Pell’s Equation and Truncated Squares

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October 31, 2018

Abstract

A number which is a square and which remains a square after removing the one’s digit is called a t-square. Such numbers include 169, 256, and 361. In this paper we study these numbers using the theory of Pell equations and prove several theorems regarding their distribution in $\mathbb{Z}$.

1 Motivation

I once graded a college algebra exam which asked the student to find all roots of the polynomial $p(x) = 256x - x^3$. After factoring $x$ from the polynomial, most students recognized $256 - x^2$ as a difference of squares and gave the appropriate factorization $p(x) = x(16 - x)(16 + x)$, from which the roots are easily deduced. Several students, however, incorrectly wrote the polynomial as $p(x) = 25x - x^3$, which, incidentally, leads also to a difference of squares. And so the questions arose. How many such integers are there? Do there exist arbitrarily long sequences of such integers (i.e. the sequence 25, 256 consists of two numbers, while 1, 16, 169 consists of three)? These questions are the topic of the present paper.

For a list of the integers studied in this paper, see A023110 (roots of truncatable squares) and A031150 (roots of truncated squares) on the OEIS.

2 Definitions and Background Theory

The following theorems and definitions are taken from [1]. Throughout this paper, $D$ will denote the set of digits $D = \{0, 1, \ldots, 9\}$.

Definition. Let $D$ be a positive integer that is not a perfect square. The equation

$$u^2 - Dv^2 = 1 \quad (2.1)$$

is called a Pell’s resolvent. For an integer $N \neq 0, 1$, the equation

$$x^2 - Dy^2 = N \quad (2.2)$$

is called a general Pell’s equation. We denote a solution to (2.1) by $u + v\sqrt{D}$ and a solution to (2.2) by $x + y\sqrt{D}$. We denote the trivial solution $u_0 = 1, v_0 = 0$ to (2.1) by $u_0 + v_0\sqrt{D}$ and the fundamental (minimal non-trivial) solution by $u_1 + v_1\sqrt{D}$.
Definition. Two solutions \( x + y\sqrt{D} \) and \( x' + y'\sqrt{D} \) to (2.2) are associated if there is an integer \( n \) and a sign \( \pm 1 \) so that
\[
x' + y'\sqrt{D} = \pm(x + y\sqrt{D})(u_0 + v_0\sqrt{D})^n.
\]
Equivalently, \( x + y\sqrt{D} \) and \( x' + y'\sqrt{D} \) are associated if and only if
\[
xx' \equiv yy' D \pmod{N} \quad \text{and} \quad xy' \equiv x'y \pmod{N}.
\] (2.3)

It is easy that the relation “associated” defined above is an equivalence relation on the set of solutions to (2.2). Note that if \( x + y\sqrt{D} \) is a solution to (2.2), then so are \( x - y\sqrt{D} \) and \( -x - y\sqrt{D} \). The solutions \( x + y\sqrt{D} \) and \( -x - y\sqrt{D} \) always belong to the same class. However, the classes to which \( x + y\sqrt{D} \) and \( -x - y\sqrt{D} \) belong are in general distinct. If they are not, we say that their class is ambiguous.

Definition. If \( N > 1 \), the fundamental solution in a class of solutions to (2.2) is the solution with the minimal nonnegative \( y \). If the class is not ambiguous, the corresponding value of \( x \) is determined uniquely. Otherwise, we prescribe that \( x > 0 \).

Definition. A square which remains a square when its last digit is removed is called a \( t \)-square. A square to which a digit may be appended (to the one’s place) to produce another square is called an \( a \)-square.

Theorem 2.1. Every Pell resolvent \( u^2 - Dv^2 = 1 \) has a minimal nontrivial solution \( u_1 + v - 1\sqrt{D} \). Furthermore, all nontrivial solutions are given by
\[
\pm(u_1 + v\sqrt{D})^n,
\]
where \( n \) runs through both the positive and negative integers.

