the natural transformation $H^0(r)$, formed by cohomology classes $[{}_Xr_0s^{-1}] \in H^0(\mathcal{B}(Xf, Xg), m_1)$. The composition of monoidal functors

$$(H^0(\Sigma), H^0(\varsigma)) = \bigl[\mathbb{Z} \xrightarrow{(\Sigma,\varsigma)} \overline{A^u_\infty}(\mathcal{C}, \mathcal{C}) \xrightarrow{H^0} \Bbbk\text{-}\mathcal{C}at(H^0(\mathcal{C}), H^0(\mathcal{C}))\bigr]$$

determines a translation structure on $H^0(\mathcal{C})$. $\square$

13.8 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category. Then the following equation holds*

$$\bigl(\mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]} \xrightarrow{\Sigma^n} \mathcal{B}^{[]}\bigr) = \bigl(\mathcal{B}^{[][]} \xrightarrow{\Sigma^n} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]}\bigr). \tag{13.8.1}$$

Proof. The algebra $\mathcal{Z}$ satisfies the identity

$$\begin{aligned}
&\bigl[\mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_\mathcal{Z}} \mathcal{Z} \xrightarrow{\boldsymbol{\sigma}^n} \mathcal{Z}\bigr] \\
&= \bigl[\mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_\mathcal{Z}} \mathcal{Z} \xrightarrow{\lambda^{\mathsf{I}\,\cdot}} \mathcal{Z} \boxtimes \mathbb{1}_p \xrightarrow{1\boxtimes\dot{n}} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_\mathcal{Z}} \mathcal{Z}\bigr] \\
&= \bigl[\mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\lambda^{\mathsf{II}\,\cdot}} \mathcal{Z} \boxtimes \mathcal{Z} \boxtimes \mathbb{1}_p \xrightarrow{1\boxtimes1\boxtimes\dot{n}} \mathcal{Z} \boxtimes \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes^{\mathbf{3}}_\mathcal{Z}} \mathcal{Z}\bigr] \\
&= \bigl[\mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{1\boxtimes\lambda^{\mathsf{I}\,\cdot}} \mathcal{Z} \boxtimes (\mathcal{Z} \boxtimes \mathbb{1}_p) \xrightarrow{1\boxtimes(1\boxtimes\dot{n})} \mathcal{Z} \boxtimes (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxtimes\otimes_\mathcal{Z}} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_\mathcal{Z}} \mathcal{Z}\bigr] \\
&= \bigl[\mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{1\boxtimes\boldsymbol{\sigma}^n} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_\mathcal{Z}} \mathcal{Z}\bigr].
\end{aligned}$$

Hence, (13.8.1) holds:

$$\begin{aligned}
m_{[]} \cdot \Sigma^n &= \bigl[(\mathcal{B} \boxdot \mathcal{Z}) \boxdot \mathcal{Z} \xrightarrow{\alpha^{\mathbf{2}}} \mathcal{B} \boxdot (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxdot\otimes_{\mathcal{Z}}} \mathcal{B} \boxdot \mathcal{Z} \xrightarrow{1\boxdot\boldsymbol{\sigma}^n} \mathcal{B} \boxdot \mathcal{Z}\bigr] \\
&= \bigl[(\mathcal{B} \boxdot \mathcal{Z}) \boxdot \mathcal{Z} \xrightarrow{\alpha^{\mathbf{2}}} \mathcal{B} \boxdot (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxdot(1\boxtimes\boldsymbol{\sigma}^n)} \mathcal{B} \boxdot (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxdot\otimes_{\mathcal{Z}}} \mathcal{B} \boxdot \mathcal{Z}\bigr] \\
&= \bigl[(\mathcal{B} \boxdot \mathcal{Z}) \boxdot \mathcal{Z} \xrightarrow{1\boxdot\boldsymbol{\sigma}^n} (\mathcal{B} \boxdot \mathcal{Z}) \boxdot \mathcal{Z} \xrightarrow{\alpha^{\mathbf{2}}} \mathcal{B} \boxdot (\mathcal{Z} \boxtimes \mathcal{Z}) \xrightarrow{1\boxdot\otimes_{\mathcal{Z}}} \mathcal{B} \boxdot \mathcal{Z}\bigr] \\
&= \Sigma^n \cdot m_{[]}
\end{aligned}$$

by naturality of $\alpha^{\mathbf{2}}$. □

13.9 Example. Let $\mathcal{B}$ be a unital A_∞-category. Then $\mathcal{B}^{[]}$ has the canonical translation structure $\Sigma^n = 1 \boxdot \boldsymbol{\sigma}^n$, $\varsigma = \mathrm{id}$. The canonical translation structure on $\mathcal{B}^{[][]}$ could be transferred to an a priori different translation structure $(\tilde{\Sigma}, \tilde{\varsigma})$ on $\mathcal{B}^{[]}$ along the A_∞-equivalence $u_{[]} : \mathcal{B}^{[]} \to \mathcal{B}^{[][]}$. However, choosing $m_{[]} : \mathcal{B}^{[][]} \to \mathcal{B}^{[]}$ as the A_∞-equivalence quasi-inverse to $u_{[]}$, choosing $\mathrm{id} : \mathrm{id}_{\mathcal{B}^{[]}} \to u_{[]} \cdot m_{[]}$ as the unit 2-isomorphism η we get $\tilde{\Sigma}^n = \Sigma^n$, $\tilde{\varsigma} = \varsigma = \mathrm{id}$. Indeed, there is a unique 2-isomorphism $\varepsilon : m_{[]} \cdot u_{[]} \to \mathrm{id}_{\mathcal{B}^{[][]}}$ in $\overline{A^u_\infty}$ which makes $(u_{[]}, m_{[]}, \mathrm{id}, \varepsilon)$ into an adjunction–equivalence. The construction of Remark 13.4 gives

$$\tilde{\Sigma}^n_{\mathcal{B}^{[]}} = u_{[]}\Sigma^n_{\mathcal{B}^{[][]}}m_{[]} = u_{[]}m_{[]}\Sigma^n_{\mathcal{B}^{[]}} = \Sigma^n_{\mathcal{B}^{[]}} : \mathcal{B}^{[]} \to \mathcal{B}^{[]}$$

by Lemma 13.8. Since η and ε satisfy property (13.1.2) the following equation

$$\bigl(m_{[]} = m_{[]} \cdot u_{[]} \cdot m_{[]} \xrightarrow{\varepsilon m_{[]}} m_{[]}\bigr) = \mathrm{id}_{m_{[]}} : \mathcal{B}^{[][]} \to \mathcal{B}^{[]}$$

holds. Therefore,

$$\begin{aligned}
\tilde{\varsigma}^{\mathcal{B}^{[]}}_{m,n} &= \bigl(\Sigma^m_{\mathcal{B}^{[]}}\Sigma^n_{\mathcal{B}^{[]}} = u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}m_{[]}u_{[]}\Sigma^n_{\mathcal{B}^{[][]}}m_{[]} \xrightarrow{u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}\varepsilon\Sigma^n_{\mathcal{B}^{[][]}}m_{[]}} \\
&\qquad\qquad u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}\Sigma^n_{\mathcal{B}^{[][]}}m_{[]} = u_{[]}\Sigma^{m+n}_{\mathcal{B}^{[][]}}m_{[]} = \Sigma^{m+n}_{\mathcal{B}^{[]}}\bigr) \\
&= \bigl(\Sigma^m_{\mathcal{B}^{[]}}\Sigma^n_{\mathcal{B}^{[]}} = u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}m_{[]}u_{[]}m_{[]}\Sigma^n_{\mathcal{B}^{[]}} \xrightarrow{u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}\varepsilon m_{[]}\Sigma^n_{\mathcal{B}^{[]}}} \\
&\qquad\qquad u_{[]}\Sigma^m_{\mathcal{B}^{[][]}}m_{[]}\Sigma^n_{\mathcal{B}^{[]}} = \Sigma^m_{\mathcal{B}^{[]}}\Sigma^n_{\mathcal{B}^{[]}} = \Sigma^{m+n}_{\mathcal{B}^{[]}}\bigr)
\end{aligned}$$

is the identity morphism, and coincides with $\varsigma^{\mathcal{B}^{[]}}_{m,n}$.

Introduce the full subcategory $A^{[]\text{-closed}}_\infty$ of the $\Bbbk$-$\mathcal{C}at$-category (2-category) $\overline{A^u_\infty}$ from Section 9.21. Its objects are unital A_∞-categories closed under shifts.

13.10 Proposition. *The embedding $A^{[]\text{-closed}}_\infty \hookrightarrow \overline{A^u_\infty}$ lifts naturally to a 2-functor $\tilde{-} : A^{[]\text{-closed}}_\infty \to \mathrm{Trans}\,\overline{A^u_\infty}$, identity on 2-morphisms.*

Proof. It suffices to lift the 2-functor on the full 2-subcategory of $\widetilde{A_\infty^{[]\text{-closed}}}$, consisting of $\mathcal{C}^{[]}$, $\mathcal{C} \in \operatorname{Ob} A_\infty^u$, since this 2-subcategory is equivalent to $A_\infty^{[]\text{-closed}}$. Define $\widetilde{\mathcal{C}^{[]}} = (\mathcal{C}^{[]}, \Sigma)$.

Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories. As shown in Proposition 10.39 the A_∞-categories $A_\infty^u(\mathcal{A}, \mathcal{B}^{[]})$ and $A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]})$ are equivalent. An A_∞-equivalence (constructed in the proof) is given by $Fh = h^{[]}m_{[]}$,

$$F = \bigl[A_\infty^u(\mathcal{A}, \mathcal{B}^{[]}) \xrightarrow[S']{-^{[]}} A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[][]}) \xrightarrow{A_\infty^u(1,m_{[]})} A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]})\bigr].$$

It is a strict A_∞-functor since S' and $m_{[]}$ are strict by Section 10.30.2.

13.11 Lemma. *The both following compositions are equal for the above F and any $n \in \mathbb{Z}$:*

$$A_\infty^u(\mathcal{A}, \mathcal{B}^{[]}) \xrightarrow{F} A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]}) \underset{A_\infty^u(\Sigma^n,1)}{\overset{A_\infty^u(1,\Sigma^n)}{\rightrightarrows}} A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]}). \tag{13.11.1}$$

Proof. We are going to prove that the following compositions in multicategory $\mathsf{A}_\infty^\mathrm{u}$ are equal:

$$\begin{aligned}
&\bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{1,S'} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]};\mathcal{B}^{[][]}) \xrightarrow{1,\underline{\mathsf{A}_\infty^\mathrm{u}}(1;m_{[]}\cdot\Sigma^n)} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}} \mathcal{B}^{[]}\bigr] \\
&= \bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{1,F} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \xrightarrow{1,\underline{\mathsf{A}_\infty^\mathrm{u}}(\Sigma^n;1)} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}} \mathcal{B}^{[]}\bigr].
\end{aligned} \tag{13.11.2}$$

Using the presentation $S' = u_{[]} \cdot \underline{S}$ and definitions (4.12.2), (4.12.3), (4.18.3) we transform the left hand side to

$$\begin{aligned}
&\bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{1,u_{[]}} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]})^{[]} \xrightarrow{1,\underline{S}} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]},\mathcal{B}^{[][]}) \xrightarrow{\mathrm{ev}} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]} \xrightarrow{\Sigma^n} \mathcal{B}^{[]}\bigr] \\
&= \bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{1,u_{[]}} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]})^{[]} \xrightarrow{\mathrm{ev}^{[]}} \mathcal{B}^{[][]} \xrightarrow{\Sigma^n} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]}\bigr] \\
&= \bigl[\mathcal{A}^{[]}, \mathcal{C} \xrightarrow{1,u_{[]}} \mathcal{A}^{[]}, \mathcal{C}^{[]} \xrightarrow{\mathrm{ev}^{[]}} \mathcal{B}^{[][]} \xrightarrow{\Sigma^n} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]}\bigr],
\end{aligned} \tag{13.11.3}$$

where $\mathcal{C}$ denotes $\underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]})$. We have used commutation (13.8.1).

The right hand side of (13.11.2) transforms to

$$\begin{aligned}
&\bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{1,F} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]},\mathcal{B}^{[]}) \xrightarrow{\Sigma^n,1} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A}^{[]};\mathcal{B}^{[]}) \xrightarrow{\mathrm{ev}} \mathcal{B}^{[]}\bigr] \\
&= \bigl[\mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]}) \xrightarrow{\Sigma^n,u_{[]}} \mathcal{A}^{[]}, \underline{\mathsf{A}_\infty^\mathrm{u}}(\mathcal{A};\mathcal{B}^{[]})^{[]} \xrightarrow{\mathrm{ev}^{[]}} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]}\bigr] \\
&= \bigl[\mathcal{A}^{[]}, \mathcal{C} \xrightarrow{\Sigma^n,u_{[]}} \mathcal{A}^{[]}, \mathcal{C}^{[]} \xrightarrow{\mathrm{ev}^{[]}} \mathcal{B}^{[][]} \xrightarrow{m_{[]}} \mathcal{B}^{[]}\bigr]
\end{aligned}$$

similarly to the left hand side. Actually, this equals (13.11.3) for an arbitrary morphism $\mathrm{ev} : \mathcal{A}, \mathcal{C} \to \mathcal{B}^{[]}$, not only for evaluation. Indeed, $u_{[]} = \bigl(\mathcal{C} \xrightarrow[\sim]{\lambda^{\mathsf{I}}\cdot} \mathcal{C} \boxdot \mathbb{1}_p \xrightarrow{1\boxdot\eta_{\mathcal{Z}}} \mathcal{C} \boxdot \mathcal{Z}\bigr)$,

where $\eta_{\mathcal{Z}} = \dot{0} : \mathbb{1}_p \to \mathcal{Z}$ is the unit of the algebra $\mathcal{Z}$, $\mathrm{ev}^{[]} = \mathrm{ev} \boxdot \otimes_{\mathcal{Z}}$ by (10.17.1), and the equation to prove reads:

$$\big[\mathcal{A} \boxdot \mathcal{Z}, \mathcal{C} \boxdot \mathbb{1}_p \xrightarrow{1\boxdot 1, 1\boxdot \eta_{\mathcal{Z}}} \mathcal{A} \boxdot \mathcal{Z}, \mathcal{C} \boxdot \mathcal{Z} \xrightarrow{\mathrm{ev}\boxdot\otimes_{\mathcal{Z}}} \mathcal{B}^{[]} \boxdot \mathcal{Z} \xrightarrow{1\boxdot\boldsymbol{\sigma}^n} \mathcal{B}^{[]} \boxdot \mathcal{Z}\big]$$
$$= \big[\mathcal{A} \boxdot \mathcal{Z}, \mathcal{C} \boxdot \mathbb{1}_p \xrightarrow{1\boxdot\boldsymbol{\sigma}^n, 1\boxdot \eta_{\mathcal{Z}}} \mathcal{A} \boxdot \mathcal{Z}, \mathcal{C} \boxdot \mathcal{Z} \xrightarrow{\mathrm{ev}\boxdot\otimes_{\mathcal{Z}}} \mathcal{B}^{[]} \boxdot \mathcal{Z}\big].$$

Since the action $\boxdot$ is a multifunctor this equation reduces to

$$\big[\mathcal{Z} \boxtimes \mathbb{1}_p \xrightarrow{1\boxtimes\eta_{\mathcal{Z}}} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_{\mathcal{Z}}} \mathcal{Z} \xrightarrow{\boldsymbol{\sigma}^n} \mathcal{Z}\big] = \big[\mathcal{Z} \boxtimes \mathbb{1}_p \xrightarrow{\boldsymbol{\sigma}^n\boxtimes\eta_{\mathcal{Z}}} \mathcal{Z} \boxtimes \mathcal{Z} \xrightarrow{\otimes_{\mathcal{Z}}} \mathcal{Z}\big].$$

Composing it with $\lambda^{\mathsf{I}\cdot} : \mathcal{Z} \to \mathcal{Z} \boxtimes \mathbb{1}_p$ we reduce it to obvious identity $\boldsymbol{\sigma}^0 \cdot \boldsymbol{\sigma}^n = \boldsymbol{\sigma}^n \cdot \boldsymbol{\sigma}^0$ since $\eta_{\mathcal{Z}} = \dot{0}$. The lemma is proven. $\square$

Therefore, for any A_∞-functor $g : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$ which belongs to $\operatorname{Im} \operatorname{Ob} F$, we have $\Sigma^n g = g\Sigma^n$. Since F and Σ^n are strict, all A_∞-functors in (13.11.1) are strict. Applying their first components to an arbitrary A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}^{[]}$ we get $\Sigma^n(rF_1) = (rF_1)\Sigma^n : (fF)\Sigma^n \to (gF)\Sigma^n : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$. Thus, the lifting $\tilde{-}$ can be defined by identity transformations on the images of $\operatorname{Ob} F$ and F_1. Since F is an A_∞-equivalence one can extend the lifting to the whole $A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]})$. Let us supply the details.

Let $f : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$ be a unital A_∞-functor. There is an A_∞-functor $g : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$, $g = g'F$, isomorphic to f. Let $\alpha = rs^{-1} : f \to g$, $\beta = ps^{-1} : g \to f$ be 2-morphisms of $\overline{A_\infty^u}$, inverse to each other. Define $\tilde{f} = (f, \phi_n) : (\mathcal{A}^{[]}, \Sigma) \to (\mathcal{B}^{[]}, \Sigma)$, where

$$\phi_n = \big(\Sigma^n f \xrightarrow{\Sigma^n\alpha} \Sigma^n g = g\Sigma^n \xrightarrow{\alpha^{-1}\Sigma^n} f\Sigma^n\big),$$

the composition is taken in 2-category $\overline{A_\infty^u}$. Since $\Sigma^0 = \mathrm{id}$, we have $\phi_0 = \mathrm{id}$, hence, equation (13.2.5) is satisfied. The upper row in commutative diagram

$$\begin{tikzcd}
\Sigma^m\Sigma^n f \arrow[r, "\Sigma^m\phi_n"] \arrow[d, "\Sigma^m\Sigma^n\alpha"'] & \Sigma^m f\Sigma^n \arrow[r, "\phi_m\Sigma^n"] \arrow[d, shift left, "\Sigma^m\alpha\Sigma^n"] & f\Sigma^m\Sigma^n \arrow[d, "\alpha^{-1}\Sigma^m\Sigma^n"] \\
\Sigma^m\Sigma^n g \arrow[r, equal] & \Sigma^m g\Sigma^n \arrow[r, equal] \arrow[u, shift left, "\Sigma^m\alpha^{-1}\Sigma^n"] & g\Sigma^m\Sigma^n
\end{tikzcd}$$

composes to ϕ_{m+n}, hence, (13.2.4) is satisfied.

Let us prove that ϕ_n does not depend on the choice of g and α. If $\gamma : f \to h$ is a 2-isomorphism with $h = h'F$, the 2-isomorphism $\delta = \gamma^{-1} \cdot \alpha : h \to g$ can be represented by $(rF_1)s^{-1}$ for some natural A_∞-transformation $r : h' \to g' : \mathcal{A} \to \mathcal{B}^{[]}$. Indeed, being an equivalence $H^0(F) : H^0(A_\infty^u(\mathcal{A}, \mathcal{B}^{[]})) \to H^0(A_\infty^u(\mathcal{A}^{[]}, \mathcal{B}^{[]}))$ is full and faithful. The equation $\Sigma^n(rF_1) = (rF_1)\Sigma^n$ between natural A_∞-transformations implies the equation $\Sigma^n\delta = \delta\Sigma^n$ between 2-morphisms in $\overline{A_\infty^u}$, see [Lyu03, Proposition 7.1]. The obtained equation $(\Sigma^n\gamma^{-1}) \cdot (\Sigma^n\alpha) = (\gamma^{-1}\Sigma^n) \cdot (\alpha\Sigma^n)$ implies that $\phi_n = (\Sigma^n\alpha) \cdot (\alpha^{-1}\Sigma^n) = (\Sigma^n\gamma) \cdot (\gamma^{-1}\Sigma^n)$ does not depend on the choices made.

Let $k : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$ be another unital A_∞-functor with the lifting $\tilde{k} = (k, \kappa_n) : (\mathcal{A}^{[]}, \Sigma) \to (\mathcal{B}^{[]}, \Sigma)$. Let $\nu : f \to k : \mathcal{A}^{[]} \to \mathcal{B}^{[]}$ be a 2-morphism in $\overline{A^u_\infty}$. We claim that it is also a 2-morphism $\nu : (f, \phi_n) \to (k, \kappa_n) : (\mathcal{A}^{[]}, \Sigma) \to (\mathcal{B}^{[]}, \Sigma)$ in $\operatorname{Trans} \overline{A^u_\infty}$. Indeed, construct ϕ_n (resp. κ_n) using a 2-isomorphism $\alpha : f \to g$ (resp. $\gamma : h \to k$) with $g, h \in \operatorname{Im} \operatorname{Ob} F$. Then there is a unique 2-morphism $\delta : g \to h$ such that $\nu = \bigl(f \xrightarrow{\alpha} g \xrightarrow{\delta} h \xrightarrow{\gamma} k\bigr)$. The above equation $\Sigma^n \delta = \delta \Sigma^n$ implies commutativity of the following diagram

$$
\begin{tikzcd}
\Sigma^n f \arrow[rrr, "\Sigma^n \nu"] \arrow[dr, "\Sigma^n \alpha"'] \arrow[dd, "\phi_n"'] & & & \Sigma^n k \arrow[dd, "\kappa_n"] \\
& \Sigma^n g \arrow[r, "\Sigma^n \delta"] \arrow[d, equal] & \Sigma^n h \arrow[ur, "\Sigma^n \gamma"'] \arrow[d, equal] & \\
& g \Sigma^n \arrow[r, "\delta \Sigma^n"] & h \Sigma^n \arrow[dr, "\gamma \Sigma^n"] & \\
f \Sigma^n \arrow[rrr, "\nu \Sigma^n"'] \arrow[ur, "\alpha \Sigma^n"] & & & k \Sigma^n
\end{tikzcd}
$$

Its exterior gives diagram (13.2.6) for ν. The proposition is proven. □

13.12 Corollary. *There is a composite 2-functor*

$$\tilde{H}^0 = \bigl(A_\infty^{[]\text{-closed}} \xrightarrow{\tilde{}} \operatorname{Trans} \overline{A^u_\infty} \xrightarrow{\operatorname{Trans} H^0} \operatorname{Trans} \Bbbk\text{-}\mathcal{C}at\bigr).$$

Given an A_∞-category $\mathcal{B}$ closed under shifts, we denote by $\tilde{H}^0(\mathcal{B})$ the $\Bbbk$-linear category $H^0(\mathcal{B})$ equipped with the natural translation structure, obtained via the above corollary. Furthermore, a unital A_∞-functor $f : \mathcal{B} \to \mathcal{C}$ between such A_∞-categories induces a translation preserving functor $\tilde{H}^0(f) = (H^0(f), \phi_n) : \tilde{H}^0(\mathcal{B}) \to \tilde{H}^0(\mathcal{C})$. Similarly, any natural A_∞-transformation $p : g \to h : \mathcal{B} \to \mathcal{C}$ of unital A_∞-functors between A_∞-categories closed under shifts induces a translation preserving transformation $\tilde{H}^0(p) : \tilde{H}^0(g) \to \tilde{H}^0(h) : \tilde{H}^0(\mathcal{B}) \to \tilde{H}^0(\mathcal{C})$ in zeroth homology.

Now we are going to describe the translation structure on $\mathcal{C}^{\mathsf{tr}} = \mathcal{C}^{[]\mathsf{mc}}$.

13.13 Lemma. *Let $\mathcal{C}$ be an A_∞-category. Then for each $n \in \mathbb{Z}$ the following equation holds:*

$$\bigl(\mathcal{C}^{\mathsf{mc}[]} \xrightarrow{\Sigma^n_{\mathcal{C}^{\mathsf{mc}[]}}} \mathcal{C}^{\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{C}^{[]\mathsf{mc}}\bigr) = \bigl(\mathcal{C}^{\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{C}^{[]\mathsf{mc}} \xrightarrow{(\Sigma^n_{\mathcal{C}^{[]}})^{\mathsf{mc}}} \mathcal{C}^{[]\mathsf{mc}}\bigr). \tag{13.13.1}$$

Proof. Since all A_∞-functors in equation (13.13.1) are strict and their first components are identity maps, it suffices to check that both compositions act identically on objects. Let $X : I \to \mathcal{C}$, $i \mapsto X_i$, $x = (x_{ij})_{i,j \in I}$, $x_{ij} \in [s\mathcal{C}(X_i, X_j)]^0$ be an object of $\mathcal{C}^{\mathsf{mc}}$ and let $m \in \mathbb{Z}$. The first functor in the left hand side of (13.13.1) maps the object $X[m]$ of $\mathcal{C}^{\mathsf{mc}[]}$ to the object $X[m{+}n]$, the second functor maps it further to the object $X' = X[m{+}n]\mathfrak{c} : I \to \mathcal{C}^{[]}$, $i \mapsto X_i[m{+}n]$, $x' = x = (x_{ij})$, $x'_{ij} \in \mathcal{C}^{[]}(X_i[m{+}n], X_j[m{+}n])$ of $\mathcal{C}^{[]\mathsf{mc}}$. The first functor in the right hand side of (13.13.1) maps the object $X[m]$ to the object $X'' = X[m]\mathfrak{c} : I \to \mathcal{C}^{[]}$, $i \mapsto X_i[m]$, $x'' = x = (x_{ij})$, $x''_{ij} \in s\mathcal{C}^{[]}(X_i[m], X_j[m])$ of $\mathcal{C}^{[]\mathsf{mc}}$. The second functor maps

X'' to the object $X''(\Sigma^n_{\mathcal{C}[]})^{\mathsf{mc}} = (I \xrightarrow{X''} \mathcal{C}^{[]} \xrightarrow{\Sigma^n_{\mathcal{C}[]}} \mathcal{C}^{[]})$ of $\mathcal{C}^{[]\mathsf{mc}}$. The functor $\Sigma^n_{\mathcal{C}[]}$ is strict and its first component is the identity map, therefore the objects X' and $X''(\Sigma^n_{\mathcal{C}[]})^{\mathsf{mc}}$ are given by the same data. Thus, the objects $X[m]\Sigma^n_{\mathcal{C}^{\mathsf{mc}}[]}\mathfrak{c}$ and $X[m]\mathfrak{c}(\Sigma^n_{\mathcal{C}[]})^{\mathsf{mc}}$ coincide. □

Let $\mathcal{B}$ be a unital A_∞-category. The A_∞-equivalence $u_{[]} : \mathcal{B}^{[]\mathsf{mc}} \to \mathcal{B}^{[]\mathsf{mc}[]}$ has a one-sided inverse

$$U_{[]} = \big(\mathcal{B}^{[]\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{B}^{[][]\mathsf{mc}} \xrightarrow{m^{\mathsf{mc}}_{[]}} \mathcal{B}^{[]\mathsf{mc}}\big),$$

that is, $u_{[]} \cdot U_{[]} = \mathrm{Id}_{\mathcal{B}^{[]\mathsf{mc}}}$. Indeed, by (12.4.1) and Proposition 11.26

$$\begin{aligned} u_{[]} \cdot U_{[]} &= \big(\mathcal{B}^{[]\mathsf{mc}} \xrightarrow{u_{[]}} \mathcal{B}^{[]\mathsf{mc}[]} \xrightarrow{\mathfrak{c}} \mathcal{B}^{[][]\mathsf{mc}} \xrightarrow{m^{\mathsf{mc}}_{[]}} \mathcal{B}^{[]\mathsf{mc}}\big) \\ &= \big(\mathcal{B}^{[]\mathsf{mc}} \xrightarrow{u^{\mathsf{mc}}_{[]}} \mathcal{B}^{[][]\mathsf{mc}} \xrightarrow{m^{\mathsf{mc}}_{[]}} \mathcal{B}^{[]\mathsf{mc}}\big) = (u_{[]}m_{[]})^{\mathsf{mc}} = (\mathrm{Id}_{\mathcal{B}^{[]}})^{\mathsf{mc}} = \mathrm{Id}_{\mathcal{B}^{[]\mathsf{mc}}} . \end{aligned}$$

13.14 Lemma. *For each $n \in \mathbb{Z}$ the following equation holds:*

$$\big(\mathcal{B}^{[]\mathsf{mc}[]} \xrightarrow{\Sigma^n_{\mathcal{B}[]\mathsf{mc}[]}} \mathcal{B}^{[]\mathsf{mc}[]} \xrightarrow{U_{[]}} \mathcal{B}^{[]\mathsf{mc}}\big) = \big(\mathcal{B}^{[]\mathsf{mc}[]} \xrightarrow{U_{[]}} \mathcal{B}^{[]\mathsf{mc}} \xrightarrow{(\Sigma^n_{\mathcal{B}[]})^{\mathsf{mc}}} \mathcal{B}^{[]\mathsf{mc}}\big). \tag{13.14.1}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} \mathcal{B}^{[]\mathsf{mc}[]} \arrow[r, "\mathfrak{c}"] \arrow[d, "\Sigma^n_{\mathcal{B}[]\mathsf{mc}[]}"'] & \mathcal{B}^{[][]\mathsf{mc}} \arrow[r, "m^{\mathsf{mc}}_{[]}"] \arrow[d, "(\Sigma^n_{\mathcal{B}[][]})^{\mathsf{mc}}"'] & \mathcal{B}^{[]\mathsf{mc}} \arrow[d, "(\Sigma^n_{\mathcal{B}[]})^{\mathsf{mc}}"] \\ \mathcal{B}^{[]\mathsf{mc}[]} \arrow[r, "\mathfrak{c}"] & \mathcal{B}^{[][]\mathsf{mc}} \arrow[r, "m^{\mathsf{mc}}_{[]}"] & \mathcal{B}^{[]\mathsf{mc}} \end{tikzcd}$$

The left square commutes by (13.13.1). By Proposition 11.26, $-^{\mathsf{mc}}$ is an A^u_∞-2-functor, therefore commutativity of the right square is a consequence of equation (13.8.1). The exterior of the diagram gives equation (13.14.1). □

The quadruple $(u_{[]}, U_{[]}, \mathrm{id}, \varepsilon)$ is an adjunction–equivalence for a unique natural isomorphism $\varepsilon : U_{[]} \cdot u_{[]} \to \mathrm{id}_{\mathcal{B}^{[]\mathsf{mc}[]}}$. Equations (13.1.1) and (13.1.2) take the form

$$\begin{aligned} &\big(u_{[]} = u_{[]} \cdot U_{[]} \cdot u_{[]} \xrightarrow{u_{[]}\varepsilon} u_{[]}\big) = \mathrm{id}_{u_{[]}} : u_{[]} \to u_{[]} : \mathcal{B}^{[]\mathsf{mc}} \to \mathcal{B}^{[]\mathsf{mc}[]}, \\ &\big(U_{[]} = U_{[]} \cdot u_{[]} \cdot U_{[]} \xrightarrow{\varepsilon U_{[]}} U_{[]}\big) = \mathrm{id}_{U_{[]}} : U_{[]} \to U_{[]} : \mathcal{B}^{[]\mathsf{mc}[]} \to \mathcal{B}^{[]\mathsf{mc}}. \end{aligned} \tag{13.14.2}$$

Given these data, equip the A_∞-category $\mathcal{B}^{[]\mathsf{mc}}$ with a translation structure as described in Remark 13.4. For each $n \in \mathbb{Z}$ the A_∞-functor $\Sigma^n_{\mathcal{B}[]\mathsf{mc}}$ is given by

$$\Sigma^n_{\mathcal{B}[]\mathsf{mc}} = u_{[]}\Sigma^n_{\mathcal{B}[]\mathsf{mc}[]}U_{[]} = u_{[]}U_{[]}(\Sigma^n_{\mathcal{B}[]})^{\mathsf{mc}} = (\Sigma^n_{\mathcal{B}[]})^{\mathsf{mc}} : \mathcal{B}^{[]\mathsf{mc}} \to \mathcal{B}^{[]\mathsf{mc}}. \tag{13.14.3}$$

The second equality holds due to (13.14.1). For each pair $n, m \in \mathbb{Z}$ the 2-isomorphism $\varsigma_{m,n}^{\mathcal{B}[]^{\mathsf{mc}}}$ is given by the composite

$$\begin{aligned}
&\big(\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}}\varSigma^n_{\mathcal{B}[]^{\mathsf{mc}}} = u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]}u_{[]}\varSigma^n_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]} \xrightarrow{u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}\varepsilon\varSigma^n_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]}} u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}\varSigma^n_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]} \\
&\qquad\qquad\qquad\qquad \xRightarrow{u_{[]}\varsigma_{m,n}^{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]}} u_{[]}\varSigma^{m+n}_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]} = \varSigma^{m+n}_{\mathcal{B}[]^{\mathsf{mc}}}\big) \\
&= \big(\varSigma^m_{\mathcal{B}^{\mathsf{tr}}}\varSigma^n_{\mathcal{B}^{\mathsf{tr}}} = u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]}u_{[]}U_{[]}(\varSigma^n_{\mathcal{B}[]})^{\mathsf{mc}} \xrightarrow{u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}\varepsilon U_{[]}(\varSigma^n_{\mathcal{B}[]})^{\mathsf{mc}}} u_{[]}\varSigma^m_{\mathcal{B}[]^{\mathsf{mc}}[]}U_{[]}(\varSigma^n_{\mathcal{B}[]})^{\mathsf{mc}} = \varSigma^{m+n}_{\mathcal{B}^{\mathsf{tr}}}\big).
\end{aligned}$$

The equality holds by (13.14.1). By (13.14.2) the latter 2-morphism is the identity.

Thus, $(\varSigma^n_{\mathcal{B}^{\mathsf{tr}}}, \mathrm{id}) = ((\varSigma^n_{\mathcal{B}[]})^{\mathsf{mc}}, \mathrm{id}) : \mathbb{Z} \to \overline{A^u_\infty}(\mathcal{B}^{\mathsf{tr}}, \mathcal{B}^{\mathsf{tr}})$ is a strict monoidal functor. The strict A_∞-functor $\varSigma^n_{\mathcal{B}^{\mathsf{tr}}} = (\varSigma^n_{\mathcal{B}[]})^{\mathsf{mc}}$ takes an object $(I \ni i \mapsto X_i[m_i], (x_{ij}))$ of $\mathcal{B}^{\mathsf{tr}}$ to the object $(I \ni i \mapsto X_i[m_i+n], (x_{ij}))$. Its first component acts on morphisms as identity map.

13.15 Systems of n-triangles. Let us recall the definition of an n-triangle in a category given by Maltsiniotis [Mal06]. To large extent it was extracted from the work of Neeman on K-theory for triangulated categories, see survey [Nee05, Sections 6–8] and references therein. A different but similar notion of an n-triangle is used by Künzer [Kün07a].

Consider $\mathbb{Z}$ as a category, whose objects are integers and $\mathbb{Z}(i,j) = \{e_{ij}\}$ are one-element sets if $i \leqslant j$, otherwise $\mathbb{Z}(i,j)$ are empty. The square $\mathbb{Z} \times \mathbb{Z}$ of this category is associated with the partially ordered set $\mathbb{Z} \times \mathbb{Z}$. View its subsets as full subcategories of $\mathbb{Z} \times \mathbb{Z}$. For an integer $n \geqslant 0$ consider the following subsets:

$$\begin{aligned}
D_n &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x \leqslant y \leqslant x+n+1\}, \\
\overset{\circ}{D}_n &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x < y < x+n+1\}, \\
\partial^0 D_n &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x = y\}, \\
\partial^1 D_n &= \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y = x+n+1\}.
\end{aligned}$$

For any non-decreasing map $\psi : [m] \to [n]$ define a non-decreasing map $\overline{\psi} : \mathbb{Z} \to \mathbb{Z}$ via

$$\overline{\psi}(k) = (n+1)q + \psi(r),$$

where integers q, r are uniquely determined by k from the conditions

$$k = (m+1)q + r, \qquad 0 \leqslant r \leqslant m.$$

In other words, q and r are the quotient and the remainder for k divided by $m+1$. A non-decreasing map and a functor are defined as follows:

$$D_\psi : D_m \to D_n, \qquad (x,y) \mapsto (\overline{\psi}(x), \overline{\psi}(y)).$$

Introduce also the increasing bijections (category automorphisms) $I_n, J_n : D_n \to D_n$ via:

$$I_n(x,y) = (y, x+n+1), \qquad J_n(x,y) = (x+1, y+1).$$

They satisfy the relations

$$I_n J_n = J_n I_n, \qquad I_n^2 = J_n^{n+1}, \qquad D_\psi \circ I_m = I_n \circ D_\psi$$

for an arbitrary non-decreasing map $\psi : [m] \to [n]$. Make D_n into a category with translation structure by equipping it with the automorphisms $\Sigma^p = I_n^p$, $p \in \mathbb{Z}$, which form a strict representation of $\mathbb{Z}$ on D_n.

Assume given from now on a category with translation structure $(\mathcal{T}, \Sigma^p, \varsigma)$ which has a zero object 0 (initial and final object). All zero objects are isomorphic. Any category equivalence $E : \mathcal{T} \to \mathcal{T}'$ takes a zero object to a zero object.

13.16 Definition. Define an *n-triangle* of $\mathcal{T}$ as a translation preserving functor $(F, \phi^p) : D_n \to \mathcal{T}$ such that $(x,x)F$ are zero objects of $\mathcal{T}$ for all $x \in \mathbb{Z}$. Thus, $F : D_n \to \mathcal{T}$ is a functor, and $\phi_p : I_n^p F \xrightarrow{\sim} F\Sigma^p$, $p \in \mathbb{Z}$, are functorial isomorphisms related as in (13.2.4), (13.2.5). A *morphism of n-triangles* is a translation preserving morphism of functors, thus, (13.2.6) holds. A *system of n-triangles* in $\mathcal{T}$ is a collection of n-triangles, called *distinguished n-triangles*, which satisfies the axiom

(FTR0) Each n-triangle isomorphic to a distinguished n-triangle is distinguished.

Let $\mathcal{B}$, $\mathcal{C}$ be categories equipped with a system of n-triangles. A translation preserving functor $(G, \psi_p) : \mathcal{B} \to \mathcal{C}$ is called *triangulated* if for every distinguished n-triangle $F : D_n \to \mathcal{B}$ the composite n-triangle $F \cdot G : D_n \to \mathcal{C}$ is distinguished.

The condition $(x,y)F \simeq 0$ holds for $(x,y) \in \partial^0 D_n$ by definition. It implies also that $(x,y)F \simeq 0$ for $(x,y) \in \partial^1 D_n$.

Diagrams introduced by Beilinson, Bernstein and Deligne in [BBD82, Remark 1.1.14] are the same as hypersimplices considered by Gelfand and Manin in [GM03, Exercise IV.2.1c], and these are essentially examples of n-triangles.

Let us look at examples for small n. If $n = 0$, then $(x,y)F \simeq 0$ for all $(x,y) \in D_0$. When $n = 1$, a 1-triangle is unambiguously recovered from a sequence of objects $X_k = (k, k+1)F$, $k \in \mathbb{Z}$, and a family of isomorphisms $\phi_p : X_p \xrightarrow{\sim} X_0 \Sigma^p$, $p \neq 0$. Clearly, the category of 1-triangles in $\mathcal{T}$ is equivalent to $\mathcal{T}$ via the functor $(X_k, \phi_p)_{k,p \in \mathbb{Z}}^{p \neq 0} \mapsto X_0$.

A 2-triangle is precisely a candidate triangle in the sense of ordinary triangulated categories (up to irrelevant choice of zero objects $(x,y)F$ for $(x,y) \in \partial^\bullet D_2$). That is, essentially, a 2-triangle consists of three objects $(0,1)F$, $(0,2)F$, $(1,2)F$ and three mor-

phisms u, v, $w \cdot \phi_1$ as shown below,

$$\begin{tikzcd}
(0,1)F \arrow[r, "u"] & (0,2)F \arrow[d, "v"'] & & \\
& (1,2)F \arrow[r, "w"] & (1,3)F \arrow[r, "\phi_1", "\sim"'] \arrow[d, "\phi_1\cdot(u\Sigma^1)\cdot\phi_1^{-1}"'] & (0,1)F\Sigma^1 \arrow[d, "u\Sigma^1"] \\
& & (2,3)F \arrow[r, "\phi_1", "\sim"'] & (0,2)F\Sigma^1
\end{tikzcd}$$

such that the three consecutive compositions, $u \cdot v$, $v \cdot (w \cdot \phi_1)$ and $(w \cdot \phi_1) \cdot (u\Sigma^1)$ vanish (factor through 0).

A 3-triangle is an octahedron from the point of view of ordinary triangulated categories. Its essential data are described on the following diagram:

$$\begin{tikzcd}
X_1 \arrow[r, "f_{12}"] & X_2 \arrow[r, "f_{23}"] \arrow[d, "i_2"'] & X_3 \arrow[d, "i_3"'] & \\
& C_{12} \arrow[r, "v_{23}"] & C_{13} \arrow[r, "j_1"] \arrow[d, "u_{12}"'] & X_1\Sigma^1 \arrow[d, "f_{12}\Sigma^1"] \\
& & C_{23} \arrow[r, "j_2"] & X_2\Sigma^1 \arrow[d, "f_{23}\Sigma^1"] \\
& & & X_3\Sigma^1
\end{tikzcd}$$

Here $X_k = (0,k)F$, $C_{12} = (1,2)F$, $C_{13} = (1,3)F$ and $C_{23} = (2,3)F$. Two manifest and four hidden squares have to commute. The latter mean squares obtained after adding zero objects at the sides of the strip. General n-triangles are higher generalizations of octahedra.

13.17 The principal distinguished n-triangle. Define an A_∞-category (a non-unital differential graded category) $\mathcal{A}_n$ for $n \geqslant 0$ as follows. Objects set is $\operatorname{Ob}\mathcal{A}_n = \{\underline{1}, \underline{2}, \dots, \underline{n}\}$, morphisms sets are $\mathcal{A}_n(\underline{i}, \underline{j}) = \Bbbk\{e_{ij}\} \simeq \Bbbk$ if $i < j$, and $\mathcal{A}_n(\underline{i}, \underline{j}) = 0$ if $i \geqslant j$. Here $\deg e_{ij} = 0$, so morphism complexes are concentrated in degree 0. The only possibly non-vanishing component of the multiplication is m_2. It is given by $(e_{ij} \otimes e_{jk})m_2 = e_{ik}$.

Denote by $\mathcal{D}_n$ the differential graded category $\mathcal{A}_n^{\mathsf{su}}$ (with the identity morphisms added). The identity morphism of $\underline{k}$ is denoted also e_{kk}, so that the above multiplication formula still holds. Let us construct the principal distinguished n-triangle in $H^0(\mathcal{D}_n^{\mathsf{tr}})$.

