CONFORMAL DIMENSION:
CANTOR SETS AND FUGLEDE MODULUS

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ABSTRACT. In this paper we give several conditions for a space to be minimal for conformal dimension. We show that there are sets of zero length and conformal dimension 1 thus answering a question of Bishop and Tyson. Another sufficient condition for minimality is given in terms of a modulus of a system of measures in the sense of Fuglede [7]. It implies in particular that if \( E \subset \mathbb{R} \) is minimal for conformal dimension and supports a measure \( \lambda \) such that for every \( \varepsilon > 0 \) there is a constant \( 0 < C < \infty \) such that \( C^{-1}r^{1+\varepsilon} \leq \lambda(E \cap B(x,r)) \leq Cr^{1-\varepsilon} \) then \( X \times Y \) is minimal for conformal dimension for every compact \( Y \).

Contents

1. Introduction 1
2. Background 4
3. Thick Cantor sets 5
4. Proof of Theorem 3.2 7
5. Modulus of measures & Conformal dimension 13
6. Proof of Theorem 5.5 15
7. Remarks and Problems 18
References 20

1. Introduction

Given a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) a map \( f \) between metric spaces \((X,d_X)\) and \((Y,d_Y)\) is called \( \eta\)-quasisymmetric if for all distinct triples \( x, y, z \in X \) and \( t > 0 \)

\[
\frac{d_X(x,y)}{d_X(y,z)} \leq t \quad \Rightarrow \quad \frac{d_Y(f(x),f(y))}{d_Y(f(y),f(z))} \leq \eta(t).
\]

If \( \eta(t) \leq C \max\{t^K, t^{1/K}\} \) for some \( K \geq 1 \) and \( C > 0 \) then \( f \) is said to be power quasisymmetric. We will denote by \( QS(X) \) the collection of all quasisymmetric maps defined on \( X \).

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Conformal dimension of a metric space, a concept introduced by Pansu in [14], is the infimal Hausdorff dimension of quasisymmetric images of $X$,

$$C \dim X = \inf_{f \in \mathcal{QS}(X)} \dim_H f(X).$$

We say $X$ is minimal for conformal dimension or just minimal if $C \dim X = \dim_H X$. Euclidean spaces with standard metric are the simplest examples of minimal spaces.

In [12] Kovalev proved a conjecture of Tyson that if $\dim_H X < 1$ then $C \dim X = 0$. In other words a minimal space cannot have Hausdorff dimension strictly between 0 and 1.

When Hausdorff dimension of $X$ is 1 its conformal dimension can be either 0 or 1. First such examples of conformal dimension 0 were given by Tukia in [18]. On the other hand Staples and Ward showed in [17] the existence of (totally disconnected) quasisymmetrically thick sets, i.e. sets $E \subset \mathbb{R}$ with the property that $f(E)$ has positive Lebesgue measure for every quasisymmetric $f : \mathbb{R} \to \mathbb{R}$. It is not known whether all quasisymmetrically thick sets are minimal or not. In [3] Bishop and Tyson constructed minimal Cantor sets of dimension $\alpha$ for every $\alpha \geq 1$. They also noticed that some of the quasisymmetrically thick sets considered by Staples and Ward were in fact minimal. All the sets considered in [3] were of positive Hausdorff 1-measure and Bishop and Tyson asked if there are sets of conformal dimension 1 which are not quasisymmetrically thick. The following result gives an affirmative answer to the question above, by showing that not only a minimal set $E \subset \mathbb{R}$ can be taken to be not quasisymmetrically thick but even of zero Lebesgue measure.

**Theorem 1.1.** There is a set $E \subset \mathbb{R}$ of zero length and conformal dimension 1.

This result is a consequence of Theorem [3,2] which gives a sufficient condition for a set $E \subset \mathbb{R}$ to have conformal dimension $\geq 1$. Previously (see [8]) the present author proved that all middle interval Cantor sets $E \subset \mathbb{R}$ of Hausdorff dimension 1 have the property that $\dim_H f(E) = 1$ for all quasisymmetric selfmaps $f$ of the line $\mathbb{R}$. In [11] Hu and Wen generalized this result to include a larger class of uniform Cantor sets (see Section 7 for the definitions). We remark that these results do not imply Theorem 1.1, since in the definition of the conformal dimension of $E \subset \mathbb{R}$ one considers all quasisymmetric mappings of $E$, which may have no quasisymmetric extension to $\mathbb{R}$ and the images $f(E)$ are not necessarily subsets of $\mathbb{R}$. In fact we do not know whether all middle interval Cantor sets are minimal. However, Theorem 3.2 implies that all uniformly perfect examples considered in [8] and [11] are minimal not only for the self maps of the line but for arbitrary quasisymmetric homeomorphisms. It would be interesting to know if there are subsets of the line which have conformal dimension 0 but are minimal for quasisymmetries of $\mathbb{R}$. We refer to Section 7 and in particular to Corollary...
7.1 for a further discussion of uniform and more general Cantor sets and related problems.