Lemma 2.2 (Multiplication Principle). If \( x + y\sqrt{D} \) is a solution to the general Pell equation, then so are
\[
(x + y\sqrt{D})(u_1 + v\sqrt{D}) = (xu_1 + Dvy_1) + (yu_1 + xv_1)\sqrt{D},
\]
\[
(x + y\sqrt{D})(u_1 - v\sqrt{D}) = (xu_1 - Dvy_1) + (yu_1 - xv_1)\sqrt{D}.
\]

The previous lemma shows that a fundamental solution generates an infinite family of solutions, and as we have already seen, these families are equivalence classes of the solutions to (2.2)

Theorem 2.3. If \( N > 1 \) and \( x + y\sqrt{D} \) is the fundamental solution in a class of solutions 2.2, then
\[
(i) \sqrt{N} \leq x < \sqrt{\frac{1}{2}N(u_1 + 1)},
\]

\[
(ii) x = \sqrt{\frac{1}{2}N(t + 1)} \quad \text{and} \quad y = u_1\sqrt{\frac{N}{2(t+1)}}.
\]
\[
\begin{array}{|c|c|}
\hline
N & x_1 + y_1 \sqrt{10} \\
\hline
1 & 19 + 6\sqrt{10} \\
4 & 2 \\
6 & 4 + \sqrt{10} \\
6 & 16 + 5\sqrt{10} \\
9 & 3 \\
9 & 7 + 2\sqrt{10} \\
9 & 13 + 4\sqrt{10} \\
\hline
\end{array}
\]

Table 1: Fundamental solutions to \( x^2 - 10y^2 = N \)

3 Classification of t-squares

If \( x \) is a t-square with corresponding a-square \( y \), then \( x + y\sqrt{10} \) is a solution to the general Pell equation
\[
x^2 - 10y^2 = N \tag{3.1}
\]
for some \( N \in \mathcal{D} \). We can determine the number of classes of solutions to (3.1) as well as the fundamental solution of each class by using Theorem 2.3 and testing each possible combination of \( x \) and \( y \). The case of \( N = 1 \) is a Pell resolvent whose solutions are described completely by Theorem 2.1. The remaining fundamental solutions, along with that of the Pell resolvent, are summarized in Table 1 and are obtained using Theorem 2.3 and the conditions (2.3). We summarize these results in the following theorem.

**Theorem 3.1.** Of the values \( N \in \mathcal{D} \), the equation \( x^2 - 10y^2 = N \) is solvable only for \( N = 1, 4, 6, 9 \). The classes of each \( N \) are as follows.

- **One class** for \( N = 1 \) consisting of the solutions to the Pell resolvent \( u^2 - 10v^2 = 1 \), generated by \( 19 + 6\sqrt{10} \).

- **One class** for \( N = 4 \) generated by \( 2 \).

- **Two classes** for \( N = 6 \) generated by \( 4 + 2\sqrt{10} \) and \( 16 + 5\sqrt{10} \).

- **Three classes** for \( N = 9 \) generated by \( 3, 7 + 2\sqrt{10}, \) and \( 13 + 4\sqrt{10} \).

Since we only are only interested in positive values of \( x \) and \( y \), we consider only solutions of the form \( (x + y\sqrt{10})(19 + 6\sqrt{10})^n \) for \( n \geq 0 \). In particular, we do not consider solutions of the form \( \pm (x + y\sqrt{10})(19 - 6\sqrt{10})^n \), as these only give conjugates of our positive solutions. As such, the full set of positive solutions in a class are obtained by multiplying the minimal solution of the class by a power of \( 19 + 6\sqrt{10} \), and we say that the class is generated by this minimal solution. We also exclude \( 4 - \sqrt{10} \) and \( 7 - 2\sqrt{10} \) from their respective classes so that these classes are then generated by \( 16 + 5\sqrt{10} \) and \( 13 + 4\sqrt{10} \), respectively. Our next theorem gives the exact structure of the digits which are appended to a-squares, or equivalently, the one’s digits of t-squares.
Theorem 3.2. If $A$ is the sequence of non-zero t-squares, then

$$A \pmod{10} = 1, 4, 9, 6, 9, 9, 6.$$  \hspace{1cm} (3.2)

In particular, the sequence of one’s digits of non-zero t-squares is periodic with period 7.