Consider the objects

$$C(e_{kl}) = \operatorname{Cone}(e_{kl}) = \Bigl(1 \mapsto \underline{k}[1], 2 \mapsto \underline{l}[0], \begin{pmatrix} 0 & e_{kl} \\ & 0 \end{pmatrix}\Bigr) = \begin{pmatrix} \underline{k}[1] & e_{kl} \\ & \underline{l} \end{pmatrix}$$

of $\mathcal{D}_n^{\mathsf{tr}}$. There are cycles in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$ between them:

$$u_m(e_{kl}) = \begin{pmatrix} e_{kl}[1] & 0 \\ 0 & 1_{\underline{m}} \end{pmatrix} : C(e_{km}) = \begin{pmatrix} \underline{k}[1] & e_{km} \\ & \underline{m} \end{pmatrix} \to \begin{pmatrix} \underline{l}[1] & e_{lm} \\ & \underline{m} \end{pmatrix} = C(e_{lm}) \text{ for } k < l < m,$$
$$v_j(e_{kl}) = \begin{pmatrix} 1_{\underline{j}[1]} & 0 \\ 0 & e_{kl} \end{pmatrix} : C(e_{jk}) = \begin{pmatrix} \underline{j}[1] & e_{jk} \\ & \underline{k} \end{pmatrix} \to \begin{pmatrix} \underline{j}[1] & e_{jl} \\ & \underline{l} \end{pmatrix} = C(e_{jl}) \text{ for } j < k < l.$$

For homogeneity we consider also the zero maps $e_{0l} = 0 : 0 \to \underline{l}$, $e_{k,n+1} = 0 : \underline{k} \to 0$ in $\mathcal{D}_n^{\mathsf{tr}}$ for $1 \leqslant k, l \leqslant n$ and their cones descended to the same **dg**-category

$$C(e_{0l}) = \operatorname{Cone}(e_{0l})m_{\mathsf{tr}} = \begin{pmatrix} 0 & 0 \\ & \underline{l} \end{pmatrix} m_{\mathsf{tr}} = \underline{l}.$$
$$C(e_{k,n+1}) = \operatorname{Cone}(e_{k,n+1})m_{\mathsf{tr}} = \begin{pmatrix} \underline{k}[1] & 0 \\ & 0 \end{pmatrix} m_{\mathsf{tr}} = \underline{k}[1].$$

Then the morphisms defined above identify with

$$u_m(e_{0l}) = (01) : C(e_{0m}) = \underline{m} \to \begin{pmatrix} \underline{l}[1] & e_{lm} \\ & \underline{m} \end{pmatrix} = C(e_{lm}) \text{ for } 0 < l < m,$$
$$v_0(e_{kl}) = e_{kl} : C(e_{0k}) = \underline{k} \to \underline{l} = C(e_{0l}) \text{ for } 0 < k < l,$$
$$u_{n+1}(e_{kl}) = e_{kl}[1] : C(e_{k,n+1}) = \underline{k}[1] \to \underline{l}[1] = C(e_{l,n+1}) \text{ for } k < l,$$
$$v_j(e_{k,n+1}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C(e_{jk}) = \begin{pmatrix} \underline{j}[1] & e_{jk} \\ & \underline{k} \end{pmatrix} \to \underline{j}[1] = C(e_{j,n+1}) \text{ for } j < k.$$

Notice that for arbitrary $i \leqslant j \leqslant k \leqslant l$ we have the relations

$$\bigl[C(e_{il}) \xrightarrow{u_l(e_{ij})} C(e_{jl}) \xrightarrow{u_l(e_{jk})} C(e_{kl})\bigr] = u_l(e_{ik}),$$
$$\bigl[C(e_{ij}) \xrightarrow{v_i(e_{jk})} C(e_{ik}) \xrightarrow{v_i(e_{kl})} C(e_{il})\bigr] = v_i(e_{il}).$$

The following square commutes in the category $Z^0(\mathcal{D}_n^{\mathsf{tr}})$ for all $j < k \leqslant l < m$:

$$\begin{tikzcd} C(e_{jl}) \arrow[r, "v_j(e_{lm})"] \arrow[d, "u_l(e_{jk})"'] & C(e_{jm}) \arrow[d, "u_m(e_{jk})"] \\ C(e_{kl}) \arrow[r, "v_k(e_{lm})"] & C(e_{km}) \end{tikzcd}$$

A forteriori it commutes in $H^0(\mathcal{D}_n^{\mathsf{tr}})$.

Let us summarize the above morphisms in the following commutative diagram, where we put $C(e_{kl})$ to the point with the coordinates $(x, y) = (k, l)$. Note that the x-axis goes down, and the y-axis goes right. We depict the case $n = 4$. This diagram will become a part of the interior of an n-triangle defined below.

$$
\begin{tikzcd}
\underline{1} \arrow[r, "e_{12}"] & \underline{2} \arrow[r, "e_{23}"] \arrow[d, "(01)"'] & \underline{3} \arrow[r, "e_{34}"] \arrow[d, "(01)"'] & \underline{4} \arrow[d, "(01)"'] & \\
& C(e_{12}) \arrow[r, "v_1(e_{23})"] & C(e_{13}) \arrow[r, "v_1(e_{34})"] \arrow[d, "u_3(e_{12})"'] & C(e_{14}) \arrow[r, "\binom{1}{0}"] \arrow[d, "u_4(e_{12})"'] & \underline{1}[1] \arrow[d, "e_{12}[1]"'] \\
& & C(e_{23}) \arrow[r, "v_2(e_{34})"] & C(e_{24}) \arrow[r, "\binom{1}{0}"] \arrow[d, "u_4(e_{23})"'] & \underline{2}[1] \arrow[d, "e_{23}[1]"'] \\
& & & C(e_{34}) \arrow[r, "\binom{1}{0}"] & \underline{3}[1] \arrow[d, "e_{34}[1]"'] \\
& & & & \underline{4}[1]
\end{tikzcd}
$$

The boundary of D_n is filled with the zero object $(\varnothing \to \mathcal{D}_n^{[]})$ of $H^0(\mathcal{D}_n^{\mathsf{tr}})$. The same diagram can be given a homogeneous form:

$$
\begin{tikzcd}
C(e_{01}) \arrow[r, "v_0(e_{12})"] & C(e_{02}) \arrow[r, "v_0(e_{23})"] \arrow[d, "u_2(e_{01})"'] & C(e_{03}) \arrow[r, "v_0(e_{34})"] \arrow[d, "u_3(e_{01})"'] & C(e_{04}) \arrow[d, "u_4(e_{01})"'] & \\
& C(e_{12}) \arrow[r, "v_1(e_{23})"] & C(e_{13}) \arrow[r, "v_1(e_{34})"] \arrow[d, "u_3(e_{12})"'] & C(e_{14}) \arrow[r, "v_1(e_{45})"] \arrow[d, "u_4(e_{12})"'] & C(e_{15}) \arrow[d, "u_5(e_{12})"'] \\
& & C(e_{23}) \arrow[r, "v_2(e_{34})"] & C(e_{24}) \arrow[r, "v_2(e_{45})"] \arrow[d, "u_4(e_{23})"'] & C(e_{25}) \arrow[d, "u_5(e_{23})"'] \\
& & & C(e_{34}) \arrow[r, "v_3(e_{45})"] & C(e_{35}) \arrow[d, "u_5(e_{34})"'] \\
& & & & C(e_{45})
\end{tikzcd}
$$

Define a translation preserving functor $\triangle_n : D_n \to \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}})$, the principal distinguished n-triangle in $\tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}})$. Any point (x, y) of $D_n \subset \mathbb{Z} \times \mathbb{Z}$ equals $I_n^p(k, l)$ for some uniquely determined $p \in \mathbb{Z}$, $0 \leqslant k \leqslant l \leqslant n$. These inequalities define the fundamental domain

of the group generated by I_n on vertices. Set $\triangle_n(x,y) = C(e_{kl})[p]$. Applying I_n and $[1]$ (or I_n^{-1} and $[-1]$) simultaneously several times we cover the whole strip D_n with the commutative diagram of the above type, where the rightmost column is removed. It remains commutative, thus it gives a functor $\triangle_n$. By construction this functor satisfies the identity $I_n \cdot \triangle_n = \triangle_n \cdot [1]$ and, in general, $I_n^p \cdot \triangle_n = \triangle_n \cdot [p]$, $p \in \mathbb{Z}$. Together with the identity natural transformations $\phi^p = \mathrm{id} : I_n^p \cdot \triangle_n \to \triangle_n \cdot [p]$ the functor $\triangle_n$ is an n-triangle of $\tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}})$, called the *principal distinguished n-triangle.*

13.18 Definition. Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let $f : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$ be a unital A_∞-functor. The composition of translation preserving functors

$$\triangle_n \cdot \tilde{H}^0(f) = \big(D_n \xrightarrow{\triangle_n} \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \xrightarrow{\tilde{H}^0(f)} \tilde{H}^0(\mathcal{B})\big) \tag{13.18.1}$$

is called a *standard distinguished n-triangle* in $\tilde{H}^0(\mathcal{B})$. *Distinguished n-triangles* in $\tilde{H}^0(\mathcal{B})$ are defined as those isomorphic to a standard distinguished n-triangle for some f. This is the system of n-triangles associated with $\mathcal{B}$.

The definition implies that axiom (FTR0) holds for distinguished n-triangles in $\tilde{H}^0(\mathcal{B})$.

13.19 Proposition. *Any unital A_∞-functor $g : \mathcal{B} \to \mathcal{C}$ between A_∞-categories closed under shifts induces a triangulated functor $\tilde{H}^0(g) : \tilde{H}^0(\mathcal{B}) \to \tilde{H}^0(\mathcal{C})$ in zeroth homology.*

Proof. Since $\tilde{H}^0(g)$ maps isomorphic n-triangles to isomorphic ones, it suffices to check that the image of a standard distinguished n-triangle under $\tilde{H}^0(g) = (H^0(g), \psi_n)$ is a standard distinguished n-triangle as well. Let $f : \mathcal{D}_2^{\mathsf{tr}} \to \mathcal{B}$ be a unital A_∞-functor, it gives rise to the standard n-triangle $\triangle_n \cdot \tilde{H}^0(f) = \triangle_n \cdot (H^0(f), \phi_n)$ in $\tilde{H}^0(\mathcal{B})$, given by (13.18.1). Its image under $\tilde{H}^0(g)$ coincides with $\triangle_n \cdot \tilde{H}^0(fg)$, which is a standard distinguished n-triangle in $\tilde{H}^0(\mathcal{C})$. $\square$

13.20 Definition. *The base of an n-triangle* $(F, \phi^p) : D_n \to \mathcal{T}$ is the composite functor $\mathbf{n} \xrightarrow{\mathrm{in}_2} D_n \xrightarrow{F} \mathcal{T}$, where the increasing map in_2 takes k to $(0,k) \in D_n$. For $n > 0$ this is the sequence of $n-1$ composable morphisms of $\mathcal{T}$

$$F(0,1) \to F(0,2) \to \cdots \to F(0,n).$$

We are interested in such systems of distinguished n-triangles which satisfy the property

(FTR1) Each functor $\mathbf{n} \to \mathcal{T}$ is the base of a distinguished n-triangle.

13.21 Proposition. *Let $\mathcal{B}$ be a pretriangulated A_∞-category. Then the axiom (FTR1) holds for distinguished n-triangles in $\tilde{H}^0(\mathcal{B})$.*

Proof. First of all, we establish a lemma.

13.22 Lemma. *Let $\mathcal{C}$ be a unital A_∞-category. Assume that*

$$X_1 \xrightarrow{g^1} X_2 \xrightarrow{g^2} \dots \xrightarrow{g^{n-1}} X_n \tag{13.22.1}$$

is a sequence of composable morphisms of $H^0\mathcal{C}$. Then there is an A_∞-functor $f : \mathcal{A}_n \to \mathcal{C}$ such that $\underline{k}f = X_k$, $k \leqslant n$, and the cycle $e_{k,k+1}sf_1s^{-1}$ belongs to the homology class g^k for all $k < n$.

Proof. By the Yoneda Lemma there is an A_∞-equivalence $\phi : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{D}$ is a differential graded category with $\operatorname{Ob}\mathcal{D} = \operatorname{Ob}\mathcal{C}$ and $\operatorname{Ob}\phi = \mathrm{id}$. Then $H^0\phi : H^0\mathcal{D} \to H^0\mathcal{C}$ is an isomorphism of categories, identity on objects. Take the sequence of morphisms of $H^0\mathcal{D}$ that corresponds to (13.22.1). Lift it to a sequence of cycles

$$X_1 \xrightarrow{h^1} X_2 \xrightarrow{h^2} \dots \xrightarrow{h^{n-1}} X_n$$

in $Z^0\mathcal{D}$. Define a differential graded functor (between non-unital differential graded categories) $F : \mathcal{A}_n \to \mathcal{D}$ by $F(\underline{k}) = X_k$, $k \leqslant n$, and $F(e_{k,k+1}) = h^k \in Z^0\mathcal{D}(X_k, X_{k+1})$ for all $k < n$. View F as an A_∞-functor and set $f = F \cdot \phi : \mathcal{A}_n \to \mathcal{C}$. This A_∞-functor satisfies the requirements of the lemma. $\square$

As shown in lemma any sequence (13.22.1) comes from the first component of an A_∞-functor $f : \mathcal{A}_n \to \mathcal{B}$. The restriction maps

$$\mathsf{A}^{\mathrm{u}}_\infty(\mathcal{D}^{\mathsf{tr}}_n; \mathcal{B}) \longrightarrow \mathsf{A}^{\mathrm{u}}_\infty(\mathcal{D}_n; \mathcal{B}) \longrightarrow \mathsf{A}_\infty(\mathcal{A}_n; \mathcal{B})$$

are surjective for any pretriangulated A_∞-category $\mathcal{B}$ by Theorem 9.31 and Proposition 12.15.2. Therefore, the given sequence (13.22.1) is the base of any triangle $\triangle_n \cdot \widetilde{H}^0(g)$, where $g : \mathcal{D}^{\mathsf{tr}}_n \to \mathcal{B}$ restricts to f. $\square$

A *morphism between the bases of n-triangles* $F, G : D_n \to \mathcal{T}$ is by definition a natural transformation $p : \mathrm{in}_2 \cdot F \to \mathrm{in}_2 \cdot G : \mathbf{n} \to \mathcal{T}$. In other terms, this is a commutative diagram

$$\begin{tikzcd}
F(0,1) \arrow[r] \arrow[d, "p"'] & F(0,2) \arrow[r] \arrow[d, "p"'] & \dots \arrow[r] \arrow[d, "p"'] & F(0,n) \arrow[d, "p"'] \\
G(0,1) \arrow[r] & G(0,2) \arrow[r] & \dots \arrow[r] & G(0,n)
\end{tikzcd}$$

13.23 Remark. The A_∞-category $\mathcal{A}_1 \equiv \mathbb{1}_u$ has a single object $\underline{1}$ and $\mathcal{A}_1(\underline{1}, \underline{1}) = 0$. An A_∞-functor $F : \mathcal{A}_1 \to \mathcal{B}$ has no non-vanishing components and amounts to an object $X = \underline{1}F$ of $\mathcal{B}$. Furthermore, there is an isomorphism of A_∞-categories $A_\infty(\mathcal{A}_1, \mathcal{B}) \cong \mathcal{B}$. Let $e : \mathcal{A}_1 \to \mathcal{A}^{\mathsf{tr}}_2$ be the A_∞-functor that corresponds to the object $\operatorname{Cone}(e_{12})$ of $\mathcal{A}^{\mathsf{tr}}_2$. Consider the composite A_∞-functor

$$\operatorname{Cone} = \bigl(A_\infty(\mathcal{A}_2, \mathcal{B}) \xrightarrow[S']{-[]} A_\infty(\mathcal{A}^{[]}_2, \mathcal{B}^{[]}) \xrightarrow{-\mathsf{mc}} A_\infty(\mathcal{A}^{\mathsf{tr}}_2, \mathcal{B}^{\mathsf{tr}}) \xrightarrow{A_\infty(e,1)} A_\infty(\mathcal{A}_1, \mathcal{B}^{\mathsf{tr}}) \cong \mathcal{B}^{\mathsf{tr}}\bigr).$$

It is strict as the composition of strict A_∞-functors. An A_∞-functor $F : \mathcal{A}_2 \to \mathcal{B}$ identifies with an arrow of $Z^0\mathcal{B}$, namely with $e_{12}sF_1s^{-1} : \underline{1}F \to \underline{2}F$, which can be arbitrary. It is mapped to the object

$$\mathrm{Cone}(F) = \mathrm{Cone}(e_{12})F^{\mathsf{tr}} = \big(1 \mapsto \underline{1}F[1], 2 \mapsto \underline{2}F[0], \left(\begin{smallmatrix} 0 & e_{12}sF_1s^{-1} \\ 0 & 0 \end{smallmatrix}\right)\big)$$

of $\mathcal{B}^{\mathsf{tr}}$. By (10.19.5) an A_∞-transformation $r : F \to G : \mathcal{A}_2 \to \mathcal{B}$ is mapped to

$$\begin{aligned} \mathrm{Cone}_1(r) = {}_{\mathrm{Cone}(e_{12})}r_0^{\mathsf{tr}} = \sum_{n\geqslant 0} \begin{pmatrix} 0 & e_{12} \\ 0 & 0 \end{pmatrix}^{\otimes n} r_n^{[]} = \begin{pmatrix} {}_{\underline{1}[1]}r_0^{[]} & e_{12}r_1^{[]} \\ 0 & {}_{\underline{2}}r_0^{[]} \end{pmatrix} \\ = \begin{pmatrix} {}_{\underline{1}}r_0 & (-)^r e_{12}sr_1s^{-1} \\ 0 & {}_{\underline{2}}r_0 \end{pmatrix} \in s\mathcal{B}^{\mathsf{tr}}(\mathrm{Cone}(F), \mathrm{Cone}(G)). \end{aligned}$$

Since r_0 and r_1 are the only non-vanishing components of r, the mapping $r \mapsto \mathrm{Cone}_1(r)$ is an embedding. This embedding identifies the A_∞-category $A_\infty(\mathcal{A}_2, \mathcal{B})$ of arrows in $\mathcal{B}$ with the subcategory of $\mathcal{B}^{\mathsf{tr}}$ whose objects are A_∞-functors $\mathbf{2} \to \mathcal{B}$ and morphisms are given by matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. It is important to stress that unlike the case of ordinary triangulated categories so defined $\mathrm{Cone} : A_\infty(\mathcal{A}_2, \mathcal{B}) \to \mathcal{B}^{\mathsf{tr}}$ **is an A_∞-functor**.

If $\mathcal{B}$ is pretriangulated, the A_∞-functor Cone can be extended to the composition $A_\infty(\mathcal{A}_2, \mathcal{B}) \xrightarrow{\mathrm{Cone}} \mathcal{B}^{\mathsf{tr}} \xrightarrow{U_{\mathsf{tr}}} \mathcal{B}$, which can also be given the name of cone.

13.24 A category of strips. Following Maltsiniotis [Mal06] denote by $\boldsymbol{D}$ the subcategory of the category of categories, whose objects are D_n, $n \geqslant 0$, and whose morphisms are generated by the functors D_ψ, $I_n^{\pm 1}$, $J_n^{\pm 1}$. Notice that any morphism $h : D_m \to D_n$ of $\boldsymbol{D}$ satisfies the relation $I_m \cdot h = h \cdot I_n$. Thus, $(h, \mathrm{id}) : D_m \to D_n$ is a translation preserving functor. This observation allows us to view $\boldsymbol{D}$ as a subcategory of the category of categories with translation structure.

For any morphism $h : D_m \to D_n$ of $\boldsymbol{D}$ there is a unique $\varepsilon_h \in \{0, 1\}$ such that $(\partial^0 D_m)h \subset \partial^{\varepsilon_h} D_n$. For instance, $\varepsilon_{D_\psi} = 0$, $\varepsilon_{I_n^{\pm 1}} = 1$ and $\varepsilon_{J_n^{\pm 1}} = 0$. The sign $(-1)^{\varepsilon_h}$ is multiplicative in h, in other terms, there is a functor $\varepsilon : \boldsymbol{D} \to \mathbb{Z}/2$, $h \mapsto \varepsilon_h$, where $\mathbb{Z}/2$ is the category with one object, whose endomorphism monoid is $\mathbb{Z}/2$.

Let $(\mathcal{T}, \Sigma^p, \varsigma)$ be a $\Bbbk$-linear category with translation structure (an object of $\mathrm{Trans}\,\Bbbk\text{-}\mathcal{C}at$). If $(F, \phi^p) : (\mathcal{D}, \Sigma^p, \varsigma) \to (\mathcal{T}, \Sigma^p, \varsigma)$ is a translation preserving functor (not necessarily $\Bbbk$-linear), then so is $(F, a^p\phi^p)$, where $a \in \Bbbk$ is an invertible element, for instance, $a = -1$. Actually, $(F, a^p\phi^p) = (F, \phi^p) \cdot (\mathrm{Id}, a^p)$ is the composition of the original functor with the twisted identity functor $(\mathrm{Id}, a^p) : (\mathcal{T}, \Sigma^p, \varsigma) \to (\mathcal{T}, \Sigma^p, \varsigma)$.

Let $h : D_m \to D_n$ be a morphism of $\boldsymbol{D}$, and let $(F, \phi^p) : D_n \to \mathcal{T}$ be an n-triangle of $\mathcal{T}$. Then $(h, \mathrm{id}) \cdot (F, \phi^p) = (h \cdot F, h \cdot \phi^p)$ is always an m-triangle of $\mathcal{T}$. However, we are interested in its twisting

$$h^*(F, \phi^p) = (h, \mathrm{id}) \cdot (F, \phi^p) \cdot (\mathrm{Id}, (-1)^{p\varepsilon_h}) = (h \cdot F, (-1)^{p\varepsilon_h} h \cdot \phi^p),$$

which is called *the inverse image of the n-triangle (F, ϕ^p) by the morphism h.*

13.25 Definition (Strongly triangulated categories, Maltsiniotis [Mal06])**.** A *strongly triangulated category* is an additive $\Bbbk$-linear category $\mathcal{C}$ with a translation structure on it and a system of n-triangles for each $n \geqslant 0$ which satisfies axioms (FTR0), (FTR1) and the following axioms:

(FTR2) Any morphism of the base $\mathbf{n} \xrightarrow{\mathrm{in}_2} D_n \xrightarrow{F} \mathcal{T}$ to the base $\mathbf{n} \xrightarrow{\mathrm{in}_2} D_n \xrightarrow{G} \mathcal{T}$ equals $\mathrm{in}_2 \cdot r$ for some morphism $r : F \to G$ of distinguished n-triangles.

(FTR3) Let $h : D_m \to D_n$ be a morphism of $\boldsymbol{D}$. Let (F, ϕ^p) be a distinguished n-triangle in $\mathcal{T}$. Then the inverse image m-triangle $h^*(F, \phi^p) = (h \cdot F, (-1)^{p\varepsilon_h} h \cdot \phi^p)$ is distinguished.

Maltsiniotis proves in [Mal06] that a strongly triangulated category equipped with the class of distinguished 2-triangles is an ordinary triangulated category in the sense of Verdier [Ver77]. This has the following corollary: *Any morphism $r : F \to G$ of distinguished n-triangles, whose restriction to bases is invertible, is invertible itself.* Indeed, each object $F(i, j)$ for $0 < i < j \leqslant n$ is the third vertex of a distinguished 2-triangle $F(0, i) \to F(0, j) \to F(i, j) \to$, whose first two vertices belong to the base. The morphism r includes, in particular, a morphism of the above 2-triangle to the distinguished 2-triangle $G(0, i) \to G(0, j) \to G(i, j) \to$. Being invertible on the first two places, this morphism is invertible on the third place as well, by the property of the usual triangulated categories. Thus, axiom (FTR2) implies that any isomorphism of the base of one distinguished n-triangle with the base of another distinguished n-triangle is the restriction of some isomorphism of these n-triangles. In other words, isomorphism classes of n-triangles are in bijection with isomorphism classes of bases.

Künzer gives in [Kün07b] an example of two non-isomorphic octahedra on the same base in an ordinary (not strongly) triangulated category. Any sub-2-triangle of these octahedra obtained via inverse image from a morphism $h : D_2 \to D_3$ of $\boldsymbol{D}$ is distinguished (there are essentially four of them). In this example the identity morphism between the bases cannot be prolonged to a morphism between the whole Verdier octahedra. Clearly, both mentioned octahedra can not be distinguished 3-triangles simultaneously.

13.26 Proposition. *Let $\mathcal{C}$ be an* mc*-closed A_∞-category. Then the $\Bbbk$-linear category $H^0(\mathcal{C})$ is additive.*

Proof. The A_∞-functor $u_{\mathsf{mc}} : \mathcal{C} \to \mathcal{C}^{\mathsf{mc}}$ is an equivalence by Proposition 11.32. The category $H^0(\mathcal{C}^{\mathsf{mc}})$ admits a zero object (initial and final object), namely, the A_∞-functor $\varnothing \to \mathcal{C}$. Since $H^0(u_{\mathsf{mc}}) : H^0(\mathcal{C}) \to H^0(\mathcal{C}^{\mathsf{mc}})$ is an equivalence, the category $H^0(\mathcal{C})$ has a zero object 0 as well.

Let X, Y be objects of $\mathcal{C}$. Consider the object $Z = \big(1 \mapsto X, 2 \mapsto Y, \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\big)$ of $\mathcal{C}^{\mathsf{mc}}$.

There are cycles of degree 0

$$i_X = ({}_X\mathbf{i}_0 s^{-1}, 0) \in \mathcal{C}^{\mathsf{mc}}(Xu_{\mathsf{mc}}, Z), \quad i_Y = (0, {}_Y\mathbf{i}_0 s^{-1}) \in \mathcal{C}^{\mathsf{mc}}(Yu_{\mathsf{mc}}, Z),$$
$$p_X = \begin{pmatrix} {}_X\mathbf{i}_0 s^{-1} \\ 0 \end{pmatrix} \in \mathcal{C}^{\mathsf{mc}}(Z, Xu_{\mathsf{mc}}), \quad p_Y = \begin{pmatrix} 0 \\ {}_Y\mathbf{i}_0 s^{-1} \end{pmatrix} \in \mathcal{C}^{\mathsf{mc}}(Z, Yu_{\mathsf{mc}}).$$

They satisfy equations

$$\begin{aligned}
(i_X \otimes p_X)m_2 &= ({}_X\mathbf{i}_0 \otimes {}_X\mathbf{i}_0)b_2 s^{-1} \equiv {}_X\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} s^{-1} \in \mathcal{C}^{\mathsf{mc}}(Xu_{\mathsf{mc}}, Xu_{\mathsf{mc}}), \\
(i_X \otimes p_Y)m_2 &= 0 \in \mathcal{C}^{\mathsf{mc}}(Xu_{\mathsf{mc}}, Yu_{\mathsf{mc}}), \\
(i_Y \otimes p_X)m_2 &= 0 \in \mathcal{C}^{\mathsf{mc}}(Yu_{\mathsf{mc}}, Xu_{\mathsf{mc}}), \\
(i_Y \otimes p_Y)m_2 &= ({}_Y\mathbf{i}_0 \otimes {}_Y\mathbf{i}_0)b_2 s^{-1} \equiv {}_Y\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} s^{-1} \in \mathcal{C}^{\mathsf{mc}}(Yu_{\mathsf{mc}}, Yu_{\mathsf{mc}}), \\
(p_X \otimes i_X)m_2 + (p_Y \otimes i_Y)m_2 &= \begin{pmatrix} ({}_X\mathbf{i}_0 \otimes {}_X\mathbf{i}_0)b_2 s^{-1} & 0 \\ 0 & ({}_Y\mathbf{i}_0 \otimes {}_Y\mathbf{i}_0)b_2 s^{-1} \end{pmatrix} \\
&\equiv \begin{pmatrix} {}_X\mathbf{i}_0 s^{-1} & 0 \\ 0 & {}_Y\mathbf{i}_0 s^{-1} \end{pmatrix} = {}_Z\mathbf{i}_0^{\mathcal{C}^{\mathsf{mc}}} s^{-1} \in \mathcal{C}^{\mathsf{mc}}(Z, Z)
\end{aligned}$$

due to (11.23.1). Thus, the object Z is a direct sum of Xu_{mc} and Yu_{mc} in $H^0(\mathcal{C}^{\mathsf{mc}})$. Since $H^0(u_{\mathsf{mc}}) : H^0(\mathcal{C}) \to H^0(\mathcal{C}^{\mathsf{mc}})$ is an equivalence, the objects X and Y admit a direct sum $X \oplus Y$ in $H^0(\mathcal{C})$. □

13.27 Theorem. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Then axiom (FTR3) holds true for $\tilde{H}^0(\mathcal{B})$. If $\mathcal{B}$ is pretriangulated, then $\tilde{H}^0(\mathcal{B})$ is strongly triangulated.*

Proof. The proof is given in several lemmata. We begin with particular cases of property (FTR3) which imply it in general case.

13.28 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let $\psi : [m] \to [n]$ be a non-decreasing map. Let F be a distinguished n-triangle in $\tilde{H}^0(\mathcal{B})$. Then the inverse image m-triangle $D_\psi^* F = D_\psi \cdot F$ is distinguished.*

Proof. It suffices to prove the statement for the standard distinguished n-triangle $F = \triangle_n \cdot \tilde{H}^0(f)$, where $f : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$ is a unital A_∞-functor. Consider the **dg**-functor $\mathcal{D}_\psi : \mathcal{D}_m \to \mathcal{D}_n^{\mathsf{tr}}$, determined by ψ. It takes an object $\underline{k} \in \operatorname{Ob} \mathcal{D}_m$ to the object $\operatorname{Cone}(e_{\psi 0, \psi k}) = \begin{pmatrix} \underline{\psi 0}[1] & e_{\psi 0,\psi k} \\ & \underline{\psi k} \end{pmatrix}$ of $\mathcal{D}_n^{\mathsf{tr}}$. A morphism e_{kl} of $\mathcal{D}_m$ goes to morphism

$$v_{\psi 0}(e_{\psi k, \psi l}) = \begin{pmatrix} 1 & 0 \\ 0 & e_{\psi k, \psi l} \end{pmatrix} : C(e_{\psi 0, \psi k}) = \begin{pmatrix} \underline{\psi 0}[1] & e_{\psi 0,\psi k} \\ & \underline{\psi k} \end{pmatrix} \to \begin{pmatrix} \underline{\psi 0}[1] & e_{\psi 0,\psi l} \\ & \underline{\psi l} \end{pmatrix} = C(e_{\psi 0, \psi l}).$$

Extend the **dg**-functor $\mathcal{D}_\psi$ to the **dg**-functor $\widehat{\mathcal{D}}_\psi = \big(\mathcal{D}_m^{\mathsf{tr}} \xrightarrow{\mathcal{D}_\psi^{\mathsf{tr}}} \mathcal{D}_m^{\mathsf{tr}\,\mathsf{tr}} \xrightarrow{m_{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}}\big)$. Denote

$$g = \big[\mathcal{D}_m^{\mathsf{tr}} \xrightarrow{\widehat{\mathcal{D}}_\psi} \mathcal{D}_n^{\mathsf{tr}} \xrightarrow{f} \mathcal{B}\big].$$

We claim that $D_\psi^*[\triangle_n \cdot \tilde{H}^0(f)] = D_\psi \cdot \triangle_n \cdot \tilde{H}^0(f)$ is isomorphic to $\triangle_m \cdot \tilde{H}^0(g)$. In detail, there are mutually inverse natural transformations α, β shown in the following diagram:

$$\begin{tikzcd} D_m \arrow[r, "D_\psi"] \arrow[d, "\triangle_m"'] & D_n \arrow[d, "\triangle_n"] \arrow[dl, Rightarrow, "\alpha"'] & \\ \tilde{H}^0(\mathcal{D}_m^{\mathsf{tr}}) \arrow[r, "\tilde{H}^0(\widehat{\mathcal{D}}_\psi)"'] \arrow[ur, Rightarrow, "\beta"'] & \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \arrow[r, "\tilde{H}^0(f)"'] & \tilde{H}^0(\mathcal{B}) \end{tikzcd}$$

Their value on the point $(k,l) \in D_m$ of the fundamental domain of I_m, $0 \leqslant k < l \leqslant m$, is given by the following matrices. Denote $a = \psi 0$, $b = \psi k$, $c = \psi l$. If all these are distinct, the transformations are chosen equal

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : C(e_{bc}) = \begin{pmatrix} \underline{b}[1] & e_{bc} \\ & \underline{c} \end{pmatrix} \to \operatorname{Cone}\bigl(v_a(e_{bc}) : C(e_{ab}) \to C(e_{ac})\bigr), \qquad (13.28.1)$$

$$\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ e_{ab}[1] & 0 \\ 0 & 1 \end{pmatrix} : \operatorname{Cone}\bigl(v_a(e_{bc}) : C(e_{ab}) \to C(e_{ac})\bigr) = \begin{pmatrix} \underline{a}[2] & e_{ab}[1] & -1 & 0 \\ & \underline{b}[1] & 0 & e_{bc} \\ & & \underline{a}[1] & e_{ac} \\ & & & \underline{c} \end{pmatrix} \to C(e_{bc}).$$

When $a = b < c$, we choose $\alpha, \beta : C(e_{ac}) \to C(e_{ac})$ to be the identity morphisms. If $b = c$, then $\alpha, \beta : 0 \to 0$ are obvious.

We see immediately that α and β are cycles and that $\alpha\beta = 1$. An easy calculation shows that $1 - \beta\alpha = hm_1$, where the element h of degree -1 is given by the matrix

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \underline{a}[2] & e_{ab}[1] & -1 & 0 \\ & \underline{b}[1] & 0 & e_{bc} \\ & & \underline{a}[1] & e_{ac} \\ & & & \underline{c} \end{pmatrix} \to \begin{pmatrix} \underline{a}[2] & e_{ab}[1] & -1 & 0 \\ & \underline{b}[1] & 0 & e_{bc} \\ & & \underline{a}[1] & e_{ac} \\ & & & \underline{c} \end{pmatrix}.$$

Therefore, the cycles α and β induce isomorphisms in $H^0(\mathcal{D}_n^{\mathsf{tr}})$ inverse to each other.

Let us show that α, β are natural transformations. Let us prove that α, β satisfy the naturality relation for any vertical arrow $(j,l) \to (k,l)$ within the fundamental domain, thus $0 \leqslant j < k < l \leqslant m$. Denote $a = \psi 0$, $b = \psi j$, $c = \psi k$, $d = \psi l$. Notice that the following diagram commutes in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$:

$$\begin{tikzcd} C(e_{bd}) \arrow[r, shift left, "\alpha"] \arrow[d, "u_d(e_{bc})"'] & \operatorname{Cone}\bigl(v_a(e_{bd}) : C(e_{ab}) \to C(e_{ad})\bigr) \arrow[l, shift left, "\beta"] \arrow[d, "u_l(v_a(e_{bc}))"] \\ C(e_{cd}) \arrow[r, shift left, "\alpha"] & \operatorname{Cone}\bigl(v_a(e_{cd}) : C(e_{ac}) \to C(e_{ad})\bigr) \arrow[l, shift left, "\beta"] \end{tikzcd}$$

due to explicit computation using the matrices

$$u_d(e_{bc}) = \begin{pmatrix} e_{bc}[1] & 0 \\ 0 & 1 \end{pmatrix}, \qquad u_l(v_a(e_{bc})) = \begin{pmatrix} v_a(e_{bc})[1] & 0 \\ 0 & 1 \end{pmatrix} = \operatorname{diag}(1, e_{bc}[1], 1, 1).$$

Verification is left to the reader as an exercise. Commutativity of this diagram is the desired result if all four numbers a, b, c, d are distinct. If some of them coincide, the diagram still commutes in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$, but contains cones of identity morphisms, which we have to replace with 0. Such cones are isomorphic to the zero object of $H^0(\mathcal{D}_n^{\mathsf{tr}})$, and Cone is functorial by Remark 13.23. Hence, the required diagram commutes in the homotopy category. Besides, when $b = c$ or $c = d$, the required relation is obvious. When $a = b < c < d$, the equation to prove is the following:

$$\begin{tikzcd}
C(e_{ad}) \arrow[r, equal] \arrow[d, "u_d(e_{ac})"'] & C(e_{ad}) \arrow[d, "(01)"] \\
C(e_{cd}) \arrow[r, shift left, "\alpha"] & \operatorname{Cone}\big(v_a(e_{cd}) : C(e_{ac}) \to C(e_{ad})\big) \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

It takes in $H^0(\mathcal{D}_n^{\mathsf{tr}})$ the following explicit form:

$$\begin{tikzcd}
\begin{pmatrix} \underline{a}[1] & e_{ad} \\ & \underline{d} \end{pmatrix} \arrow[r, equal] \arrow[d, "\begin{pmatrix} e_{ac}[1] & 0 \\ 0 & 1 \end{pmatrix}"'] & \begin{pmatrix} \underline{a}[1] & e_{ad} \\ & \underline{d} \end{pmatrix} \arrow[d, "\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}"] \\
\begin{pmatrix} \underline{c}[1] & e_{cd} \\ & \underline{d} \end{pmatrix} \arrow[r, shift left, "\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}"] & \begin{pmatrix} \underline{a}[2] & e_{ac}[1] & -1 & 0 \\ & \underline{c}[1] & 0 & e_{cd} \\ & & \underline{a}[1] & e_{ad} \\ & & & \underline{d} \end{pmatrix} \arrow[l, shift left, "\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ e_{ac}[1] & 0 \\ 0 & 1 \end{pmatrix}"]
\end{tikzcd}$$

The naturality condition for β holds already in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$, while the naturality condition for α holds true up to the boundary of the element

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \underline{a}[1] & e_{ad} \\ & \underline{d} \end{pmatrix} \to \begin{pmatrix} \underline{a}[2] & e_{ac}[1] & -1 & 0 \\ & \underline{c}[1] & 0 & e_{cd} \\ & & \underline{a}[1] & e_{ad} \\ & & & \underline{d} \end{pmatrix}$$

of degree -1.

Now let us prove that α, β give a commutative square for each horizontal arrow $(j, k) \to (j, l)$ within the fundamental domain, thus $0 \leqslant j < k < l \leqslant m$. Denote $a = \psi 0$, $b = \psi j$, $c = \psi k$, $d = \psi l$. Notice that the following diagram commutes in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$:

$$\begin{tikzcd}
C(e_{bc}) \arrow[r, shift left, "\alpha"] \arrow[d, "v_b(e_{cd})"'] & \operatorname{Cone}\big(v_a(e_{bc}) : C(e_{ab}) \to C(e_{ac})\big) \arrow[l, shift left, "\beta"] \arrow[d, "v_j(v_a(e_{cd}))"] \\
C(e_{bd}) \arrow[r, shift left, "\alpha"] & \operatorname{Cone}\big(v_a(e_{bd}) : C(e_{ab}) \to C(e_{ad})\big) \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

due to explicit computation using the matrices

$$v_b(e_{cd}) = \begin{pmatrix}1 & 0\\ 0 & e_{cd}\end{pmatrix}, \qquad v_j(v_a(e_{cd})) = \begin{pmatrix}1 & 0\\ 0 & v_a(e_{cd})\end{pmatrix} = \operatorname{diag}(1,1,1,e_{cd}).$$

If some of the numbers a, b, c, d coincide, the required square in $H^0(\mathcal{D}_n^{\mathsf{tr}})$ differs form the above, however, its commutativity is obvious.

It remains to verify naturality of α, β on the horizontal arrows $(k,m) \to (k,m+1)$, $0 < k \leqslant m$, which join the fundamental domain of D_m with its image under I_m. Denote $a = \psi 0$, $b = \psi k$, $c = \psi m$. The points $D_\psi(k,m) = (\psi k, \psi m) = (b,c)$ and $D_\psi(k,m+1) = (\psi k, n+1+\psi 0) = (b, n+1+a)$ lie in the fundamental domain of D_n and its image under I_n, respectively. They are joined by the composition of two arrows: $(b,c) \to (b,n+1) \to (b,n+1+a)$. The corresponding diagram to verify is the following:

$$\begin{tikzcd}
C(e_{bc}) \arrow[rr, shift left, "\alpha"] \arrow[d, "\binom{1}{0}"'] & & \operatorname{Cone}\big(v_a(e_{bc}) : C(e_{ab}) \to C(e_{ac})\big) \arrow[ll, shift left, "\beta"] \arrow[d, "\binom{1}{0}"] \\
\underline{b}[1] \arrow[r, "(01)"] & C(e_{ab})[1] \arrow[r, equal, "\alpha"', "\beta"] & C(e_{ab})[1]
\end{tikzcd}$$

Its explicit form in $H^0(\mathcal{D}_n^{\mathsf{tr}})$ is the following:

$$\begin{tikzcd}
\begin{pmatrix}\underline{b}[1] & e_{bc}\\ & \underline{c}\end{pmatrix} \arrow[rr, shift left, "\left(\begin{smallmatrix}0&1&0&0\\0&0&0&1\end{smallmatrix}\right)"] \arrow[d, "\binom{1}{0}"'] & & \begin{pmatrix}\underline{a}[2] & e_{ab}[1] & -1 & 0\\ & \underline{b}[1] & 0 & e_{bc}\\ & & \underline{a}[1] & e_{ac}\\ & & & \underline{c}\end{pmatrix} \arrow[ll, shift left, "\left(\begin{smallmatrix}0&0\\1&0\\e_{ab}[1]&0\\0&1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\\0&0\end{smallmatrix}\right)"] \\
\underline{b}[1] \arrow[rr, "(01)"] & & \begin{pmatrix}\underline{a}[2] & e_{ab}[1]\\ & \underline{b}[1]\end{pmatrix}
\end{tikzcd}$$

This square commutes up to a boundary. □

13.29 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let F be a distinguished n-triangle in $\tilde{H}^0(\mathcal{B})$. Then the inverse image n-triangle $J_n^* F = J_n \cdot F$ is distinguished.*

Proof. As in the proof of the preceding lemma, it suffices to check the assertion for the standard distinguished n-triangle $F = \triangle_n \cdot H^0(f)$, where $f : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$ is a unital A_∞-functor. Consider the **dg**-functor $j : \mathcal{D}_n \to \mathcal{D}_n^{\mathsf{tr}}$ defined as follows. It maps an object

$\underline{k} \in \operatorname{Ob} \mathcal{D}_n$ to the object $\operatorname{Cone}(e_{1,k+1}) = \begin{pmatrix} \underline{1}[1] & e_{1,k+1} \\ & \underline{k+1} \end{pmatrix}$ of $\mathcal{D}_n^{\mathsf{tr}}$ if $0 \leqslant k < n$ and to the object $\underline{1}[1]$ if $k = n$. A morphism e_{kl} of $\mathcal{D}_n$ is mapped to the morphism

$$v_1(e_{k+1,l+1}) = \begin{pmatrix} 1_{\underline{1}[1]} & 0 \\ 0 & e_{k+1,l+1} \end{pmatrix} : C(e_{1,k+1}) = \begin{pmatrix} \underline{1}[1] & e_{1,k+1} \\ & \underline{k+1} \end{pmatrix} \to \begin{pmatrix} \underline{1}[1] & e_{1,l+1} \\ & \underline{l+1} \end{pmatrix} = C(e_{1,l+1})$$

if $0 \leqslant k \leqslant l < n$ and to the morphism

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} : C(e_{1,k+1}) = \begin{pmatrix} \underline{1}[1] & e_{1,k+1} \\ & \underline{k+1} \end{pmatrix} \to \underline{1}[1]$$

if $0 \leqslant k < l = n$. The restriction of j to $\mathcal{D}_{n-1}$ coincides with the **dg**-functor $\mathcal{D}_\psi : \mathcal{D}_{n-1} \to \mathcal{D}_n^{\mathsf{tr}}$, determined by the shift map $\psi : [n-1] \to [n]$, $m \mapsto m+1$ in the proof of Lemma 13.28.