Already in the works of Pansu and Bourdon it was realized that the presence of large families of curves gives lower bounds for the conformal dimension (see [5], [15]). In [19] Tyson proved that if $X$ is an Ahlfors $Q$-regular space then $C \dim X = Q$ if there is a curve family $\Gamma$ in $X$ of positive $Q$ modulus (see Section 5 for the definitions and precise statement of Tyson’s theorem). In particular $(0, 1) \times Y$ is minimal for every Borel metric space $Y$. In [3] it was also shown that $E \times Y$ is minimal for every compact $Y \subset \mathbb{R}^n$ if the Hausdorff 1-contents of quasisymmetric images of $E$ are uniformly bounded away from 0. The second main results of this paper, Theorem 5.5, is a sufficient condition for a space $X$ to be minimal in terms of a certain modulus of a system of measures in the sense of Fuglede [7]. The following result is a consequence of Theorem 5.5 and shows that even sets of zero measure can have the property of having minimal products, provided they are minimal themselves.

**Theorem 1.2.** If $E \subset \mathbb{R}$ is minimal and supports a measure $\lambda$ s.t. for every $\varepsilon > 0$

$$r^{1+\varepsilon} \lesssim \lambda(E \cap B_r(x)) \lesssim r^{1-\varepsilon}$$

for all $x \in E$ and all $r > 0$ then $E \times Y$ is minimal for every nonempty compact $Y$.

Examples of sets $E$ satisfying Theorems 1.1 and 1.2 are easily obtained. Consider the so called middle interval Cantor sets constructed as follows. Start from the unit interval on the line. Remove its $c_1$-st middle part to obtain two intervals of equal length. By induction, in the $i$-th step remove $c_i$-th middle part of every remaining component from the previous step to obtain $2^i$ intervals of equal length. If $c_i \rightarrow 0$ and $\sum_{i=1}^{\infty} c_i = \infty$ then the resulting Cantor set $E$ would satisfy the conclusions of Theorems 1.1 and 1.2. In fact we will show that all uniformly perfect middle interval Cantor sets are minimal if (and only if) they have Hausdorff dimension 1. If the sequence $c_i$ is a constant sequence, $c_i = c, \forall i$ for some $c \in [0, 1)$, we will denote the corresponding middle interval Cantor set by $E_c$. The proof of Theorem 3.2 also proves the following result, see Remark 4.9.

**Theorem 1.3.** Let $\eta$ be as in the definition of quasisymmetric maps. For every $0 < d < 1$ there is a $c > 0$ such that $\dim_H f(E_{c'}) \geq d$ for every $c' < c$ and $f$ which is $\eta$ quasisymmetric (note that $\dim E_c < 1$ and therefore $C \dim E_c = 0$).

Theorem 1.3 gives other examples for Theorem 1.1.

**Corollary 1.4.** If $c_i > 0$ and $c_i \rightarrow 0$ as $i \rightarrow \infty$ then the set $E = \bigcup_i E_{c_i}$ has zero length and conformal dimension 1.
Proof. Since \( \dim_H E_{c_i} < 1 \) for every \( i \in \mathbb{N} \) it is clear that \( E \) has zero length. To show that \( E \) is of conformal dimension 1 we suppose there is an \( \eta \)-quasisymmetric map \( f \) such that \( \dim_H f(E) < 1 \). By Theorem 1.3 we can choose \( d \in (\dim_H f(E), 1) \) and \( i_0 \in \mathbb{N} \) so that for every \( \eta \)-quasisymmetric map \( g \) we have \( \dim_H g(E_{c_i}) \geq d > \dim_H f(E) \) for \( i > i_0 \) since \( c_i \to 0 \) as \( i \to \infty \). Hence \( \dim_H f(E) \geq \dim_H f(E_{c_{i_0+1}}) > \dim_H f(E) \), a contradiction. \( \square \)

This paper is organized as follows. In Section 2 we provide some background material and fix the notations. In Section 3 we state Theorem 3.2 and explain how Theorem 1.1 follows from it. In Section 4 we prove Theorem 3.2 and Theorem 1.3. In Section 5 we recall the definitions of the modulus of a system of measures and discrete modulus and deduce Theorem 1.2 from Theorem 5.5. We prove Theorem 5.5 in Section 6. Section 7 is devoted to some further remarks and open problems.