Proof. Let $(a_n), (b_n), (c_n), (d_n), (e_n), (f_n), (g_n)$ for $n \geq 1$ be defined as follows.

$(a_n)$ = the solutions to Pell’s resolvent, including the trivial solution.

$(b_n)$ = the class generated by 2.

$(c_n)$ = the class generated by 3.

$(d_n)$ = the class generated by $4 + \sqrt{10}$.

$(e_n)$ = the class generated by $7 + 2\sqrt{10}$.

$(f_n)$ = the class generated by $13 + 4\sqrt{10}$.

$(g_n)$ = the class generated by $16 + 5\sqrt{10}$.

We denote the general class by $(s_n)$ and denote the components of each solution by

$s_n = s_n(x) + s_n(y)\sqrt{10}$. We shall prove via induction that the following inequalities hold for all $n$:

$$a_n(x) < b_n(x) < c_n(x) < d_n(x) < e_n(x) < f_n(x) < g_n(x) < a_{n+1}(x),$$

$$a_n(y) \leq b_n(y) \leq c_n(y) \leq d_n(y) \leq e_n(y) \leq f_n(y) \leq g_n(y) \leq a_{n+1}(y).$$ \hspace{1cm} (3.3)

That is, each line holds simultaneously for each $n$. Since we include the trivial solution in

$(a_n)$, $a_1 = 1$ and $a_2 = 19 + 6\sqrt{10}$. This and the list above defining each sequence imply that

the inequalities (3.3) hold for $n = 1$. Suppose then that (3.3) holds for some $n \geq 1$. We

will show only that the inequalities between $(a_n)$ and $(b_n)$ are satisfied, as the rest follow

by an identical argument. By the multiplication principle, $s_{n+1}(x) = 19 s_n(x) + 60 s_n(y)$ and

$s_{n+1}(y) = 6 s_n(x) + 19 s_n(y)$. Then we have

$$a_{n+1}(y) = 19 a_n(y) + 6 a_n(x) \leq 6 b_n(x) + 19 b_n(y) = b_{n+1}(y),$$

from which it follows that

$$a_{n+1}(x) = 19 a_n(x) + 60 a_n(y) < 19 b_n(x) + 60 b_n(y) = b_{n+1}(y).$$

It follows by the induction hypothesis that the inequalities (3.3) for all $n$. The values of $N$

corresponding to the sequences are exactly those of (3.2) in the same order. Since the one’s
digits of t-squares are precisely these values of $N$, the theorem is proved.

\[ \square \]

Next, we give closed form expressions for the solutions in each class. This will allow us
to compute an asymptotic formula for the sequences of t- and a-squares. We have already
noted that advancing from one solution in a class to the next is accomplished by multiplying
the solution by $19 + 6\sqrt{10}$. Consequently, if $s_0(x) + s_0(y)\sqrt{10}$ is the fundamental solution of
a class \( s_n = s_n(x) + s_n(y)\sqrt{10} \), then all solutions with positive \( s_n(x) \) and \( s_n(y) \) are given by \( (s_n(x) + s_n(y)\sqrt{10})(19 + 6\sqrt{10})^n \). We may encode this information in a matrix as

\[
\begin{pmatrix}
  s_n(x) \\
  s_n(y)
\end{pmatrix}
= 
\begin{pmatrix}
  19 & 60 \\
  6 & 19
\end{pmatrix}^n
\begin{pmatrix}
  s_0(x) \\
  s_0(y)
\end{pmatrix}.
\]