Extend the **dg**-functor j to the **dg**-functor

$$\hat{\jmath} = \bigl(\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{j^{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}} \xrightarrow{m_{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}}\bigr).$$

Denote by g the composite $g = \bigl[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{\hat{\jmath}} \mathcal{D}_n^{\mathsf{tr}} \xrightarrow{f} \mathcal{B}\bigr]$. We claim that $J_n^*[\triangle_n \cdot H^0(f)] = J_n \cdot \triangle_n \cdot H^0(f)$ is isomorphic to $\triangle_n \cdot H^0(g)$. We are going to construct a pair of reciprocal natural isomorphisms α and β as in the following diagram

$$\begin{tikzcd}
D_n \arrow[r, "J_n"] \arrow[d, "\triangle_n"'] & D_n \arrow[d, "\triangle_n"] \arrow[dl, Rightarrow, "\alpha"', shift right] & \\
H^0(\mathcal{D}_n^{\mathsf{tr}}) \arrow[r, "H^0(\hat{\jmath})"'] \arrow[ur, Rightarrow, "\beta"', shift right] & H^0(\mathcal{D}_n^{\mathsf{tr}}) \arrow[r, "H^0(f)"'] & H^0(\mathcal{B})
\end{tikzcd}$$

Let $(k, l) \in D_n$ be a point of the fundamental domain of I_n, $0 \leqslant k < l \leqslant n$. In order to define α and β, we need to distinguish the following cases. If $0 \leqslant k < l < n$, we define the morphisms α, β by the same formulae

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} \underline{k+1}[1] & e_{k+1,l+1} \\ & \underline{l+1} \end{pmatrix} \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 & 0 \\ & \underline{k+1}[1] & 0 & e_{k+1,l+1} \\ & & \underline{1}[1] & e_{1,l+1} \\ & & & \underline{l+1} \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ e_{1,k+1}[1] & 0 \\ 0 & 1 \end{pmatrix}$$

as in Lemma 13.28 for $\psi(m) = m + 1$. See equations (13.28.1) with $a = 1$, $b = k + 1$, $c = l + 1$.

If $0 = k < l < n$, both functors $J_n \cdot \triangle_n$ and $\triangle_n \cdot H^0(\hat{\jmath})$ map the point $(0, l)$ to the object $C(e_{1,l+1})$, and we choose α and β to be the identities. Also if $k = 0$, $l = n$, both

functors map the point $(0, n)$ to the object $\underline{1}[1]$, and we take $\alpha = \beta = 1_{\underline{1}[1]}$. Finally, if $0 < k < l = n$, we define the morphisms

$$\underline{k+1}[1] \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix}$$

by the matrices

$$\alpha = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \\ e_{1,k+1}[1] \end{pmatrix}.$$

Again, it is obvious that α and β are cycles such that $\alpha\beta = 1$. Moreover, $1 - \beta\alpha = tm_1$, where the element t of degree -1 is given by the matrix

$$t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \to \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix}.$$

Let us check the naturality of α and β. First, let us check that α and β satisfy the naturality condition for any vertical arrow $(i, l) \to (k, l)$ in the fundamental domain, which means that $0 \leqslant i < k < l \leqslant n$. Due to peculiarities in the definition of α and β, it is necessary to distinguish the following cases. If $0 = i < k < l < n$, then the required diagrams follow from the proof of Lemma 13.28.

If $0 < i < k < l = n$, the required diagrams for α and β

$$\begin{tikzcd}
\underline{i+1}[1] \arrow[r, shift left, "\alpha"] \arrow[d, "e_{i+1,k+1}[1]"'] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1} & -1 \\ & \underline{i+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[l, shift left, "\beta"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & e_{i+1,k+1}[1] & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \\
\underline{k+1}[1] \arrow[r, shift left, "\alpha"] & \begin{pmatrix} \underline{1}[2] & e_{1,k+1} & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

commute in $Z^0(\mathcal{D}_n^{\mathsf{tr}})$. If $0 = i < k < l = n$, then the corresponding diagram for β

$$\begin{tikzcd}
\underline{1}[1] \arrow[rr, equal] \arrow[d, "(0\ 0\ 1)"'] & & \underline{1}[1] \arrow[d, "e_{1,k+1}[1]"] \\
\begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[rr, "\beta"] & & \underline{k+1}[1]
\end{tikzcd}$$

commutes already in $Z^0(\mathcal{D}_n^{\mathrm{tr}})$, while the diagram for α

$$\begin{tikzcd}
\underline{1}[1] \arrow[r, equal] \arrow[d, "{e_{1,k+1}[1]}"'] & \underline{1}[1] \arrow[d, "(0\ 0\ 1)"] \\
\underline{k+1}[1] \arrow[r, "\alpha"] & \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix}
\end{tikzcd}$$

commutes up to the boundary of the element

$$\begin{pmatrix} -1 & 0 & 0 \end{pmatrix} : \underline{1}[1] \to \begin{pmatrix} \underline{1}[2] & e_{1,k+1}[1] & -1 \\ & \underline{k+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix}$$

of degree -1.

Now let us check that α and β satisfy the naturality condition for any horizontal arrow $(i,k) \to (i,l)$ in the fundamental domain, thus $0 \leqslant i < k < l \leqslant n+1$. We need to distinguish the following cases. If $0 < i < k < l < n$, then the required diagrams commute in $Z^0(\mathcal{D}_n^{\mathrm{tr}})$, hence a fortiori in $H^0(\mathcal{D}_n^{\mathrm{tr}})$, due to the proof of Lemma 13.28.

If $0 = i < k < l < n$, the naturality conditions are trivially satisfied since both α and β are identities. The same is true if $0 = i < k < l = n$. If $0 < i < k < l = n$, the required naturality condition expressed by the commutativity of the diagrams

$$\begin{tikzcd}
\begin{pmatrix} \underline{i+1}[1] & e_{i+1,k+1} \\ & \underline{k+1} \end{pmatrix} \arrow[r, shift left, "\alpha"] \arrow[d, "\begin{pmatrix} 1 & 0 \end{pmatrix}"'] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1} & -1 & 0 \\ & \underline{i+1} & 0 & e_{i+1,k+1} \\ & & \underline{1}[1] & e_{1,k+1} \\ & & & \underline{k+1} \end{pmatrix} \arrow[l, shift left, "\beta"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}"] \\
\underline{i+1}[1] \arrow[r, shift left, "\alpha"] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1} & -1 \\ & \underline{i+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[l, shift left, "\beta"]
\end{tikzcd}$$

is satisfied already in $Z^0(\mathcal{D}_n^{\mathrm{tr}})$. Finally, it remains to verify the naturality of α and β on the horizontal arrows $(i,n) \to (i,n+1)$, $0 < i \leqslant n$, which join the fundamental domain

of D_n with its image under I_n. The naturality condition for α takes the form

$$\begin{tikzcd}
\underline{i+1}[1] \arrow[r, "\alpha"] \arrow[d, "(0\ 1)"'] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] & -1 \\ & \underline{i+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)"] \\
\begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] \\ & \underline{i+1}[1] \end{pmatrix} \arrow[r, equal] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] \\ & \underline{i+1}[1] \end{pmatrix}
\end{tikzcd}$$

It holds true on the nose (i.e., in the category $Z^0(\mathcal{D}_n^{\mathsf{tr}})$). The naturality condition for β reads

$$\begin{tikzcd}
\begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] & -1 \\ & \underline{i+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \arrow[r, "\beta"] \arrow[d, "\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)"'] & \underline{i+1} \arrow[d, "(0\ 1)"] \\
\begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] \\ & \underline{i+1}[1] \end{pmatrix} \arrow[r, equal] & \begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] \\ & \underline{i+1}[1] \end{pmatrix}
\end{tikzcd}$$

It holds true up to the boundary of the element

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} : \begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] & -1 \\ & \underline{i+1}[1] & 0 \\ & & \underline{1}[1] \end{pmatrix} \to \begin{pmatrix} \underline{1}[2] & e_{1,i+1}[1] \\ & \underline{i+1}[1] \end{pmatrix}$$

of degree -1. □

13.30 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let F be a distinguished n-triangle in $\tilde{H}^0(\mathcal{B})$. Then the inverse image n-triangle $(J_n^{-1})^*F = J_n^{-1} \cdot F$ is distinguished.*

Proof. As in Lemma 13.29, it suffices to check the assertion for the standard distinguished n-triangle $F = \triangle_n \cdot H^0(f)$, where $f : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$ is a unital A_∞-functor. Consider the differential graded functor $i : \mathcal{D}_n \to \mathcal{D}_n^{\mathsf{tr}}$ defined as follows. It maps an object $\underline{k} \in \operatorname{Ob} \mathcal{D}_n$ to the object $C(e_{k-1,n})[-1] = \begin{pmatrix} \underline{k-1} & e_{k-1,n}[-1] \\ & \underline{n}[-1] \end{pmatrix}$ of $\mathcal{D}_n^{\mathsf{tr}}$ if $k > 1$ and to the object $\underline{n}[-1]$ if $k = 1$. A morphism e_{kl} of $\mathcal{D}_n$ is mapped to the morphism

$$u_n(e_{k-1,l-1})[-1] = \begin{pmatrix} e_{k-1,l-1} & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} \underline{k-1} & e_{k-1,n}[-1] \\ & \underline{n}[-1] \end{pmatrix} \to \begin{pmatrix} \underline{l-1} & e_{l-1,n}[-1] \\ & \underline{n}[-1] \end{pmatrix}$$

if $1 < k < l \leqslant n$ and to the morphism

$$\begin{pmatrix} 0 & 1 \end{pmatrix} : \underline{n}[-1] \to \begin{pmatrix} \underline{l-1} & e_{l-1,n}[-1] \\ & \underline{n}[-1] \end{pmatrix}$$

if $1 = k < l \leqslant n$. Extend the **dg**-functor i to the **dg**-functor $\hat{\imath} = \bigl(\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{i^{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}} \xrightarrow{m_{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}}\bigr)$. Denote by g the composite

$$g = \bigl[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{\hat{\imath}} \mathcal{D}_n^{\mathsf{tr}} \xrightarrow{f} \mathcal{B}\bigr].$$

Similarly to Lemma 13.29 one can prove that $(J_n^{-1})^*[\triangle_n \cdot H^0(f)] = J_n^{-1} \cdot \triangle_n \cdot H^0(f)$ is isomorphic to $\triangle_n \cdot H^0(g)$. □

Recall that $[1] = 1 \boxdot \boldsymbol{\sigma}^1 : \mathcal{D}_n^{[]} \to \mathcal{D}_n^{[]}$ denotes the shift functor in the **dg**-category $\mathcal{D}_n^{[]}$. Its restriction to $\mathcal{D}_n$ is denoted by $[1] : \mathcal{D}_n \to \mathcal{D}_n^{[]}$ by abuse of notation. There is a natural isomorphism γ of differential graded functors, unique up to a constant factor, as shown on the following diagram:

$$\begin{tikzcd} \mathcal{D}_n^{[]} \arrow[r, "{[1]}"] \arrow[d, "{[1]^{[]}}"'] & \mathcal{D}_n^{[]} \arrow[d, "\mathrm{Id}"] \arrow[dl, Rightarrow, "\gamma"'] \\ \mathcal{D}_n^{[][]} \arrow[r, "m_{[]}"] & \mathcal{D}_n^{[]} \end{tikzcd} \tag{13.30.1}$$

For each $k \in \{1, 2, \dots, n\}$ and $q \in \mathbb{Z}$, we set $\gamma_{\underline{k}[q]} = (-1)^q : \underline{k}[q+1] \to \underline{k}[q+1]$. One can check that the isomorphism γ induces an isomorphism of translation preserving functors

$$\begin{tikzcd} \tilde{H}^0(\mathcal{D}_n^{[]}) \arrow[r, "\tilde{H}^0({[1]})"] \arrow[d, "\tilde{H}^0({[1]^{[]}})"'] & \tilde{H}^0(\mathcal{D}_n^{[]}) \arrow[d, "{(\mathrm{Id},(-)^p)}"] \arrow[dl, Rightarrow, "\sim"'] \\ \tilde{H}^0(\mathcal{D}_n^{[][]}) \arrow[r, "\tilde{H}^0(m_{[]})"] & \tilde{H}^0(\mathcal{D}_n^{[]}) \end{tikzcd}$$

This amounts to the commutative diagram of natural differential graded transformations

$$\begin{tikzcd} \Sigma^p \cdot [1] \cdot \mathrm{Id} \arrow[r, "\gamma"] \arrow[d, "(-1)^p"'] & \Sigma^p \cdot [1]^{[]} \cdot m_{[]} \arrow[d, equal] \\ {[1]} \cdot \mathrm{Id} \cdot \Sigma^p \arrow[r, "\gamma \cdot \Sigma^p"] & {[1]^{[]}} \cdot m_{[]} \cdot \Sigma^p \end{tikzcd} \tag{13.30.2}$$

where $\Sigma^p = \Sigma^p_{\mathcal{D}_n^{[]}} = [p]$ and we use a different notation to stress the rôle of Σ^p as part of the translation structure on the differential graded category $\mathcal{D}_n^{[]}$. Applying the A_∞^u-2-functor

$-^{\mathsf{mc}}$ to transformation (13.30.1), we obtain an isomorphism of differential graded functors

$$\begin{tikzcd} \mathcal{D}_n^{\mathsf{tr}} \arrow[r, "{[1]}"] \arrow[d, "\nu = {[1]^{\mathsf{tr}}}"'] & \mathcal{D}_n^{\mathsf{tr}} \arrow[d, "\mathrm{Id}"] \arrow[dl, Rightarrow, "\gamma^{\mathsf{mc}}"'] \\ \mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}} \arrow[r, "m_{\mathsf{tr}}"] & \mathcal{D}_n^{\mathsf{tr}} \end{tikzcd}$$

In fact, for each A_∞-functor $j : \mathcal{D}_n \to \mathcal{D}_n^{\mathsf{tr}}$ of the form

$$j = \bigl[\mathcal{D}_n \xrightarrow{h} \mathcal{D}_n^{[]} \xrightarrow{u_{\mathsf{mc}}} \mathcal{D}_n^{\mathsf{tr}}\bigr]$$

for an A_∞-functor $h : \mathcal{D}_n \to \mathcal{D}_n^{[]}$, the equation

$$\bigl[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{j^{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}} \xrightarrow{m_{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}}\bigr] = \bigl[\mathcal{D}_n^{[]} \xrightarrow{h^{[]}} \mathcal{D}_n^{[][]} \xrightarrow{m_{[]}} \mathcal{D}_n^{[]}\bigr]^{\mathsf{mc}} \tag{13.30.3}$$

holds true. Indeed, the left hand side expands out as

$$\bigl[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{h^{\mathsf{tr}}} \mathcal{D}_n^{[]\mathsf{tr}} = \mathcal{D}_n^{[][]\mathsf{mc}} \xrightarrow{u_{\mathsf{mc}}^{[]\mathsf{mc}}} \mathcal{D}_n^{[]\mathsf{mc}[]\mathsf{mc}} \xrightarrow{\mathfrak{c}^{\mathsf{mc}}} \mathcal{D}_n^{[][]\mathsf{mc}\,\mathsf{mc}} \xrightarrow{m_{\mathsf{mc}}} \mathcal{D}_n^{[][]\mathsf{mc}} \xrightarrow{m_{[]}^{\mathsf{mc}}} \mathcal{D}_n^{\mathsf{tr}}\bigr],$$

which is equal to the right hand side of (13.30.3) due to equation (12.5.1). Applying the A_∞^u-2-functor $-^{\mathsf{mc}}$ to diagram (13.30.2) we obtain a diagram of natural differential graded transformations

$$\begin{tikzcd} \Sigma^p \cdot [1] \cdot \mathrm{Id} \arrow[r, "\nu"] \arrow[d, "(-1)^p"'] & \Sigma^p \cdot [1]^{\mathsf{tr}} \cdot m_{\mathsf{tr}} \arrow[d, equal] \\ {[1]} \cdot \mathrm{Id} \cdot \Sigma^p \arrow[r, "\nu\cdot\Sigma^p"] & {[1]^{\mathsf{tr}}} \cdot m_{\mathsf{tr}} \cdot \Sigma^p \end{tikzcd}$$

where $\Sigma^p = (\Sigma^p_{\mathcal{D}[]})^{\mathsf{mc}} = \Sigma^p_{\mathcal{D}_n^{\mathsf{tr}}}$. Therefore the transformation ν induces a natural isomorphism of translation preserving functors $H^0(\nu)$ shown as the bottom square in the diagram

$$\begin{tikzcd} D_n \arrow[r, "{(I_n,\mathrm{id})}"] \arrow[d, "{(\triangle_n,\mathrm{id})}"'] \arrow[dr, phantom, "="] & D_n \arrow[d, "{(\triangle_n,\mathrm{id})}"] \\ \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \arrow[r, "{\tilde{H}^0([1])}"] \arrow[d, "{\tilde{H}^0([1]^{\mathsf{tr}})}"'] & \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \arrow[d, "{(\mathrm{Id},(-)^p)}"] \arrow[dl, Rightarrow, "H^0(\nu)"'] \\ \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}}) \arrow[r, "\tilde{H}^0(m_{\mathsf{tr}})"] & \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \end{tikzcd} \tag{13.30.4}$$

The upper square is commutative by the definition of $\triangle_n$.

13.31 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let F be a distinguished n-triangle in $\tilde{H}^0(\mathcal{B})$. Then the inverse image n-triangle $I_n^* F = (I_n \cdot F, (-1)^p)$ is distinguished.*

Proof. Again, it suffices to check the assertion for the standard distinguished n-triangle $F = \triangle_n \cdot H^0(f)$, where $f : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$ is a unital A_∞-functor. Consider the **dg**-functor $S = \big[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{[1]^{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}\,\mathsf{tr}} \xrightarrow{m_{\mathsf{tr}}} \mathcal{D}_n^{\mathsf{tr}}\big]$ and denote by g the composite $\big[\mathcal{D}_n^{\mathsf{tr}} \xrightarrow{S} \mathcal{D}_n^{\mathsf{tr}} \xrightarrow{f} \mathcal{B}\big]$. We claim that $I_n^*[\triangle_n \cdot H^0(f)] = I_n \cdot \triangle_n \cdot (\mathrm{Id}, (-)^p) \cdot H^0(f)$ is isomorphic to $\triangle_n \cdot H^0(g)$. Indeed, the required isomorphism is obtained, for example, by composing diagram (13.30.4) with the functor $\tilde{H}^0(f) : \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \to \tilde{H}^0(\mathcal{B})$. □

Similarly the following lemma is proven:

13.32 Lemma. *Let $\mathcal{B}$ be a unital A_∞-category closed under shifts. Let F be a distinguished n-triangle in $\tilde{H}^0(\mathcal{B})$. Then the inverse image n-triangle $I_n^{-1*} F = (I_n^{-1} \cdot F, (-1)^p)$ is distinguished.*

Summing up results of Lemmata 13.28–13.32 we obtain property (FTR3) for $\tilde{H}^0(\mathcal{B})$.

From now on we suppose that the A_∞-category $\mathcal{B}$ is pretriangulated. We prove axiom (FTR2) for $\tilde{H}^0(\mathcal{B})$.

In [LM06a], with a differential graded quiver $\mathcal{Q}$ we associated the free A_∞-category $\mathcal{F}\mathcal{Q}$ generated by $\mathcal{Q}$. The freeness of $\mathcal{F}\mathcal{Q}$ is expressed by an A_∞-equivalence

$$\mathrm{restr} : A_\infty(\mathcal{F}\mathcal{Q}, \mathcal{C}) \to A_1(\mathcal{Q}, \mathcal{C}),$$

for each unital A_∞-category $\mathcal{C}$. We recall that the objects of the target A_∞-category are A_1-functors $\mathcal{Q} \to \mathcal{C}$, which are nothing but morphisms of differential graded quivers. For an A_1-functor $f : \mathcal{Q} \to \mathcal{C}$, denote by f_1 its first component, which is a $\Bbbk$-linear map $f_1 : s\mathcal{Q} \to s\mathcal{C}$ of degree 0. For a pair of A_1-functors $f, g : \mathcal{Q} \to \mathcal{C}$ and an integer k, the k-th component of the graded $\Bbbk$-module $sA_1(\mathcal{Q}, \mathcal{C})(f, g)$ consists of A_1-transformations $f \to g : \mathcal{Q} \to \mathcal{C}$ of degree k. An A_1-transformation $r : f \to g : \mathcal{Q} \to \mathcal{C}$ is unambiguously determined by its components ${}_X r_0 : \Bbbk \to s\mathcal{C}(Xf, Xg)$, for each object $X \in \operatorname{Ob} \mathcal{Q}$, and $r_1 : s\mathcal{Q}(X, Y) \to s\mathcal{C}(Xf, Yg)$, for each pair of objects $X, Y \in \operatorname{Ob} \mathcal{Q}$. We refer the reader to [LM06a] for the formulas of components of the codifferential $B : TsA_1(\mathcal{Q}, \mathcal{C}) \to TsA_1(\mathcal{Q}, \mathcal{C})$. Here we reproduce the formula for B_1, since we are going to make use of it in the sequel. For each pair of A_1-functors $f, g : \mathcal{Q} \to \mathcal{C}$, the differential $B_1 : sA_1(\mathcal{Q}, \mathcal{C}) \to sA_1(\mathcal{Q}, \mathcal{C})$, $r \mapsto rB_1$, is given by

$$\begin{aligned} [rB_1]_0 &= r_0 b_1, \\ [rB_1]_1 &= (f_1 \otimes r_0) b_2 + (r_0 \otimes g_1) b_2 + r_1 b_1 - (-)^r b_1 r_1. \end{aligned}$$

An A_1-transformation r of degree -1 is *natural* if $rB_1 = 0$. If $\mathcal{C}$ is a unital A_∞-category, so is the A_∞-category $A_1(\mathcal{Q}, \mathcal{C})$. For each A_1-functor $f : \mathcal{Q} \to \mathcal{C}$, the unit element ${}_f\mathbf{i}_0^{A_1(\mathcal{Q},\mathcal{C})}$ of f is just $f\mathbf{i}^{\mathcal{C}}$, where $\mathbf{i}^{\mathcal{C}}$ is the unit transformation of $\mathcal{C}$.

Consider now the special graded $\Bbbk$-linear quiver $Q = \mathfrak{a}_n$, whose objects are symbols $\underline{1}$, $\underline{2}$, …, $\underline{n}$, and morphisms sets are $\mathfrak{a}_n(\underline{i}, \underline{i+1}) = \Bbbk\{e_{i,i+1}\} \simeq \Bbbk$, $1 \leqslant i < n$, and $\mathfrak{a}_n(\underline{i}, \underline{j}) = 0$ if $j \neq i+1$. Here $\deg e_{ij} = 0$. Thus $\mathfrak{a}_n$ is the graded $\Bbbk$-linear envelope of the ordinary quiver with the same vertices and edges $e_{i,i+1}$. We have $T^{\geqslant 1}\mathfrak{a}_n = \mathcal{A}_n$ and $T\mathfrak{a}_n = \mathcal{D}_n$. Denote the category corresponding to the poset $\mathbf{n}$ also by $\mathbf{n}$. Its $\Bbbk$-linear envelope $\Bbbk\mathbf{n}$ is isomorphic to the $\Bbbk$-linear category $H^0(\mathcal{D}_n) = H^0(T\mathfrak{a}_n) = TH^0(\mathfrak{a}_n)$.

It suffices to prove property (FTR2) for standard distinguished n-triangles F and G, obtained from unital A_∞-functors $f, g : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$, respectively. The base of standard distinguished triangle F given by (13.18.1) equals

$$\begin{aligned}\big[\mathbf{n} \xrightarrow{\mathrm{in}_2} D_n \xrightarrow{\triangle_n} H^0(\mathcal{D}_n^{\mathsf{tr}}) \xrightarrow{H^0(f)} H^0(\mathcal{B})\big] \\ = \big[\mathbf{n} \to \mathfrak{a}_n \xrightarrow{\mathrm{in}_1} T\mathfrak{a}_n = \mathcal{D}_n = H^0(\mathcal{D}_n) \xrightarrow{H^0(u_{\mathsf{tr}})} H^0(\mathcal{D}_n^{\mathsf{tr}}) \xrightarrow{H^0(f)} H^0(\mathcal{B})\big]. \quad (13.32.1)\end{aligned}$$

A morphism of the base of F to the base of G is the commutative diagram in $H^0(\mathcal{B})$:

$$\begin{tikzcd}
\underline{1}f \arrow[r, "{[e_{12}sf_1s^{-1}]}"] \arrow[d, "\rho(\underline{1})"'] & \underline{2}f \arrow[r, "{[e_{23}sf_1s^{-1}]}"] \arrow[d, "\rho(\underline{2})"'] & \underline{3}f \arrow[r] \arrow[d, "\rho(\underline{3})"'] & \dots \arrow[r] & \underline{n-1}f \arrow[r, "{[e_{n-1,n}sf_1s^{-1}]}"] \arrow[d, "\rho(\underline{n-1})"] & \underline{n}f \arrow[d, "\rho(\underline{n})"] \\
\underline{1}g \arrow[r, "{[e_{12}sg_1s^{-1}]}"] & \underline{2}g \arrow[r, "{[e_{23}sg_1s^{-1}]}"] & \underline{3}g \arrow[r] & \dots \arrow[r] & \underline{n-1}g \arrow[r, "{[e_{n-1,n}sg_1s^{-1}]}"] & \underline{n}g
\end{tikzcd}$$

Lifting morphisms $\rho(\underline{k})$ of $H^0(\mathcal{B})$ to cycles ${}_{\underline{k}}r_0s^{-1}$ from $Z^0(\mathcal{B})$ we obtain a diagram in $Z^0(\mathcal{B})$, commutative up to boundaries. More precisely, for each square in

$$\begin{tikzcd}
\underline{1}f \arrow[r, "e_{12}sf_1s^{-1}"] \arrow[d, "{}_{\underline{1}}r_0s^{-1}"'] & \underline{2}f \arrow[r, "e_{23}sf_1s^{-1}"] \arrow[d, "{}_{\underline{2}}r_0s^{-1}"'] & \underline{3}f \arrow[r] \arrow[d, "{}_{\underline{3}}r_0s^{-1}"'] & \dots \arrow[r] & \underline{n-1}f \arrow[r, "e_{n-1,n}sf_1s^{-1}"] \arrow[d, "{}_{\underline{n-1}}r_0s^{-1}"] & \underline{n}f \arrow[d, "{}_{\underline{n}}r_0s^{-1}"] \\
\underline{1}g \arrow[r, "e_{12}sg_1s^{-1}"] & \underline{2}g \arrow[r, "e_{23}sg_1s^{-1}"] & \underline{3}g \arrow[r] & \dots \arrow[r] & \underline{n-1}g \arrow[r, "e_{n-1,n}sg_1s^{-1}"] & \underline{n}g
\end{tikzcd}$$

that starts, say, with $\underline{k}f$, there is an element ${}_{\underline{k}}r's^{-1} \in \mathcal{B}(\underline{k}f, \underline{k+1}g)^{-1}$ such that the difference of two m_2-compositions along edges of the square is the boundary ${}_{\underline{k}}r's^{-1}m_1$. Define the $\Bbbk$-linear map

$$ {}_{\underline{k}}r_1 : \mathfrak{a}_n(\underline{k}, \underline{k+1})[1] \to \mathcal{B}(\underline{k}f, \underline{k+1}g)[1], \qquad e_{k,k+1}s \mapsto {}_{\underline{k}}r'. $$

It has degree -1. One can check that the pair $({}_{\underline{k}}r_0, {}_{\underline{k}}r_1)$ defines a natural A_1-transformation. This implies that the third of the following maps is surjective:

$$\begin{aligned} H^0(A_\infty^u(\mathcal{D}_n^{\mathsf{tr}}, \mathcal{B})(f,g)) \xrightarrow{\sim} H^0(A_\infty^u(\mathcal{D}_n, \mathcal{B})(f\big|_{\mathcal{D}_n}, g\big|_{\mathcal{D}_n})) \xrightarrow{\sim} H^0(A_1(\mathfrak{a}_n, \mathcal{B})(f\big|_{\mathfrak{a}_n}, g\big|_{\mathfrak{a}_n})) \\ \longrightarrow\!\triangleright\ \Bbbk\text{-mod-quivers}(H^0(\mathfrak{a}_n), H^0(\mathcal{B}))(H^0(f\big|_{\mathfrak{a}_n}), H^0(g\big|_{\mathfrak{a}_n})) \\ \xrightarrow{\sim} \underline{\Bbbk\text{-}\mathcal{C}at}(\Bbbk\mathbf{n}, H^0(\mathcal{B}))(H^0f\big|_{\Bbbk\mathbf{n}}, H^0g\big|_{\Bbbk\mathbf{n}}) \xrightarrow{\sim} \underline{\mathcal{C}at}(\mathbf{n}, H^0(\mathcal{B}))(H^0f\big|_{\mathbf{n}}, H^0g\big|_{\mathbf{n}}). \quad (13.32.2)\end{aligned}$$

The other maps are bijections. Note that $\Bbbk$-mod-quivers$(H^0(\mathfrak{a}_n), H^0(\mathcal{B}))$ is a category, since $H^0(\mathcal{B})$ is. Its meaning is explained better by the isomorphic expression $\underline{\Bbbk\text{-}\mathcal{C}at}(TH^0(\mathfrak{a}_n), H^0(\mathcal{B})) \simeq \underline{\Bbbk\text{-}\mathcal{C}at}(\Bbbk\mathbf{n}, H^0(\mathcal{B}))$ from the next term.

The last term of (13.32.2) consists of morphisms between bases of the standard distinguished n-triangles F and G. Take one such morphism ρ, lift it to a natural A_1-transformation $r : f\big|_{\mathfrak{a}_n} \to g\big|_{\mathfrak{a}_n} : \mathfrak{a}_n \to \mathcal{B}$ as explained above, and further to an equivalence class of a natural A_∞-transformation $\tilde{r} : f \to g : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{B}$. According to Corollary 13.12 the latter defines a translation preserving transformation $\tilde{H}^0(\tilde{r}) : \tilde{H}^0(f) \to \tilde{H}^0(g) : \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \to \tilde{H}^0(\mathcal{B})$. The morphism of distinguished n-triangles

$$\overline{r} = \triangle_n \cdot \tilde{H}^0(\tilde{r}) = \left(D_n \xrightarrow{\triangle_n} \tilde{H}^0(\mathcal{D}_n^{\mathsf{tr}}) \overset{\tilde{H}^0(f)}{\underset{\tilde{H}^0(g)}{\underset{\Downarrow \tilde{H}^0(\tilde{r})}{\rightrightarrows}}} \tilde{H}^0(\mathcal{B}) \right)$$

satisfies the requirement of axiom (FTR2). In fact, $\mathrm{in}_2 \cdot \overline{r}$ equals $\rho : \mathrm{in}_2 \cdot F \to \mathrm{in}_2 \cdot G : \mathbf{n} \to \tilde{H}^0(\mathcal{B})$ that we started with. Indeed, computation made in (13.32.1) implies that $\mathrm{in}_2 \cdot \overline{r}$ equals

$$\left[\mathbf{n} \to \mathfrak{a}_n \xrightarrow{\mathrm{in}_1} T\mathfrak{a}_n = H^0(\mathcal{D}_n) \xrightarrow{H^0(u_{\mathsf{tr}})} H^0(\mathcal{D}_n^{\mathsf{tr}}) \overset{H^0(f)}{\underset{H^0(g)}{\underset{\Downarrow H^0(\tilde{r})}{\rightrightarrows}}} H^0(\mathcal{B}) \right]$$

$$= \left[\mathbf{n} \to \mathfrak{a}_n = H^0(\mathfrak{a}_n) \overset{H^0(f|_{\mathfrak{a}_n})}{\underset{H^0(g|_{\mathfrak{a}_n})}{\underset{\Downarrow H^0(r)}{\rightrightarrows}}} H^0(\mathcal{B}) \right] = \rho.$$

This finishes the proof of Theorem 13.27. □

Recall that the derived category of an abelian category is the zero cohomology of some pretriangulated differential graded category due to Drinfeld [Dri04]. Thus, Theorem 13.27 implies that the derived category of an abelian category is strongly triangulated. Besides, this fact is already known by work of Neeman [Nee05] and Maltsiniotis [Mal06].

13.33 Corollary. *Let $\mathcal{C}$ be a pretriangulated A_∞-category. Then its homotopy category $\tilde{H}^0(\mathcal{C})$ is triangulated.*

13.34 Remark. The differential graded functors $\hat{\jmath}, \hat{\imath} : \mathcal{D}_n^{\mathsf{tr}} \to \mathcal{D}_n^{\mathsf{tr}}$ constructed in Lemmata 13.29, 13.30, are quasi-inverse to each other in $\overline{A_\infty^u}$. In particular, they are A_∞-equivalences. It is important that these two are not quasi-inverse to each other as differential graded functors! This naturally arising example shows that the **dg**-context is not sufficient and should be replaced with the A_∞-set-up.

We are going to check that the composites $\hat{\jmath} \cdot \hat{\imath}$ and $\hat{\imath} \cdot \hat{\jmath}$ are A_∞-isomorphic to the identity of $\mathcal{D}_n^{\mathsf{tr}}$. In order to do that, one may use the following argument. We observe that

$\mathcal{D}_n$ is the free differential graded category generated by the quiver $\mathfrak{a}_n$. By Corollary 9.35, the restriction A_∞-functor

$$\mathrm{restr} : A^u_\infty(\mathcal{D}_n, \mathcal{C}) \to A_1(\mathfrak{a}_n, \mathcal{C}),$$

is an A_∞-equivalence, for an arbitrary unital A_∞-category $\mathcal{C}$. Furthermore, if $\mathcal{C}$ is pretriangulated, the embedding $u_{\mathsf{tr}} : \mathcal{D}_n \hookrightarrow \mathcal{D}^{\mathsf{tr}}_n$ induces an A_∞-equivalence

$$A^u_\infty(u_{\mathsf{tr}}, 1) = (u_{\mathsf{tr}} \boxtimes 1)M : A^u_\infty(\mathcal{D}^{\mathsf{tr}}_n, \mathcal{C}) \to A^u_\infty(\mathcal{D}_n, \mathcal{C}).$$

Taking $\mathcal{C} = \mathcal{D}^{\mathsf{tr}}_n$, we find that the composite

$$A^u_\infty(\mathcal{D}^{\mathsf{tr}}_n, \mathcal{D}^{\mathsf{tr}}_n) \xrightarrow{A^u_\infty(u_{\mathsf{tr}},1)} A^u_\infty(\mathcal{D}_n, \mathcal{D}^{\mathsf{tr}}_n) \xrightarrow{\mathrm{restr}} A_1(\mathfrak{a}_n, \mathcal{D}^{\mathsf{tr}}_n)$$

is an A_∞-equivalence. Note that A_∞-equivalences reflect isomorphisms. In order to show that the A_∞-functors $\hat{\jmath} \cdot \hat{\imath}$ and $\hat{\imath} \cdot \hat{\jmath}$ are isomorphic to the identity of $\mathcal{D}^{\mathsf{tr}}_n$, one shows that the restrictions of these A_∞-functors to the differential graded subquiver $\mathfrak{a}_n$ of $\mathcal{D}^{\mathsf{tr}}_n$ are isomorphic (as A_1-functors) to the restriction of the A_∞-functor $u_{\mathsf{tr}} : \mathcal{D}_n \hookrightarrow \mathcal{D}^{\mathsf{tr}}_n$.

13.35 (Pre)triangulatedness and A_∞-modules. Let $\mathcal{A}$ be an A_∞-category. Then $A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ is pretriangulated by Lemma 12.19. The Yoneda homotopy fully faithful A_∞-functor $\mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ makes $H^0\mathcal{A}$ into a full subcategory of the strongly triangulated category$H^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$. By definition, the *replete closure* of a full subcategory $\mathcal{B} \hookrightarrow \mathcal{C}$ is a full subcategory of $\mathcal{C}$ consisting of objects isomorphic to an object of $\mathcal{B}$.

13.36 Proposition. *Let $\mathcal{A}$ be an A_∞-category. Then the following are equivalent:*

(i) the A_∞-category $\mathcal{A}$ is closed under shifts;

(ii) the replete closure of the subcategory $H^0\mathcal{A} \hookrightarrow H^0(\mathcal{A}^{[]})$ is closed under translation;

(iii) the replete closure of the subcategory $H^0\mathcal{A} \hookrightarrow H^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$ is closed under translation.

Let $\mathcal{A}$ be an A_∞-category closed under shifts. Then the following are equivalent:

(iv) the A_∞-category $\mathcal{A}$ is pretriangulated;

(v) $\mathrm{Ob}\,\mathcal{A}$ *is not empty and the replete closure of the subcategory $H^0\mathcal{A} \hookrightarrow H^0(\mathcal{A}^{\mathsf{mc}})$ is closed under taking cones; (If two objects from a distinguished triangle are in* $\mathrm{Ob}\,H^0\mathcal{A}$*, then the third is isomorphic to an object of $H^0\mathcal{A}$.)*

(vi) $\mathrm{Ob}\,\mathcal{A}$ *is not empty and the replete closure of the subcategory $H^0\mathcal{A} \hookrightarrow H^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$ is closed under taking cones.*

Proof. (i) $\Longrightarrow$ (ii), (iii). If $\mathcal{A}$ is closed under shifts, the unital A_∞-functors $u_{[]} : \mathcal{A} \to \mathcal{A}^{[]}$, $\mathscr{Y} : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ induce fully faithful functors $\tilde{H}^0(u_{[]}) : \tilde{H}^0(\mathcal{A}) \to \tilde{H}^0(\mathcal{A}^{[]})$, $\tilde{H}^0(\mathscr{Y}) : \tilde{H}^0(\mathcal{A}) \to \tilde{H}^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$ between categories with translation structure by Corollary 13.12.

(ii) $\Longrightarrow$ (i). The translation structure $H^0(\Sigma^n)$ of $H^0(\mathcal{A}^{[]})$ explicitly constructed in Proposition 13.7 takes an object $X.u_{[]} = X[0]$ to $X[n]$ which has to be isomorphic to $Y[0]$ for some object Y of $\mathcal{A}$.

(iv) $\Longrightarrow$ (v), (vi). If $\mathcal{A}$ is pretriangulated, the unital A_∞-functors $u_{\mathsf{mc}} : \mathcal{A} \to \mathcal{A}^{\mathsf{mc}}$, $\mathscr{Y} : \mathcal{A} \to A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)$ induce fully faithful triangulated functors $\tilde{H}^0(u_{\mathsf{mc}}) : \tilde{H}^0(\mathcal{A}) \to \tilde{H}^0(\mathcal{A}^{\mathsf{mc}})$, $\tilde{H}^0(\mathscr{Y}) : \tilde{H}^0(\mathcal{A}) \to \tilde{H}^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$ by Proposition 13.19.

(v) $\Longrightarrow$ (iv). Since $\mathcal{A}$ contains an object X, $H^0\mathcal{A}$ contains a zero object (isomorphic to $\mathrm{Cone}(\mathrm{id} : X \to X)$). Therefore, $\mathcal{A}$ contains a contractible object (see Definition 11.31(c2)). For any cycle $f \in \mathcal{A}(X,Y)^0$ the object $\mathrm{Cone}(f)$ of $H^0(\mathcal{A}^{\mathsf{mc}})$ appears in distinguished triangle

$$\triangle_f \overset{\text{def}}{=} \bigl(X \xrightarrow{f} Y \xrightarrow{i_Y} \mathrm{Cone}(f) \xrightarrow{j_X} X[1]\bigr). \tag{13.36.1}$$

Hence, it is isomorphic to Cu_{mc} for some object C of $\mathcal{A}$. We conclude by Proposition 11.32 that $\mathcal{A}$ is mc-closed.

(iii) $\Longrightarrow$ (ii). Similarly to Remark 12.20 there is a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\mathscr{Y}"] \arrow[d, "u_{[]}"'] & & A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk) \\
\mathcal{A}^{[]} \arrow[rru, "\mathscr{Y}^{[]}.U_{[]}"] \arrow[rr, "\mathscr{Y}^{[]}"'] & & A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk)^{[]} \arrow[u, "U_{[]}"']
\end{tikzcd}$$

where $U_{[]}$ is described in Proposition 10.37. The top triangle part implies the equation

$$H^0\mathscr{Y} = \bigl[H^0\mathcal{A} \xrightarrow{H^0(u_{[]})} H^0(\mathcal{A}^{[]}) \xrightarrow{H^0(\mathscr{Y}^{[]}.U_{[]})} H^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))\bigr],$$

where all functors are full embeddings, and the second is triangulated. This makes the implication obvious.

(vi) $\Longrightarrow$ (v). Similar to the above with mc in place of $[]$. $\square$

Notice that properties (iii) and (vi) of the above proposition are taken as part of the definition of *triangulatedness* of a differential graded category by Toën and Vaquié [TV05]. These two properties distinguish *pretriangulated* (or *exact*) differential graded categories in terms of Keller [Kel06b, Section 4.5]. They are equivalent to pretriangulatedness in our sense due to above proposition. Thus, some properties of A_∞-categories can be defined in terms of properties of triangulated categories.

A triangulated differential graded category in the sense of Toën and Vaquié [TV05] in addition to the above is required to have $H^0\mathcal{A}$ closed under retracts in $H^0(A^u_\infty(\mathcal{A}^{\mathrm{op}}, \underline{\mathsf{C}}_\Bbbk))$.

Equivalently, $H^0\mathcal{A}$ is *idempotent complete* or *has split idempotents.* Keller calls such differential graded categories *Morita fibrant* [Kel06b, Section 4.6].

Chapter 14

Further applications

14.1 A_∞-bimodules and Serre A_∞-functors. The techniques developed in this book play an important rôle in the study of A_∞-bimodules and Serre A_∞-functors carried out in [LM08b]. Here we give only a short account of the results referring the curious reader to [loc.cit.] for further details. We begin by recalling the notion of (ordinary) Serre functor.