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2. Background

Constants in this article will be denoted by the letter \( C \) and can have different values from line to line. The notation \( A \preceq B \) means there is a constant \( C \) such that \( A \leq CB \). Given \( r > 0 \), by \( B_r \) we will denote any open ball in \( X \) of radius \( r \) and by \( B(x, r) \) the one centered at \( x \in X \). \( CB(x, r) \) will denote the ball \( B(x, Cr) \).

Recall that the Hausdorff \( t \)-measure of a metric space \( (X, d_X) \) is defined as follows. For every open cover of \( X \) by balls \( B(x_i, r_i), i \in \mathbb{N} \) let

\[
H^t_\varepsilon(X) = \inf \left\{ \sum_{i=1}^\infty r_i^t : X \subset \bigcup_{i=1}^\infty B(x_i, r_i), r_i < \varepsilon \right\},
\]

and

\[
H_t(X) = \lim_{\varepsilon \to 0} H^t_\varepsilon(X).
\]

The Hausdorff dimension of \( X \) is

\[
\dim_H(X) = \inf \{ t : H_t(X) = 0 \} = \sup \{ t : H_t(X) = \infty \}
\]

One usually gives an upper bound for the Hausdorff dimension of a set by finding explicit covers for it. Lower bounds can be obtained by finding a measure on \( X \).

**Lemma 2.1** (Mass distribution principle). If the metric space \( (X, d_X) \) supports a positive Borel measure \( \mu \) satisfying \( \mu(U) \leq C(diam U)^d \), for some fixed constant \( C > 0 \) and every \( U \subset X \) then \( \dim_H(E) \geq d \).
Proof. For every cover \( \{U_i\}_{i=1}^{\infty} \) of \( X \) we have \( \sum_i (\text{diam } U_i)^d \geq \frac{1}{C} \sum_i \mu(U_i) \geq \frac{1}{C} \mu(X) \). Therefore \( H^d(X) \geq \frac{\mu(X)}{C} > 0 \). \qed 

An important converse is the following lemma, see [13].

Lemma 2.2 (Frostman’s Lemma). If \( X \) is a metric space of Hausdorff dimension \( d \) and \( 0 < d' < d \) then there is a finite and positive measure \( \mu \) on \( X \) such that

\[ \mu(B_r) \lesssim r^{d'} \]

for every ball \( B_r \subset X \).

3. Thick Cantor sets.

The next definition is similar to the one of thick sets of Staples and Ward from [17]. We do not introduce new terminology since under the condition (3.2) of the Theorem 3.2 the following definition coincides with the one from [17].

Definition 3.1. Given a sequence \( \{c_n\}_{n=0}^{\infty} \) such that \( 0 \leq c_n < 1 \), a set \( E \subset \mathbb{R} \) is called \( \{c_n\} \)-thick if it can be constructed as follows. Let \( E_{0,1} = [0,1] \). Remove an open interval \( J_{0,1} \) from \( E_{0,1} \) such that \( \frac{\text{diam } J_{0,1}}{\text{diam } E_{0,1}} \leq c_0 \) and denote the remaining intervals \( E_{1,1} \) and \( E_{1,2} \). For \( n > 1 \) suppose the closed intervals \( E_{n,1}, \ldots, E_{n,2^n} \) have been constructed. Remove and open interval \( J_{n,j} \) from \( E_{n,j} \) for every \( j = 1, \ldots, 2^n \) such that

\[ \frac{\text{diam } J_{n,j}}{\text{diam } E_{n,j}} \leq c_n \]

and denote the remaining intervals \( E_{n+1,1}, \ldots, E_{n+1,2^{n+1}} \). Let

\[ E = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^n} E_{n,j}. \]

Clearly \( E \) has a tree structure. We will use the following terminology. Every interval \( E_{n,j} \in \mathcal{E}_n \) has:

- two children intervals \( E_{n+1,2j-1}, E_{n+1,2j} \in \mathcal{E}_{n+1} \);
- one parent interval, containing \( E_{n,j} \) and denoted by \( \tilde{E}_{n,j} \in \mathcal{E}_{n-1} \);
- one sibling interval which has the same parent as \( E_{n,j} \) and denoted by \( E'_{n,j} \in \mathcal{E}_n \). We will denote by \( \mathcal{E}_n \) the collection of intervals \( E_{n,j}, j = 1, \ldots, 2^n \) and will refer to this collections as the generation \( n \) intervals.