If we put \( \lambda_1 = 19 + 6\sqrt{10} \) and \( \lambda_2 = 19 - 6\sqrt{10} \), then with a bit of linear algebra we find that

\[
\begin{pmatrix}
  s_n(x) \\
  s_n(y)
\end{pmatrix}
= 
\begin{pmatrix}
  \frac{1}{2} (\lambda_1^n + \lambda_2^n) \\
  \frac{1}{2\sqrt{10}} (\lambda_1^n - \lambda_2^n)
\end{pmatrix}
\begin{pmatrix}
  s_0(x) \\
  s_0(y)
\end{pmatrix},
\]

from which it follows that

\[
s_n(x) = \frac{1}{2} \left[ (\lambda_1^n + \lambda_2^n) s_0(x) + \sqrt{10} (\lambda_1^n - \lambda_2^n) s_0(y) \right],
\]

\[
s_n(y) = \frac{1}{2\sqrt{10}} \left[ (\lambda_1^n - \lambda_2^n) s_0(x) + \sqrt{10} (\lambda_1^n + \lambda_2^n) s_0(y) \right].
\]

(3.4)

Using the first equation of (3.4), we can now give a formula for the square roots of t-squares. We present this formula as the following theorem. The proof, a set of straightforward calculations, is left as an exercise to the reader.

**Theorem 3.3.** Let \((t_n)\) be the sequence of t-squares with \( t_1 = 1 \). Put \( \lambda_1 = 19 + 6\sqrt{10} \) and \( \lambda_2 = 19 - 6\sqrt{10} \). Then

\[
t_n = \begin{cases}
  \frac{1}{4} \lambda_1^{2(n-1)} + \frac{1}{4} \lambda_2^{2(n-1)} + \frac{1}{2} & \text{if } n \equiv 1 \pmod{7}, \\
  \lambda_1^{\frac{n-2}{7}} + \lambda_2^{\frac{n-2}{7}} + 2 & \text{if } n \equiv 2 \pmod{7}, \\
  \frac{9}{4} \lambda_1^{\frac{2(n-3)}{7}} + \frac{9}{4} \lambda_2^{\frac{2(n-3)}{7}} + \frac{9}{2} & \text{if } n \equiv 3 \pmod{7}, \\
  \left( \frac{13}{2} + 2\sqrt{10} \right) \lambda_1^{\frac{2(n-4)}{7}} + \left( \frac{13}{2} - 2\sqrt{10} \right) \lambda_2^{\frac{2(n-4)}{7}} + 3 & \text{if } n \equiv 4 \pmod{7}, \\
  \left( \frac{89}{4} + 7\sqrt{10} \right) \lambda_1^{\frac{2(n-5)}{7}} + \left( \frac{89}{4} - 7\sqrt{10} \right) \lambda_2^{\frac{2(n-5)}{7}} + \frac{9}{2} & \text{if } n \equiv 5 \pmod{7}, \\
  \left( \frac{329}{4} + 26\sqrt{10} \right) \lambda_1^{\frac{2(n-6)}{7}} + \left( \frac{329}{4} - 26\sqrt{10} \right) \lambda_2^{\frac{2(n-6)}{7}} + \frac{9}{2} & \text{if } n \equiv 6 \pmod{7}, \\
  \left( \frac{253}{2} + 40\sqrt{10} \right) \lambda_1^{\frac{2(n-7)}{7}} + \left( \frac{253}{2} - 40\sqrt{10} \right) \lambda_2^{\frac{2(n-7)}{7}} + 3 & \text{if } n \equiv 0 \pmod{7}.
\end{cases}
\]

(4 Strings of Truncated Squares)

In addition to giving formulas for truncated squares, we can also ask about the ways in which these numbers can be “strung together.” To illustrate, consider the sequence 169, 16, 1. This is a string of 3 truncated squares where each subsequent term is a truncation of the previous term. Note that the entirety of the sequence is encoded in the single number 169. As such, we shall consider truncated square for which we may perform successive truncations, rather than study sequences themselves. There are two main qualities that we might require of a truncated square. These are enunciated in the following definition.
Definition. A truncated square is \textit{\textit{nth truncatable}} if it may be truncated \textit{n} times by removing the one’s digit while remaining a square at each step of the truncated. If the truncated square has \textit{n} digits, it is said to be \textit{completely truncatable}.