Let $\Bbbk$ be a field and let $\mathcal{C}$ be a $\Bbbk$-linear category. A $\Bbbk$-linear functor $S : \mathcal{C} \to \mathcal{C}$ is called a *right Serre functor* if there exists an isomorphism of $\Bbbk$-vector spaces

$$\mathcal{C}(X, YS) \cong \mathcal{C}(Y, X)^*,$$

natural in $X, Y \in \operatorname{Ob}\mathcal{C}$, where $*$ denotes the dual $\Bbbk$-vector space. A Serre functor, if it exists, is determined uniquely up to an isomorphism. If, moreover, S is an equivalence, it is called a *Serre functor*. The notion of Serre functor was introduced by Bondal and Kapranov [BK89], who used it to reformulate Serre–Grothendieck duality for coherent sheaves on a smooth projective variety. More precisely, Bondal and Kapranov observed that the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X of dimension n over a field $\Bbbk$ admits a Serre functor $S = - \otimes \omega_X[n]$, where ω_X is the canonical sheaf. In other words, there exists an isomorphism of $\Bbbk$-vector spaces

$$\operatorname{Hom}_{D^b(X)}(\mathscr{F}^\bullet, \mathscr{G}^\bullet \otimes \omega_X[n]) \cong \operatorname{Hom}_{D^b(X)}(\mathscr{G}^\bullet, \mathscr{F}^\bullet)^*,$$

natural in $\mathscr{F}^\bullet, \mathscr{G}^\bullet \in \operatorname{Ob} D^b(X)$. In particular, if $\mathscr{F}$ and $\mathscr{G}$ are sheaves concentrated in degrees i and 0 respectively, the above isomorphism specializes to the familiar form of Serre duality:

$$\operatorname{Ext}^{n-i}(\mathscr{F}, \mathscr{G} \otimes \omega_X) \cong \operatorname{Ext}^i(\mathscr{G}, \mathscr{F})^*.$$

Being an abstract category theory notion, Serre functors have been discovered in other contexts, for example, in Kapranov's studies of constructible sheaves on stratified spaces [Kap90]. Reiten and van den Bergh have shown that Serre functors in abelian categories of modules are related to Auslander–Reiten sequences and triangles, and they classified the noetherian hereditary Ext-finite abelian categories with Serre duality [Rv02]. Serre functors play an important role in reconstruction of a variety from its derived category of coherent sheaves [BO01].

In all cases of interest, Serre functors operate on triangulated categories rather than mere $\Bbbk$-linear categories. In the context of the program of rewriting homological algebra in the language of pretriangulated **dg**-categories or A_∞-categories as explained in Chapter 1,

it is desirable to have an analog of the notion of Serre functor at the level of **dg**-categories or A_∞-categories. This is a point where the inadequacy of **dg**-functors becomes apparent. Namely, there is a natural generalization of the notion of Serre functor to the differential graded context. However, given a pretriangulated **dg**-category $\mathcal{A}$ whose homotopy category $H^0(\mathcal{A})$ admits a Serre functor, it is hard (if possible at all) to prove the existence of a Serre **dg**-functor from $\mathcal{A}$ to itself. In contrast to a Serre **dg**-functor, a Serre A_∞-functor does exist, under some standard assumptions. Let us supply some details.

In order to define Serre A_∞-functors, we study basic properties of A_∞-bimodules. The definition of A_∞-bimodule over A_∞-algebras has been given by Tradler [Tra01, Tra02]. The notion of bimodule over some kind of A_∞-categories was introduced by Lefèvre-Hasegawa under the name of bipolydule [LH03]. Tradler's definition of A_∞-bimodules improved in [TT06] is extended in [LM08b] from graded $\Bbbk$-modules to graded quivers. The precise definition is not important at the moment; what is important is that A_∞-bimodules over A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ can be identified with A_∞-bifunctors $\mathcal{A}^{\mathrm{op}}, \mathcal{B} \to \underline{\mathsf{C}}_\Bbbk$. In particular, $\mathcal{A}$-$\mathcal{B}$-bimodules form a **dg**-category isomorphic to $\underline{\mathsf{A}_\infty}(\mathcal{A}^{\mathrm{op}}, \mathcal{B}; \underline{\mathsf{C}}_\Bbbk)$. Of course, there is nothing surprising in this statement. In fact, it is in complete analogy with the ordinary category theory: on the one hand, a bimodule over $\Bbbk$-linear categories $\mathcal{A}$ and $\mathcal{B}$ can be defined as a collection of $\Bbbk$-modules $\mathcal{P}(X,Y)$, $X \in \operatorname{Ob}\mathcal{A}$, $Y \in \operatorname{Ob}\mathcal{B}$, together with $\Bbbk$-linear action maps $\mathcal{A}(U,X) \otimes \mathcal{P}(X,Y) \otimes \mathcal{B}(Y,V) \to \mathcal{P}(U,V)$ compatible with compositions and identities; on the other hand, an $\mathcal{A}$-$\mathcal{B}$-bimodule can be defined as a $\Bbbk$-linear bifunctor $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{B} \to \Bbbk$-**Mod**. That these definitions are equivalent is a straightforward exercise. In the case of A_∞-bimodules it is still straightforward, however computations become cumbersome.

In spite of its obviousness, the interpretation of A_∞-bimodules via A_∞-functors is quite helpful. For on the one hand, it allows to apply general results about A_∞-functors to the study of A_∞-bimodules. On the other hand, A_∞-bimodules are often more suited for computations than A_∞-functors.

Explicitly, an A_∞-bimodule $\mathcal{B}$ over A_∞-categories $\mathcal{A}$ and $\mathcal{B}$ is a graded span $\mathcal{P}$ such that $\operatorname{Ob}_s \mathcal{P} = \operatorname{Ob}\mathcal{A}$ and $\operatorname{Ob}_t \mathcal{P} = \operatorname{Ob}\mathcal{B}$, equipped with $\Bbbk$-linear maps of degree 1

$$\begin{aligned} b^{\mathcal{P}}_{mn} : s\mathcal{A}(X_m, X_{m-1}) \otimes \dots \otimes s\mathcal{A}(X_1, X_0) \otimes s\mathcal{P}(X_0, Y_0) \otimes \\ \otimes s\mathcal{B}(Y_0, Y_1) \otimes \dots \otimes s\mathcal{B}(Y_{n-1}, Y_n) \to s\mathcal{P}(X_m, Y_n), \quad m, n \geqslant 0, \end{aligned}$$

satisfying identities which resemble those imposed on the components of the codifferential of an A_∞-category. An A_∞-category $\mathcal{A}$ gives rise to a regular $\mathcal{A}$-$\mathcal{A}$-bimodule denoted by $\mathcal{R}_\mathcal{A}$ or simply by $\mathcal{A}$. Its underlying graded span coincides with $\mathcal{A}$ and $b^{\mathcal{R}_\mathcal{A}}_{mn} = b^{\mathcal{A}}_{m+1+n}$, $m, n \geqslant 0$.

To an arbitrary $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{P}$ there is a dual $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{P}^*$. Its underlying graded span is given by $\mathcal{P}^*(Y,X) = \underline{\mathsf{C}}_\Bbbk(\mathcal{P}(X,Y), \Bbbk)$, $X \in \operatorname{Ob}\mathcal{A}$, $Y \in \operatorname{Ob}\mathcal{B}$.

An A_∞-functor $g : \mathcal{B} \to \mathcal{A}$ gives rise to an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{A}^g$ which is obtained from the regular $\mathcal{A}$-$\mathcal{A}$-bimodule by the restriction of scalars along the A_∞-functor g. In particular,

its underlying graded span is given by $\mathcal{A}^g(X,Y) = \mathcal{A}(X,Yg)$, $X \in \operatorname{Ob}\mathcal{A}$, $Y \in \operatorname{Ob}\mathcal{B}$. An $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{P}$ is called *representable* if it is isomorphic to $\mathcal{A}^g$ for some A_∞-functor $g : \mathcal{B} \to \mathcal{A}$, which is said to represent $\mathcal{P}$.

A unital A_∞-category $\mathcal{A}$ is said to admit a right Serre A_∞-functor if the dual $\mathcal{A}^*$ of the regular $\mathcal{A}$-$\mathcal{A}$-bimodule is representable; a representing A_∞-functor $S : \mathcal{A} \to \mathcal{A}$ is called a *right Serre A_∞-functor*. If it exists, it is determined uniquely up to an isomorphism. If, moreover, S is an A_∞-equivalence, it is called a *Serre A_∞-functor*.

It appears advantageous to introduce also an intermediate notion of Serre functor for enriched categories. The case of categories enriched in the category of $\Bbbk$-modules corresponds to the notion of ordinary Serre functor. Another case of interest is categories enriched in $\mathcal{K}$, the homotopy category of complexes of $\Bbbk$-modules. As we have seen in Chapter 9, these form a bridge from unital A_∞-categories to ordinary $\Bbbk$-linear categories. We consider also full homology of an A_∞-category, and correspondingly Serre **gr**-functors.

The main result of [LM08b] asserts the following.

14.2 Theorem (cf. [LM08b, Theorem 6.5])**.** *Suppose that the ground commutative ring $\Bbbk$ is a field. Let $\mathcal{A}$ be a unital A_∞-category.*

(a) The mappings $\mathsf{k} : S \mapsto \mathsf{k}S$, $H^\bullet : S' \mapsto H^\bullet(S')$,

$$N : S'' \mapsto (S'')^0 = \textit{restriction of } S'' \textit{ to degree } 0$$

take Serre functors to Serre functors as shown below:

$$\{\textit{Serre } A_\infty\textit{-functors in } \mathcal{A}\} \xrightarrow{\mathsf{k}} \{\textit{Serre } \mathcal{K}\textit{-functors in } \mathsf{k}\mathcal{A}\} \xrightarrow{H^\bullet}$$
$$\{\textit{Serre } \mathbf{gr}\textit{-functors in } H^\bullet\mathcal{A}\} \xrightarrow{N} \{\textit{Serre } \Bbbk\textit{-linear functors in } H^0\mathcal{A}\}. \quad (14.2.1)$$

(b) If furthermore $\mathcal{A}$ is closed under shifts, and one of the sets of Serre functors in (14.2.1) *is not empty, then the other three are not empty as well.*

In particular, if $\mathcal{A}$ is closed under shifts and $H^0(\mathcal{A})$ admits an ordinary Serre functor, then $\mathcal{A}$ admits a Serre A_∞-functor. Note that the Serre A_∞-functor whose existence is claimed in part (b) of the theorem need not be a **dg**-functor even if $\mathcal{A}$ is a **dg**-category.

An application of this theorem is the following. Let $\Bbbk$ be a field. Drinfeld's construction of quotients of pretriangulated **dg**-categories [Dri04] allows to find a pretriangulated **dg**-category $\mathcal{A}$ such that $H^0(\mathcal{A})$ is some familiar derived category (e.g. the bounded derived category $D^b(X)$ of coherent sheaves on a projective variety X). If a Serre functor exists for $H^0(\mathcal{A})$, then $\mathcal{A}$ admits a Serre A_∞-functor S by the above theorem. That is the case of $H^0(\mathcal{A}) \simeq D^b(X)$, where X is a smooth projective variety [BK89, Example 3.2].

Now we shall reinforce the statement of Theorem 14.2. First we prove a lemma.

14.3 Lemma. *Let $\mathcal{A}$, $\mathcal{B}$ be unital A_∞-categories over an arbitrary commutative ground ring $\Bbbk$, and let $f : \mathcal{A} \to \mathcal{B}$ be a unital A_∞-functor. Let $\beta : \mathsf{k}f \to j : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{B}$ be*

an isomorphism of $\mathcal{K}$-functors. Then there is a unital A_∞-functor $g : \mathcal{A} \to \mathcal{B}$ and an invertible natural A_∞-transformation $r : f \to g : \mathcal{A} \to \mathcal{B}$ such that $j = \mathsf{k}g$ and $\mathsf{k}r = \beta$. The latter means that the chain map ${}_Xr_0s^{-1} : \Bbbk \to \mathcal{B}(Xf, Xg)$ belongs to the homotopy equivalence class β_X for all objects X of $\mathcal{A}$.

Proof. Take the mapping $g = \operatorname{Ob} g = \operatorname{Ob} j : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$, and choose chain maps $g_1 : s\mathcal{A}(X,Y) \to s\mathcal{B}(Xg, Yg)$ so that sg_1s^{-1} belongs to the homotopy equivalence class $j_{X,Y} \in \mathcal{K}(\mathcal{A}(X,Y), \mathcal{B}(Xj, Yj))$. Choose chain maps ${}_Xr_0 : \Bbbk \to s\mathcal{B}(Xf, Xg)$ so that ${}_Xr_0s^{-1}$ belongs to the homotopy equivalence class $\beta_X \in \mathcal{K}(\Bbbk, \mathcal{B}(Xf, Xj))$. Naturality of the transformation β means that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\mathcal{A}(X,Y) \arrow[r, "{}_Xr_0s^{-1}\otimes sg_1s^{-1}"] \arrow[d, "sf_1s^{-1}\otimes {}_Yr_0s^{-1}"'] \arrow[dr, phantom, "\sim"] & \mathcal{B}(Xf, Xg) \otimes \mathcal{B}(Xg, Yg) \arrow[d, "m_2"] \\
\mathcal{B}(Xf, Yf) \otimes \mathcal{B}(Yf, Yg) \arrow[r, "m_2"] & \mathcal{B}(Xf, Yg)
\end{tikzcd}$$

That is, there are maps $h : \mathcal{A}(X,Y) \to \mathcal{B}(Xf, Yg)$ of degree -1 such that

$$({}_Xr_0s^{-1} \otimes sg_1s^{-1})m_2 - (sf_1s^{-1} \otimes {}_Yr_0s^{-1})m_2 = hm_1 + m_1h.$$

Define $r_1 = s^{-1}hs : s\mathcal{A}(X,Y) \to s\mathcal{B}(Xf, Yg)$. Then the above equation can be rewritten as

$$(r_0 \otimes g_1)b_2 + (f_1 \otimes r_0)b_2 + r_1b_1 + b_1r_1 = 0.$$

Together with the obvious equation $g_1b_1 = b_1g_1$ it implies that $(\operatorname{Ob} g, g_1)$ and (r_0, r_1) satisfy the equations of an A_1-functor and A_1-transformation. Invertibility of β_X means that ${}_Xr_0s^{-1}$ is invertible in $H^0\mathcal{B}$. Therefore, the hypotheses of Proposition 1.6 are satisfied for $n = 1$. Applying it we find the required g and r. □

14.4 Proposition. *Suppose that the ground commutative ring $\Bbbk$ is a field. Let $\mathcal{A}$ be a unital A_∞-category closed under shifts. Then mappings* (14.2.1) *between sets of Serre functors are surjective.*

Proof. We leave to the reader the proof that the last two maps are canonically split surjections. Let us prove that the first map is a surjection. Assume that $\overline{S} : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{A}$ is a Serre $\mathcal{K}$-functor. By Theorem 14.2 there exists a Serre A_∞-functor $S' : \mathcal{A} \to \mathcal{A}$. It gives one more Serre $\mathcal{K}$-functor $\mathsf{k}S' : \mathsf{k}\mathcal{A} \to \mathsf{k}\mathcal{A}$. Uniqueness of Serre functors in the enriched setting implies that $\mathsf{k}S'$ and $\overline{S}$ are isomorphic $\mathcal{K}$-functors, see [LM08b, Corollary 2.10]. By Lemma 14.3 there is an A_∞-functor $S : \mathcal{A} \to \mathcal{A}$ isomorphic to S' such that $\mathsf{k}S = \overline{S}$. Being isomorphic to a Serre A_∞-functor, S is a Serre A_∞-functor itself, see [LM08b, Proposition 5.22]. Thus, the first map k is a surjection, and we have provided a non-canonical splitting of this surjection. □

14.5 Remark. Let $\mathcal{C}$ be a minimal A_∞-category. Then $\mathsf{k}\mathcal{C}$ has differential 0. Hence, $\mathsf{k}\mathcal{C}$ is a graded category with zero differential. If $S : \mathcal{C} \to \mathcal{C}$ is a Serre A_∞-functor, then the Serre $\mathcal{K}$-functor $\mathsf{k}S : \mathsf{k}\mathcal{C} \to \mathsf{k}\mathcal{C}$ is a **gr**-functor. It identifies with $H^\bullet S$, so it is a Serre **gr**-functor.

14.6 Remark. Let $\Bbbk$ be a field, and let $\mathcal{A}$ be a unital A_∞-category closed under shifts. Lifting homology classes to cycles one can construct a chain quiver map $\alpha_1 : H^\bullet\mathcal{A}[1] \to \mathcal{A}[1]$ such that $\operatorname{Ob}\alpha_1 = \mathrm{id}_{\operatorname{Ob}\mathcal{A}}$ and $H^\bullet\alpha_1 = \mathrm{id}_{H^\bullet\mathcal{A}[1]}$. Here the graded quiver $H^\bullet\mathcal{A}[1]$ is equipped with the zero differential. The maps $\alpha_1 : H^\bullet\mathcal{A}(X,Y)[1] \to \mathcal{A}(X,Y)[1]$ are homotopy invertible. Applying Proposition 1.3 or a result of Kadeishvili [Kad82] we get a minimal A_∞-structure on $H^\bullet\mathcal{A}$ and an A_∞-equivalence $\alpha : H^\bullet\mathcal{A} \to \mathcal{A}$, whose first component is the above α_1. Thus, **gr**-functors $H^\bullet\alpha$ and $\mathrm{id}_{H^\bullet\mathcal{A}}$ coincide. A quasi-inverse A_∞-functor $\alpha^{-1} : \mathcal{A} \to H^\bullet\mathcal{A}$ to α can also be chosen in such a way that $\operatorname{Ob}\alpha^{-1} = \mathrm{id}_{\operatorname{Ob}\mathcal{A}}$, e.g. by Theorem 1.2. Hence, $H^\bullet\alpha^{-1} = \mathrm{id}_{H^\bullet\mathcal{A}}$.

Assume now that $S : \mathcal{A} \to \mathcal{A}$ is a Serre A_∞-functor. Consider the tautological square of $\mathcal{K}$-categories and $\mathcal{K}$-functors:

$$\begin{tikzcd}
H^\bullet\mathcal{A} \arrow[r, "(\mathsf{k}\alpha)\cdot(\mathsf{k}S)\cdot(\mathsf{k}\alpha^{-1})"] \arrow[d, "\mathsf{k}\alpha"'] \arrow[dr, phantom, "="] & H^\bullet\mathcal{A} \\
\mathsf{k}\mathcal{A} \arrow[r, "\mathsf{k}S"] & \mathsf{k}\mathcal{A} \arrow[u, "\mathsf{k}\alpha^{-1}"']
\end{tikzcd}$$

Applying to it $H^\bullet$ we find that $(\mathsf{k}\alpha)\cdot(\mathsf{k}S)\cdot(\mathsf{k}\alpha^{-1}) = H^\bullet S$. Thus, the Serre **gr**-functor $H^\bullet S = H^\bullet(\alpha\cdot S\cdot\alpha^{-1})$ comes from the Serre A_∞-functor $\alpha\cdot S\cdot\alpha^{-1}$ for the minimal A_∞-category $H^\bullet\mathcal{A}$ as in Remark 14.5.

Take now an arbitrary Serre **gr**-functor $\overline{S} : H^\bullet\mathcal{A} \to H^\bullet\mathcal{A}$. A priori it senses only **gr**-category structure of $H^\bullet\mathcal{A}$. However, the latter admits a lifting to a minimal A_∞-category structure in $H^\bullet\mathcal{A}$ along some α as above. This $\overline{S}$ equals $H^\bullet S$ for some Serre A_∞-functor $S : \mathcal{A} \to \mathcal{A}$. Hence, it equals $H^\bullet(\alpha\cdot S\cdot\alpha^{-1}) = \mathsf{k}(\alpha\cdot S\cdot\alpha^{-1})$ for the Serre A_∞-functor $\alpha\cdot S\cdot\alpha^{-1}$ in $H^\bullet\mathcal{A}$. Thus, $\overline{S}$ extends to a Serre A_∞-functor simultaneously with the extension of **gr**-structure on $H^\bullet\mathcal{A}$ to an A_∞-structure.

The proof of Theorem 14.2 relies on the Yoneda Lemma in the form of Proposition 1.5.

14.7 The Yoneda Lemma.

A version of the classical Yoneda Lemma is presented in Mac Lane's book [Mac88, Section III.2] as the following statement. For any category $\mathcal{C}$ there is an isomorphism of functors

$$\mathrm{ev}^{\mathcal{C}at} \simeq \bigl[\mathcal{C}\times\underline{\mathcal{C}at}(\mathcal{C},\mathcal{S}\mathrm{et}) \xrightarrow{\mathscr{Y}^{\mathrm{op}}\times 1} \underline{\mathcal{C}at}(\mathcal{C},\mathcal{S}\mathrm{et})^{\mathrm{op}}\times\underline{\mathcal{C}at}(\mathcal{C},\mathcal{S}\mathrm{et}) \xrightarrow{\mathrm{Hom}_{\underline{\mathcal{C}at}(\mathcal{C},\mathcal{S}\mathrm{et})}} \mathcal{S}\mathrm{et}\bigr],$$

where $\mathscr{Y} : \mathcal{C}^{\mathrm{op}} \to \underline{\mathcal{C}at}(\mathcal{C},\mathcal{S}\mathrm{et})$, $X \mapsto \mathcal{C}(X,\_)$, is the Yoneda embedding. This observation was generalized to A_∞-setting in [LM08b]. It required the use of A_∞-transformations of two variables considered in this book.

14.8 Theorem (The Yoneda Lemma [LM08b, Theorem A.1]). *For any A_∞-category $\mathcal{A}$ there is a natural A_∞-transformation*

$$\Omega : \mathrm{ev}^{\mathsf{A}_\infty} \to \Bigl[\mathcal{A}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathscr{Y}^{\mathrm{op}},1} \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}}, \underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \xrightarrow{\mathrm{Hom}_{\underline{\mathsf{A}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}} \underline{\mathsf{C}}_\Bbbk\Bigr].$$

If the A_∞-category $\mathcal{A}$ is unital, Ω restricts to an invertible *natural A_∞-transformation*

$$\begin{tikzcd}
\mathcal{A}, \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[rr, "\mathrm{ev}^{\mathsf{A}^{\mathrm{u}}_\infty}"] \arrow[dr, "\mathscr{Y}^{\mathrm{op}},1"'] & {} \arrow[d, phantom, "\Downarrow\Omega"] & \underline{\mathsf{C}}_\Bbbk \\
& \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)^{\mathrm{op}}, \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk) \arrow[ur, "\mathrm{Hom}_{\underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)}"'] &
\end{tikzcd}$$

Some previously published A_∞-versions of the Yoneda Lemma contented with the statement that for a unital A_∞-category $\mathcal{A}$, the Yoneda A_∞-functor $\mathscr{Y} : \mathcal{A}^{\mathrm{op}} \to \underline{\mathsf{A}^{\mathrm{u}}_\infty}(\mathcal{A};\underline{\mathsf{C}}_\Bbbk)$ is homotopy full and faithful [Fuk02, Theorem 9.1], [LM08c, Theorem A.11], see Corollary 1.4 and the preceding discussion. A more general form of the Yoneda Lemma cited in Proposition 1.5 was obtained in [LM08b, Proposition A.3] as a corollary of the above theorem. The same corollary was proven by Seidel [Sei08, Lemma 2.12] over a ground field $\Bbbk$.

The instruments from this book used in [LM08b] do not restrict to A_∞-functors and A_∞-transformations of several variables. The use of multifunctor $\mathsf{k} : \mathsf{A}^{\mathrm{u}}_\infty \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$ from Section 9.7 was essential. As usual, $\mathcal{K}$ denotes the homotopy category of complexes of $\Bbbk$-modules. The multifunctor was even generalized to the following.

14.9 Proposition ([LM08b, Proposition 3.5]). *There is a symmetric $\Bbbk$-$\mathcal{C}at$-multifunctor $\mathsf{k} : \overline{A^u_\infty} \to \widehat{\mathcal{K}\text{-}\mathcal{C}at}$.*

Here $\overline{A^u_\infty}$ denotes a symmetric $\Bbbk$-$\mathcal{C}at$-multicategory obtained from the symmetric $\mathsf{A}^{\mathrm{u}}_\infty$-multicategory $\underline{\mathsf{A}^{\mathrm{u}}_\infty}$ via the base change functor $H^0 : \mathsf{A}^{\mathrm{u}}_\infty \to \Bbbk$-$\mathcal{C}at$. That is, the objects of $\overline{A^u_\infty}$ are unital A_∞-categories, and for each collection $(\mathcal{A}_i)_{i\in I}$, $\mathcal{B}$ of unital A_∞-categories, there is a $\Bbbk$-linear category $\overline{A^u_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B}) = H^0\underline{\mathsf{A}^{\mathrm{u}}_\infty}((\mathcal{A}_i)_{i\in I};\mathcal{B})$, whose objects are unital A_∞-functors, and whose morphisms are equivalence classes of natural A_∞-transformations.

Appendix A

Internal Monoidal categories

In this chapter, we provide the general set-up necessary for the construction of actions of one multicategory on another, which is the subject of Appendices B, C. Instead of working inside a symmetric monoidal 2-category we develop the language of internal lax symmetric Monoidal categories which live inside a symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{C}$, interpreted as a 2-category. Mostly we are interested in the paradigmatic example of the category $\mathfrak{C}$ = sym-Mono-$\mathcal{C}at$ of symmetric Monoidal categories introduced in Chapter 2. Lax symmetric Monoidal categories have internal analogues in $\mathfrak{C}$: lax-symmetric-Monoidal-categories which are objects of $\mathfrak{C}$. We prove that these constitute a symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{C}^\star$. Moreover, the correspondence $\mathfrak{C} \mapsto \mathfrak{C}^\star$ becomes a 2-functor.

A.1 Definition. A *lax-symmetric-Monoidal-category* $\mathcal{C}$ of a symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ (viewed also as a 2-category) is

1. An object $\mathcal{C}$ of $\mathfrak{C}$.
2. A 1-morphism $\otimes^I : \boxtimes^I \mathcal{C} \to \mathcal{C}$, for every set $I \in \operatorname{Ob} \mathcal{S}$, such that $\otimes^I = \mathrm{Id}_{\mathcal{C}}$ for each 1-element set I.

   For a map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$ introduce a 1-morphism
$$\otimes^f_{\mathcal{C}} = \big(\boxtimes^I \mathcal{C} \xrightarrow{\Lambda^f_{\mathfrak{C}}} \boxtimes^{j\in J} \boxtimes^{f^{-1}j} \mathcal{C} \xrightarrow{\boxtimes^{j\in J} \otimes^{f^{-1}j}} \boxtimes^J \mathcal{C}\big).$$

3. A 2-morphism λ^f for every map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$:
$$\begin{tikzcd}
\boxtimes^I \mathcal{C} \arrow[r, "\otimes^f_{\mathcal{C}}"] \arrow[dr, "\otimes^I"'] \arrow[dr, phantom, shift left=3, "\Uparrow \lambda^f"] & \boxtimes^J \mathcal{C} \arrow[d, "\otimes^J"] \\
 & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\boxtimes^{j\in J} \boxtimes^{f^{-1}j} \mathcal{C} \arrow[r, "\boxtimes^{j\in J} \otimes^{f^{-1}j}"] \arrow[dr, phantom, "\Uparrow \lambda^f"] & \boxtimes^J \mathcal{C} \arrow[d, "\otimes^J"] \\
\boxtimes^I \mathcal{C} \arrow[u, "\Lambda^f_{\mathfrak{C}}"] \arrow[r, "\otimes^I"'] & \mathcal{C}
\end{tikzcd}$$

such that

(i) for all sets $I \in \operatorname{Ob} \mathcal{O}$, for all 1-element sets J
$$\lambda^{\mathrm{id}_I} = \mathrm{id}, \qquad \lambda^{I \to J} = \mathrm{id};$$

(ii) for any pair of composable maps $I \xrightarrow{f} J \xrightarrow{g} K$ from $\mathcal{S}$ this equation holds:

$$
\begin{tikzcd}
\boxtimes^J \mathcal{C} \arrow[r, "\otimes^g_{\mathcal{C}}"] \arrow[dr, "\otimes^J"] \arrow[dr, phantom, shift right=6, "\lambda^f"] \arrow[dr, phantom, shift left=6, "\lambda^g"] & \boxtimes^K \mathcal{C} \arrow[d, "\otimes^K"] \\
\boxtimes^I \mathcal{C} \arrow[u, "\otimes^f_{\mathcal{C}}"] \arrow[r, "\otimes^I"'] & \mathcal{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\boxtimes^J \mathcal{C} \arrow[r, "\otimes^g_{\mathcal{C}}"] & \boxtimes^K \mathcal{C} \arrow[d, "\otimes^K"] \\
\boxtimes^I \mathcal{C} \arrow[u, "\otimes^f_{\mathcal{C}}"] \arrow[ur, "\otimes^{g\circ f}_{\mathcal{C}}"] \arrow[r, "\otimes^I"'] \arrow[ur, phantom, shift left=6, "\boxtimes^{k\in K}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] \arrow[ur, phantom, shift right=6, "\lambda^{g\circ f}"] & \mathcal{C}
\end{tikzcd}
\tag{A.1.1}
$$

Here $\boxtimes^{k\in K}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}$ means the 2-morphism

$$
\begin{tikzcd}
 & \boxtimes^J \mathcal{C} \arrow[dr, "\Lambda^g_{\mathfrak{C}}"] \arrow[d, phantom, "="] & \\
\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\mathcal{C} \arrow[ur, "\boxtimes^{j\in J}\otimes^{f^{-1}j}"] \arrow[r, "\Lambda^g_{\mathfrak{C}}"] \arrow[dr, phantom, "="] & \boxtimes^{k\in K}\boxtimes^{j\in g^{-1}k}\boxtimes^{f^{-1}j}\mathcal{C} \arrow[r, "\boxtimes^{k\in K}\boxtimes^{j\in g^{-1}k}\otimes^{f^{-1}j}"] \arrow[dr, phantom, "\Uparrow\ \boxtimes^{k\in K}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] & \boxtimes^{k\in K}\boxtimes^{g^{-1}k}\mathcal{C} \arrow[d, "\boxtimes^{k\in K}\otimes^{g^{-1}k}"] \\
\boxtimes^I \mathcal{C} \arrow[u, "\Lambda^f_{\mathfrak{C}}"] \arrow[r, "\Lambda^{g\circ f}_{\mathfrak{C}}"'] & \boxtimes^{k\in K}\boxtimes^{f^{-1}g^{-1}k}\mathcal{C} \arrow[u, "\boxtimes^{k\in K}\Lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}_{\mathfrak{C}}"] \arrow[r, "\boxtimes^{k\in K}\otimes^{f^{-1}g^{-1}k}"'] & \boxtimes^K \mathcal{C}
\end{tikzcd}
$$

The top quadrilateral in above diagram is the identity 2-morphism due to the 2-transformation Λ^g being strict. The left square is equation (2.37.2).

A.2 Definition. A *lax-symmetric-Monoidal-functor* $(F,\phi^I) : (\mathcal{C},\otimes^I,\lambda^f) \to (\mathcal{D},\otimes^I,\mu^f)$ between lax-symmetric-Monoidal-categories $\mathcal{C}$, $\mathcal{D}$ of a symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C},\boxtimes^I,\Lambda^f)$ is

i) a 1-morphism $F : \mathcal{C} \to \mathcal{D}$ between objects $\mathcal{C}$, $\mathcal{D}$ of $\mathfrak{C}$,

ii) a 2-morphism for each set $I \in \operatorname{Ob}\mathcal{S}$

$$
\begin{tikzcd}
\boxtimes^I \mathcal{C} \arrow[r, "\boxtimes^I F"] \arrow[d, "\otimes^I"'] & \boxtimes^I \mathcal{D} \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\phi^I"'] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D}
\end{tikzcd}
$$

such that $\phi^I = \mathrm{id}_F$ for each 1-element set I, and for every map $f : I \to J$ of $\mathcal{S}$ the

following equation holds:

$$
\begin{tikzcd}
 & \boxtimes^I \mathcal{C} \arrow[rr, "\boxtimes^I F"] \arrow[dl, "\otimes^f"'] & & \boxtimes^I \mathcal{D} \arrow[dl, "\otimes^f"] \arrow[dd, "\otimes^I"] \arrow[lll, Rightarrow, "\boxtimes^{j\in J}\phi^{f^{-1}j}"'] \\
\boxtimes^J \mathcal{C} \arrow[rr, "\boxtimes^J F"'] \arrow[dr, "\otimes^J"'] & & \boxtimes^J \mathcal{D} \arrow[dl, Rightarrow, "\phi^J"'] \arrow[dr, "\otimes^J"] & \arrow[l, Rightarrow, "\mu^f"] \\
 & \mathcal{C} \arrow[rr, "F"'] & & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \boxtimes^I \mathcal{C} \arrow[r, "\boxtimes^I F"] \arrow[dl, "\otimes^f"'] \arrow[dd, "\otimes^I"] & \boxtimes^I \mathcal{D} \arrow[dd, "\otimes^I"] \arrow[ddl, Rightarrow, "\phi^I"] \\
\boxtimes^J \mathcal{C} \arrow[dr, "\otimes^J"'] & \arrow[l, Rightarrow, "\lambda^f"'] & \\
 & \mathcal{C} \arrow[r, "F"'] & \mathcal{D}
\end{tikzcd}
\tag{A.2.1}
$$

Here 2-morphism $\boxtimes^{j\in J}\phi^{f^{-1}j}$ means the pasting

$$
\begin{tikzcd}
\boxtimes^I \mathcal{C} \arrow[r, "\boxtimes^I F"] \arrow[d, "\Lambda^f"'] \arrow[dr, phantom, "="] & \boxtimes^I \mathcal{D} \arrow[d, "\Lambda^f"] \\
\boxtimes^{j\in J} \boxtimes^{f^{-1}j} \mathcal{C} \arrow[r, "\boxtimes^{j\in J}\boxtimes^{f^{-1}j}F"] \arrow[d, "\boxtimes^{j\in J}\otimes^{f^{-1}j}"'] & \boxtimes^{j\in J} \boxtimes^{f^{-1}j} \mathcal{D} \arrow[d, "\boxtimes^{j\in J}\otimes^{f^{-1}j}"] \arrow[dl, Rightarrow, "\boxtimes^{j\in J}\phi^{f^{-1}j}"'] \\
\boxtimes^J \mathcal{C} \arrow[r, "\boxtimes^J F"] & \boxtimes^J \mathcal{D}
\end{tikzcd}
$$

A.3 Definition. A *Monoidal-transformation* $t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \mu^f)$ between lax-symmetric-Monoidal-functors of a symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ is a 2-morphism $t : F \to G$ such that for every $I \in \operatorname{Ob}\mathcal{S}$

$$
\begin{tikzcd}
\boxtimes^I \mathcal{C} \arrow[r, bend left, "\boxtimes^I F"] \arrow[r, bend right, "\boxtimes^I G"'] \arrow[r, phantom, "\boxtimes^I t\Downarrow"] \arrow[d, "\otimes^I"'] & \boxtimes^I \mathcal{D} \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\psi^I"'] \\
\mathcal{C} \arrow[r, "G"'] & \mathcal{D}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\boxtimes^I \mathcal{C} \arrow[r, "\boxtimes^I F"] \arrow[d, "\otimes^I"'] & \boxtimes^I \mathcal{D} \arrow[d, "\otimes^I"] \arrow[dl, Rightarrow, "\phi^I"] \\
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "t\Downarrow"] & \mathcal{D}
\end{tikzcd}
\quad .
$$

For each symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{C}$ there is a $\mathcal{C}at$-category lax-sym-Mono-cat-$\mathfrak{C}$ with

0. Objects – lax-symmetric-Monoidal-categories of $\mathfrak{C}$;

1. 1-morphisms – lax-symmetric-Monoidal-functors of $\mathfrak{C}$;

2. 2-morphisms – Monoidal-transformations of $\mathfrak{C}$.

Let us put for brevity $\mathfrak{C}^\star = \text{lax-sym-Mono-cat-}\mathfrak{C}$. We are going to show that this $\mathcal{C}at$-category is equipped with a natural symmetric Monoidal structure. To achieve this, we extend $-^\star$ to symmetric Monoidal $\mathcal{C}at$-functors and Monoidal transformations as a 2-functor.

A.4 Proposition. *Let $(R, \rho^I) : (\mathfrak{C}, \boxtimes^I, \Lambda^f_{\mathfrak{C}}) \to (\mathfrak{D}, \boxtimes^I, \Lambda^f_{\mathfrak{D}})$ be a symmetric Monoidal $\mathcal{C}at$-functor between symmetric Monoidal $\mathcal{C}at$-categories.*

(a) Let $(\mathcal{C}, \otimes^I, \lambda^f)$ be a lax-symmetric-Monoidal-category in $\mathfrak{C}$. For each set $I \in \operatorname{Ob} \mathcal{S}$ put

$$\otimes^I_{R\mathcal{C}} = \big[\boxtimes^I R\mathcal{C} \xrightarrow{\rho^I} R \boxtimes^I \mathcal{C} \xrightarrow{R(\otimes^I)} R\mathcal{C}\big].$$

For each map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$, let $\lambda^f_{R\mathcal{C}}$ be a 2-morphism given by the following pasting:

$$\begin{tikzcd}
\boxtimes^{j\in J}\boxtimes^{f^{-1}j} R\mathcal{C} \arrow[r, "\boxtimes^{j\in J}\rho^{f^{-1}j}"] & \boxtimes^{j\in J} R\boxtimes^{f^{-1}j}\mathcal{C} \arrow[r, "\boxtimes^{j\in J}R(\otimes^{f^{-1}j})"] \arrow[d, "\rho^J"'] \arrow[dr, phantom, "="] & \boxtimes^J R\mathcal{C} \arrow[d, "\rho^J"] \\
{} \arrow[r, phantom, "="] & R\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\mathcal{C} \arrow[r, "R(\boxtimes^{j\in J}\otimes^{f^{-1}j})"] \arrow[dr, phantom, "\Uparrow R\lambda^f"] & R\boxtimes^J\mathcal{C} \arrow[d, "R(\otimes^J)"] \\
\boxtimes^I R\mathcal{C} \arrow[r, "\rho^I"] \arrow[uu, "\Lambda^f_{\mathfrak{D}}"] & R\boxtimes^I\mathcal{C} \arrow[r, "R(\otimes^I)"] \arrow[u, "R\Lambda^f_{\mathfrak{C}}"'] & R\mathcal{C}
\end{tikzcd}$$

in which the pentagon commutes by (2.6.1), and the upper square is commutative due to the naturality of ρ^J. Then $(R\mathcal{C}, \otimes^I_{R\mathcal{C}}, \lambda^f_{R\mathcal{C}})$ is a lax-symmetric-Monoidal-category in $\mathfrak{D}$.

(b) Let $(F, \phi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \mu^f)$ be a lax-symmetric-Monoidal-functor. For each set $I \in \operatorname{Ob} \mathcal{S}$, let ϕ^I_{RF} denote the 2-morphism given by the pasting

$$\begin{tikzcd}
\boxtimes^I R\mathcal{C} \arrow[r, "\boxtimes^I RF"] \arrow[d, "\rho^I"'] \arrow[dr, phantom, "="] & \boxtimes^I R\mathcal{D} \arrow[d, "\rho^I"] \\
R\boxtimes^I\mathcal{C} \arrow[r, "R\boxtimes^I F"] \arrow[d, "R(\otimes^I)"'] & R\boxtimes^I\mathcal{D} \arrow[d, "R(\otimes^I)"] \arrow[dl, Rightarrow, "R\phi^I"'] \\
R\mathcal{C} \arrow[r, "RF"] & R\mathcal{D}
\end{tikzcd}$$

in which the upper square is commutative due to the naturality of ρ^I. Then $(RF, \phi^I_{RF}) : (R\mathcal{C}, \otimes^I_{R\mathcal{C}}, \lambda^f_{R\mathcal{C}}) \to (R\mathcal{D}, \otimes^I_{R\mathcal{D}}, \mu^f_{R\mathcal{D}})$ is a lax-symmetric-Monoidal-functor.

(c) Let $t : (F, \phi^I) \to (G, \psi^I) : (\mathcal{C}, \otimes^I, \lambda^f) \to (\mathcal{D}, \otimes^I, \mu^f)$ be a Monoidal-transformation. Then $Rt : (RF, \phi^I_{RF}) \to (G, \psi^I_{RG}) : (R\mathcal{C}, \otimes^I_{R\mathcal{C}}, \lambda^f_{R\mathcal{C}}) \to (R\mathcal{D}, \otimes^I_{R\mathcal{D}}, \mu^f_{R\mathcal{D}})$ is a Monoidal–transformation.

(d) *The correspondence defined in (a), (b), and (c) is a $\mathcal{C}at$-functor $(R,\rho^I)^\star : \mathfrak{C}^\star \to \mathfrak{D}^\star$.*

Proof. The proof of (a), (b), and (c) is somewhat cumbersome, but truly straightforward. The last claim is obvious. $\square$

Let $t : (R,\rho^I) \to (S,\varsigma^I) : (\mathfrak{C}, \boxtimes^I, \Lambda^I_{\mathfrak{C}}) \to (\mathfrak{D}, \boxtimes^I, \Lambda^I_{\mathfrak{D}})$ be a Monoidal transformation of symmetric Monoidal $\mathcal{C}at$-functors. It gives rise to a natural transformation of $\mathcal{C}at$-functors $t^\star : (R,\rho)^\star \to (S,\varsigma^I)^\star : \mathfrak{C}^\star \to \mathfrak{D}^\star$ which we shall describe below.

Let $(\mathcal{C}, \otimes^I, \lambda^f)$ be a lax-symmetric-Monoidal-category in $\mathfrak{C}$. The following diagram commutes:

$$\begin{tikzcd}
\boxtimes^I R\mathcal{C} \arrow[r, "\boxtimes^I t"] \arrow[d, "\rho^I"'] & \boxtimes^I S\mathcal{C} \arrow[d, "\varsigma^I"] \\
R \boxtimes^I \mathcal{C} \arrow[r, "t"] \arrow[d, "R(\otimes^I)"'] & S \boxtimes^I \mathcal{C} \arrow[d, "S(\otimes^I)"] \\
R\mathcal{C} \arrow[r, "t"] & S\mathcal{C}
\end{tikzcd}$$

The lower rectangle commutes due to the naturality of t, the upper rectangle commutes due to the Monoidality of t.

A.5 Lemma. $(t, \mathrm{id}) : (R\mathcal{C}, \otimes^I_{R\mathcal{C}}, \lambda^f_{R\mathcal{C}}) \to (S\mathcal{C}, \otimes^I_{S\mathcal{C}}, \lambda^f_{S\mathcal{C}})$ *is a lax-symmetric-Monoidal-functor.*

The collection of lax-symmetric-Monoidal-functors

$$(t, \mathrm{id}) : (R\mathcal{C}, \otimes^I_{R\mathcal{C}}, \lambda^f_{R\mathcal{C}}) \to (S\mathcal{C}, \otimes^I_{S\mathcal{C}}, \lambda^f_{S\mathcal{C}}),$$

where $\mathcal{C}$ runs over lax-symmetric-Monoidal-categories, constitutes a $\mathcal{C}at$-natural transformation $t^\star : (R,\rho^I)^\star \to (S,\varsigma^I)^\star : \mathfrak{C}^\star \to \mathfrak{D}^\star$. The $\mathcal{C}at$-naturality of $t^\star$ follows from the $\mathcal{C}at$-naturality of t.

The correspondence $\mathfrak{C} \mapsto \mathfrak{C}^\star$, $(R,\rho^I) \mapsto (R,\rho^I)^\star$, $t \mapsto t^\star$ is a strict 2-functor from the 2-category of symmetric Monoidal $\mathcal{C}at$-categories, symmetric Monoidal $\mathcal{C}at$-functors and Monoidal transformations to the 2-category of $\mathcal{C}at$-categories, $\mathcal{C}at$-functors, and $\mathcal{C}at$-natural transformations. In fact, more is true.