Middle interval Cantor sets are examples of \( \{c_n\} \)-thick sets. The more general construction in the definition above allows one to remove the intervals not necessarily from the middle. This flexibility allows one to include many more examples, for instance the uniform Cantor sets discussed in Section 7.

If \( \sum_i c_i < \infty \) then a \( \{c_i\} \)-thick set has a positive Lebesgue measure on the line. It was shown in [17] that if \( \sum_i c_i^p < \infty \) for every \( p > 0 \) then \( E \) is quasisymmetrically thick, i.e. \( f(E) \) has positive Lebesgue measure.
whenever $f : \mathbb{R} \to \mathbb{R}$ is a quasisymmetric map. In the case of the middle interval Cantor sets the condition was shown to be necessary and sufficient for $E$ to be quasisymmetrically thick, see [6].

**Theorem 3.2.** Let $E$ be a $\{c_i\}$-thick set and $f$ a power quasisymmetric embedding of $E$ into some metric space. For every interval $E_{n,j} \in \mathcal{E}_n$ let $r_{n,j}$ denote the ratio of the lengths of the longer of the two components of $E_{n,j} \setminus J_{n,j}$ to the shorter one. If

$$\sqrt[n]{\prod_{i=1}^{n}(1-c_i)} \to 1, \text{ and}$$

$$r_{n,j} \leq M, \text{ for some } M < \infty$$

then $\dim_H f(E) \geq 1$.

**Corollary 3.3.** Suppose $E \subset \mathbb{R}$ is a middle interval Cantor set

(i). If $E$ is uniformly perfect then it is minimal for conformal dimension if and only if $\dim_H E = 1$.

(ii). If $\dim_H E = 1$ then $\dim_H f(E) \geq 1$ whenever $f$ extends to a quasisymmetric map of a uniformly perfect space.

Recall that a metric space is uniformly perfect if there is a constant $C \geq 1$ so that for each $x \in X$ and for all $r > 0$

$$X \setminus B(x,r) \neq \emptyset \quad \Rightarrow \quad B(x,r) \setminus B(x,\frac{r}{C}) \neq \emptyset.$$ 

This condition in a sense rules out “large gaps” in the space. Examples of uniformly perfect sets are connected sets as well as many totally disconnected sets, like middle third Cantor set or many sets arising in conformal dynamics. The importance of uniform perfectness in quasiconformal geometry comes from the following fact, see [9].

**Theorem 3.4.** Any quasisymmetric embedding of a uniformly perfect space is power-quasisymmetric.

**Proof of Corollary 3.3.** By Kovalev’s theorem for (i) we only need to show that if $\dim_H E = 1$ then $E$ is minimal. Since every quasisymmetric map of a uniformly perfect space is power quasisymmetric and in the case of middle interval Cantor sets $r_{n,j} = 1$ to prove (i) and (ii) we only need to show that $\dim_H E = 1$ implies [8.1].

Let $N(X,\varepsilon)$ be the minimal number of $\varepsilon$ balls needed to cover $X$. Recall that upper and lower Minkowski dimensions of $X$ are defined as

$$\overline{\dim}_M(X) = \limsup_{\varepsilon \to 0} \frac{\log N(X,\varepsilon)}{\log 1/\varepsilon} \quad \text{and} \quad \underline{\dim}_M(X) = \liminf_{\varepsilon \to 0} \frac{\log N(X,\varepsilon)}{\log 1/\varepsilon}$$

respectively. When these two numbers are the same the common value is called Minkowski dimension of $X$ and is denoted by $\dim_X \mathbb{X}$. Generally
\[ \dim_H(X) \leq \dim_M(X) \leq \dim_M(X), \text{ see [13]. Therefore if } X \subset \mathbb{R} \text{ and } \dim_H(X) = 1 \text{ then Minkowski dimension of } X \text{ exists, is equal 1 and} \]

\[ \dim_M(E) = \lim_{n \to \infty} \frac{\log 2^n}{\log \prod_{i=1}^{n} (1-c_i)^{1}} = 1. \]  

(3.3)

Therefore \( \dim_H E = 1 \) if and only if (3.1) holds. \( \square \)

4. PROOF OF THEOREM 3.2

We will need the following easy estimate in the proof of Theorem 3.2.