We first show that there are only a few completely truncatable squares.

\textbf{Theorem 4.1.} The only completely truncatable squares are 1, 4, 9, 16, 49, and 169.

\textit{Proof.} In order for a completely truncatable square to exist, its leading digit \(L\) must be 1, 4, or 9. As such, we divide the proof into three cases.

\textbf{Case 1:} \(L = 1\). The only digit that can be appended to 1 is 6, and the only digit which can be appended to 16 is 9. Since 160 + \(d = x^2\) has no solutions for \(d \in \mathcal{D}\), there are no other completely truncatable squares with \(L = 1\).

\textbf{Case 2:} \(L = 4\). The only digit that can be appended to 4 is 9. Since 490 + \(d = x^2\) has no solutions for \(d \in \mathcal{D}\), there are no other completely truncatable squares with \(L = 4\).

\textbf{Case 3:} \(L = 9\). Since 90 + \(d = x^2\) has no solutions for \(d \in \mathcal{D}\), there are no other completely truncatable squares with \(L = 9\).

\(\Box\)

We consider the next the more general question of \textit{\textit{n}}\textit{th truncatable squares}. In order for a square \(w^2\) to be twice truncatable to another square \(y^2\), the equations

\[w^2 - 10x^2 = d_1 \quad \text{and} \quad x^2 - 10y^2 = d_2\]

must be solvable with \(d_1, d_2 \in \mathcal{D}\). Hence if \(w^2\) is twice truncatable to \(y^2\), \(w\) and \(y\) satisfy

\[w^2 - 100y^2 = w^2 - (10y)^2 = 10d_2 + d_1.\]

From our results in the previous section, we know that there are only 16 possibilities for 10d_2 + d_1.

\section{Truncated Cubes}

In order for a non-trivial truncated cube to exist, the equation

\[x^3 - 10y^3 = N\]

must have a solution with \(x, y > 0\) for some positive \(N = 0, \ldots, 9\). Thue has shown that any such cubic equation has at most finitely many solutions. The fact that no nontrivial solutions to this equation exist for \(0 \leq N \leq 9\) follows from the work of Bennett in [2]. His work is far beyond the scope of this paper, so we state his result without proof.

\textbf{Theorem 5.1} (Bennett). For any positive integers \(p\) and \(q\), one has

\[\left| \sqrt[3]{10} - \frac{p}{q} \right| > \frac{0.15}{q^{2.45}}.\]
Corollary 5.2. The only solutions to (5.1) are the three trivial solutions \((0, 0), (1, 0), (2, 0)\) when \(N = 0, 1, 8\), respectively.

Proof. Factoring the difference of cubes, we have

\[ x^3 - 10y^3 = (x - y^{3/10})(x^2 + xy^{3/10} + y^{2+3/10}). \]

In order for (5.1) to have a solutions with \(x, y > 0\), we need \(x \geq y^{3/10}\). Using this and Theorem 5.1, we get that

\[ |x^3 - 10y^3| > \frac{0.15}{y^{1.45}} (3y^{2+3/10}) > 2\sqrt{y}. \]

Thus for \(y \geq 21\), the equation (5.1) has no solutions. Checking the for each \(1 \leq y \leq 20\), we verify by brute force that (5.1) has no solutions, and therefore the only solutions are trivial with \(y = 0\).

\[ \square \]

References