A.6 Theorem. *The 2-functor*

$$-^\star : \textit{symmetric-Monoidal-}\mathcal{C}at\textit{-categories} \to \mathcal{C}at\textit{-categories}$$

restricts to a 2-functor

$$-^\star : \textit{symmetric-Monoidal-}\mathcal{C}at\textit{-categories} \to \textit{symmetric-Monoidal-}\mathcal{C}at\textit{-categories}.$$

Sketch of proof. We only outline the crucial steps of the proof, letting the curious reader fill in details.

For a symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$, the $\mathcal{C}at$-functor $\boxtimes^I : \mathfrak{C}^I \to \mathfrak{C}$ together with the $\mathcal{C}at$-natural transformation $\sigma_{(12)} : \boxtimes^{j\in J} \boxtimes^{i\in I} \mathcal{C}_{ij} \to \boxtimes^{i\in I} \boxtimes^{j\in J} \mathcal{C}_{ij}$ is a symmetric Monoidal $\mathcal{C}at$-functor, cf. Example 4.26. Furthermore, the symmetric Monoidal $\mathcal{C}at$-category $(\mathfrak{C}^I)^\star$ identifies naturally with $(\mathfrak{C}^\star)^I$. Therefore, the symmetric Monoidal $\mathcal{C}at$-functor $(\boxtimes^I, \sigma_{(12)}) : \mathfrak{C}^I \to \mathfrak{C}$ gives rise to a $\mathcal{C}at$-functor $\boxtimes^I = (\boxtimes^I, \sigma_{(12)})^\star : (\mathfrak{C}^\star)^I \to \mathfrak{C}^\star$ by Proposition A.4,(a).

Next notice that, for each map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$, the $\mathcal{C}at$-natural transformation $\Lambda^f_{\mathfrak{C}} : \boxtimes^I \to \boxtimes^f \cdot \boxtimes^J : \mathfrak{C}^I \to \mathfrak{C}$ is Monoidal. Thus $\Lambda^f_{\mathfrak{C}}$ induces a $\mathcal{C}at$-natural transformation $(\Lambda^f_{\mathfrak{C}})^\star : \boxtimes^I \to \boxtimes^f \cdot \boxtimes^J : (\mathfrak{C}^\star)^I \to \mathfrak{C}^\star$ by Proposition A.4,(c). These transformations are invertible, since so are $\Lambda^f_{\mathfrak{C}}$, and satisfy equation (2.37.2), since so do $\Lambda^f_{\mathfrak{C}}$. Therefore, $\mathfrak{C}^\star$ becomes a symmetric Monoidal $\mathcal{C}at$-category.

A symmetric Monoidal $\mathcal{C}at$-functor $(R, \rho^I) : (\mathfrak{C}, \boxtimes^I, \Lambda^f_{\mathfrak{C}}) \to (\mathfrak{D}, \boxtimes^I, \Lambda^f_{\mathfrak{D}})$ gives rise to a $\mathcal{C}at$-functor $(R, \rho^I)^\star : \mathfrak{C}^\star \to \mathfrak{D}^\star$ by Proposition A.4,(b). It remains to show that transformation $\rho^I : R^I \cdot \boxtimes^I \to \boxtimes^I \cdot R : \mathfrak{C}^I \to \mathfrak{D}$ is Monoidal, and thus it gives rise to a transformation of $\mathcal{C}at$-functors $(\rho^I)^\star : (R^\star)^I \cdot \boxtimes^I \to \boxtimes^I \cdot R^\star : (\mathfrak{C}^\star)^I \to \mathfrak{D}^\star$ which turns $R^\star$ into a symmetric Monoidal $\mathcal{C}at$-functor. We shall leave the last assertion as an exercise for the reader.

Finally, a Monoidal transformation $t : (R, \rho^I) \to (S, \varsigma^I) : (\mathfrak{C}, \boxtimes^I, \Lambda^I_{\mathfrak{C}}) \to (\mathfrak{D}, \boxtimes^I, \Lambda^I_{\mathfrak{D}})$ of symmetric Monoidal $\mathcal{C}at$-functors gives rise to a $\mathcal{C}at$-natural transformation $t^\star : (R, \rho)^\star \to (S, \varsigma^I)^\star : \mathfrak{C}^\star \to \mathfrak{D}^\star$ by Proposition A.4,(c). The $\mathcal{C}at$-natural transformation $t^\star$ is Monoidal since the Monoidality conditions for t and $t^\star$ are just the same. □

A.7 Monoidal $\mathcal{C}at$-functor from lax Monoidal categories to multicategories. One can prove that the map which assigns a multicategory to a lax Monoidal category gives rise to a symmetric Monoidal $\mathcal{C}at$-functor.

A.8 Theorem. *The assignment $\mathcal{C} \mapsto \widehat{\mathcal{C}}$, $F \mapsto \widehat{F}$, $r \mapsto \widehat{r}$ of Propositions 3.22, 3.28, 3.29 is a symmetric Monoidal $\mathcal{C}at$-functor* lax-Mono-cat $\to$ $\mathcal{M}\mathfrak{C}$atm, lax-sym-Mono-cat $\to$ $\mathcal{S}\mathcal{M}\mathfrak{C}$atm.

Combining the results of Theorems A.6 and A.8 we obtain the following corollary.

A.9 Corollary. *The symmetric Monoidal $\mathcal{C}at$-functor* $\widehat{-}$: lax-sym-Mono-cat $\to$ $\mathcal{S}\mathcal{M}\mathfrak{C}$atm *of Theorem A.8 extends to a symmetric Monoidal $\mathcal{C}at$-functor*

$$\widehat{-}^\star : \text{lax-sym-Mono-cat-(lax-sym-Mono-cat)} \to \text{lax-sym-Mono-cat-}(\mathcal{S}\mathcal{M}\mathfrak{C}\text{atm}).$$

Let sym-Mono-cat-$\mathfrak{C}$ $\subset$ lax-sym-Mono-cat-$\mathfrak{C}$ be a full $\mathcal{C}at$-subcategory, consisting of those $(\mathcal{C}, \otimes^I, \lambda^f)$ for which all 2-morphisms λ^f are invertible, that is, of symmetric-Monoidal-categories. Its 1-morphisms are lax-symmetric-Monoidal-functors and 2-morphisms are Monoidal-transformations of $\mathfrak{C}$. Just like a commutative algebra in a symmetric Monoidal category gives rise to a commutative algebra in the symmetric Monoidal category of commutative algebras, we have the following statement.

A.10 Proposition. *There is a natural map*

$$\operatorname{Ob} \mathit{sym\text{-}Mono\text{-}cat\text{-}}\mathfrak{C} \to \operatorname{Ob} \mathit{sym\text{-}Mono\text{-}cat\text{-}}(\mathit{sym\text{-}Mono\text{-}cat\text{-}}\mathfrak{C}).$$

Sketch of proof. We only describe the map. Let $(\mathcal{C}, \otimes^I, \lambda^f)$ be a symmetric-Monoidal-category in $\mathfrak{C}$. In order to turn it into a symmetric-Monoidal-category in sym-Mono-cat-$\mathfrak{C}$, one can proceed as follows: first, show that for each set $J \in \operatorname{Ob} \mathcal{S}$ the 1-morphism $\otimes^J : \boxtimes^J \mathcal{C} \to \mathcal{C}$ can be equipped with the structure of a lax-symmetric-Monoidal-functor; second, show that for each map $f : I \to J$ in $\operatorname{Mor} \mathcal{S}$ the 2-morphism λ^f is a Monoidal-transformation $\otimes^I \to \otimes^f \cdot \otimes^J : \boxtimes^I \mathcal{C} \to \mathcal{C}$. To turn $\otimes^J : \boxtimes^J \mathcal{C} \to \mathcal{C}$ into a lax-symmetric-Monoidal-functor, we need to define a 2-morphism

$$\begin{tikzcd}
\boxtimes^K \boxtimes^J \mathcal{C} \arrow[r, "\boxtimes^K \otimes^J"] \arrow[d, "\otimes^K_{\boxtimes^J \mathcal{C}}"'] & \boxtimes^K \mathcal{C} \arrow[d, "\otimes^K"] \arrow[dl, Rightarrow, "\tau^K_J"'] \\
\boxtimes^J \mathcal{C} \arrow[r, "\otimes^J"] & \mathcal{C}
\end{tikzcd}$$

for each set $K \in \operatorname{Ob} \mathcal{S}$. Let τ^K_J be given by the pasting

$$\begin{tikzcd}
\boxtimes^K \boxtimes^J \mathcal{C} \arrow[r, equal] \arrow[d, "\boxtimes^{k \in K} (\Lambda_{\mathfrak{C}}^{\{k\} \times J \to J})^{-1}"'] & \boxtimes^K \boxtimes^J \mathcal{C} \arrow[d, "\boxtimes^K \otimes^J"] \arrow[dl, Rightarrow, "\boxtimes^{k \in K} (\lambda^{\{k\} \times J \to J})^{-1}"'] \\
\boxtimes^{k \in K} \boxtimes^{\{k\} \times J} \mathcal{C} \arrow[r, "\boxtimes^{k \in K} \otimes^{\{k\} \times J}"] \arrow[d, "(\Lambda_{\mathfrak{C}}^{K \times J \to K})^{-1}"'] & \boxtimes^K \mathcal{C} \arrow[d, "\otimes^K"] \arrow[dl, Rightarrow, "(\lambda^{K \times J \to K})^{-1}"'] \\
\boxtimes^{K \times J} \mathcal{C} \arrow[r, "\otimes^{K \times J}"] \arrow[d, "\Lambda_{\mathfrak{C}}^{K \times J \to J \times K}"'] & \mathcal{C} \arrow[d, equal] \arrow[dl, Rightarrow, "\lambda^{K \times J \to J \times K}"'] \\
\boxtimes^{J \times K} \mathcal{C} \arrow[r, "\otimes^{J \times K}"] \arrow[d, "\Lambda_{\mathfrak{C}}^{J \times K \to J}"'] \arrow[dr, phantom, "\Downarrow \lambda^{J \times K \to J}"] & \mathcal{C} \\
\boxtimes^{j \in J} \boxtimes^{\{j\} \times K} \mathcal{C} \arrow[r, "\boxtimes^{j \in J} \otimes^{\{j\} \times K}"] \arrow[d, "\boxtimes^{j \in J} \Lambda_{\mathfrak{C}}^{\{j\} \times K \to K}"'] \arrow[dr, phantom, "\Downarrow \boxtimes^{j \in J} \lambda^{\{j\} \times K \to K}"] & \boxtimes^J \mathcal{C} \arrow[u, "\otimes^J"'] \\
\boxtimes^J \boxtimes^K \mathcal{C} \arrow[r, equal] & \boxtimes^J \boxtimes^K \mathcal{C} \arrow[u, "\boxtimes^J \otimes^K"']
\end{tikzcd}$$

where $(\lambda^{K \times J \to K})^{-1}$ and $(\lambda^{\{k\} \times J \to J})^{-1}$ are abbreviations for$(\Lambda_{\mathfrak{C}}^{K \times J \to K})^{-1} \bullet (\lambda^{K \times J \to K})^{-1}$ and $(\Lambda_{\mathfrak{C}}^{\{k\} \times J \to J})^{-1} \bullet (\lambda^{\{k\} \times J \to J})^{-1}$ respectively. It can be shown that $(\otimes^J, \tau^K_J) : \boxtimes^J \mathcal{C} \to \mathcal{C}$ is a lax-symmetric-Monoidal-functor and that $\lambda^f : \otimes^L \to \otimes^f \cdot \otimes^K : \boxtimes^L \mathcal{C} \to \mathcal{C}$ is a Monoidal-transformation. The proofs of these facts are omitted. $\square$

A.11 Category $\mathfrak{C}(\mathbb{1},\mathscr{D})$ is a Monoidal category. Let $(\mathfrak{C},\boxtimes^I,\Lambda^f)$ be a symmetric Monoidal $\mathcal{C}at$-category. The $\mathcal{C}at$-functor $\mathfrak{C}(\mathbb{1},-):\mathfrak{C}\to\mathcal{C}at$ admits the following structure of a symmetric Monoidal $\mathcal{C}at$-functor. Given a family $(\mathcal{D}_i)_{i\in I}$ of objects of $\mathfrak{C}$, denote

$$\gamma^I=\Bigl[\prod_{i\in I}\mathfrak{C}(\mathbb{1},\mathcal{D}_i)\xrightarrow{\boxtimes^I}\mathfrak{C}(\boxtimes^I\mathbb{1},\boxtimes^{i\in I}\mathcal{D}_i)\xrightarrow{\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to I},1)}\mathfrak{C}(\mathbb{1},\boxtimes^{i\in I}\mathcal{D}_i)\Bigr].$$

Clearly, γ^I is a natural transformation of $\mathcal{C}at$-functors.

A.12 Proposition. *$(\mathfrak{C}(\mathbb{1},-),\gamma^I)$ is a symmetric Monoidal $\mathcal{C}at$-functor.*

Proof. Let $f:I\to J$ be a map in $\mathcal{S}$. We must prove the equation:

$$\begin{tikzcd}
\prod_{i\in I}\mathfrak{C}(\mathbb{1},\mathcal{D}_i) \arrow[rr, "\gamma^I"] \arrow[d, "\wr"'] & & \mathfrak{C}(\mathbb{1},\boxtimes^{i\in I}\mathcal{D}_i) \arrow[d, "\mathfrak{C}(\mathbb{1},\Lambda^f_{\mathfrak{C}})"'] \\
\prod_{j\in J}\prod_{i\in f^{-1}j}\mathfrak{C}(\mathbb{1},\mathcal{D}_i) \arrow[r, "\prod_{j\in J}\gamma^{f^{-1}j}"] & \prod_{j\in J}\mathfrak{C}(\mathbb{1},\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[r, "\gamma^J"] & \mathfrak{C}(\mathbb{1},\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{D}_i)
\end{tikzcd}$$

It coincides with the exterior of the diagram

$$\begin{tikzcd}
\prod_{i\in I}\mathfrak{C}(\mathbb{1},\mathcal{D}_i) \arrow[r, "\boxtimes^I"] \arrow[ddd, "\wr"'] & \mathfrak{C}(\boxtimes^I\mathbb{1},\boxtimes^{i\in I}\mathcal{D}_i) \arrow[r, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to I},1)"] \arrow[d, "\mathfrak{C}(1,\Lambda^f_{\mathfrak{C}})"'] & \mathfrak{C}(\mathbb{1},\boxtimes^{i\in I}\mathcal{D}_i) \arrow[d, "\mathfrak{C}(1,\Lambda^f_{\mathfrak{C}})"] \\
 & \mathfrak{C}(\boxtimes^I\mathbb{1},\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[r, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to I},1)"] & \mathfrak{C}(\mathbb{1},\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \\
 & \mathfrak{C}(\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\mathbb{1},\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[r, "\mathfrak{C}(\boxtimes^{j\in J}\Lambda_{\mathfrak{C}}^{\varnothing\to f^{-1}j},1)"] \arrow[u, "\mathfrak{C}(\Lambda^f_{\mathfrak{C}},1)"] & \mathfrak{C}(\boxtimes^J\mathbb{1},\boxtimes^{j\in J}\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[u, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to J},1)"'] \\
\prod_{j\in J}\prod_{i\in f^{-1}j}\mathfrak{C}(\mathbb{1},\mathcal{D}_i) \arrow[r, "\prod_{j\in J}\boxtimes^{f^{-1}j}"] & \prod_{j\in J}\mathfrak{C}(\boxtimes^{f^{-1}j}\mathbb{1},\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[r, "\prod_{j\in J}\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to f^{-1}j},1)"] \arrow[u, "\boxtimes^J"] & \prod_{j\in J}\mathfrak{C}(\mathbb{1},\boxtimes^{i\in f^{-1}j}\mathcal{D}_i) \arrow[u, "\boxtimes^J"']
\end{tikzcd}$$

The commutativity of the hexagon is nothing else but the naturality of $\Lambda^f_{\mathfrak{C}}$. The bottom square commutes since $\boxtimes^J$ is a $\mathcal{C}at$-functor, commutativity of the middle square follows from equation (2.37.2) written for the pair of maps $\varnothing\to I\xrightarrow{f}J$. $\square$

As a consequence, the functor $(\mathfrak{C}(\mathbb{1},-),\gamma^I)$ gives rise to a symmetric Monoidal $\mathcal{C}at$-functor $(\mathfrak{C}(\mathbb{1},-),\gamma^I)^\star$: lax-sym-Mono-cat-$\mathfrak{C}$ $\to$ lax-sym-Mono-cat. In particular,

if $(\mathcal{D}, \otimes^I, \lambda^f)$ is a lax-symmetric-Monoidal-category in $\mathfrak{C}$, then the category $\mathfrak{C}(\mathbb{1}, \mathscr{D})$ equipped with functors

$$\boxtimes^I : \prod_{i \in I} \mathfrak{C}(\mathbb{1}, \mathscr{D}) \xrightarrow{\boxtimes^I} \mathfrak{C}(\boxtimes^I \mathbb{1}, \boxtimes^I \mathscr{D}) \xrightarrow{\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing \to I}, \otimes^I)} \mathfrak{C}(\mathbb{1}, \mathscr{D})$$

and with transformations $\nu^f = \boxtimes^I \cdot \mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing \to I}, \lambda^f)$, presented on the following page, is a lax symmetric Monoidal category.

$$\begin{tikzcd}
\prod_{j\in J}\mathfrak{C}(\mathbb{1},\mathscr{D})^{f^{-1}j} \arrow[r, "\prod_{j\in J}\boxtimes^{f^{-1}j}"] \arrow[dr, phantom, "="] & \prod_{j\in J}\mathfrak{C}(\boxtimes^{f^{-1}j}\mathbb{1},\boxtimes^{f^{-1}j}\mathscr{D}) \arrow[r, "\prod_{j\in J}\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to f^{-1}j},1)"] \arrow[d, "\boxtimes^J"'] \arrow[dr, phantom, "="] & \prod_{j\in J}\mathfrak{C}(\mathbb{1},\boxtimes^{f^{-1}j}\mathscr{D}) \arrow[r, "\prod_{j\in J}\mathfrak{C}(1,\otimes^{f^{-1}j})"] \arrow[d, "\boxtimes^J"'] \arrow[dr, phantom, "="] & \mathfrak{C}(\mathbb{1},\mathscr{D})^J \arrow[d, "\boxtimes^J"'] \\
 & \mathfrak{C}(\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\,\mathbb{1},\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\,\mathscr{D}) \arrow[r, "\mathfrak{C}(\boxtimes^{j\in J}\Lambda_{\mathfrak{C}}^{\varnothing\to f^{-1}j},1)"] \arrow[d, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{f},1)"'] \arrow[dr, phantom, "="] & \mathfrak{C}(\boxtimes^J\mathbb{1},\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\,\mathscr{D}) \arrow[r, "\mathfrak{C}(1,\boxtimes^{j\in J}\otimes^{f^{-1}j})"] \arrow[d, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to J},1)"'] \arrow[dr, phantom, "="] & \mathfrak{C}(\boxtimes^J\mathbb{1},\boxtimes^J\mathscr{D}) \arrow[d, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to J},1)"'] \\
 & \mathfrak{C}(\boxtimes^I\mathbb{1},\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\,\mathscr{D}) \arrow[r, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to I},1)"] \arrow[dr, phantom, "="] & \mathfrak{C}(\mathbb{1},\boxtimes^{j\in J}\boxtimes^{f^{-1}j}\,\mathscr{D}) \arrow[r, "\mathfrak{C}(1,\boxtimes^{j\in J}\otimes^{f^{-1}j})"] \arrow[dr, phantom, "\Uparrow\,\mathfrak{C}(1,\lambda^f)"] & \mathfrak{C}(\mathbb{1},\boxtimes^J\mathscr{D}) \arrow[d, "\mathfrak{C}(1,\otimes^J)"'] \\
\mathfrak{C}(\mathbb{1},\mathscr{D})^I \arrow[uuu, "\sim"] \arrow[r, "\boxtimes^I"] & \mathfrak{C}(\boxtimes^I\mathbb{1},\boxtimes^I\mathscr{D}) \arrow[r, "\mathfrak{C}(\Lambda_{\mathfrak{C}}^{\varnothing\to I},1)"] \arrow[u, "\mathfrak{C}(1,\Lambda_{\mathfrak{C}}^{f})"] & \mathfrak{C}(\mathbb{1},\boxtimes^I\mathscr{D}) \arrow[r, "\mathfrak{C}(1,\otimes^I)"] \arrow[u, "\mathfrak{C}(1,\Lambda_{\mathfrak{C}}^{f})"] & \mathfrak{C}(\mathbb{1},\mathscr{D})
\end{tikzcd}$$

Appendix B

Actions of categories

We define an action of a Monoidal-category $\mathscr{D}$ on an object $\mathscr{A}$ of a Monoidal $\mathcal{C}at$-category $\mathfrak{C}$. The definition is obtained by internalizing the notion of action of a Monoidal category on a category. The examples of interest are actions of the category of graded $\Bbbk$-linear categories by tensoring on the categories of graded quivers $\mathscr{Q}_p$ and $\mathscr{Q}_u$. We conclude by considering the following question: given a symmetric multicategory C with a symmetric multicomonad T on it, when does an action of a Monoidal-category D in the symmetric Monoidal $\mathcal{C}at$-category of multicategories on the symmetric multicategory C lift to an action on the Kleisli multicategory C^T? We describe the additional datum, an intertwiner of the action and the multicomonad, which ensures the existence of a lifting.

B.1 Actions of lax-Monoidal-categories. We define actions of Monoidal–categories on objects of a symmetric Monoidal $\mathcal{C}at$-category in a style similar to the definition of a Monoidal-category.

B.2 Definition. Let $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ be a Monoidal $\mathcal{C}at$-category. Let $\mathscr{A} \in \operatorname{Ob}\mathfrak{C}$ and let $(\mathscr{D}, \otimes^I, \lambda^f_{\mathsf{D}})$ be a Monoidal-category of $\mathfrak{C}$. A *right action* of $\mathscr{D}$ on $\mathscr{A}$ is

1. A 1-morphism $\boxdot^{[I]} : \boxtimes^{[I]}(\mathscr{A}, (\mathscr{D})_I) \to \mathscr{A}$, for each $I \in \operatorname{Ob}\mathcal{O}$, such that $\boxdot^{[0]} = \mathrm{Id}_{\mathscr{A}}$.

   For each isotonic map $f : [I] \to [J]$ such that $f(0) = 0$, introduce a 1-morphism

$$\boxdot^f = \bigl[\boxtimes^{[I]}(\mathscr{A}, (\mathscr{D})_I) \xrightarrow{\Lambda^f} \boxtimes^{[J]}\left(\boxtimes^{f^{-1}(0)}(\mathscr{A}, \mathscr{D}, \dots, \mathscr{D}), (\boxtimes^{f^{-1}(j)}\mathscr{D})_{j\in J}\right)$$
$$\xrightarrow{\boxtimes^{[J]}(\boxdot^{f^{-1}(0)}, (\otimes^{f^{-1}(j)})_{j\in J})} \boxtimes^{[J]}\left(\mathscr{A}, (\mathscr{D})_J\right)\bigr].$$

2. A 2-isomorphism $\lambda^f_{\cdot}$ for every isotonic map $f : [I] \to [J]$ such that $f(0) = 0$:

$$\begin{tikzcd}
\boxtimes^{[I]}(\mathscr{A}, (\mathscr{D})_I) \arrow[r, "\boxdot^f"] \arrow[dr, "\boxdot^{[I]}"'] \arrow[r, phantom, shift right=4ex, "\Uparrow \lambda^f_{\cdot}"'] & \boxtimes^{[J]}(\mathscr{A}, (\mathscr{D})_J) \arrow[d, "\boxdot^{[J]}"] \\
 & \mathscr{A}
\end{tikzcd}$$

such that

$$\lambda^{[I]\to[0]}_{\cdot} = \mathrm{id}, \qquad \lambda^{\mathrm{id}_{[I]}}_{\cdot} = \mathrm{id},$$

and for any pair of composable isotonic maps $[I] \xrightarrow{f} [J] \xrightarrow{g} [K]$ such that $f(0) = 0$, $g(0) = 0$, an equation holds:

$$
\begin{tikzcd}
\boxtimes^{[J]}(\mathscr{A},(\mathscr{D})_J) \arrow[r, "\boxdot^g"] \arrow[rd, "\boxdot^{[J]}" description] \arrow[rd, phantom, shift right=4ex, "\lambda^f_{\cdot}\ \Uparrow"] & \boxtimes^{[K]}(\mathscr{A},(\mathscr{D})_K) \arrow[d, "\boxdot^{[K]}"] \arrow[ld, phantom, shift right=4ex, "\lambda^g_{\cdot}\ \Uparrow"] \\
\boxtimes^{[I]}(\mathscr{A},(\mathscr{D})_I) \arrow[u, "\boxdot^f"] \arrow[r, "\boxdot^{[I]}"'] & \mathscr{A}
\end{tikzcd}
=
\begin{tikzcd}
\boxtimes^{[J]}(\mathscr{A},(\mathscr{D})_J) \arrow[r, "\boxdot^g"] \arrow[rd, phantom, shift left=4ex, "\boxtimes^{k\in[K]}\lambda^{f_k}\ \Uparrow"] & \boxtimes^{[K]}(\mathscr{A},(\mathscr{D})_K) \arrow[d, "\boxdot^{[K]}"] \arrow[ld, phantom, shift left=4ex, "\lambda^{g\circ f}_{\cdot}\ \Uparrow"] \\
\boxtimes^{[I]}(\mathscr{A},(\mathscr{D})_I) \arrow[u, "\boxdot^f"] \arrow[r, "\boxdot^{[I]}"'] \arrow[ru, "\boxdot^{g\circ f}" description] & \mathscr{A}
\end{tikzcd}
\tag{B.2.1}
$$

where $f_k = f : f^{-1}g^{-1}k \to g^{-1}k$.

Here $\reflectbox{$\lambda$}^{(f,g)} \overset{\text{def}}{=} \boxtimes^{k\in[K]}\lambda^{f_k} : \boxdot^{f\cdot g} \to \boxdot^f \cdot \boxdot^g$ denotes the 2-morphism

$$
\begin{tikzcd}[column sep=small]
& \boxtimes^{[J]}(\mathscr{A},(\mathscr{D})_J) \arrow[rd, "\Lambda^g_{\mathfrak{C}}"] \arrow[d, phantom, "="] & \\
\boxtimes^{j\in[J]}\boxtimes^{i\in f^{-1}j}\mathcal{C}_i \arrow[ru, "\boxtimes^{j\in[J]}\odot^{f^{-1}(j)}"] \arrow[r, "\Lambda^g_{\mathfrak{C}}"] \arrow[rd, phantom, "="] & \boxtimes^{k\in[K]}\boxtimes^{j\in g^{-1}k}\boxtimes^{i\in f^{-1}j}\mathcal{C}_i \arrow[r, "\boxtimes^{k\in[K]}\boxtimes^{j\in g^{-1}k}\odot^{f^{-1}j}"] \arrow[rd, phantom, "\Uparrow \boxtimes^{k\in[K]}\lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}"] & \boxtimes^{k\in[K]}\boxtimes^{j\in g^{-1}k}\mathcal{C}_j \arrow[d, "\boxtimes^{k\in[K]}\odot^{g^{-1}k}"] \\
\boxtimes^{[I]}(\mathscr{A},(\mathscr{D})_I) \arrow[u, "\Lambda^f_{\mathfrak{C}}"] \arrow[r, "\Lambda^{g\circ f}_{\mathfrak{C}}"'] & \boxtimes^{k\in[K]}\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}_i \arrow[u, "\boxtimes^{k\in[K]}\Lambda^{f:f^{-1}g^{-1}k\to g^{-1}k}_{\mathfrak{C}}"] \arrow[r, "\boxtimes^{k\in[K]}\odot^{f^{-1}g^{-1}k}"'] & \boxtimes^{[K]}(\mathscr{A},(\mathscr{D})_K)
\end{tikzcd}
$$

where $\mathcal{C}_0 = \mathscr{A}$, $\mathcal{C}_i = \mathscr{D}$ for $i > 0$. In this diagram $\odot^{f^{-1}j}$ means $\boxdot^{f^{-1}j}$ if $j = 0$, and $\otimes^{f^{-1}j}$ if $j > 0$. The 2-morphism λ means either $\lambda_{\cdot}$ or $\lambda_{\mathscr{D}}$. The top quadrilateral in above diagram is the identity 2-morphism due to the 2-transformation $\Lambda^g_{\mathfrak{C}}$ being strict. The left square is equation (2.37.2). Notice that $\reflectbox{$\lambda$}^{(f,\triangleright)} = \lambda^f$.

B.3 Proposition. *Let $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$ be a Monoidal $\mathcal{C}at$-category. Let Monoidal-category $(\mathscr{D}, \otimes^I, \lambda^f_{\mathcal{D}})$ of $\mathfrak{C}$ act on object $\mathscr{A}$ of $\mathfrak{C}$. Assume that non-decreasing mappings (from $\mathcal{O}$) $[I] \xrightarrow{f} [J] \xrightarrow{g} [K] \xrightarrow{h} [L]$ map 0 to 0. Then the following associativity equation holds:*

$$
\begin{tikzcd}
\boxtimes^{[J]}(\mathscr{A},(\mathscr{D})_J) \arrow[r, "\boxdot^g"] \arrow[rd, "\boxdot^{gh}" description] \arrow[rd, phantom, shift right=4ex, "\reflectbox{λ}^{(f,gh)}\ \Uparrow"] & \boxtimes^{[K]}(\mathscr{A},(\mathscr{D})_K) \arrow[d, "\boxdot^h"] \arrow[ld, phantom, shift right=4ex, "\reflectbox{λ}^{(g,h)}\ \Uparrow"] \\
\boxtimes^{[I]}(\mathscr{A},(\mathscr{D})_I) \arrow[u, "\boxdot^f"] \arrow[r, "\boxdot^{fgh}"'] & \boxtimes^{[L]}(\mathscr{A},(\mathscr{D})_L)
\end{tikzcd}
=
\begin{tikzcd}
\boxtimes^{[J]}(\mathscr{A},(\mathscr{D})_J) \arrow[r, "\boxdot^g"] \arrow[rd, phantom, shift left=4ex, "\reflectbox{λ}^{(f,g)}\ \Uparrow"] & \boxtimes^{[K]}(\mathscr{A},(\mathscr{D})_K) \arrow[d, "\boxdot^h"] \arrow[ld, phantom, shift left=4ex, "\reflectbox{λ}^{(fg,h)}\ \Uparrow"] \\
\boxtimes^{[I]}(\mathscr{A},(\mathscr{D})_I) \arrow[u, "\boxdot^f"] \arrow[r, "\boxdot^{fgh}"'] \arrow[ru, "\boxdot^{fg}" description] & \boxtimes^{[L]}(\mathscr{A},(\mathscr{D})_L)
\end{tikzcd}
$$

Proof is similar to the proof of Proposition 2.14.

For every element $[f(0)] \xrightarrow{f_1} [f(1)] \xrightarrow{f_2} \dots \xrightarrow{f_n} [f(n)]$ of height $n > 2$ in $\mathsf{N}_n\mathcal{O}$ such that $f_i(0) = 0$, we define a 1-morphism

$$\begin{tikzcd} \boxtimes^{[f(1)]}(\mathscr{A}, (\mathscr{D})_{f(1)}) \arrow[r, "\boxdot^{f_2}"] & \dots \arrow[r, "\boxdot^{f_{n-1}}"] & \boxtimes^{[f(n-1)]}(\mathscr{A}, (\mathscr{D})_{f(n-1)}) \arrow[d, "\boxdot^{f_n}"] \\ \boxtimes^{[f(0)]}(\mathscr{A}, (\mathscr{D})_{f(0)}) \arrow[u, "\boxdot^{f_1}"] \arrow[rr, "\boxdot^{f_1 \cdot \dots \cdot f_n}"'] \arrow[ur, phantom, "\Uparrow \curlywedge^{(f_i)_{i\in\mathbf{n}}}"] & & \boxtimes^{[f(n)]}(\mathscr{A}, (\mathscr{D})_{f(n)}) \end{tikzcd}$$

as the composition of $(n-1)$ morphisms $\curlywedge^{(-,-)}$ corresponding to any triangulation of the above $(n+1)$-gon, in particular as the composition of $\curlywedge^{(f_1,f_2)}, \curlywedge^{(f_1 f_2, f_3)}, \dots, \curlywedge^{(f_1 \cdots f_{n-1}, f_n)}$. According to Proposition B.3 the result is independent of triangulation. For $n = 2$ transformation $\curlywedge^{(f_1,f_2)}$ was defined above. We define $\curlywedge$ also for element $[I] \xrightarrow{f} [J]$ of height 1 of the nerve as $\curlywedge^{(f)} = \mathrm{id} : \boxdot^f_{\mathcal{C}} \to \boxdot^f_{\mathcal{C}}$, and for element $[I]$ of height 0 in $\mathsf{N}_0\mathcal{O}$ as $\curlywedge^{()} = \mathrm{id} : \boxdot^{\mathrm{id}_I}_{\mathcal{C}} \to \mathrm{Id}$. The independence of triangulation implies the following

B.4 Proposition. *Let $\phi : I \to J$ be an isotonic map, and let $(f_i)_{i \in I} \in \mathsf{N}_I\mathcal{O}$. Then*

$$\curlywedge^{(f_i)_{i\in I}} = \Big[\boxdot^{\bullet^{i\in I} f_i} = \boxdot^{\bullet^{j\in J}(\bullet^{i\in\phi^{-1}j} f_i)} \xrightarrow{\curlywedge^{(\bullet^{i\in\phi^{-1}j} f_i)_{j\in J}}} \bullet^{j\in J} \boxdot^{\bullet^{i\in\phi^{-1}j} f_i}$$
$$\xrightarrow{\bullet^{j\in J} \curlywedge^{(f_i)_{i\in\phi^{-1}j}}} \bullet^{j\in J} \bullet^{i\in\phi^{-1}j} \boxdot^{f_i} = \bullet^{i\in I} \boxdot^{f_i}\Big].$$

Proof. If in equation (B.2.1) the map $f = \mathrm{id}$, then both sides of this equation are equal to λ^g and $\curlywedge^{(\mathrm{id},g)} = \mathrm{id}$. If in the same equation the map $g = \mathrm{id}$, then both sides of this equation are equal to λ^f and $\curlywedge^{(f,\mathrm{id})} = \mathrm{id}$. Consequently, if in assumptions of Proposition B.3 one of maps equals the identity map, then both sides of the equation considered there are equal to $\curlywedge$. Namely, $f = \mathrm{id}$ implies that both sides equal $\curlywedge^{(g,h)}$, $g = \mathrm{id}$ implies that both sides equal $\curlywedge^{(f,h)}$, and $h = \mathrm{id}$ implies that both sides equal $\curlywedge^{(f,g)}$. The rest of the proof uses the independence of triangulation analogously to the proof of Proposition 2.15. $\square$

B.5 Action of graded categories on graded quivers. As an example of a symmetric Monoidal $\mathcal{C}at$-category we take $(\mathfrak{C}, \times^I, \Lambda^f) = $ sym-Mono-$\mathcal{C}at$. Each symmetric Monoidal category $\mathscr{D}$ defines a symmetric–Monoidal–category in sym-Mono-$\mathcal{C}at$. In particular, this holds for $\mathscr{D} = (\mathbf{gr}\text{-}\mathcal{C}at, \boxtimes^I, \lambda^f)$, the category of graded $\Bbbk$-linear categories, $\mathbf{gr} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$. We claim that $\mathscr{D}$ acts on the symmetric Monoidal categories of graded quivers $\mathscr{A} = \mathscr{Q}_p = (\mathscr{Q}, \boxtimes^I, \lambda^f)$ and $\mathscr{A} = \mathscr{Q}_u = (\mathscr{Q}, \boxtimes^I_u, \lambda^f_u)$ (p stands for plain, and u for unital). Particular cases of action (for $I = 1$) are lax Monoidal functors

$$\boxdot = (\boxtimes, \tilde{\zeta}^I) : \mathscr{Q}_p \times \mathbf{gr}\text{-}\mathcal{C}at \to \mathscr{Q}_p,$$
$$\boxdot = (\boxtimes, \zeta^I) : \mathscr{Q}_u \times \mathbf{gr}\text{-}\mathcal{C}at \to \mathscr{Q}_u,$$

where natural transformations are given by the following expressions:

$$\tilde{\zeta}^I = \sigma_{(12)} : \boxtimes^{i\in I}(\mathcal{A}_i \boxtimes \mathcal{C}_i) \to (\boxtimes^{i\in I}\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I}\mathcal{C}_i),$$

for graded quivers $\mathcal{A}_i$ and graded categories $\mathcal{C}_i$, and

$$\begin{aligned}\zeta^I = \Big[\boxtimes_u^{i\in I}(\mathcal{A}_i\boxtimes\mathcal{C}_i) = \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}T^{\chi(i\in S)}(\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow[\sim]{\oplus\boxtimes^I\overline{\varkappa}} \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}(T^{\chi(i\in S)}\mathcal{A}_i\boxtimes T^{\chi(i\in S)}\mathcal{C}_i) \\ \xrightarrow[\sim]{\oplus\sigma_{(12)}} \bigoplus_{\varnothing\neq S\subset I}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i) \xrightarrow{\oplus 1\boxtimes(\boxtimes^I\mu)} \bigoplus_{\varnothing\neq S\subset I}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}\mathcal{C}_i) \\ = (\boxtimes_u^{i\in I}\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}\mathcal{C}_i)\Big],\end{aligned}$$

where the composition μ is the identity morphism $\mu = \mathrm{id} : T^1\mathcal{C}_i \to \mathcal{C}_i$, or the unit $\mu = \eta : T^0\mathcal{C}_i \to \mathcal{C}_i$.

Let us describe the constructed action $\boxdot^{[J]} : \prod_{[J]}(\mathscr{A}, (\mathscr{D})_J) \to \mathscr{A}$ in general. As a functor it coincides with $\boxtimes^{[J]}$ and assigns to a graded quiver $\mathcal{A}$ and graded categories $\mathcal{C}^j$, $j \in J$, the quiver $\boxdot^{[J]}(\mathcal{A}, (\mathcal{C}^j)_{j\in J}) = \boxtimes^{[J]}(\mathcal{A}, (\mathcal{C}^j)_{j\in J})$. Thus, $\boxdot^{[J]} = (\boxtimes^{[J]}, \tilde{\zeta}^I)$ and $\boxdot^{[J]} = (\boxtimes^{[J]}, \zeta^I)$ in plain (resp. unital) case, where natural transformations are

$$\tilde{\zeta}^I = \sigma_{(12)} : \boxtimes^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \to \boxtimes^{[J]}(\boxtimes^{i\in I}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}),$$

for graded quivers $\mathcal{A}_i$ and graded categories $\mathcal{C}_i^j$, and

$$\begin{aligned}\zeta^I = \Big[\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) &= \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}T^{\chi(i\in S)}\boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \\ &\xrightarrow[\sim]{\oplus\boxtimes^I\overline{\varkappa}} \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{i\in I}\boxtimes^{[J]}(T^{\chi(i\in S)}\mathcal{A}_i, (T^{\chi(i\in S)}\mathcal{C}_i^j)_{j\in J}) \\ &\xrightarrow[\sim]{\oplus\sigma_{(12)}} \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{[J]}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i, (\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i^j)_{j\in J}) \\ &\xrightarrow{\oplus\boxtimes^{[J]}(1,(\boxtimes^I\mu)_J)} \bigoplus_{\varnothing\neq S\subset I}\boxtimes^{[J]}(\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) = \boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\Big].\end{aligned}$$

B.6 Lemma. *The morphisms ζ^I and $\tilde{\zeta}^I$ are related by the following embeddings:*

$$\begin{tikzcd}
\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[rr, "\zeta^I"] \arrow[d, hook, "\mathrm{in}"'] \arrow[drr, phantom, "="] & & \boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[d, hook, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] \\
\boxtimes^{i\in I}T^{\leqslant 1}\boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes^I\xi'"] & \boxtimes^{i\in I}\boxtimes^{[J]}(T^{\leqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\tilde{\zeta}^I"] & \boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})
\end{tikzcd}$$

where

$$\xi' = \bigl[T^{\leqslant 1} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\overline{\varkappa}} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}, (T^{\leqslant 1}\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu)_J)} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J})\bigr]$$

has restrictions $\xi'\big|_{\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J})} = \boxtimes^{[J]}(\mathrm{in}_1, (1)_J)$ *and*

$$\xi'\big|_{T^0\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J})} = \bigl[T^0 \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\overline{\varkappa}} \boxtimes^{[J]} (T^0\mathcal{A}, (T^0\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(\mathrm{in}_0,(\eta)_J)} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J})\bigr].$$

Proof. This is the exterior of the following diagram:

$$\begin{tikzcd}
 & \boxtimes_u^{i\in I} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[dl, hook', "\mathrm{in}"'] \arrow[dr, "(\boxtimes^I\overline{\varkappa})\sigma_{(12)}"] & \\
\boxtimes^{i\in I}T^{\leqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[dr, "(\boxtimes^I\overline{\varkappa})\sigma_{(12)}"] \arrow[dd, "\boxtimes^I\overline{\varkappa}"'] & & \bigoplus_{\varnothing\neq S\subset I} \boxtimes^{[J]} (\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i, (\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i^j)_{j\in J}) \arrow[dl, hook'] \arrow[ddd, "\boxtimes^{[J]}(1,(\boxtimes^I\mu)_J)"] \\
 & \bigoplus_{S\subset I} \boxtimes^{[J]} (\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{A}_i, (\boxtimes^{i\in I}T^{\chi(i\in S)}\mathcal{C}_i^j)_{j\in J}) \arrow[ddd, "\boxtimes^{[J]}(1,(\boxtimes^I\mu)_J)"] & \\
\boxtimes^{i\in I} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}_i, (T^{\leqslant 1}\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^I\boxtimes^{[J]}(1,(\mu)_J)"'] & & \\
\boxtimes^{i\in I} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[dr, "\sigma_{(12)}"'] & & \boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[dl, hook', "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] \\
 & \boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant 1}\mathcal{A}_i, (\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) &
\end{tikzcd}$$

in which all cells commute. □

B.7 Proposition. *The both pairs*

$$(\boxtimes^{[J]}, \tilde{\zeta}^I) : \prod_{[J]}(\mathscr{Q}_p, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_p, \qquad (\boxtimes^{[J]}, \zeta^I) : \prod_{[J]}(\mathscr{Q}_u, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_u, \quad \text{(B.7.1)}$$

are lax symmetric Monoidal functors.

Proof. Lax Monoidality of $(\boxtimes^{[J]}, \tilde{\zeta}^I)$ expressed by equation (2.17.2) for a map $f : I \to K$

of $\mathcal{S}$ takes the form

$$
\begin{aligned}
&\bigl[\boxtimes^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\lambda^f} \boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\boxtimes^{k\in K}\tilde\zeta^{f^{-1}k}} \\
&\boxtimes^{k\in K}\boxtimes^{[J]}(\boxtimes^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\tilde\zeta^K} \boxtimes^{[J]}(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J})\bigr] \\
&= \bigl[\boxtimes^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\tilde\zeta^I} \boxtimes^{[J]}(\boxtimes^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \\
&\qquad \xrightarrow{\boxtimes^{[J]}(\lambda^f,(\lambda^f)_J)} \boxtimes^{[J]}(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J})\bigr].
\end{aligned}
\tag{B.7.2}
$$

Since $\tilde\zeta = \sigma_{(12)}$ can be expressed in terms of λ, the equation follows from coherence principle of Remark 2.34. Therefore, $(\boxtimes^{[J]},\tilde\zeta^I)$ is a lax symmetric Monoidal functor.

To be a lax symmetric Monoidal functor $(\boxtimes^{[J]},\zeta^I)$ has to satisfy the following equation:

$$
\begin{aligned}
&\bigl[\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\lambda_u^f} \boxtimes_u^{k\in K}\boxtimes_u^{i\in f^{-1}k}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\boxtimes_u^{k\in K}\zeta^{f^{-1}k}} \\
&\boxtimes_u^{k\in K}\boxtimes^{[J]}(\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\zeta^K} \boxtimes^{[J]}(\boxtimes_u^{k\in K}\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J})\bigr] \\
&= \bigl[\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\zeta^I} \boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \\
&\qquad \xrightarrow{\boxtimes^{[J]}(\lambda_u^f,(\lambda^f)_J)} \boxtimes^{[J]}(\boxtimes_u^{k\in K}\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i,(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J})\bigr].
\end{aligned}
\tag{B.7.3}
$$

Its right hand side is related to the right hand side of (B.7.2) by the following maps:

$$
\begin{tikzcd}
& \boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[dddd, hook, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] \arrow[dr, "\boxtimes^{[J]}(\lambda_u^f,(\lambda^f)_J)"] & \\
\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[ur, "\zeta^I"] \arrow[d, hook, "\mathrm{in}"'] & & \boxtimes^{[J]}(\boxtimes_u^{k\in K}\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i, (\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \arrow[d, hook, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"'] \\
\boxtimes^{i\in I}T^{\leqslant1}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^I\xi'"'] \arrow[r, phantom, "="] & \arrow[r, phantom, "="] & \boxtimes^{[J]}(\boxtimes^{k\in K}T^{\leqslant1}\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i, (\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{[J]}(\boxtimes^{k\in K}(\vartheta^{f^{-1}k})^{-1},(1)_J)"'] \\
\boxtimes^{i\in I}\boxtimes^{[J]}(T^{\leqslant1}\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[dr, "\tilde\zeta^I"'] & & \boxtimes^{[J]}(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}T^{\leqslant1}\mathcal{A}_i, (\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \\
& \boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[ur, "\boxtimes^{[J]}\lambda^f"'] &
\end{tikzcd}
\tag{B.7.4}
$$

Indeed, the left pentagon is proven in Lemma B.6, and the right pentagon follows from commutative diagram (10.7.3).

The left hand side of (B.7.3) is related to the left hand side of (B.7.2) by the same maps as for the right hand sides. Indeed, for any map $f : I \to K$ the diagram on the next page is commutative by Lemma B.6, diagram (10.7.3) and due to the following equation, which holds for an arbitrary subset $L \subset I$,

$$
\begin{tikzcd}
T^{\leqslant 1} \boxtimes_u^{i\in L} \boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "T^{\leqslant 1}\zeta^L"] \arrow[d, "(\vartheta^L)^{-1}"', "\wr"] & T^{\leqslant 1} \boxtimes^{[J]} (\boxtimes_u^{i\in L}\mathcal{A}_i, (\boxtimes^{i\in L}\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\xi'"] \\
\boxtimes^{i\in L} T^{\leqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{i\in L}\xi'"'] & \boxtimes^{[J]}(T^{\leqslant 1} \boxtimes_u^{i\in L} \mathcal{A}_i, (\boxtimes^{i\in L}\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{[J]}((\vartheta^L)^{-1},(1)_J)"] \\
\boxtimes^{i\in L} \boxtimes^{[J]} (T^{\leqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\tilde{\zeta}^L"] & \boxtimes^{[J]}(\boxtimes^{i\in L} T^{\leqslant 1}\mathcal{A}_i, (\boxtimes^{i\in L}\mathcal{C}_i^j)_{j\in J})
\end{tikzcd}
$$

in particular, for $L = f^{-1}k$, $k \in K$. Indeed, on the summand $\boxtimes_u^{i\in L} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$ of the left top corner the equation reduces to the statement of Lemma B.6. On the summand $T^0 \boxtimes_u^{i\in L} \boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$ the above equation expands to exterior of the diagram

$$
\begin{tikzcd}
T^0 \boxtimes^{i\in L} \boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "T^0\sigma_{(12)}"] \arrow[d, "\overline{\varkappa}"'] & T^0 \boxtimes^{[J]} (\boxtimes^{i\in L}\mathcal{A}_i, (\boxtimes^{i\in L}\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\overline{\varkappa}"] \\
\boxtimes^{i\in L} T^0 \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^L\overline{\varkappa}"'] & \boxtimes^{[J]}(T^0 \boxtimes^{i\in L} \mathcal{A}_i, (T^0 \boxtimes^{i\in L} \mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{[J]}(\overline{\varkappa},(\overline{\varkappa})_J)"] \\
\boxtimes^{i\in L} \boxtimes^{[J]} (T^0\mathcal{A}_i, (T^0\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\sigma_{(12)}"] \arrow[d, "\boxtimes^L\boxtimes^{[J]}(1,(\eta)_J)"'] & \boxtimes^{[J]}(\boxtimes^{i\in L} T^0\mathcal{A}_i, (\boxtimes^{i\in L} T^0\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{[J]}(1,(\boxtimes^L\eta)_J)"] \\
\boxtimes^{i\in L} \boxtimes^{[J]} (T^0\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\sigma_{(12)}"] & \boxtimes^{[J]}(\boxtimes^{i\in L} T^0\mathcal{A}_i, (\boxtimes^{i\in L}\mathcal{C}_i^j)_{j\in J})
\end{tikzcd}
$$

It is commutative due to coherence principle.

This proves equation (B.7.3) and the fact that $(\boxtimes, \zeta^I)$ is a lax symmetric Monoidal functor. □

B.8 Proposition. *Both lax symmetric Monoidal functors* (B.7.1)

$$\boxdot^{[J]} = (\boxtimes^{[J]}, \tilde{\zeta}^I) : \prod_{[J]} (\mathscr{Q}_p, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_p, \ \boxdot^{[J]} = (\boxtimes^{[J]}, \zeta^I) : \prod_{[J]} (\mathscr{Q}_u, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_u,$$

equipped with the 2-isomorphism $\lambda^f_. = \lambda^f : \boxdot^{[I]} \to \boxdot^f \boxdot^{[J]} : \prod_{[I]}(\mathscr{Q}, (\mathbf{gr}\text{-}\mathcal{C}at)_I) \to \mathscr{Q}$ *for each isotonic map* $f : [I] \to [J]$ *such that* $f(0) = 0$, *define an action of* $\mathbf{gr}$-$\mathcal{C}at$ *on* $\mathscr{Q}_p$ *and on* $\mathscr{Q}_u$.

$$
\begin{tikzcd}
\boxtimes_u^{i\in I} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\lambda_u^f"] \arrow[dd, hook, "\mathrm{in}"'] & \boxtimes_u^{k\in K} \boxtimes_u^{i\in f^{-1}k} \boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes_u^{k\in K}\zeta^{f^{-1}k}"] \arrow[d, hook, "\mathrm{in}"'] & \begin{matrix} \boxtimes_u^{k\in K} \boxtimes^{[J]} (\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i, \\ (\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[r, "\zeta^K"] \arrow[d, hook, "\mathrm{in}"'] & \begin{matrix} \boxtimes^{[J]}(\boxtimes_u^{k\in K} \boxtimes_u^{i\in f^{-1}k} \mathcal{A}_i, \\ (\boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[dd, hook, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] \\
 & \boxtimes^{k\in K}\boldsymbol{T}^{\leqslant 1} \boxtimes_u^{i\in f^{-1}k} \boxtimes^{[J]}(\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes^{k\in K}\boldsymbol{T}^{\leqslant 1}\zeta^{f^{-1}k}"] \arrow[d, "\boxtimes^{k\in K}(\vartheta^{f^{-1}k})^{-1}"', "\wr"] & \begin{matrix} \boxtimes^{k\in K}\boldsymbol{T}^{\leqslant 1} \boxtimes^{[J]} (\boxtimes_u^{i\in f^{-1}k}\mathcal{A}_i, \\ (\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[d, "\boxtimes^K\xi'"'] & \\
\boxtimes^{i\in I}\boldsymbol{T}^{\leqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\lambda^f"] \arrow[d, "\boxtimes^I\xi'"'] & \boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \boldsymbol{T}^{\leqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[d, "\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}k}\xi'"'] & \begin{matrix} \boxtimes^{k\in K} \boxtimes^{[J]} (\boldsymbol{T}^{\leqslant 1} \boxtimes_u^{i\in f^{-1}k} \mathcal{A}_i, \\ (\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[r, "\tilde{\zeta}^K"] \arrow[d, "\boxtimes^{k\in K}\boxtimes^{[J]}((\vartheta^{f^{-1}k})^{-1},(1)_J)"'] & \begin{matrix} \boxtimes^{[J]}(\boxtimes^{k\in K}\boldsymbol{T}^{\leqslant 1} \boxtimes_u^{i\in f^{-1}k} \mathcal{A}_i, \\ (\boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[d, "\boxtimes^{[J]}(\boxtimes^{k\in K}(\vartheta^{f^{-1}k})^{-1},(1)_J)"'] \\
\boxtimes^{i\in I} \boxtimes^{[J]} (\boldsymbol{T}^{\leqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\lambda^f"] & \boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \boxtimes^{[J]}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes^{k\in K}\tilde{\zeta}^{f^{-1}k}"] & \begin{matrix} \boxtimes^{k\in K} \boxtimes^{[J]} (\boxtimes^{i\in f^{-1}k}\boldsymbol{T}^{\leqslant 1}\mathcal{A}_i, \\ (\boxtimes^{i\in f^{-1}k}\mathcal{C}_i^j)_{j\in J}) \end{matrix} \arrow[r, "\tilde{\zeta}^K"] & \begin{matrix} \boxtimes^{[J]}(\boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \boldsymbol{T}^{\leqslant 1}\mathcal{A}_i, \\ (\boxtimes^{k\in K} \boxtimes^{i\in f^{-1}k} \mathcal{C}_i^j)_{j\in J}) \end{matrix}
\end{tikzcd}
$$

Proof. We have to prove that $\lambda^f_\cdot$ is a Monoidal transformation for each isotonic map $f:[I]\to[J]$ such that $f(0)=0$. First we show this in plain case for $\boxdot^{[J]}=(\boxtimes^{[J]},\tilde{\zeta}^I)$. In fact, let $\mathcal{C}^0_k$ be graded $\Bbbk$-linear quivers, $k\in K\in\operatorname{Ob}\mathcal{O}$, and let $\mathcal{C}^i_k$ be graded $\Bbbk$-linear categories, $i\in I$, $k\in K$. Then the diagram

$$
\begin{tikzcd}
\boxtimes^{k\in K}\boxtimes^{i\in[I]}\mathcal{C}^i_k \arrow[rr, "\tilde{\zeta}^K", "\sigma_{(12)}"'] \arrow[d, "\boxtimes^K\lambda^f"] & & \boxtimes^{i\in[I]}\boxtimes^{k\in K}\mathcal{C}^i_k \arrow[d, "\lambda^f"'] \\
\boxtimes^{k\in K}\boxtimes^{j\in[J]}\boxtimes^{i\in f^{-1}j}\mathcal{C}^i_k \arrow[r, "\tilde{\zeta}^K", "\sigma_{(12)}"'] & \boxtimes^{j\in[J]}\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}j}\mathcal{C}^i_k \arrow[r, "\boxtimes^{[J]}\tilde{\zeta}^K", "\boxtimes^{[J]}\sigma_{(12)}"'] & \boxtimes^{j\in[J]}\boxtimes^{i\in f^{-1}j}\boxtimes^{k\in K}\mathcal{C}^i_k
\end{tikzcd}
\tag{B.8.1}
$$

commutes due to coherence principle.

In the unital case for $\boxdot^{[J]}=(\boxtimes^{[J]},\zeta^I)$ denote graded $\Bbbk$-linear quivers by $\mathcal{A}_k$, $k\in K$, and let $\mathcal{C}^i_k$ be graded $\Bbbk$-linear categories, $i\in I$, $k\in K$. We have to prove the equation

$$
\begin{aligned}
&\big[\boxtimes^{k\in K}_u\boxtimes^{[I]}(\mathcal{A}_k,(\mathcal{C}^i_k)_{i\in I})\xrightarrow{\zeta^K}\boxtimes^{[I]}(\boxtimes^{k\in K}_u\mathcal{A}_k,(\boxtimes^{k\in K}\mathcal{C}^i_k)_{i\in I})\\
&\qquad\xrightarrow{\lambda^f}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boxtimes^{k\in K}_u\mathcal{A}_k,(\boxtimes^{k\in K}\mathcal{C}^i_k)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\boxtimes^{k\in K}\mathcal{C}^i_k)_{j\in J}]\big]\\
&=\big[\boxtimes^{k\in K}_u\boxtimes^{[I]}(\mathcal{A}_k,(\mathcal{C}^i_k)_{i\in I})\xrightarrow{\boxtimes^K_u\lambda^f}\boxtimes^{k\in K}_u\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}^i_k)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\mathcal{C}^i_k)_{j\in J}]\\
&\qquad\xrightarrow{\zeta^K}\boxtimes^{[J]}[\boxtimes^{k\in K}_u\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}^i_k)^{fi=0}_{i\in I}),(\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}j}\mathcal{C}^i_k)_{j\in J}]\\
&\qquad\xrightarrow{\boxtimes^{[J]}(\zeta^K,(\tilde{\zeta}^K)_J)}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boxtimes^{k\in K}_u\mathcal{A}_k,(\boxtimes^{k\in K}\mathcal{C}^i_k)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\boxtimes^{k\in K}\mathcal{C}^i_k)_{j\in J}]\big].
\end{aligned}
\tag{B.8.2}
$$

This is the back wall pentagon of diagram on the following page, where $\mathcal{C}^0_k$ denotes $T^{\leqslant1}\mathcal{A}_k$. Here the front wall commutes due to equation (B.8.1). The ceiling, the right wall and the floor commute due to Lemma B.6. Since the arrow $\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\mathrm{in},(1)_I),(1)_J]$ is a split embedding, commutativity of the left wall would imply that the back wall pentagon commutes as well. This is precisely equation (B.8.2).

It remains to prove that the left wall commutes. This is a corollary of the following equation, which holds for a graded $\Bbbk$-linear quiver $\mathcal{A}$ and graded $\Bbbk$-linear categories $\mathcal{C}^i$, $i\in I$:

$$
\begin{tikzcd}
T^{\leqslant1}\boxtimes^{[I]}(\mathcal{A},(\mathcal{C}^i)_{i\in I}) \arrow[r, "\xi'"] \arrow[dd, "T^{\leqslant1}\lambda^f"'] \arrow[dr, phantom, "="] & \boxtimes^{[I]}(T^{\leqslant1}\mathcal{A},(\mathcal{C}^i)_{i\in I}) \arrow[d, "\lambda^f"] \\
 & \boxtimes^{[J]}[\boxtimes^{f^{-1}0}(T^{\leqslant1}\mathcal{A},(\mathcal{C}^i)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}] \\
T^{\leqslant1}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\mathcal{A},(\mathcal{C}^i)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}] \arrow[r, "\xi'"] & \boxtimes^{[J]}[T^{\leqslant1}\boxtimes^{f^{-1}0}(\mathcal{A},(\mathcal{C}^i)^{fi=0}_{i\in I}),(\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}] \arrow[u, "\boxtimes^{[J]}[\xi',(1)_J]"']
\end{tikzcd}
$$

Indeed, expanding the maps ξ' we get the commutative diagram on page 443. In particular,

$$
\begin{tikzcd}
\boxtimes_u^{k\in K}\boxtimes^{[I]}(\mathcal{A}_k,(\mathcal{C}_k^i)_{i\in I}) \arrow[rr, "\zeta^K"] \arrow[dr, "\mathrm{in}\cdot\boxtimes^K\xi'"] \arrow[dddd, "\boxtimes_u^K\lambda^f"'] & & \boxtimes^{[I]}(\boxtimes_u^{k\in K}\mathcal{A}_k,(\boxtimes^{k\in K}\mathcal{C}_k^i)_{i\in I}) \arrow[dr, "\boxtimes^{[I]}(\mathrm{in},(1)_I)"] \arrow[dd, "\lambda^f"] & \\
& \boxtimes^{k\in K}\boxtimes^{i\in[I]}\mathcal{C}_k^i \arrow[rr, "\tilde{\zeta}^K"'] \arrow[ddddd, "\boxtimes^K\lambda^f"'] & & \boxtimes^{i\in[I]}\boxtimes^{k\in K}\mathcal{C}_k^i \arrow[dd, "\lambda^f"] \\
& & \begin{array}{c}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boxtimes_u^{k\in K}\mathcal{A}_k,(\boxtimes^{k\in K}\mathcal{C}_k^i)_{i\in I}^{fi=0}),\\ (\boxtimes^{i\in f^{-1}j}\boxtimes^{k\in K}\mathcal{C}_k^i)_{j\in J}]\end{array} \arrow[dr, "{\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\mathrm{in},(1)_I),(1)_J]}"] & \\
& & & \boxtimes^{j\in[J]}\boxtimes^{i\in f^{-1}j}\boxtimes^{k\in K}\mathcal{C}_k^i \\
\begin{array}{c}\boxtimes_u^{k\in K}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}_k^i)_{i\in I}^{fi=0}),\\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i)_{j\in J}]\end{array} \arrow[rr, "\zeta^K"] \arrow[d, "\mathrm{in}\cdot\boxtimes^K\xi'"'] & & \begin{array}{c}\boxtimes^{[J]}[\boxtimes_u^{k\in K}\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}_k^i)_{i\in I}^{fi=0}),\\ (\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i)_{j\in J}]\end{array} \arrow[uu, "{\boxtimes^{[J]}(\zeta^K,(\tilde{\zeta}^K)_J)}"'] \arrow[d, "\boxtimes^{[I]}(\mathrm{in},(1)_I)"] & \\
\begin{array}{c}\boxtimes^{k\in K}\boxtimes^{[J]}[T^{\leqslant 1}\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}_k^i)_{i\in I}^{fi=0}),\\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i)_{j\in J}]\end{array} \arrow[rr, "\tilde{\zeta}^K"] \arrow[dr, "{\boxtimes^K\boxtimes^{[J]}[\xi',(1)_J]}"'] & & \begin{array}{c}\boxtimes^{[J]}[\boxtimes_u^{k\in K}T^{\leqslant 1}\boxtimes^{f^{-1}0}(\mathcal{A}_k,(\mathcal{C}_k^i)_{i\in I}^{fi=0}),\\ (\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i)_{j\in J}]\end{array} \arrow[dr, "{\boxtimes^{[J]}[\boxtimes^K\xi',(1)_J]}"] & \\
& \boxtimes^{k\in K}\boxtimes^{j\in[J]}\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i \arrow[rr, "\tilde{\zeta}^K"] & & \boxtimes^{j\in[J]}\boxtimes^{k\in K}\boxtimes^{i\in f^{-1}j}\mathcal{C}_k^i \arrow[uuu, "\boxtimes^{[J]}\tilde{\zeta}^K"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boldsymbol{T}^{\leqslant 1} \boxtimes^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \arrow[rr, "\varkappa"] \arrow[d, "\boldsymbol{T}^{\leqslant 1}\lambda^f"'] \arrow[ddrr, "\xi'"] & & \boxtimes^{[I]}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}, (\boldsymbol{T}^{\leqslant 1}\mathcal{C}^i)_{i\in I}) \arrow[dd, "\boxtimes^{[I]}[1,(\mu)_I]"] \arrow[ddddll, "\lambda^f"] \\
\begin{array}{c}\boldsymbol{T}^{\leqslant 1} \boxtimes^{[J]} [\boxtimes^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}]\end{array} \arrow[d, "\varkappa"'] \arrow[ddr, "\xi'"] & & \\
\begin{array}{c}\boxtimes^{[J]}[\boldsymbol{T}^{\leqslant 1} \boxtimes^{f^{-1}0} (\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boldsymbol{T}^{\leqslant 1} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{j\in J}]\end{array} \arrow[d, "\boxtimes^{[J]}[1,(\varkappa)_J]"'] & & \boxtimes^{[I]}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \arrow[dd, "\lambda^f"] \\
\begin{array}{c}\boxtimes^{[J]}[\boldsymbol{T}^{\leqslant 1} \boxtimes^{f^{-1}0} (\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\boldsymbol{T}^{\leqslant 1}\mathcal{C}^i)_{j\in J}]\end{array} \arrow[r, "\boxtimes^{[J]}[1,(\boxtimes^{i\in f^{-1}j}\mu)_J]"] \arrow[d, "\boxtimes^{[J]}[\varkappa,(1)_J]"'] & \begin{array}{c}\boxtimes^{[J]}[\boldsymbol{T}^{\leqslant 1} \boxtimes^{f^{-1}0} (\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}]\end{array} \arrow[d, "\boxtimes^{[J]}[\varkappa,(1)_J]"'] \arrow[dr, "\boxtimes^{[J]}[\xi',(1)_J]"] & \\
\begin{array}{c}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}, (\boldsymbol{T}^{\leqslant 1}\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\boldsymbol{T}^{\leqslant 1}\mathcal{C}^i)_{j\in J}]\end{array} \arrow[r, "\boxtimes^{[J]}[1,(\boxtimes^{i\in f^{-1}j}\mu)_J]"] \arrow[rr, bend right, "\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(1,(\mu)),(\boxtimes^{i\in f^{-1}j}\mu)_J]"'] & \begin{array}{c}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}, (\boldsymbol{T}^{\leqslant 1}\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}]\end{array} \arrow[r, "\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(1,(\mu)),(1)_J]"] & \begin{array}{c}\boxtimes^{[J]}[\boxtimes^{f^{-1}0}(\boldsymbol{T}^{\leqslant 1}\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), \\ (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}]\end{array}
\end{tikzcd}
$$

central pentagon gives the required equation. Therefore, $\lambda^f_\cdot$ is a Monoidal transformation in both cases.

Clearly, the 2-isomorphism $\lambda^f_\cdot$ satisfies the necessary equations, and thus defines an action in both cases. $\square$

Applying symmetric Monoidal $\mathcal{C}at$-functor $\widehat{-}$: lax-sym-Mono-cat $\to$ $\mathfrak{SMCatm}$ from Theorem A.8 to the action $(\boxtimes^{[J]}, \zeta^I) : \prod_{[J]}(\mathscr{Q}_u, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_u$ we get an action $\boxdot^{[J]} : \prod_{[J]}(\widehat{\mathscr{Q}_u}, (\widehat{\mathbf{gr}\text{-}\mathcal{C}at})_J) \to \widehat{\mathscr{Q}_u}$ in the symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{SMCatm}$ of symmetric multicategories.

B.9 Actions on Kleisli multicategories. Let $\mathfrak{C}$ be the symmetric Monoidal $\mathcal{C}at$-category $\mathfrak{SMCatm}$ of symmetric multicategories. Let C, D be symmetric multicategories with Monoidal-category structure $(\mathsf{D}, \boxtimes^I, \lambda^f_{\mathsf{D}})$ and an action $\boxdot^{[J]} : \boxtimes^{[J]}(\mathsf{C}, (\mathsf{D})_J) \to \mathsf{C}$. Let (T, Δ, ε) be a symmetric multicomonad in C. An additional datum, an intertwiner of the action and the comonad, will ensure that the given action lifts to an action on the Kleisli multicategory C^T.

Let $\xi : T \boxdot^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \to \boxdot^{[J]}(T\mathcal{A}, (\mathcal{C}^j)_{j\in J})$, $\mathcal{A} \in \operatorname{Ob}\mathsf{C}$, $\mathcal{C}_j \in \operatorname{Ob}\mathsf{D}$, $j \in J$, be a multinatural transformation. We call it an *intertwiner* if the following conditions are satisfied:

(a) compatibility with comultiplication:

$$\begin{tikzcd}
T \boxdot^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \arrow[r, "\Delta"] \arrow[d, "\xi"'] & TT \boxdot^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \arrow[r, "T\xi"] & T \boxdot^{[J]} (T\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \arrow[d, "\xi"] \\
\boxdot^{[J]}(T\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \arrow[rr, "\boxdot^{[J]}(\Delta,(1)_{j\in J})"] & & \boxdot^{[J]}(TT\mathcal{A}, (\mathcal{C}^j)_{j\in J})
\end{tikzcd} \tag{B.9.1}$$

(b) compatibility with counit:

$$\bigl[T\boxdot^{[J]}(\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\xi} \boxdot^{[J]}(T\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxdot^{[J]}(\varepsilon,(1)_{j\in J})} \boxdot^{[J]}(\mathcal{A}, (\mathcal{C}^j)_{j\in J})\bigr] = \varepsilon. \tag{B.9.2}$$

We claim that in this case the action $\boxdot^{[J]}$ gives rise to an action $\boxdot^{[J]}_T : \boxtimes^{[J]}(\mathsf{C}^T, (\mathsf{D})_J) \to \mathsf{C}^T$ of D on the Kleisli multicategory C^T. Let us describe it as a multifunctor. Its action on objects coincides with that of $\boxdot^{[J]}$, that is, $\boxdot^{[J]}_T(\mathcal{A}, (\mathcal{C}^j)_{j\in J}) = \boxdot^{[J]}(\mathcal{A}, (\mathcal{C}^j)_{j\in J})$, $\mathcal{A} \in \operatorname{Ob}\mathsf{C}$, $\mathcal{C}_j \in \operatorname{Ob}\mathsf{D}$, $j \in J$. Given $f \in \mathsf{C}^T((\mathcal{A}_i)_{i\in I}; \mathcal{B}) = \mathsf{C}((T\mathcal{A}_i)_{i\in I}; \mathcal{B})$, $g^j \in \mathsf{D}((\mathcal{C}^j_i)_{i\in I}; \mathcal{D}^j)$, $j \in J$, we define

$$\boxdot^{[J]}_T (f, (g^j)_{j\in J}) = \bigl[\bigl(T \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{C}^j_i)_{j\in J})\bigr)_{i\in I} \xrightarrow{(\xi)_{i\in I}} \bigl(\boxdot^{[J]}(T\mathcal{A}_i, (\mathcal{C}^j_i)_{j\in J})\bigr)_{i\in I} \xrightarrow{\boxdot^{[J]}(f,(g^j)_{j\in J})} \boxdot^{[J]} (\mathcal{B}, (\mathcal{D}^j)_{j\in J})\bigr]. \tag{B.9.3}$$

B.10 Lemma. *Suppose that equations* (B.9.1) *and* (B.9.2) *hold. Then* (B.9.3) *defines a multifunctor.*

Proof. Let us show that $\boxdot_T^{[J]}$ is compatible with multiplications. Let $\phi : I \to L$ be a map in $\operatorname{Mor} \mathcal{S}$. Let $f_l \in \mathsf{C}^T((\mathcal{A}_i)_{i\in\phi^{-1}l};\mathcal{B}_l) = \mathsf{C}((T\mathcal{A}_i)_{i\in\phi^{-1}l};\mathcal{B}_l)$, $l \in L$, $h \in \mathsf{C}^T((\mathcal{B}_l)_{l\in L};\mathcal{C}) = \mathsf{C}((T\mathcal{B}_l)_{l\in L};\mathcal{C})$, $g_l^j \in \mathsf{D}((\mathcal{D}_i^j)_{i\in\phi^{-1}l};\mathcal{E}_l^j)$, $l \in L$, $k^j \in \mathsf{D}((\mathcal{E}_l^j)_{l\in L};\mathcal{F}^j)$, $j \in J$. We must show that

$$\boxdot_T^{[J]}\left((f_l)_{l\in L} \cdot_{\mathsf{C}^T} h, ((g_l^j)_{l\in L} \cdot_{\mathsf{D}} k^j)_{j\in J}\right) = (\boxdot_T^{[J]}(f_l, (g_l^j)_{j\in J}))_{l\in L} \cdot_{\mathsf{C}^T} \boxdot_T^{[J]}(h, (k^j)_{j\in J}). \quad \text{(B.10.1)}$$

The left hand side of (B.10.1) equals

$$\begin{aligned}\big[\big(T \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} &\xrightarrow{(\xi)_{i\in I}} \big(\boxdot^{[J]}(T\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \\ &\xrightarrow{(\boxdot^{[J]}(\Delta,(1)_{j\in J}))_{i\in I}} \big(\boxdot^{[J]}(TT\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \xrightarrow{\boxdot^{[J]}(Tf_l,(g_l^j)_{j\in J})_{l\in L}} \\ &\big(\boxdot^{[J]}(T\mathcal{B}_l, (\mathcal{E}_l^j)_{j\in J})\big)_{l\in L} \xrightarrow{\boxdot^{[J]}(h,(k^j)_{j\in J})} \boxdot^{[J]} (\mathcal{C}, (\mathcal{F}^j)_{j\in J})\big].\end{aligned} \quad \text{(B.10.2)}$$

The right hand side of (B.10.1) is

$$\begin{aligned}\big[\big(T \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} &\xrightarrow{(\Delta)_{i\in I}} \big(TT \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \xrightarrow{(T\xi)_{i\in I}} \\ &\big(T \boxdot^{[J]} (T\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \xrightarrow{(T\boxdot^{[J]}(f_l,(g_l^j)_{j\in J}))_{l\in L}} \big(T \boxdot^{[J]} (\mathcal{B}_l, (\mathcal{E}_l^j)_{j\in J})\big)_{l\in L} \\ &\xrightarrow{(\xi)_{l\in L}} \big(\boxdot^{[J]}(T\mathcal{B}_l, (\mathcal{E}_l^j)_{j\in J})\big)_{l\in L} \xrightarrow{\boxdot^{[J]}(h,(k^j)_{j\in J})} \boxdot^{[J]} (\mathcal{C}, (\mathcal{F}^j)_{j\in J})\big].\end{aligned} \quad \text{(B.10.3)}$$

Using multinaturality of ξ we can write (B.10.3) as follows:

$$\begin{aligned}\big[\big(T \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} &\xrightarrow{(\Delta)_{i\in I}} \big(TT \boxdot^{[J]} (\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \xrightarrow{(T\xi)_{i\in I}} \\ &\big(T \boxdot^{[J]} (T\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \xrightarrow{(\xi)_{i\in I}} \big(\boxdot^{[J]}(TT\mathcal{A}_i, (\mathcal{D}_i^j)_{j\in J})\big)_{i\in I} \\ &\xrightarrow{(\boxdot^{[J]}(Tf_l,(g_l^j)_{j\in J}))_{l\in L}} \big(\boxdot^{[J]}(T\mathcal{B}_l, (\mathcal{E}_l^j)_{j\in J})\big)_{l\in L} \xrightarrow{\boxdot^{[J]}(h,(k^j)_{j\in J})} \boxdot^{[J]} (\mathcal{C}, (\mathcal{F}^j)_{j\in J})\big].\end{aligned} \quad \text{(B.10.4)}$$

This coincides with (B.10.2) by (B.9.1).

The compatibility with units is expressed by equation (B.9.2). □

B.11 Lemma. *There is a natural multifunctor* $\mathsf{C} \to \mathsf{C}^T$*, identity on objects, whose action on morphisms is given by*

$$((\mathcal{A}_i)_{i\in I} \xrightarrow{f} \mathcal{B}) \mapsto ((T\mathcal{A}_i)_{i\in I} \xrightarrow{(\varepsilon)_{i\in I}} (\mathcal{A}_i)_{i\in I} \xrightarrow{f} \mathcal{B}).$$

Proof. Let us show that the described map is compatible with multiplications. Let $\phi : I \to J$ be a map in $\operatorname{Mor} \mathcal{S}$, $f_j \in \mathsf{C}((\mathcal{A}_i)_{i\in\phi^{-1}j};\mathcal{B}_j)$, $j \in J$, $g \in \mathsf{C}((\mathcal{B}_j)_{j\in J};\mathcal{C})$. The equation

to prove is

$$\big[(T\mathcal{A}_i)_{i\in I} \xrightarrow{(\varepsilon)_{i\in I}} (\mathcal{A}_i)_{i\in I} \xrightarrow{(f_j)_{j\in J}} (\mathcal{B}_j)_{j\in J} \xrightarrow{g} \mathcal{C}\big] = \big[(T\mathcal{A}_i)_{i\in I} \xrightarrow{(\Delta)_{i\in I}} (TT\mathcal{A}_i)_{i\in I} \\ \xrightarrow{(T\varepsilon)_{i\in I}} (T\mathcal{A}_i)_{i\in I} \xrightarrow{(Tf_j)_{j\in J}} (T\mathcal{B}_j)_{j\in J} \xrightarrow{(\varepsilon)_{j\in J}} (\mathcal{B}_j)_{j\in J} \xrightarrow{g} \mathcal{C}\big]. \qquad \text{(B.11.1)}$$

It follows from the axioms of comonad and from the multinaturality of ε.

The compatibility with units is obvious (recall that the unit of an object $\mathcal{C}$ in the multicategory C^T coincides with $\varepsilon : T\mathcal{C} \to \mathcal{C}$). □

For every map $f : [I] \to [J]$ in $\operatorname{Mor} \mathcal{O}$ with $f(0) = 0$ define

$$\lambda^f_T = \big[T \boxdot^{[J]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\varepsilon} \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \\ \xrightarrow{\lambda^f_{\cdot}} \boxdot^{[J]} (\boxdot^{f^{-1}(0)}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J})\big].$$

Since both ε and $\lambda^f_{\cdot}$ are multinatural transformations, so is λ^f_T. Moreover, Lemma B.11 implies that λ^f_T is invertible (it coincides with the image of $\lambda^f_{\cdot}$ under the multifunctor described in the lemma).

B.12 Proposition. *The collection of multifunctors $\boxdot^J_T$ and multinatural transformations λ^f_T defines an action of D on C^T.*

Proof. Let $[I] \xrightarrow{f} [J] \xrightarrow{g} [K]$ be maps in $\operatorname{Mor}\mathcal{O}$ such that $f(0) = 0$ and $g(0) = 0$. Let f_k denote the restriction $f| : f^{-1}g^{-1}k \to g^{-1}k$, $k \in [K]$. We must prove the following equation:

$$\lambda^f_T \cdot \lambda^g_T = \lambda^{fg}_T \cdot \boxdot^{[K]}(\lambda^{f_0}_T, (\lambda^{f_k}_{\mathsf{D}})_{k\in K}). \qquad \text{(B.12.1)}$$

We have

$$\boxdot^{[K]} (\lambda^{f_0}_T, (\lambda^{f_k}_{\mathsf{D}})_{k\in K}) = \big[T \boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\xi} \\ \boxdot^{[K]} (T \boxdot^{f^{-1}g^{-1}0} (\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\boxdot^{[K]}(\varepsilon,(1)_{k\in K})} \\ \boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\boxdot^{[K]}(\lambda^{f_0}_{\cdot},(\lambda^{f_k}_{\mathsf{D}})_{k\in K})} \\ \boxdot^{[K]} (\boxdot^{g^{-1}0}(\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)^{gj=0}_{j\in J}), (\boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{k\in K})\big]$$

$$= \big[T \boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\varepsilon} \\ \boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\boxdot^{[K]}(\lambda^{f_0}_{\cdot},(\lambda^{f_k}_{\mathsf{D}})_{k\in K})} \\ \boxdot^{[K]} (\boxdot^{g^{-1}0}(\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)^{gj=0}_{j\in J}), (\boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{k\in K})\big]$$

by (B.9.2). Therefore, the right hand side of equation (B.12.1) can be written as

$$\big[T \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\Delta} TT \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{T\varepsilon} T \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I})$$
$$\xrightarrow{T\lambda^{fg}_{\cdot}} T \boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\varepsilon}$$
$$\boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\boxdot^{[K]}(\lambda^{f_0}_{\cdot},(\lambda^{f_k}_{\mathsf{D}})_{k\in K})}$$
$$\boxdot^{[K]} (\boxdot^{g^{-1}0}(\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)^{gj=0}_{j\in J}), (\boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{k\in K})\big].$$

This equals

$$\big[T \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\varepsilon} \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\lambda^{fg}_{\cdot}}$$
$$\boxdot^{[K]} (\boxdot^{f^{-1}g^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{gfi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}g^{-1}k}\mathcal{C}^i)_{k\in K}) \xrightarrow{\boxdot^{[K]}(\lambda^{f_0}_{\cdot},(\lambda^{f_k}_{\mathsf{D}})_{k\in K})}$$
$$\boxdot^{[K]} (\boxdot^{g^{-1}0}(\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)^{gj=0}_{j\in J}), (\boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{k\in K})\big]$$

by the axioms of comonad and by the naturality of ε. From Lemma B.11 it follows that

$$\lambda^f_T \cdot \lambda^g_T = \big[T \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\varepsilon} \boxdot^{[I]} (\mathcal{A}, (\mathcal{C}^i)_{i\in I}) \xrightarrow{\lambda^f_{\cdot}}$$
$$\boxdot^{[J]} (\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)_{j\in J}) \xrightarrow{\lambda^g_{\cdot}}$$
$$\boxdot^{[K]} (\boxdot^{g^{-1}0}(\boxdot^{f^{-1}0}(\mathcal{A}, (\mathcal{C}^i)^{fi=0}_{i\in I}), (\boxtimes^{i\in f^{-1}j}\mathcal{C}^i)^{gj=0}_{j\in J}), (\boxtimes^{j\in g^{-1}k} \boxtimes^{i\in f^{-1}j} \mathcal{C}^i)_{k\in K})\big].$$

Equation (B.12.1) follows now from the analogous equation for $\lambda_{\cdot}$'s. $\square$

Appendix C

An action of graded categories on Kleisli multicategory of quivers

In this chapter we construct the main examples of actions: the action of the symmetric Monoidal category $\mathbf{gr}$-$\mathcal{C}at$ of graded $\Bbbk$-linear categories (or rather of symmetric multicategory $\widehat{\mathbf{gr}\text{-}\mathcal{C}at}$) on the Kleisli multicategory of quivers and on Q. We construct the necessary intertwiner between the action on $\mathscr{Q}_u$ and the lax Monoidal comonad $T^{\geqslant 1}$. The action on Q lifts to an action of $\widehat{\mathbf{dg}\text{-}\mathcal{C}at}$ on the multicategories A_∞, $\mathsf{A}_\infty^{\mathrm{u}}$ of (unital) A_∞-categories. The latter is a generalization of the tensor product of differential graded categories. Tensor multiplication with an algebra in the category $\mathbf{dg}$-$\mathcal{C}at$ (a 'strictly' monoidal differential graded $\Bbbk$-linear category) gives a multifunctor $\mathsf{A}_\infty^{\mathrm{u}} \to \mathsf{A}_\infty^{\mathrm{u}}$. For instance, the algebra $\mathcal{Z}$ from Section 10.1 produces the multifunctor of shifts $-^{[]} : \mathsf{A}_\infty^{\mathrm{u}} \to \mathsf{A}_\infty^{\mathrm{u}}$.

C.1 Intertwiner ξ for action of graded categories and $T^{\geqslant 1}$. Let us define an intertwiner ξ for the action of graded categories on quivers and the comonad $T^{\geqslant 1}$:

$$\xi : (T^{\geqslant 1}, \tau) \circ (\boxtimes^{[J]}, \zeta) \to (\boxtimes^{[J]}, \zeta) \circ [(T^{\geqslant 1}, \tau), (\mathrm{Id})_J] : \prod_{[J]} (\mathscr{Q}_u, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_u.$$

This natural transformation is defined as

$$\xi = \bigl[T^{\geqslant 1} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\varkappa} \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}, (T^{\geqslant 1}\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu)_J)} \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J})\bigr].$$

In order to prove that it is an intertwiner, we introduce another natural transformation:

$$\widetilde{\xi} : (T, \widetilde{\tau}) \circ (\boxtimes^{[J]}, \zeta) \to (\boxtimes^{[J]}, \tilde{\zeta}) \circ [(T, \widetilde{\tau}), (\mathrm{Id})_J] : \prod_{[J]} (\mathscr{Q}_u, (\mathbf{gr}\text{-}\mathcal{C}at)_J) \to \mathscr{Q}_p,$$

which is defined as

$$\widetilde{\xi} = \bigl[T \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\varkappa} \boxtimes^{[J]} (T\mathcal{A}, (T\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu)_J)} \boxtimes^{[J]} (T\mathcal{A}, (\mathcal{C}^j)_{j\in J})\bigr].$$

C.2 Lemma. *The natural transformation $\widetilde{\xi}$ is Monoidal.*

Proof. We have to prove the equation

$$\begin{tikzcd}[column sep=small]
\boxtimes^{i\in I} T \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\widetilde{\tau}"] \arrow[d, "\boxtimes^I \widetilde{\xi}"'] & T \boxtimes_u^{i\in I} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \arrow[r, "T\zeta^I"] \arrow[d, phantom, "="] & T \boxtimes^{[J]} (\boxtimes_u^{i\in I} \mathcal{A}, (\boxtimes^{i\in I} \mathcal{C}^j)_{j\in J}) \arrow[d, "\widetilde{\xi}"] \\
\boxtimes^{i\in I} \boxtimes^{[J]} (T\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\widetilde{\zeta}^I"] & \boxtimes^{[J]} (\boxtimes^{i\in I} T\mathcal{A}_i, (\boxtimes^{i\in I} \mathcal{C}_j^i)_{j\in J}) \arrow[r, "\boxtimes^{[J]}(\widetilde{\tau},(1)_J)"] & \boxtimes^{[J]} (T \boxtimes_u^{i\in I} \mathcal{A}_i, (\boxtimes^{i\in I} \mathcal{C}_i^j)_{j\in J})
\end{tikzcd}$$

It suffices to prove the equation obtained from this one by composing both paths with the inclusion $\boxtimes^{[J]}(T\,\mathrm{in},(1)_{j\in J}):\boxtimes^{[J]}(T\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\hookrightarrow\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})$. Using the diagram

$$\begin{array}{ccc}
T\boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) & \xhookrightarrow{T\boxtimes^{[J]}(\mathrm{in},(1)_J)} & T\boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \\
\Big\downarrow{\scriptstyle\widetilde{\xi}} & & \Big\downarrow{\scriptstyle\widetilde{\xi}} \\
\boxtimes^{[J]}(T\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) & \xhookrightarrow{\boxtimes^{[J]}(T\,\mathrm{in},(1)_J)} & \boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})
\end{array}$$

which commutes by the naturality of $\widetilde{\xi}$, we can reduce the initial equation to the following one:

$$\begin{aligned}
&\bigl[\boxtimes^{i\in I}T\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{\widetilde{\tau}}T\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J})\xrightarrow{T\zeta^I}T\boxtimes^{[J]}(\boxtimes_u^{i\in I}\mathcal{A},(\boxtimes^{i\in I}\mathcal{C}^j)_{j\in J})\\
&\quad\xhookrightarrow{T\boxtimes^{[J]}(\mathrm{in},(1)_J)}T\boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\xrightarrow{\widetilde{\xi}}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\bigr]\\
&=\bigl[\boxtimes^{i\in I}T\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^I\widetilde{\xi}}\boxtimes^{i\in I}\boxtimes^{[J]}(T\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{\widetilde{\zeta}^I}\boxtimes^{[J]}(\boxtimes^{i\in I}T\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_j^i)_{j\in J})\\
&\quad\xrightarrow{\boxtimes^{[J]}(\widetilde{\tau},(1)_J)}\boxtimes^{[J]}(T\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\xhookrightarrow{\boxtimes^{[J]}(T\,\mathrm{in},(1)_J)}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\bigr].
\end{aligned}\tag{C.2.1}$$

Applying Lemma B.6 we replace the left hand side of (C.2.1) by

$$\begin{aligned}
&\bigl[\boxtimes^{i\in I}T\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{\widetilde{\tau}}T\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{T\,\mathrm{in}}T\boxtimes^{i\in I}T^{\leqslant1}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\\
&\quad\xrightarrow{T\boxtimes^I\widetilde{\varkappa}}T\boxtimes^{i\in I}\boxtimes^{[J]}(T^{\leqslant1}\mathcal{A}_i,(T^{\leqslant1}\mathcal{C}_i^j)_{j\in J})\xrightarrow{T\sigma_{(12)}}T\boxtimes^{[J]}(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}T^{\leqslant1}\mathcal{C}_i^j)_{j\in J})\\
&\xrightarrow{\varkappa}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^{[J]}(1,(\varkappa)_J)}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}TT^{\leqslant1}\mathcal{C}_i^j)_{j\in J})\\
&\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^IT\mu)_J)}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}T\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^I\mu)_J)}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\bigr].
\end{aligned}\tag{C.2.2}$$

The right hand side of (C.2.1) equals

$$\begin{aligned}
&\bigl[\boxtimes^{i\in I}T\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^I\varkappa}\boxtimes^{i\in I}\boxtimes^{[J]}(T\mathcal{A}_i,(T\mathcal{C}_i^j)_{j\in J})\xrightarrow{\sigma_{(12)}}\\
&\quad\boxtimes^{[J]}(\boxtimes^{i\in I}T\mathcal{A}_i,(\boxtimes^{i\in I}T\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^{[J]}(\widetilde{\tau},(1)_J)}\boxtimes^{[J]}(T\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}T\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^{[J]}(T\,\mathrm{in},(1)_J)}\\
&\quad\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}T\mathcal{C}_i^j)_{j\in J})\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^I\mu)_J)}\boxtimes^{[J]}(T\boxtimes^{i\in I}T^{\leqslant1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})\bigr].
\end{aligned}\tag{C.2.3}$$

Let $m\in\mathbb{Z}_{\geqslant0}$ and let $S\subset I\times\mathbf{m}$ be a subset that satisfies the condition $\mathrm{pr}_2 S=\mathbf{m}$. Let $S_i=\{p\in\mathbf{m}\mid(i,p)\in S\}$, $m_i=|S_i|$, $i\in I$. Restricting (C.2.2) and (C.2.3) to the direct

summand $\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$ of $\boxtimes^{i\in I} T \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$ we arrive at the following equation:

$$\begin{aligned}
&\big[\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{P} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^I \otimes^{\mathbf{m}} \mu)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^I \mu)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \mathcal{C}_i^j)_{j\in J})\big] \\
= &\big[\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{R} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} T^{m_i} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^I \mu)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \mathcal{C}_i^j)_{j\in J})\big],
\end{aligned} \tag{C.2.4}$$

where

$$\begin{aligned}
P = &\big[\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\boxtimes^{i\in I} \lambda^{S_i \hookrightarrow \mathbf{m}}} \boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\widetilde{\varkappa}^{-1}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\otimes^{\mathbf{m}} \boxtimes^I \widetilde{\varkappa}} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} \boxtimes^{[J]} (T^{\chi((i,p)\in S)} \mathcal{A}_i, (T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\otimes^{\mathbf{m}} \sigma_{(12)}} \otimes^{p\in\mathbf{m}} \boxtimes^{[J]} (\boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\widetilde{\varkappa}} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\widetilde{\varkappa})_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J})\big],
\end{aligned}$$

$$\begin{aligned}
R = &\big[\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\boxtimes^I \widetilde{\varkappa}} \boxtimes^{i\in I} \boxtimes^{[J]} (T^{m_i} \mathcal{A}_i, (T^{m_i} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\sigma_{(12)}} \boxtimes^{[J]} (\boxtimes^{i\in I} T^{m_i} \mathcal{A}_i, (\boxtimes^{i\in I} T^{m_i} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(\boxtimes^{i\in I} \lambda^{S_i \hookrightarrow \mathbf{m}},(1)_J)} \boxtimes^{[J]} (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} T^{m_i} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(\widetilde{\varkappa}^{-1},(1)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} T^{m_i} \mathcal{C}_i^j)_{j\in J})\big].
\end{aligned}$$

The equation

$$\begin{aligned}
P = &\big[\boxtimes^{i\in I} T^{m_i} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{R} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} T^{m_i} \mathcal{C}_i^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\boxtimes^{i\in I} \lambda^{S_i \hookrightarrow \mathbf{m}})_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in I} T^{\chi((i,p)\in S)} \mathcal{A}_i, (\boxtimes^{i\in I} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \mathcal{C}_i^j)_{j\in J})\big]
\end{aligned}$$

holds by the coherence principle. Therefore, equation (C.2.4) reduces to the equation

$$\big[T^{m_i} \mathcal{C}_i^j \xrightarrow{\lambda^{S_i \hookrightarrow \mathbf{m}}} \otimes^{p\in\mathbf{m}} T^{\chi((i,p)\in S)} \mathcal{C}_i^j \xrightarrow{\otimes^{\mathbf{m}} \mu} T^m \mathcal{C}_i^j \xrightarrow{\mu} \mathcal{C}_i^j\big] = \mu,$$

which expresses the associativity of μ. $\square$

$$
\begin{tikzcd}[column sep=small]
\boxtimes_u^{i\in I}T^{\geqslant 1}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[rr, "\tau^I"] \arrow[dr, "\mathrm{in}"] \arrow[dd, "\boxtimes_u^I\xi"'] & & T^{\geqslant 1}\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[r, "T^{\geqslant 1}\zeta^I"] \arrow[d, "\mathrm{in}"] & T^{\geqslant 1}\boxtimes^{[J]}[\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}] \arrow[dr, "\mathrm{in}"] \arrow[dd, "\xi"] & \\
& \boxtimes^{i\in I}T\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\widetilde{\tau}^I"] \arrow[ddl, "\boxtimes^I T^{\leqslant 1}\xi"] \arrow[dd, "\boxtimes^I\widetilde{\xi}"] & T\boxtimes_u^{i\in I}\boxtimes^{[J]}(\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[rr, "T\zeta^I"] & & T\boxtimes^{[J]}[\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}] \arrow[dd, "\widetilde{\xi}"] \\
\boxtimes_u^{i\in I}\boxtimes^{[J]}(T^{\geqslant 1}\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[rr, "\zeta^I"] \arrow[d, "\mathrm{in}"] & & \boxtimes^{[J]}(\boxtimes_u^{i\in I}T^{\geqslant 1}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes^{[J]}(\tau^I,(1)_J)"] \arrow[d, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] & \boxtimes^{[J]}(T^{\geqslant 1}\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[dr, "\boxtimes^{[J]}(\mathrm{in},(1)_J)"] & \\
\boxtimes^{i\in I}T^{\leqslant 1}\boxtimes^{[J]}(T^{\geqslant 1}\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\boxtimes^{i\in I}\xi'"] & \boxtimes^{i\in I}\boxtimes^{[J]}(T\mathcal{A}_i,(\mathcal{C}_i^j)_{j\in J}) \arrow[r, "\widetilde{\zeta}^I"] & \boxtimes^{[J]}(\boxtimes^{i\in I}T\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J}) \arrow[rr, "\boxtimes^{[J]}(\widetilde{\tau}^I,(1)_J)"] & & \boxtimes^{[J]}(T\boxtimes_u^{i\in I}\mathcal{A}_i,(\boxtimes^{i\in I}\mathcal{C}_i^j)_{j\in J})
\end{tikzcd}
$$

C.3 Proposition. *The natural transformation ξ is Monoidal.*

Proof. In the diagram displayed on the facing page the quadrilateral on the floor, the ceiling, the left and the right walls-quadrilaterals clearly commute. The pentagon on the floor commutes by Lemma B.6. Commutativity of the left triangle follows from the equation

$$\widetilde{\xi} = \big[T \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) = T^{\leqslant 1}T^{\geqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{T^{\leqslant 1}\xi} T^{\leqslant 1} \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J}) \xrightarrow{\xi'} \boxtimes^{[J]} (T\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})\big],$$

which is due to the fact that $\widetilde{\xi}$ coincides with ξ on $T^{\geqslant 1} \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$ and it coincides with ξ' on $T^0 \boxtimes^{[J]} (\mathcal{A}_i, (\mathcal{C}_i^j)_{j\in J})$. The front wall commutes due to $\widetilde{\xi}$ being Monoidal. Since $\boxtimes^{[J]}(\mathrm{in}, (1)_J)$ is an embedding, the back wall commutes as well, that is, ξ is Monoidal. □

C.4 Proposition. *ξ satisfies conditions* (B.9.1) *and* (B.9.2).

Proof. Let us show that ξ satisfies condition (B.9.1). Consider the diagram displayed at Fig. C.1. Here the floor, the ceiling, the left and the right walls clearly commute. The commutativity of the back wall has to be proven. We prove instead that the front wall commutes. Since $\boxtimes^{[J]}(\mathrm{in}, (1)_J)$ is an embedding, the back wall will have to commute as well.

The front wall expands to the equation

$$\begin{aligned}
&\big[T \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\varkappa} \boxtimes^{[J]} (T\mathcal{A}, (T\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{\boxtimes^{[J]}(1,(\mu)_J)} \boxtimes^{[J]} (T\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(\widetilde{\Delta},(1)_J)} \boxtimes^{[J]} (TT^{\geqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J})\big] \\
&= \big[T \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\widetilde{\Delta}} TT^{\geqslant 1} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{T\varkappa} T \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}, (T^{\geqslant 1}\mathcal{C}^j)_{j\in J}) \xrightarrow{T\boxtimes^{[J]}(1,(\mu)_J)} T \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{\varkappa} \boxtimes^{[J]} (TT^{\geqslant 1}\mathcal{A}, (T\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu)_J)} \boxtimes^{[J]} (TT^{\geqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J})\big],
\end{aligned}$$

which we are going to verify. It is equivalent to the following equation between matrix elements:

$$\begin{aligned}
&\big[T^m \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\widetilde{\varkappa}} \boxtimes^{[J]} (T^m\mathcal{A}, (T^m\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{\boxtimes^{[J]}(1,(\mu^m)_J)} \boxtimes^{[J]} (T^m\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(\lambda^g,(1)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\mathcal{C}^j)_{j\in J})\big] \\
&= \big[T^m \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\otimes^{\mathbf{n}}\widetilde{\varkappa}} \\
&\qquad \otimes^{p\in\mathbf{n}} \boxtimes^{[J]}(\otimes^{g^{-1}}\mathcal{A}, (\otimes^{g^{-1}p}\mathcal{C}^j)_{j\in J}) \xrightarrow{\otimes^{\mathbf{n}}\boxtimes^{[J]}(1,(\mu^{g^{-1}p})_J)} \otimes^{p\in\mathbf{n}} \boxtimes^{[J]}(\otimes^{g^{-1}p}\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{\widetilde{\varkappa}} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\otimes^{\mathbf{n}}\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu^n)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\mathcal{C}^j)_{j\in J})\big]
\end{aligned}$$

$$
\begin{tikzcd}
\boldsymbol{T}^{\geqslant 1}\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[rr, "\boldsymbol{\xi}"] \arrow[dr, "\mathbf{in}"] \arrow[dd, "\boldsymbol{\Delta}"'] & & \boxtimes^{[J]}(\boldsymbol{T}^{\geqslant 1}\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[dr, "\boxtimes^{[J]}(\mathbf{in},(1)_J)"] \arrow[dd, "\boxtimes^{[J]}(\boldsymbol{\Delta},(1)_J)"' near end] & \\
& \boldsymbol{T}\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[rr, "\widetilde{\boldsymbol{\xi}}" near start] \arrow[ddl, "\widetilde{\boldsymbol{\Delta}}"' near start] & & \boxtimes^{[J]}(\boldsymbol{T}\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[dd, "\boxtimes^{[J]}(\widetilde{\boldsymbol{\Delta}},(1)_J)"] \\
\boldsymbol{T}^{\geqslant 1}\boldsymbol{T}^{\geqslant 1}\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[r, "\boldsymbol{T}^{\geqslant 1}\boldsymbol{\xi}"'] \arrow[d, "\mathbf{in}"] & \boldsymbol{T}^{\geqslant 1}\boxtimes^{[J]}(\boldsymbol{T}^{\geqslant 1}\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[r, "\boldsymbol{\xi}"] \arrow[d, "\mathbf{in}"] & \boxtimes^{[J]}(\boldsymbol{T}^{\geqslant 1}\boldsymbol{T}^{\geqslant 1}\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[dr, "\boxtimes^{[J]}(\mathbf{in},(1)_J)"] & \\
\boldsymbol{T}\boldsymbol{T}^{\geqslant 1}\boxtimes^{[J]}(\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[r, "\boldsymbol{T}\boldsymbol{\xi}"] & \boldsymbol{T}\boxtimes^{[J]}(\boldsymbol{T}^{\geqslant 1}\mathcal{A},(\mathcal{C}^j)_{j\in J}) \arrow[rr, "\widetilde{\boldsymbol{\xi}}"] & & \boxtimes^{[J]}(\boldsymbol{T}\boldsymbol{T}^{\geqslant 1}\mathcal{A},(\mathcal{C}^j)_{j\in J})
\end{tikzcd}
$$

Figure C.1:

for an arbitrary non-decreasing surjection $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$. Since a graded category with objects set S is an algebra in $(\mathscr{Q}/S, \otimes_S^I, \lambda^f)$, it satisfies equation (2.25.1). This allows to rewrite the above equation in the form:

$$\begin{aligned}
&\big[T^m \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\widetilde{\varkappa}} \boxtimes^{[J]} (T^m\mathcal{A}, (T^m\mathcal{C}^j)_{j\in J}) \\
&\qquad \xrightarrow{\boxtimes^{[J]}(\lambda^g,(\lambda^g)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{C}^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\otimes^{p\in\mathbf{p}}\mu^{g^{-1}p})_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\otimes^{p\in\mathbf{n}}\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu^n)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\mathcal{C}^j)_{j\in J})\big] \\
&= \big[T^m \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \\
&\quad \xrightarrow{\otimes^{\mathbf{n}}\widetilde{\varkappa}} \otimes^{p\in\mathbf{n}} \boxtimes^{[J]}(\otimes^{g^{-1}p}\mathcal{A}, (\otimes^{g^{-1}p}\mathcal{C}^j)_{j\in J}) \xrightarrow{\widetilde{\varkappa}} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{C}^j)_{j\in J}) \\
&\xrightarrow{\boxtimes^{[J]}(1,(\otimes^{p\in\mathbf{n}}\mu^{g^{-1}p})_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p} \mathcal{A}, (\otimes^{p\in\mathbf{n}}\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(1,(\mu^n)_J)} \boxtimes^{[J]} (\otimes^{p\in\mathbf{n}}\otimes^{g^{-1}p} \mathcal{A}, (\mathcal{C}^j)_{j\in J})\big].
\end{aligned}$$

The last two arrows in both sides coincide. The previous compositions ending in $\boxtimes^{[J]}(\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{A}, (\otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p} \mathcal{C}^j)_{j\in J})$ coincide due to coherence principle of Remark 2.34. Therefore, the considered diagram is commutative, and ξ satisfies equation (B.9.1).

Let us show that ξ satisfies condition (B.9.2). The equation to prove is

$$\big[T^{\geqslant 1} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\xi} \boxtimes^{[J]} (T^{\geqslant 1}\mathcal{A}, (\mathcal{C}^j)_{j\in J}) \xrightarrow{\boxtimes^{[J]}(\varepsilon,(1)_J)} \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_J)\big] = \varepsilon.$$

It is obvious, since both sides vanish on the summand $T^m \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J})$ of the source, and give the identity morphism on $T^0 \boxtimes^{[J]} (\mathcal{A}, (\mathcal{C}^j)_{j\in J})$. This accomplishes the proof of the proposition. □

Relations between lax Monoidal subjects translate to multicategory setting due to Theorem A.8. Thus, for the action of multicategory $\widehat{\mathbf{gr}\text{-}\mathcal{C}at}$ on multicategory $\widehat{\mathscr{Q}_u}$ and the multicomonad $T^{\geqslant 1}$ there is an intertwiner ξ, a multinatural transformation that satisfies conditions (B.9.1) and (B.9.2) of Section B.9. Therefore, there is an action of the multicategory $\widehat{\mathbf{gr}\text{-}\mathcal{C}at}$ on the Kleisli multicategory $\widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$. Shifting the latter by [1] we get an action of $\widehat{\mathbf{gr}\text{-}\mathcal{C}at}$ on Q.

C.5 An action of a symmetric-Monoidal-category. Each symmetric Monoidal category $\mathscr{D}$ defines a symmetric-Monoidal-category in sym-Mono-$\mathcal{C}at$, hence, a symmetric-Monoidal-category $\widehat{\mathscr{D}}$ in $\mathfrak{SMC}$atm. In particular, this holds for $\mathscr{D} = (\mathcal{V}\text{-}\mathcal{C}at, \boxtimes^I, \lambda^f)$, the category of graded $\Bbbk$-linear categories, $\mathcal{V} = \mathbf{gr}(\Bbbk\text{-}\mathbf{Mod})$. So $\widehat{\mathscr{D}} = \widehat{\mathcal{V}\text{-}\mathcal{C}at}$ is a symmetric-Monoidal-category in $\mathfrak{SMC}$atm.

We have seen that $\widehat{\mathscr{D}}$ acts on the symmetric multicategory $\mathscr{A} = \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$, the Kleisli multicategory of graded quivers. It is constructed via the comonad $\widehat{T^{\geqslant 1}} : \widehat{\mathscr{Q}_u} \to \widehat{\mathscr{Q}_u}$ denoted also $T^{\geqslant 1}$. Objects of $\mathscr{A}$ are graded quivers. Multimaps $f : (\mathcal{A}_i)_{i\in I} \to \mathcal{B}$ are

morphisms of quivers $f : \boxtimes_u^{i\in I} T^{\geqslant 1}\mathcal{A}_i \to \mathcal{B}$, or equivalently, morphisms of quivers $\overline{f} : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B}$ such that $f\big|_{\boxtimes^{i\in I}T^0\mathcal{A}_i} = 0$. The bijection between the two presentations is established in (7.9.1) and (7.9.2). The composition in $\mathscr{A}$ is given by (7.11.2), (7.11.3).

We describe the action in particular case $n = 2$, it is a symmetric multifunctor $\boxdot : \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}} \boxtimes \widehat{\mathscr{D}} \to \widehat{\mathscr{Q}_u}^{T^{\geqslant 1}}$. On objects $\mathcal{A} \in \operatorname{Ob}\mathscr{Q}$ and $\mathcal{C} \in \operatorname{Ob}\mathscr{D}$ we have $\mathcal{A}\boxdot\mathcal{C} = \mathcal{A}\boxtimes\mathcal{C} \in \operatorname{Ob}\mathscr{Q}$. To multimorphisms $\overline{f} : \boxtimes^{i\in I} T\mathcal{A}_i \to \mathcal{B} \in \mathscr{A}$, $g : \boxtimes^{i\in I}\mathcal{C}_i \to \mathcal{D} \in \mathscr{D}$ the action $\boxdot$ assigns the multimorphism

$$\begin{aligned}\overline{f \boxdot g} = \big[\boxtimes^{i\in I} T(\mathcal{A}_i \boxtimes \mathcal{C}_i) &\xrightarrow{\boxtimes^I \varkappa} \boxtimes^{i\in I}(T\mathcal{A}_i \boxtimes T\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} \\ &(\boxtimes^{i\in I} T\mathcal{A}_i) \boxtimes (\boxtimes^{i\in I} T\mathcal{C}_i) \xrightarrow{\overline{f}\boxtimes(\boxtimes^I \mu_i)} \mathcal{B} \boxtimes (\boxtimes^{i\in I}\mathcal{C}_i) \xrightarrow{1\boxtimes g} \mathcal{B}\boxtimes\mathcal{D}\big], \quad (C.5.1)\end{aligned}$$

where $\mu_i : T\mathcal{C}_i \to \mathcal{C}_i$ are iterated composition maps for the category $\mathcal{C}_i$.

The extension to $n > 1$ is the following. On objects $\mathcal{C}^0 \in \operatorname{Ob}\mathscr{Q}$ and $\mathcal{C}^p \in \operatorname{Ob}\mathscr{D}$, $1 \leqslant p \leqslant n$, we have $\boxdot^{p\in[n]}\mathcal{C}^p = \boxtimes^{p\in[n]}\mathcal{C}^p \in \operatorname{Ob}\mathscr{Q}$. To multimorphisms $\overline{f} : \boxtimes^{i\in I} T\mathcal{C}_i^0 \to \mathcal{A}^0 \in \mathscr{A}$, $g^p : \boxtimes^{i\in I}\mathcal{C}_i^p \to \mathcal{A}^p \in \mathscr{D}$, $1 \leqslant p \leqslant n$, the action $\boxdot^{[n]}$ assigns the multimorphism

$$\begin{aligned}\overline{f \boxdot g^1 \boxtimes \cdots \boxtimes g^n} = \big[\boxtimes^{i\in I} T(\boxtimes^{p\in[n]}\mathcal{C}_i^p) &\xrightarrow{\boxtimes^{i\in I}\varkappa^{[n]}} \boxtimes^{i\in I}\boxtimes^{p\in[n]} T\mathcal{C}_i^p \xrightarrow{\sigma_{(12)}} \boxtimes^{p\in[n]}\boxtimes^{i\in I} T\mathcal{C}_i^p \\ &\xrightarrow{\boxtimes^{[n]}(\overline{f},\boxtimes^I\mu_i^1,\dots,\boxtimes^I\mu_i^n)} \boxtimes^{[n]}(\mathcal{A}_0, \boxtimes^{i\in I}\mathcal{C}_i^1, \dots, \boxtimes^{i\in I}\mathcal{C}_i^n) \xrightarrow{\boxtimes^{[n]}(1,g^1,\dots,g^n)} \boxtimes^{p\in[n]}\mathcal{A}^p\big],\end{aligned}$$

where $\mu_i^p : T\mathcal{C}_i^p \to \mathcal{C}_i^p$ is the composition map for the category $\mathcal{C}_i^p$.

C.6 Another presentation of an action. Let $\mathscr{A}$ be an object of $\mathcal{C}at$-category $\mathfrak{C}$. Then, as in any 2-category, $\mathfrak{C}(\mathscr{A},\mathscr{A})$ is a strict Monoidal category. We reformulate an action of a Monoidal-category $\mathscr{D}$ on $\mathscr{A}$ in terms of the Monoidal category $\mathfrak{C}(\mathscr{A},\mathscr{A})$.

C.7 Proposition. *Let $(\boxdot^{[n]}, \lambda^f)$ be an action of a Monoidal-category $(\mathscr{D}, \otimes^I, \lambda_{\mathcal{D}}^f)$ on object $\mathscr{A}$ of a Monoidal $\mathcal{C}at$-category $(\mathfrak{C}, \boxtimes^I, \Lambda^f)$. Then the functor*

$$\begin{aligned}\aleph : \mathfrak{C}(\mathbb{1},\mathscr{D}) &\longrightarrow \mathfrak{C}(\mathscr{A},\mathscr{A}), \\ A &\longmapsto \big(\mathscr{A} \xrightarrow[\sim]{\Lambda^{\shortmid\cdot}} \mathscr{A}\boxtimes\mathbb{1} \xrightarrow{1\boxtimes A} \mathscr{A}\boxtimes\mathscr{D} \xrightarrow{\boxdot^{[1]}} \mathscr{A}\big) = (-\boxdot A), \\ h : A \to B &\longmapsto \big(\mathscr{A} \xrightarrow[\sim]{\Lambda^{\shortmid\cdot}} \mathscr{A}\boxtimes\mathbb{1} \overset{1\boxtimes A}{\underset{1\boxtimes B}{\underset{1\boxtimes h\Downarrow}{\rightrightarrows}}} \mathscr{A}\boxtimes\mathscr{D} \xrightarrow{\boxdot^{[1]}} \mathscr{A}\big),\end{aligned}$$

extends to a Monoidal functor $(\aleph, \alpha^{\mathbf{n}})$, where the source category is equipped with the Monoidal structure described in Section A.11.

Proof. Let C_1, …, C_n be objects of $\mathfrak{C}(\mathbb{1},\mathscr{D})$. We have to construct an isomorphism

$$\alpha^{\mathbf{n}} : (((-\boxdot C_1)\boxdot C_2)\boxdot\dots)\boxdot C_n \to -\boxdot(\boxtimes^{i\in\mathbf{n}} C_i).$$

For $m > 0$ define maps $p_m : [m] \to [m-1]$, $p_m(0) = p_m(1) = 0$, $p_m(k) = k-1$ for $0 < k \leqslant m$. Then for $n \geqslant 0$ we have

$$(((- \boxdot C_1) \boxdot C_2) \boxdot \dots) \boxdot C_n = \boxdot^{p_1}(\cdots \boxdot^{p_{n-1}} (\boxdot^{p_n}(-, C_1, C_2, \dots, C_n))).$$

Notice that $\boxdot^{p_1} = \boxdot^{[1]}$. Define a map $q_n : [n] \to [1]$, $q_n(0) = 0$, $q_n(k) = 1$ for $0 < k \leqslant n$. Then

$$- \boxdot (\boxtimes^{i \in \mathbf{n}} C_i) = \boxdot^{[1]}(\boxdot^{q_n}(-, C_1, C_2, \dots, C_n)).$$

Define $\alpha^{\mathbf{n}}$ as the unique 2-isomorphism such that the following equation between 2-isomorphisms holds:

$$\lambda^{q_n} = \bigl[\boxdot^{[n]} \xrightarrow{\lambda^{(p_i)_{i=n}^{1}}} \boxdot^{p_n} \cdot \boxdot^{p_{n-1}} \cdot \ldots \cdot \boxdot^{p_3} \cdot \boxdot^{p_2} \cdot \boxdot^{[1]} \xrightarrow{\alpha^{\mathbf{n}}} \boxdot^{q_n} \cdot \boxdot^{[1]}\bigr] : \boxtimes^{[n]}(\mathscr{A}, \mathscr{D}, \dots, \mathscr{D}) \to \mathscr{A}.$$

For instance, $q_0 = \mathsf{I}\cdot\ : [0] \to [1]$, $q_0(0) = 0$ and $\alpha^{\mathbf{0}} = \lambda^{\mathsf{I}\cdot}\ : \mathrm{Id} = \boxdot^{[0]} \to \boxdot^{q_0} \cdot \boxdot^{[1]}$, $X \to X \boxdot \mathbb{1}$. For $n = 1$ we get $\alpha^{\mathbf{1}} = \mathrm{id} : \boxdot^{[1]} \to \boxdot^{[1]}$. For $n = 2$ we obtain

$$\begin{aligned}\alpha^{\mathbf{2}} = \bigl[(- \boxdot C_1) \boxdot C_2 = \boxdot^{[1]}(\boxdot^{\mathsf{VI}}(-, C_1, C_2)) &\xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} \boxdot^{[2]} (-, C_1, C_2) \\ &\xrightarrow{\lambda^{\mathsf{IV}}} \boxdot^{[1]} (\boxdot^{\mathsf{IV}}(-, C_1, C_2)) = - \boxdot (C_1 \boxtimes C_2)\bigr], \quad \text{(C.7.1)}\end{aligned}$$

that is,

$$\alpha^{\mathbf{2}} = \bigl[\boxdot^{\mathsf{VI}} \cdot \boxdot^{[1]} \xrightarrow{(\lambda^{\mathsf{VI}})^{-1}} \boxdot^{[2]} \xrightarrow{\lambda^{\mathsf{IV}}} \boxdot^{\mathsf{IV}} \cdot \boxdot^{[1]}\bigr].$$

We have to prove that 2-isomorphisms α satisfy for each non-decreasing map $\phi : I \to J$ the equation

$$\begin{tikzcd} \bullet^{j\in J} \bullet^{i\in\phi^{-1}j}(- \boxdot C_i) \arrow[r, equal] \arrow[d, "\bullet^{j\in J}\alpha^{\phi^{-1}j}"'] & \bullet^{i\in I}(- \boxdot C_i) \arrow[r, "\alpha^I"] \arrow[d, phantom, "="] & - \boxdot (\boxtimes^{i\in I} C_i) \arrow[d, "-\boxdot\nu^{\phi}"] \\ \bullet^{j\in J}(- \boxdot \boxtimes^{i\in\phi^{-1}j} C_i) \arrow[rr, "\alpha^J"] & & - \boxdot (\boxtimes^{j\in J} \boxtimes^{i\in\phi^{-1}j} C_i) \end{tikzcd}$$

Clearly, $\bullet^{j\in J}\alpha^{\phi^{-1}j}$ is the horizontal composition of 2-morphisms $\alpha^{\phi^{-1}j}$.

Denote in this proof by $\phi' : [I] \to [J]$ the map $\phi'(0) = 0$, $\phi'(i) = \phi(i)$ for $i \in I$. Denote by r_j the map $r_j = \mathrm{id}_{\mathbf{1}} \sqcup \triangleright \sqcup \mathrm{id} : \mathbf{1} \sqcup \phi^{-1}j \sqcup \bigsqcup_{k>j} \phi^{-1}k \to \mathbf{1} \sqcup \mathbf{1} \sqcup \bigsqcup_{k>j} \phi^{-1}k$ for $j \in J$. We may assume that $I = \mathbf{n}$, $J = \mathbf{m}$. The above equation can be equivalently written as follows:

$$\begin{tikzcd} \bullet^{j\in\mathbf{m}} \bullet^{i\in\phi^{-1}j} \boxdot^{p_{n+1-i}} \arrow[r, equal] \arrow[d, "\bullet^{j\in\mathbf{m}}\alpha^{\phi^{-1}j}"'] & \bullet^{i\in\mathbf{n}} \boxdot^{p_{n+1-i}} \arrow[r, "\alpha^{\mathbf{n}}"] \arrow[d, phantom, "="] & \boxdot^{q_n} \cdot \boxdot^{[1]} \arrow[d, "(\lambda^{(\phi', q_m)})\boxdot^{[1]}"] \\ \bullet^{j\in\mathbf{m}}(\boxdot^{r_j} \cdot \boxdot^{p_{1\sqcup\phi^{-1}]j,m]}}) \arrow[rr, "\alpha^{\mathbf{m}}"] & & \boxdot^{\phi'} \cdot \boxdot^{q_m} \cdot \boxdot^{[1]} \end{tikzcd}$$

To prove this equation we compose it with

$$\leftthreetimes^{(p_i)_{i=n}^{1}} = \Bigl[\boxdot^{[n]} \xrightarrow{\leftthreetimes^{(v_j)_{j\in\mathbf{m}}}} \bullet^{j\in\mathbf{m}} \boxdot^{v_j} \xrightarrow{\bullet^{j\in\mathbf{m}}\leftthreetimes^{(p_{n+1-i})_{i\in\phi^{-1}j}}} \bullet^{j\in\mathbf{m}} \bullet^{i\in\phi^{-1}j} \boxdot^{p_{n+1-i}} = \bullet^{i\in\mathbf{n}} \boxdot^{p_{n+1-i}}\Bigr],$$

where

$$\begin{aligned} v_j = \bullet^{i\in\phi^{-1}j} p_{n+1-i} &= \triangleright \sqcup \mathrm{id} \\ &= \Bigl[\mathbf{1} \sqcup \phi^{-1}j \sqcup \bigsqcup_{k>j} \phi^{-1}k \xrightarrow[\mathrm{id}_{\mathbf{1}} \sqcup \triangleright \sqcup \mathrm{id}]{r_j} \mathbf{1} \sqcup \mathbf{1} \sqcup \bigsqcup_{k>j} \phi^{-1}k \xrightarrow[\triangleright \sqcup \mathrm{id}]{p_{1\sqcup\phi^{-1}]j,m]}} \mathbf{1} \sqcup \bigsqcup_{k>j} \phi^{-1}k\Bigr], \end{aligned}$$

see Proposition B.4.

Using the definition of $\alpha^{\mathbf{n}}$ we reduce the equation to prove to the following commutative diagram:

$$\begin{tikzcd}
 & & \boxdot^{[n]} \arrow[dll, "\leftthreetimes^{(v_j)_{j\in\mathbf{m}}}"'] \arrow[d, "\lambda^{\phi'}"'] \arrow[drr, "\lambda^{q_n}"] & & \\
\bullet^{j\in\mathbf{m}} \boxdot^{v_j} \arrow[d, "\bullet^{j\in\mathbf{m}}\leftthreetimes^{(r_j, p_{1\sqcup\phi^{-1}]j,m]})}"'] & & \boxdot^{\phi'} \cdot \boxdot^{[m]} \arrow[d, "\leftthreetimes^{(p_j)_{j=m}^{1}}"'] \arrow[drr, "\lambda^{q_m}"] & & \boxdot^{q_n} \cdot \boxdot^{[1]} \arrow[d, "(\leftthreetimes^{(\phi', q_m)})\boxdot^{[1]}"] \\
\bullet^{j\in\mathbf{m}} \bigl(\boxdot^{r_j} \cdot \boxdot^{p_{1\sqcup\phi^{-1}]j,m]}}\bigr) \arrow[rr, equal] & & \boxdot^{\phi'} \cdot \bigl(\bullet^{j\in\mathbf{m}} \boxdot^{p_j}\bigr) \arrow[rr, "\alpha^{\mathbf{m}}"] & & \boxdot^{\phi'} \cdot \boxdot^{q_m} \cdot \boxdot^{[1]}
\end{tikzcd}$$

The right square commutes due to condition (B.2.1). The left pentagon commutes due to the independence of $\leftthreetimes$ of triangulation of a polygon, as explained before Proposition B.4. The proposition is proven. □

C.8 An algebra produces a multifunctor. As above, let $\mathscr{D}$ denote the symmetric Monoidal category $\mathscr{D} = (\mathbf{gr}\text{-}\mathcal{C}at, \boxtimes^I, \lambda^f)$ of graded $\Bbbk$-linear categories. Recall that $\mathsf{Q} = {}^{[1]}\widehat{\mathscr{Q}}_u^{T^{\geqslant 1}}$ is the symmetric multicategory, whose objects are graded $\Bbbk$-linear quivers, and multimaps are

$$\begin{aligned} \mathsf{Q}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) = \widehat{\mathscr{Q}}_u^{T^{\geqslant 1}}((\mathcal{A}_i[1])_{i\in I}; \mathcal{B}[1]) &= \mathscr{Q}(\boxtimes_u^{i\in I} T^{\geqslant 1}(\mathcal{A}_i[1]), \mathcal{B}[1]) \\ &\simeq \{f \in \mathscr{Q}(\boxtimes^{i\in I} T(\mathcal{A}_i[1]), \mathcal{B}[1]) \mid f\big|_{\boxtimes^{i\in I} T^0\mathcal{A}_i} = 0\}. \end{aligned}$$

The multiquiver Q is isomorphic to $\mathscr{A} = \widehat{\mathscr{Q}}_u^{T^{\geqslant 1}}$ via the shift map $\mathcal{A} \mapsto s\mathcal{A} = \mathcal{A}[1]$. The action $(\boxdot^{[n]} : \boxtimes^{[n]}(\mathscr{A}, \widehat{\mathscr{D}}, \dots, \widehat{\mathscr{D}}) \to \mathscr{A}, \lambda^f)$ translates via this isomorphism to the action $(\boxdot^{[n]} : \boxtimes^{[n]}(\mathsf{Q}, \widehat{\mathscr{D}}, \dots, \widehat{\mathscr{D}}) \to \mathsf{Q}, \lambda^f)$. In particular, as quivers $(\mathcal{A} \boxdot \mathcal{C})[1] = \mathcal{A}[1] \boxtimes \mathcal{C}$ and $\mathcal{A} \boxdot \mathcal{C} = (\mathcal{A}[1] \boxtimes \mathcal{C})[-1]$. In general, on objects $\mathcal{A} \in \operatorname{Ob}\mathsf{Q}$, $\mathcal{C}^p \in \operatorname{Ob}\mathscr{D}$, $1 \leqslant p \leqslant n$, we have

$$\begin{gathered} \boxdot^{[n]}(\mathcal{A}, \mathcal{C}^1, \dots, \mathcal{C}^n) = \{\boxtimes^{[n]}(\mathcal{A}[1], \mathcal{C}^1, \dots, \mathcal{C}^n)\}[-1], \\ \{\boxdot^{[n]}(\mathcal{A}, \mathcal{C}^1, \dots, \mathcal{C}^n)\}[1] = \boxtimes^{[n]}(\mathcal{A}[1], \mathcal{C}^1, \dots, \mathcal{C}^n) \end{gathered}$$

as quivers.

Suppose that $A : \mathbb{1} \to \mathscr{D}$ is an algebra in $\mathscr{D}$. In details: $\mathbb{1}$ is the Monoidal category with one object $*$ and one morphism 1, A is a graded $\Bbbk$-linear category, equipped with a strictly associative multiplication functor $\otimes_A : A \boxtimes A \to A$ and the strict unit functor $\eta_A : \mathbb{1}_p \to A$. In other words, A is a strictly monoidal graded $\Bbbk$-linear category such that the tensor product factors as $A \times A \to A \boxtimes A \xrightarrow{\otimes_A} A$, through a graded functor $\otimes_A$.

The algebra A defines an algebra $\widehat{A} : \widehat{\mathbb{1}} \to \widehat{\mathscr{D}}$ in the sense of multicategories (a multifunctor, see Definition 3.20). Any multifunctor $B : \widehat{\mathbb{1}} \to \widehat{\mathscr{D}}$ (an algebra in $\widehat{\mathscr{D}}$) equals $\widehat{A}$ for some algebra $A : \mathbb{1} \to \mathscr{D}$ (Proposition 3.30). The functor of Proposition C.7 $\mathcal{MC}\mathrm{atm}(\widehat{\mathbb{1}}, \widehat{\mathscr{D}}) \to \mathcal{MC}\mathrm{atm}(\mathsf{Q}, \mathsf{Q})$ takes an object $\widehat{A}$ of the source to the object $F = 1 \boxdot \widehat{A} : \mathsf{Q} \to \mathsf{Q}$ of the target, which is a multifunctor.

In details: the multifunctor $\widehat{A}$ takes the only object $* \in \mathrm{Ob}\,\mathbb{1}$ to the object $A \in \mathrm{Ob}\,\mathscr{D}$. The map

$$\widehat{A} = \bigl[\widehat{\mathbb{1}}((*)_{i\in I}; *) \xrightarrow{\eta_A} \mathscr{D}(A, A) \xrightarrow{\mathscr{D}(\otimes_A^I, 1)} \mathscr{D}(\boxtimes^I A, A) = \widehat{\mathscr{D}}((A)_{i\in I}; A)\bigr]$$

is found as $1 \mapsto \otimes_A^I$ from (3.28.1). The resulting multifunctor F is the composition

$$F = 1 \boxdot \widehat{A} = \bigl(\mathsf{Q} \xrightarrow[\sim]{\Lambda^{\mathsf{I}}\cdot} \mathsf{Q} \boxtimes \widehat{\mathbb{1}} \xrightarrow{1\boxtimes\widehat{A}} \mathsf{Q} \boxtimes \widehat{\mathscr{D}} \xrightarrow{\boxdot} \mathsf{Q}\bigr).$$

It takes a quiver $\mathcal{Q} \in \mathrm{Ob}\,\mathsf{Q}$ to the quiver $\mathcal{Q} \boxdot A$. The multifunctor F operates on morphisms via the map

$$1 \boxdot \otimes_A^I = \bigl[\mathsf{Q}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \xrightarrow[\sim]{\lambda^{\mathsf{I}}\cdot} \mathsf{Q}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \times \widehat{\mathbb{1}}((*)_{i\in I}; *) \xrightarrow{1\times\widehat{A}}$$
$$\mathsf{Q}((\mathcal{A}_i)_{i\in I}; \mathcal{B}) \times \widehat{\mathscr{D}}((A)_{i\in I}; A) \xrightarrow{\boxdot} \mathsf{Q}((\mathcal{A}_i \boxdot A)_{i\in I}; \mathcal{B} \boxdot A)\bigr], \qquad f \mapsto f \boxdot (\otimes_A^I). \quad \text{(C.8.1)}$$

If A were commutative, then $\widehat{A}$ would be a commutative algebra in the source of $\mathcal{SMC}\mathrm{atm}(\widehat{\mathbb{1}}, \widehat{\mathscr{D}}) \to \mathcal{SMC}\mathrm{atm}(\mathsf{Q}, \mathsf{Q})$. So it would define a symmetric multifunctor $1 \boxdot \widehat{A} : \mathsf{Q} \to \mathsf{Q}$ which is a commutative multimonad. Our main example of Section 10.1 is not of this kind.

C.9 Remark. One can show that there is no algebra morphism $\theta : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z}$, $m \times n \mapsto m + n$, with invertible maps $\theta : (\mathcal{Z} \boxtimes \mathcal{Z})(n \times m, k \times l) \to \mathcal{Z}(n + m, k + l)$. In particular, $\mathcal{Z}$ is not commutative. The shift multifunctor $-^{[]} = 1 \boxdot \widehat{\mathcal{Z}} : \mathsf{Q} \to \mathsf{Q}$ corresponds to $\mathcal{Z}$ via construction of Section C.8. The algebra morphism $\eta_{\mathcal{Z}} : \mathbb{1} \to \mathcal{Z}$ gives a multinatural transformation of multifunctors $u_{[]} : \mathrm{Id}_{\mathsf{Q}} \to -^{[]}$. However, no multinatural transformation of multifunctors $-^{[][]} \to -^{[]}$ comes from algebra $\mathcal{Z}$.

Nevertheless, there is a graded functor $\otimes = \otimes^1 : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z}$ – a morphism of $\mathscr{D}$. It can be viewed formally as a (non-Monoidal) morphism between functors $\mathcal{Z} \boxtimes \mathcal{Z} : \mathbb{1} \to \mathscr{D}$ and $\mathcal{Z} : \mathbb{1} \to \mathscr{D}$, that is, an element of $\mathcal{C}at(\mathbb{1}, \mathscr{D})(\mathcal{Z} \boxtimes \mathcal{Z}, \mathcal{Z})$. The Monoidal functor

$$\mathcal{C}at(\mathbb{1}, \mathscr{D}) \to \mathcal{C}at(\mathsf{Q}_1, \mathsf{Q}_1)$$

constructed in Proposition C.7 takes this morphism to a morphism of functors $m_{[]} : -^{[][]} = (1 \boxdot \widehat{\mathcal{Z}})^2 \to -^{[]} : \mathsf{Q}_1 \to \mathsf{Q}_1$, where Q_1 is the underlying category of the multicategory Q (the shifted Kleisli category of quivers). The action functor $\boxdot : \mathsf{Q}_1 \boxtimes \mathscr{D} \to \mathsf{Q}_1$ is the restriction of the action multifunctor $\boxdot : \mathsf{Q} \boxtimes \widehat{\mathscr{D}} \to \mathsf{Q}$. The associativity and the unitality of $\otimes^1 : \mathcal{Z} \boxtimes \mathcal{Z} \to \mathcal{Z}$ imply the associativity and the unitality of $m_{[]}$. Therefore, $(-^{[]}, m_{[]}, u_{[]})$ is a monad in Q_1 such that the functor $-^{[]}$ lies under a multifunctor $-^{[]} : \mathsf{Q} \to \mathsf{Q}$, and $u_{[]} : \mathrm{Id}_{\mathsf{Q}} \to -^{[]}$ is a multinatural transformation of multifunctors. Moreover, it is shown in Section 10.20 that $m_{[]}$ satisfies condition (4.35.3). According to Corollary 4.35 the algebra $\mathcal{Z}$ in $\mathscr{D}$ produces a Q-multifunctor $-^{[]} = (1 \boxdot \widehat{\mathcal{Z}})' : \underline{\mathsf{Q}} \to \underline{\mathsf{Q}}$ and a Q-monad $(-^{[]}, m_{[]}, u_{[]}) : \underline{\mathsf{Q}} \to \underline{\mathsf{Q}}$.

C.10 Action of differential graded categories on A_∞-categories. The action $\boxdot : \mathsf{Q} \boxtimes \widehat{\mathbf{gr}\text{-}\mathcal{C}at} \to \mathsf{Q}$ in $\mathcal{SMC}$atm constructed in Section C.8 extends to the action in $\mathcal{SMC}$atm

$$\boxdot : \mathsf{A}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}_\infty .$$

To objects $\mathcal{A} \in \operatorname{Ob} \mathsf{A}_\infty$ and $\mathcal{C} \in \operatorname{Ob} \mathbf{dg}\text{-}\mathcal{C}at$ this multifunctor assigns the quiver $\mathcal{A} \boxdot \mathcal{C} \in \operatorname{Ob} \mathscr{Q}$, equipped with the differential $b^{\mathcal{A}\boxdot\mathcal{C}} : \mathrm{id} \to \mathrm{id} : \mathcal{A} \boxdot \mathcal{C} \to \mathcal{A} \boxdot \mathcal{C}$, which is an element of $\underline{\mathscr{Q}_p}(T(s\mathcal{A} \boxtimes \mathcal{C}), s\mathcal{A} \boxtimes \mathcal{C})(\mathrm{id}, \mathrm{id})$, specified by its components:

$$b_1^{\mathcal{A}\boxdot\mathcal{C}} = b_1 \boxtimes 1 - 1 \boxtimes d : s\mathcal{A} \boxtimes \mathcal{C} \to s\mathcal{A} \boxtimes \mathcal{C}, \tag{C.10.1}$$
$$b_n^{\mathcal{A}\boxdot\mathcal{C}} = \bigl[T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\varkappa} T^n s\mathcal{A} \boxtimes T^n \mathcal{C} \xrightarrow{b_n^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr], \quad n > 1.$$

C.11 Proposition. *$b^{\mathcal{A}\boxdot\mathcal{C}}$ is a codifferential.*

Proof. Let us prove that $\mathcal{A} \boxdot \mathcal{C}$ is a differential A_∞-category. This means the existence for $s\mathcal{E} = s\mathcal{A} \boxtimes \mathcal{C}$ of two anticommuting codifferentials $\hat{b}' : Ts\mathcal{E} \to Ts\mathcal{E}$, $\hat{d}' : Ts\mathcal{E} \to Ts\mathcal{E}$, where the components d'_n of the second codifferential vanish if $n \neq 1$. In our case d' is specified by $d'_1 = 1 \boxtimes d : s\mathcal{A} \boxtimes \mathcal{C} \to s\mathcal{A} \boxtimes \mathcal{C}$, and $b' : T^{\geqslant 1}(s\mathcal{A} \boxtimes \mathcal{C}) \to s\mathcal{A} \boxtimes \mathcal{C}$ is given by $b'_1 = b_1^{\mathcal{A}} \boxtimes 1$, $b'_n = b_n^{\mathcal{A}\boxdot\mathcal{C}}$.

As $d^2 = 0$ we have $(\hat{d}')^2 = 0$. The anticommutativity condition $\hat{b}'\hat{d}' + \hat{d}'\hat{b}' = 0$ reduces to equations

$$b'_n d + \sum_{i=1}^{n} (1^{\otimes(i-1)} \otimes d \otimes 1^{\otimes(n-i)}) b'_n = 0$$

for all $n \geqslant 1$. In our case we find in $\underline{\mathscr{Q}_p}$:

$$\sum_{i=1}^{n}(1^{\otimes(i-1)} \otimes d \otimes 1^{\otimes(n-i)})b'_n$$
$$= \sum_{i=1}^{n}\bigl[T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},1\otimes d,(1)_{p>i}]} T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\overline{\varkappa}} T^n s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{b_n^{\mathcal{A}}\boxtimes\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr]$$
$$= -\sum_{i=1}^{n}\bigl[T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\overline{\varkappa}} T^n s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{b_n^{\mathcal{A}}\boxtimes\otimes^{\mathbf{n}}[(1)_{p<i},d,(1)_{p>i}]} s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{1\boxtimes\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr]$$
$$= -\bigl[T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\overline{\varkappa}} T^n s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{b_n^{\mathcal{A}}\boxtimes\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C} \xrightarrow{1\boxtimes d} s\mathcal{A} \boxtimes \mathcal{C}\bigr] = -b'_n d$$

due to $\mathcal{C}$ being a differential graded category.

Let us show that $\hat{b}'$ is a codifferential, that is,

$$\sum_{i-1+j+k=m}(1^{\otimes(i-1)} \otimes b'_j \otimes 1^{\otimes k})b'_{i+k} = 0 : T^n(s\mathcal{A} \boxtimes \mathcal{C}) \to s\mathcal{A} \boxtimes \mathcal{C}.$$

This would imply that the difference $\hat{b}' - \hat{d}'$ is a codifferential. Define the map $g = \mathrm{id} \sqcup \triangleright \sqcup \mathrm{id} : \mathbf{m} = \mathbf{i-1} \sqcup \mathbf{j} \sqcup \mathbf{k} \to \mathbf{i-1} \sqcup \mathbf{1} \sqcup \mathbf{k} = \mathbf{n}$, depending on m, i, j, k, n, such that $m = i-1+j+k$ and $n = i+k$. The summands of the above sum are in bijection with pairs (g, i), where $g : \mathbf{m} \twoheadrightarrow \mathbf{n}$ is a surjection and $i \in \mathbf{n}$ is such that $g| : \mathbf{m} \smallsetminus g^{-1}\{i\} \to \mathbf{n} \smallsetminus \{i\}$ is bijective. Such a summand is

$$\bigl\{T^m(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p}(s\mathcal{A} \boxtimes \mathcal{C})$$
$$\xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},\varkappa,(1)_{p>i}]} \otimes^{\mathbf{n}} \bigl[(s\mathcal{A} \boxtimes \mathcal{C})_{p<i}, \otimes^{g^{-1}i}s\mathcal{A} \boxtimes \otimes^{g^{-1}i}\mathcal{C}, (s\mathcal{A} \boxtimes \mathcal{C})_{p>i}\bigr]$$
$$\xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},b_j^{\mathcal{A}}\boxtimes\mu_{\mathcal{C}}^{g^{-1}i},(1)_{p>i}]} T^n(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\varkappa} T^n s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{b_n^{\mathcal{A}}\boxtimes\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr\}$$
$$= \bigl\{T^m(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\lambda^g} \otimes^{p\in\mathbf{n}} \otimes^{g^{-1}p}(s\mathcal{A} \boxtimes \mathcal{C})$$
$$\xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},\varkappa,(1)_{p>i}]} \otimes^{\mathbf{n}} \bigl[(s\mathcal{A} \boxtimes \mathcal{C})_{p<i}, \otimes^{g^{-1}i}s\mathcal{A} \boxtimes \otimes^{g^{-1}i}\mathcal{C}, (s\mathcal{A} \boxtimes \mathcal{C})_{p>i}\bigr]$$
$$\xrightarrow{\varkappa} \otimes^{\mathbf{n}} \bigl[(s\mathcal{A})_{p<i}, \otimes^{g^{-1}i}s\mathcal{A}, (s\mathcal{A})_{p>i}\bigr] \boxtimes \otimes^{\mathbf{n}}\bigl[(\mathcal{C})_{p<i}, \otimes^{g^{-1}i}\mathcal{C}, (\mathcal{C})_{p>i}\bigr]$$
$$\xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},b_j^{\mathcal{A}},(1)_{p>i}]\boxtimes\otimes^{\mathbf{n}}[(1)_{p<i},\mu_{\mathcal{C}}^{g^{-1}i},(1)_{p>i}]} T^n s\mathcal{A} \boxtimes T^n\mathcal{C} \xrightarrow{b_n^{\mathcal{A}}\boxtimes\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr\}$$
$$= \bigl\{T^m(s\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\varkappa} T^m s\mathcal{A} \boxtimes T^m\mathcal{C}$$
$$\xrightarrow{\lambda^g\boxtimes\lambda^g} \otimes^{\mathbf{n}} \bigl[(s\mathcal{A})_{p<i}, \otimes^{g^{-1}i}s\mathcal{A}, (s\mathcal{A})_{p>i}\bigr] \boxtimes \otimes^{\mathbf{n}}\bigl[(\mathcal{C})_{p<i}, \otimes^{g^{-1}i}\mathcal{C}, (\mathcal{C})_{p>i}\bigr]$$
$$\xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},b_j^{\mathcal{A}},(1)_{p>i}]b_n^{\mathcal{A}}\boxtimes\otimes^{\mathbf{n}}[(1)_{p<i},\mu_{\mathcal{C}}^{g^{-1}i},(1)_{p>i}]\mu_{\mathcal{C}}^{\mathbf{n}}} s\mathcal{A} \boxtimes \mathcal{C}\bigr\} \quad \text{(C.11.1)}$$

due to coherence principle of Remark 2.34. Since the category $\mathcal{C}$ is an algebra in the Monoidal category of $\Bbbk$-quivers with the object set $\operatorname{Ob}\mathcal{C}$, we have

$$\bigl[T^m\mathcal{C} \xrightarrow{\lambda^g} \otimes^{\mathbf{n}}[(\mathcal{C})_{p<i}, \otimes^{g^{-1}i}\mathcal{C}, (\mathcal{C})_{p>i}] \xrightarrow{\otimes^{\mathbf{n}}[(1)_{p<i},\mu_{\mathcal{C}}^{g^{-1}i},(1)_{p>i}]} T^n\mathcal{C} \xrightarrow{\mu_{\mathcal{C}}^{\mathbf{n}}} \mathcal{C}\bigr] = \mu_{\mathcal{C}}^{\mathbf{m}}$$

(that is, composition in $\mathcal{C}$ is associative). Thus (C.11.1) equals

$$\bigl\{T^m(s\mathcal{A}\boxtimes\mathcal{C}) \xrightarrow{\varkappa} T^m s\mathcal{A}\boxtimes T^m\mathcal{C} \xrightarrow{(\lambda^g\cdot\otimes^{\mathbf{n}}[(1)_{p<i},b_j^{\mathcal{A}},(1)_{p>i}]\cdot b_n^{\mathcal{A}})\boxtimes\mu_{\mathcal{C}}^{\mathbf{m}}} s\mathcal{A}\boxtimes\mathcal{C}\bigr\}.$$

The sum of such expressions over all allowed pairs (g,i) gives

$$(\hat{b}')^2\operatorname{pr}_1\big|_{T^m(s\mathcal{A}\boxtimes\mathcal{C})} = \bigl\{T^m(s\mathcal{A}\boxtimes\mathcal{C}) \xrightarrow{\varkappa} T^m s\mathcal{A}\boxtimes T^m\mathcal{C} \xrightarrow{(\hat{b}^{\mathcal{A}})^2\operatorname{pr}_1\boxtimes\mu_{\mathcal{C}}^{\mathbf{m}}} s\mathcal{A}\boxtimes\mathcal{C}\bigr\} = 0,$$

hence, b' is a codifferential. □

To A_∞-functor $\overline{f}: \boxtimes^{i\in I}Ts\mathcal{A}_i \to s\mathcal{B} \in \mathsf{A}_\infty$ and differential graded functor $g: \boxtimes^{i\in I}\mathcal{C}_i \to \mathcal{D} \in \mathscr{D} \overset{\text{def}}{=} \mathbf{dg}\text{-}\mathcal{C}at$ the action $\boxdot$ assigns multimorphism $\overline{f\boxdot g}$ in Q, given by (C.5.1):

$$\overline{f\boxdot g} = \bigl[\boxtimes^{i\in I}T(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\boxtimes^I\varkappa} \boxtimes^{i\in I}(Ts\mathcal{A}_i\boxtimes T\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in I}Ts\mathcal{A}_i)\boxtimes(\boxtimes^{i\in I}T\mathcal{C}_i) \xrightarrow{\overline{f}\boxtimes(\boxtimes^I\mu)} s\mathcal{B}\boxtimes(\boxtimes^{i\in I}\mathcal{C}_i) \xrightarrow{1\boxtimes g} s\mathcal{B}\boxtimes\mathcal{D}\bigr].$$

C.12 Proposition. *The multimorphism $f\boxdot g$ is an A_∞-functor.*

Proof. The A_∞-structures in question are differences of two A_∞-structures, b' and d'. Let us prove that $f\boxdot g$ is an A_∞-functor with respect to both these A_∞-structures, hence for their difference as well. First we show this for d'. The only non-vanishing component of $\hat{d}'$ is the first one, thus, equation (8.8.1) in $\underline{\mathscr{Q}_p}$ reads

$$\begin{aligned}
&\bigl[\boxtimes^{i\in\mathbf{n}}T^{m_i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\overline{f\boxdot g}} s\mathcal{B}\boxtimes\mathcal{D} \xrightarrow{1\boxtimes d} s\mathcal{B}\boxtimes\mathcal{D}\bigr] \\
&= \bigl[\boxtimes^{i\in\mathbf{n}}T^{m_i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\sum_{i=1}^n 1^{\boxtimes(i-1)}\boxtimes(\sum_{j=1}^{m_i}1^{\otimes(j-1)}\otimes(1\boxtimes d)\otimes 1^{\otimes(m_i-j)})\boxtimes 1^{\boxtimes(n-i)}} \\
&\qquad \boxtimes^{i\in\mathbf{n}}T^{m_i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\overline{f\boxdot g}} s\mathcal{B}\boxtimes\mathcal{D}\bigr]. \qquad \text{(C.12.1)}
\end{aligned}$$

Since g is a chain functor, the left hand side equals

$$\begin{aligned}
&\bigl[\boxtimes^{i\in\mathbf{n}}T^{m_i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}(T^{m_i}s\mathcal{A}_i\boxtimes T^{m_i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}}T^{m_i}s\mathcal{A}_i)\boxtimes(\boxtimes^{i\in\mathbf{n}}T^{m_i}\mathcal{C}_i) \\
&\qquad \xrightarrow{\overline{f}\boxtimes(\boxtimes^{\mathbf{n}}\mu)} s\mathcal{B}\boxtimes(\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes\sum_{i=1}^n 1^{\boxtimes(i-1)}\boxtimes d\boxtimes 1^{\otimes(n-i)}} s\mathcal{B}\boxtimes(\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes g} s\mathcal{B}\boxtimes\mathcal{D}\bigr] \\
&= \bigl[\boxtimes^{i\in\mathbf{n}}T^{m_i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}(T^{m_i}s\mathcal{A}_i\boxtimes T^{m_i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}}T^{m_i}s\mathcal{A}_i)\boxtimes(\boxtimes^{i\in\mathbf{n}}T^{m_i}\mathcal{C}_i) \\
&\qquad \xrightarrow{1\boxtimes\sum_{i=1}^n 1^{\boxtimes(i-1)}\boxtimes(\sum_{j=1}^{m_i}1^{\otimes(j-1)}\otimes d\otimes 1^{\otimes(m_i-j)})\boxtimes 1^{\boxtimes(n-i)}} (\boxtimes^{i\in\mathbf{n}}T^{m_i}s\mathcal{A}_i)\boxtimes(\boxtimes^{i\in\mathbf{n}}T^{m_i}\mathcal{C}_i) \\
&\qquad \xrightarrow{\overline{f}\boxtimes(\boxtimes^{\mathbf{n}}\mu)} s\mathcal{B}\boxtimes(\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes g} s\mathcal{B}\boxtimes\mathcal{D}\bigr],
\end{aligned}$$

which is equal to the right hand side of (C.12.1).

Now we shall prove that $f \boxdot g$ is an A_∞-functor with respect to the A_∞-structure b'. Let $(\ell^i)_{i\in\mathbf{n}} \in (\mathbb{Z}_{\geqslant 0})^n \smallsetminus \{0\}$. A partition $j_1+\dots+j_m = (\ell^i)_{i\in\mathbf{n}}$, where $j_1,\dots,j_m \in (\mathbb{Z}_{\geqslant 0})^n \smallsetminus \{0\}$, determines a collection of isotonic maps $g_i : \ell^{\mathbf{i}} \to \mathbf{m}$, $i \in \mathbf{n}$, such that $|g_i^{-1}(p)| = j^i_p$, $p \in \mathbf{m}$, $i \in \mathbf{n}$. The summand in the left hand side of (8.8.1) corresponding to the chosen partition reads

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{i\in\mathbf{n}}\lambda^{g_i}} \boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{m}} \otimes^{g_i^{-1}(p)} (s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\overline{\varkappa}^{-1}}$$
$$\otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} \otimes^{g_i^{-1}(p)} (s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\otimes^{p\in\mathbf{m}}\boxtimes^{\mathbf{n}}\varkappa} \otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} (T^{g_i^{-1}(p)} s\mathcal{A}_i \boxtimes T^{g_i^{-1}(p)}\mathcal{C}_i) \xrightarrow{\otimes^{p\in\mathbf{m}}\sigma_{(12)}}$$
$$\otimes^{p\in\mathbf{m}} ((\boxtimes^{i\in\mathbf{n}} T^{g_i^{-1}(p)} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{g_i^{-1}(p)}\mathcal{C}_i)) \xrightarrow{\boxtimes^{p\in\mathbf{m}}(f_{j_p}\boxtimes(\boxtimes^{i\in\mathbf{n}}\mu^{g_i^{-1}(p)}))} \otimes^{p\in\mathbf{m}} (s\mathcal{B} \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i))$$
$$\xrightarrow{\otimes^{p\in\mathbf{m}}(1\boxtimes g)} \otimes^{p\in\mathbf{m}} (s\mathcal{B} \boxtimes \mathcal{D}) \xrightarrow{\varkappa} T^m s\mathcal{B} \boxtimes T^m\mathcal{D} \xrightarrow{b_m\boxtimes\mu^{\mathbf{m}}} s\mathcal{B} \boxtimes \mathcal{D}\Big].$$

Due to coherence principle of Remark 2.34 and the fact that g is a functor, it is equal to

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}} (T^{\ell^i} s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}} T^{\ell_i} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i)$$
$$\xrightarrow{(\boxtimes^{i\in\mathbf{n}}\lambda^{g_i})\boxtimes(\boxtimes^{i\in\mathbf{n}}\lambda^{g_i})} (\boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{m}} T^{g_i^{-1}(p)} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{m}} T^{g_i^{-1}(p)}\mathcal{C}_i)$$
$$\xrightarrow{\overline{\varkappa}^{-1}\boxtimes\overline{\varkappa}^{-1}} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} T^{g_i^{-1}(p)} s\mathcal{A}_i) \boxtimes (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} T^{g_i^{-1}(p)}\mathcal{C}_i)$$
$$\xrightarrow{\otimes^{p\in\mathbf{m}} f_{j_p}\boxtimes(\otimes^{p\in\mathbf{m}}\boxtimes^{i\in\mathbf{n}}\mu^{g_i^{-1}(p)})} (\otimes^{p\in\mathbf{m}} s\mathcal{B}) \boxtimes (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} \mathcal{C}_i) \xrightarrow{1\boxtimes\varkappa} (\otimes^{p\in\mathbf{m}} s\mathcal{B}) \boxtimes (\boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{m}} \mathcal{C}_i)$$
$$\xrightarrow{1\boxtimes(\boxtimes^{i\in\mathbf{n}}\mu^{\mathbf{m}})} (\otimes^{p\in\mathbf{m}} s\mathcal{B}) \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes g} (\otimes^{p\in\mathbf{m}} s\mathcal{B}) \boxtimes \mathcal{D} \xrightarrow{b_m\boxtimes 1} s\mathcal{B} \boxtimes \mathcal{D}\Big]. \quad \text{(C.12.2)}$$

Since the category $\mathcal{C}_i$ is an algebra in the Monoidal category of $\Bbbk$-quivers with the object set $\operatorname{Ob}\mathcal{C}_i$, we have

$$\big[T^{\ell^i}\mathcal{C}_i \xrightarrow{\lambda^{g_i}} \otimes^{i\in\mathbf{m}} \otimes^{g_i^{-1}(p)}\mathcal{C}_i \xrightarrow{\otimes^{i\in\mathbf{m}}\mu^{g_i^{-1}(p)}} T^m\mathcal{C}_i \xrightarrow{\mu^{\mathbf{m}}} \mathcal{C}_i\big] = \mu^{\ell^{\mathbf{i}}},$$

therefore (C.12.2) equals

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}} (T^{\ell^i} s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}} T^{\ell^i} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i) \xrightarrow{(\boxtimes^{i\in\mathbf{n}}\lambda^{g_i})\boxtimes 1}$$
$$(\boxtimes^{i\in\mathbf{n}} \otimes^{p\in\mathbf{m}} \otimes^{g_i^{-1}(p)} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i) \xrightarrow{\overline{\varkappa}^{-1}\boxtimes(\boxtimes^{i\in\mathbf{n}}\mu^{\ell^{\mathbf{i}}})} (\otimes^{p\in\mathbf{m}} \boxtimes^{i\in\mathbf{n}} \otimes^{g_i^{-1}(p)} s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i)$$
$$\xrightarrow{(\otimes^{p\in\mathbf{m}} f_{j_p})\boxtimes g} (\otimes^{p\in\mathbf{m}} s\mathcal{B}) \boxtimes \mathcal{D} \xrightarrow{b_m\boxtimes 1} s\mathcal{B} \boxtimes \mathcal{D}\Big]. \quad \text{(C.12.3)}$$

Let us transform the right hand side of equation (8.8.1). Let $1 \leqslant q \leqslant n$ be fixed, and let r, k, t be integers such that $r+k+t = l^q$, $r,t \geqslant 0$, $k \geqslant 1$. Put $p = r+1+t$ and denote

$\phi = \mathrm{id} \sqcup \triangleright \sqcup \mathrm{id} : \ell^{\mathbf{q}} = \mathbf{r} \sqcup \mathbf{k} \sqcup \mathbf{t} \to \mathbf{r} \sqcup \mathbf{1} \sqcup \mathbf{t} = \mathbf{p}$. The corresponding summand in the right hand side of (8.8.1) is

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\lambda^\phi,(1)_{i>q}]}$$
$$\boxtimes^{\mathbf{n}}\big[(T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i<q}, \otimes^{\mathbf{p}}[(s\mathcal{A}_q \boxtimes \mathcal{C}_q)_{j\in\mathbf{r}}, T^k(s\mathcal{A}_q \boxtimes \mathcal{C}_q), (s\mathcal{A}_q \boxtimes \mathcal{C}_q)_{j\in\mathbf{t}}], (T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i>q}\big]$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},\varkappa,(1)_{j\in\mathbf{t}}],(1)_{i>q}]}$$
$$\boxtimes^{\mathbf{n}}\big[(T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i<q}, \otimes^{\mathbf{p}}[(s\mathcal{A}_q \boxtimes \mathcal{C}_q)_{j\in\mathbf{r}}, T^k s\mathcal{A}_q \boxtimes T^k\mathcal{C}_q, (s\mathcal{A}_q \boxtimes \mathcal{C}_q)_{j\in\mathbf{t}}], (T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i>q}\big]$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},b_k\boxtimes\mu^{\mathbf{k}},(1)_{j\in\mathbf{t}}],(1)_{i>q}]} \boxtimes^{\mathbf{n}}\big[(T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i<q}, T^p(s\mathcal{A}_q \boxtimes \mathcal{C}_q), (T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i))_{i>q}\big]$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i)_{i<q}, T^p s\mathcal{A}_q \boxtimes T^p\mathcal{C}_q, (T^{\ell^i}s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i)_{i>q}\big]$$
$$\xrightarrow{\sigma_{(12)}} \boxtimes^{\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i)_{i<q}, T^p s\mathcal{A}_q, (T^{\ell^i}s\mathcal{A}_i)_{i>q}\big] \boxtimes \boxtimes^{\mathbf{n}}[(T^{\ell^i}\mathcal{C}_i)_{i<q}, T^p\mathcal{C}_q, (T^{\ell^i}\mathcal{C}_i)_{i>q}]$$
$$\xrightarrow{f_{(\ell^1,\dots,\ell^{q-1},p,\ell^{q+1},\dots,\ell^n)}\boxtimes(\boxtimes^{\mathbf{n}}[(\mu^{\ell^{\mathbf{i}}})_{i<q},\mu^{\mathbf{p}},(\mu^{\ell^{\mathbf{i}}})_{i>q}])} s\mathcal{B} \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes g} s\mathcal{B} \boxtimes \mathcal{D}\Big].$$

By coherence principle of Remark 2.34 it equals

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}(T^{\ell^i}s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}}$$
$$(\boxtimes^{i\in\mathbf{n}} T^{\ell^i}s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\lambda^\phi,(1)_{i>q}]\boxtimes\boxtimes^{\mathbf{n}}[(1)_{i<q},\lambda^\phi,(1)_{i>q}]}$$
$$\boxtimes^{\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i)_{i<q}, \otimes^{\mathbf{p}}[(s\mathcal{A}_q)_{j\in\mathbf{r}}, T^k s\mathcal{A}_q, (s\mathcal{A}_q)_{j\in\mathbf{t}}], (T^{\ell^i}s\mathcal{A}_i)_{i>q}\big]$$
$$\boxtimes \boxtimes^{\mathbf{n}}[(T^{\ell^i}\mathcal{C}_i)_{i<q}, \otimes^{\mathbf{p}}[(\mathcal{C}_q)_{j\in\mathbf{r}}, T^k\mathcal{C}_q, (\mathcal{C}_q)_{j\in\mathbf{t}}], (T^{\ell^i}\mathcal{C}_i)_{i>q}]$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},b_k,(1)_{j\in\mathbf{t}}],(1)_{i>q}]\boxtimes\boxtimes^{\mathbf{n}}[(1)_{i<q},\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},\mu^{\mathbf{k}},(1)_{j\in\mathbf{t}}],(1)_{i>q}]}$$
$$\boxtimes^{\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i)_{i<q}, T^p s\mathcal{A}_q, (T^{\ell^i}s\mathcal{A}_i)_{i>q}\big] \boxtimes \boxtimes^{\mathbf{n}}[(T^{\ell^i}\mathcal{C}_i)_{i<q}, T^p\mathcal{C}_q, (T^{\ell^i}\mathcal{C}_i)_{i>q}]$$
$$\xrightarrow{f_{(\ell^1,\dots,\ell^{q-1},p,\ell^{q+1},\dots,\ell^n)}\boxtimes(\boxtimes^{\mathbf{n}}[(\mu^{\ell^{\mathbf{i}}})_{i<q},\mu^{\mathbf{p}},(\mu^{\ell^{\mathbf{i}}})_{i>q}])} s\mathcal{B} \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i) \xrightarrow{1\boxtimes g} s\mathcal{B} \boxtimes \mathcal{D}\Big]. \quad \text{(C.12.4)}$$

Since the category $\mathcal{C}_q$ is an algebra in the Monoidal category of $\Bbbk$-quivers with the object set $\operatorname{Ob}\mathcal{C}_q$, it follows that

$$\big[T^{\ell^q}\mathcal{C}_q \xrightarrow{\lambda^\phi} \otimes^{\mathbf{p}}\big[(\mathcal{C}_q)_{j\in\mathbf{r}}, T^k\mathcal{C}_q, (\mathcal{C}_q)_{j\in\mathbf{t}}\big] \xrightarrow{\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},\mu^{\mathbf{k}},(1)_{j\in\mathbf{t}}]} T^p\mathcal{C}_q \xrightarrow{\mu^{\mathbf{p}}} \mathcal{C}_q\big] = \mu^{\ell^{\mathbf{q}}}.$$

Thus (C.12.4) is equal to

$$\Big[\boxtimes^{i\in\mathbf{n}} T^{\ell^i}(s\mathcal{A}_i \boxtimes \mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}(T^{\ell^i}s\mathcal{A}_i \boxtimes T^{\ell^i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}s\mathcal{A}_i) \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i)$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\lambda^\phi,(1)_{i>q}]\boxtimes 1} \boxtimes^{\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i)_{i<q}, \otimes^{\mathbf{p}}[(s\mathcal{A}_q)_{j\in\mathbf{r}}, T^k s\mathcal{A}_q, (s\mathcal{A}_q)_{j\in\mathbf{t}}], (T^{\ell^i}s\mathcal{A}_i)_{i>q}\big] \boxtimes (\boxtimes^{i\in\mathbf{n}} T^{\ell^i}\mathcal{C}_i)$$
$$\xrightarrow{\boxtimes^{\mathbf{n}}[(1)_{i<q},\otimes^{\mathbf{p}}[(1)_{j\in\mathbf{r}},b_k,(1)_{j\in\mathbf{t}}],(1)_{i>q}]\boxtimes(\boxtimes^{i\in\mathbf{n}}\mu^{\ell^{\mathbf{i}}})} \boxtimes^{\mathbf{n}}\big[(T^{\ell^i}s\mathcal{A}_i)_{i<q}, T^p s\mathcal{A}_q, (T^{\ell^i}s\mathcal{A}_i)_{i>q}\big] \boxtimes (\boxtimes^{i\in\mathbf{n}}\mathcal{C}_i)$$

$$\xrightarrow{f_{(\ell^1,\dots,\ell^{q-1},p,\ell^{q+1},\dots,\ell^n)}\boxtimes g} s\mathcal{B}\boxtimes\mathcal{D}\Big]. \quad (C.12.5)$$

Comparing (C.12.3) and (C.12.5) we conclude that equation (8.8.1) for $f \boxdot g$ is obtained from the corresponding equation for f by tensoring the latter with

$$\boxtimes^{i\in\mathbf{n}}T^{\ell^i}\mathcal{C}_i \xrightarrow{\boxtimes^{i\in\mathbf{n}}\mu^{\ell^{\mathbf{i}}}} \boxtimes^{i\in\mathbf{n}}\,\mathcal{C}_i \xrightarrow{g} \mathcal{D}$$

and by composing both side of the obtained equation with

$$\boxtimes^{i\in\mathbf{n}}T^{\ell^i}(s\mathcal{A}_i\boxtimes\mathcal{C}_i) \xrightarrow{\boxtimes^{\mathbf{n}}\varkappa} \boxtimes^{i\in\mathbf{n}}\,(T^{\ell^i}s\mathcal{A}_i\boxtimes T^{\ell^i}\mathcal{C}_i) \xrightarrow{\sigma_{(12)}} (\boxtimes^{i\in\mathbf{n}}T^{\ell^i}s\mathcal{A}_i)\boxtimes(\boxtimes^{i\in\mathbf{n}}T^{\ell^i}\mathcal{C}_i).$$

This proves the proposition. □

C.13 Action of differential graded categories on unital A_∞-categories. Let us prove that the action constructed in Section C.10 restricts to the action in $\mathcal{SMC}$atm

$$\boxdot : \mathsf{A}^{\mathrm{u}}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}^{\mathrm{u}}_\infty\,.$$

C.14 Lemma. *Let $\mathcal{A}$ be a (strictly) unital A_∞-category and let $\mathcal{C}$ be a differential graded category. Then the A_∞-category $\mathcal{A}\boxdot\mathcal{C}$ is (strictly) unital.*

Proof. For any object X of $\mathcal{A}$ and U of $\mathcal{C}$ pick up the element ${}_X\mathbf{i}^{\mathcal{A}}_0\otimes 1_U \in s\mathcal{A}(X,X)\otimes\mathcal{C}(U,U)$. We claim that it is a unit element for the object (X,U) of $\mathcal{A}\boxdot\mathcal{C}$. Indeed, $({}_X\mathbf{i}^{\mathcal{A}}_0\otimes 1_U)b_1^{\mathcal{A}\boxdot\mathcal{C}} = {}_X\mathbf{i}^{\mathcal{A}}_0 b_1\otimes 1_U - {}_X\mathbf{i}^{\mathcal{A}}_0\otimes 1_U d = 0$ and for any $Y\in\operatorname{Ob}\mathcal{A}$, $W\in\operatorname{Ob}\mathcal{C}$ and for any morphisms $p\in s\mathcal{A}(Y,X)$, $q\in\mathcal{C}(W,U)$ we have

$$\begin{aligned}(p\otimes q).(1\otimes({}_X\mathbf{i}^{\mathcal{A}}_0\otimes 1_U))b_2^{\mathcal{A}\boxdot\mathcal{C}} &= (-)^q[(p\otimes{}_X\mathbf{i}^{\mathcal{A}}_0)\otimes(q\otimes 1_U)](b_2^{\mathcal{A}}\otimes\mu^{\mathbf{2}}_{\mathcal{C}}) = (p\otimes{}_X\mathbf{i}^{\mathcal{A}}_0)b_2^{\mathcal{A}}\otimes(q\otimes 1_U)\mu^{\mathbf{2}}_{\mathcal{C}}\\ &= p\otimes q + p.(hb_1^{\mathcal{A}}+b_1^{\mathcal{A}}h)\otimes q = p\otimes q + (p\otimes q).[(h\otimes 1)b_1^{\mathcal{A}\boxdot\mathcal{C}} + b_1^{\mathcal{A}\boxdot\mathcal{C}}(h\otimes 1)],\end{aligned}$$

where the homotopy $h : s\mathcal{A}(Y,X)\to s\mathcal{A}(Y,X)$ satisfies $(1\otimes{}_X\mathbf{i}^{\mathcal{A}}_0)b_2^{\mathcal{A}} = 1 + hb_1^{\mathcal{A}} + b_1^{\mathcal{A}}h$. Similarly, $(({}_X\mathbf{i}^{\mathcal{A}}_0\otimes 1_U)\otimes 1)b_2^{\mathcal{A}\boxdot\mathcal{C}}$ is homotopic to -1. Therefore, $\mathcal{A}\boxdot\mathcal{C}$ is unital. □

C.15 Proposition. *Let $\overline{f} : \boxtimes^{i\in I}Ts\mathcal{A}_i\to s\mathcal{B}$ be a unital A_∞-functor, and let $g : \boxtimes^{i\in I}\mathcal{C}_i\to\mathcal{D}$ be a differential graded functor. Then the A_∞-functor $f\boxdot g$ is unital. This gives the action in $\mathcal{SMC}$atm*

$$\boxdot : \mathsf{A}^{\mathrm{u}}_\infty \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}^{\mathrm{u}}_\infty\,.$$

Proof. The A_∞-categories $\mathcal{A}_i\boxdot\mathcal{C}_i$, $\mathcal{B}\boxdot\mathcal{D}$ are unital due to Lemma C.14. If $I=\varnothing$, the statement is trivial.

Assume that I is a 1-element set, and consider $\overline{f} : Ts\mathcal{A}\to s\mathcal{B}$, $g : \mathcal{C}\to\mathcal{D}$ and

$$\overline{f\boxdot g} = \big[T(s\mathcal{A}\boxtimes\mathcal{C}) \xrightarrow{\varkappa} Ts\mathcal{A}\boxtimes T\mathcal{C} \xrightarrow{\overline{f}\boxtimes\mu} s\mathcal{B}\boxtimes\mathcal{C} \xrightarrow{1\boxtimes g} s\mathcal{B}\boxtimes\mathcal{D}\big].$$

For arbitrary objects X of $\mathcal{A}$ and U of $\mathcal{C}$ the first component $(\overline{f \boxdot g})_1$ takes ${}_X\mathbf{i}_0^{\mathcal{A}} \otimes 1_U$ to $({}_X\mathbf{i}_0^{\mathcal{A}} f_1) \otimes (1_U g) = ({}_X\mathbf{i}_0^{\mathcal{A}} f_1) \otimes 1_{g(U)}$, which is homologous to ${}_{Xf}\mathbf{i}_0^{\mathcal{B}} \otimes 1_{g(U)}$. Thus, $f \boxdot g$ is unital.

Let $|I| > 1$, let $j \in I$, and let $X_i \in \operatorname{Ob} \mathcal{A}_i$, $U_i \in \operatorname{Ob} \mathcal{C}_i$ for $i \in I$, $i \neq j$. Then

$$(f \boxdot g)\big|_j^{(X_i)_{i\neq j},(U_i)_{i\neq j}} = f\big|_j^{(X_i)_{i\neq j}} \boxdot g\big|_j^{(U_i)_{i\neq j}} : \mathcal{A}_j \boxdot \mathcal{C}_j \to \mathcal{B} \boxdot \mathcal{D},$$

where the differential graded functor $g\big|_j^{(U_i)_{i\neq j}} = \mu_{j\hookrightarrow I}^{\widehat{\mathbf{dg}\text{-}\mathcal{C}at}}((U_i)_{i<j}, \mathrm{id}_{\mathcal{A}_j}, (U_i)_{i>j}, g) : \mathcal{C}_j \to \mathcal{D}$ is defined as in (8.18.1). On objects and morphisms of $\mathcal{C}_j$ it gives the expected expression $(g\big|_j^{(U_i)_{i\neq j}})(\_) = g((U_i)_{i<j}, \_, (U_i)_{i>j})$. Since the A_∞-functor $f\big|_j^{(X_i)_{i\neq j}}$ is unital, so is $(f \boxdot g)\big|_j^{(X_i)_{i\neq j},(U_i)_{i\neq j}}$ by the above $I = \mathbf{1}$ case. The A_∞-functor $f \boxdot g$ is unital by Proposition 9.13. □

Let us show that the action $\boxdot : \mathsf{A}_\infty^{\mathrm{u}} \boxtimes \widehat{\mathbf{dg}\text{-}\mathcal{C}at} \to \mathsf{A}_\infty^{\mathrm{u}}$ is a generalization of the tensor product $\boxtimes : \mathbf{dg}\text{-}\mathcal{C}at \boxtimes \mathbf{dg}\text{-}\mathcal{C}at \to \mathbf{dg}\text{-}\mathcal{C}at$ of differential graded categories. Let a unital A_∞-category come from a differential graded category. This means that $b_n^{\mathcal{A}} = 0$ for $n > 2$ and the unit elements ${}_X\mathbf{i}_0^{\mathcal{A}}$ are strict. Let $\mathcal{C}$ be a differential graded category. Then the quivers $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{A} \boxdot \mathcal{C} = (\mathcal{A}[1] \boxtimes \mathcal{C})[-1]$ are isomorphic via degree 0 map

$$\phi = \bigl(\mathcal{A} \boxtimes \mathcal{C} \xrightarrow{s\boxtimes 1} \mathcal{A}[1] \boxtimes \mathcal{C} \xrightarrow{s^{-1}} (\mathcal{A}[1] \boxtimes \mathcal{C})[-1]\bigr).$$

We claim that this is an isomorphism of differential graded categories

$$\phi : (\mathcal{A} \boxtimes \mathcal{C}, m_1^{\mathcal{A}} \boxtimes 1 + 1 \boxtimes d, \overline{\varkappa} \cdot (m_2^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}})) \to (\mathcal{A} \boxdot \mathcal{C}, m_1^{\mathcal{A}\boxdot\mathcal{C}}, m_2^{\mathcal{A}\boxdot\mathcal{C}}),$$

where composition in $\mathcal{A} \boxtimes \mathcal{C}$ is

$$(\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{A} \boxtimes \mathcal{C}) \xrightarrow{\overline{\varkappa}} (\mathcal{A} \otimes \mathcal{A}) \boxtimes (\mathcal{C} \otimes \mathcal{C}) \xrightarrow{m_2^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}}} \mathcal{A} \boxtimes \mathcal{C}.$$

Indeed, ϕ is a chain map,

$$\begin{aligned} \phi m_1^{\mathcal{A}\boxdot\mathcal{C}} &= (s \boxtimes 1)s^{-1} s b_1^{\mathcal{A}\boxdot\mathcal{C}} s^{-1} = (s \boxtimes 1)(b_1^{\mathcal{A}} \boxtimes 1 - 1 \boxtimes d)s^{-1} \\ &= (m_1^{\mathcal{A}} \boxtimes 1 + 1 \boxtimes d)(s \boxtimes 1)s^{-1} = (m_1^{\mathcal{A}} \boxtimes 1 + 1 \boxtimes d)\phi : \mathcal{A} \boxtimes \mathcal{C} \to \mathcal{A} \boxdot \mathcal{C}, \end{aligned}$$

compatible with the composition:

$$\begin{aligned} (\phi \otimes \phi) m_2^{\mathcal{A}\boxdot\mathcal{C}} &= \bigl[((s \boxtimes 1)s^{-1}) \otimes ((s \boxtimes 1)s^{-1})\bigr](s \otimes s) b_2^{\mathcal{A}\boxdot\mathcal{C}} s^{-1} \\ &= [(s \boxtimes 1) \otimes (s \boxtimes 1)]\overline{\varkappa}(b_2^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}})s^{-1} = \overline{\varkappa}[(s \boxtimes s) \otimes (1 \boxtimes 1)](b_2^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}})s^{-1} \\ &= \overline{\varkappa}(m_2^{\mathcal{A}} s \boxtimes \mu_{\mathcal{C}})s^{-1} = \overline{\varkappa}(m_2^{\mathcal{A}} \boxtimes \mu_{\mathcal{C}})\phi : (\mathcal{A} \boxtimes \mathcal{C}) \otimes (\mathcal{A} \boxtimes \mathcal{C}) \to \mathcal{A} \boxdot \mathcal{C}. \end{aligned}$$

Since the compositions in $\mathcal{A} \boxtimes \mathcal{C}$ and $\mathcal{A} \boxdot \mathcal{C}$ are identified by ϕ, the identity morphisms of $\mathcal{A} \boxtimes \mathcal{C}$ are mapped by ϕ to identity morphisms of $\mathcal{A} \boxdot \mathcal{C}$.

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