Remark 4.1. For a given \( a > 0 \) let \( S_a = S_a \{ \{ c_i \} \} = \{ i \in \mathbb{N} | c_i < a \} \), and \( s_n = \#(S_a \cap \{ i \leq n \}) \). If condition (3.1) holds then

\[ s_n \rightarrow 1 \text{ as } n \rightarrow \infty. \]  

(4.1)

Proof. From the usual inequality between geometric and arithmetic means

\[ \sqrt[n]{\prod_{i=1}^{n} (1-c_i)} \leq \frac{1}{n} \sum_{i=1}^{n} (1-c_i) \leq 1 \]

we get that \( \frac{1}{n} \sum_{i=1}^{n} (1-c_i) \rightarrow 1 \) or, equivalently, \( \frac{1}{n} \sum_{i=1}^{n} c_i \rightarrow 0. \) \( \square \)

Remark 4.2. If we take \( c_i \rightarrow 0 \) such that \( \sum_i c_i = \infty \) then the corresponding middle interval Cantor set would be an example of a set from Theorem 1.1. Indeed, if \( c_i \rightarrow 0 \) then \( \log (1-c_i) \rightarrow 0 \) and \( \frac{1}{n} \sum_{i=1}^{n} \log (1-c_i) \rightarrow 0 \) as \( n \rightarrow \infty. \)

Therefore by (3.3) \( \dim_H E(\{ \{ c_i \} \}) = 1. \) Since \( \sum_i c_i = \infty, \) it follows that the set has zero measure. Also, a middle interval Cantor set \( E(c) \) is uniformly perfect if and only if there is a constant \( C \) such that \( c_i < C < 1, \forall i \in \mathbb{N}. \]

One of the main tools for proving Theorem 3.2 will be the following lemma from [9].

Lemma 4.3. If \( f : X \rightarrow Y \) is \( \eta \)-quasisymmetric and if \( A \subset B \subset X \) are such that \( 0 < \text{diam} A \leq \text{diam} B < \infty, \) then \( \text{diam} f(B) \) is finite and

\[ \frac{1}{2\eta \left( \frac{\text{diam} f(A)}{\text{diam} f(B)} \right)} \leq \text{diam} f(A) \leq \eta \left( \frac{2\text{diam} A}{\text{diam} B} \right). \]  

(4.2)

By distance between sets below we mean Hausdorff distance: if \( Y, Z \subset X \) then

\[ \text{dist}_X(Y, Z) = \inf \{ \text{dist}_X(y, z) | y \in Y, z \in Z \}. \]

We will need a different version of (4.2).
Lemma 4.4. Suppose $X = X_1 \cup X_2$, with $X_1, X_2$ compact and $\text{dist}(X_1, X_2) > 0$. Then

\begin{equation}
\frac{1}{2\eta \left( \frac{\text{diam}X}{\text{dist}(X_1, X_2)} \right)} \leq \frac{\text{dist}(f(X_1), f(X_2))}{\text{diam}f(X)} \leq \eta \left( \frac{2\text{dist}(X_1, X_2)}{\text{diam}X} \right).
\end{equation}

Proof. Suppose $x_1 \in X_1$ and $x_2 \in X_2$ are such that $\text{dist}(X_1, X_2) = d_X(x_1, x_2)$. This is possible since $X_1$ and $X_2$ are compact. Let $A = \{x_1, x_2\}$ then right hand inequality in (4.2) implies

\begin{equation}
\frac{\text{dist}(f(X_1), f(X_2))}{\text{diam}f(X)} \leq \eta \left( \frac{2\text{dist}(x_1, x_2)}{\text{diam}X} \right).
\end{equation}

To obtain the other inequality of (4.3) take $y_1 \in f(X_1), y_2 \in f(X_2)$ in such a way that $\text{dist}(f(X_1), f(X_2)) = d_Y(y_1, y_2)$. Let $x_i' = f^{-1}(y_i)$. Now take $A = \{x_1', x_2'\}$. Then again using 4.2 we get

\begin{equation}
\frac{\text{dist}(f(X_1), f(X_2))}{\text{diam}f(X)} \geq \frac{1}{2\eta \left( \frac{\text{diam}X}{\text{dist}(x_1', x_2')} \right)}
\end{equation}

Since $d_X(x_1', x_2') \geq \text{dist}(X_1, X_2)$ and since $\eta$ is increasing we obtain

\begin{equation}
\frac{1}{2\eta \left( \frac{\text{diam}X}{\text{dist}(x_1', x_2')} \right)} \geq \frac{1}{2\eta \left( \frac{\text{diam}X}{\text{dist}(X_1, X_2)} \right)}.
\end{equation}

Combining this with the previous inequality gives (4.3). \qed

Theorem 3.2 follows from the following result and the mass distribution principle.

Lemma 4.5. Let $E$ be as in Theorem 3.2. If $f : E \to Y$ is a power-quasisymmetric homeomorphism then for every $d < 1$ there is a measure $\mu$ on $Y$ satisfying

\[ \mu(B(y, r)) \leq C r^d \]

for some constant $C > 0$, all $r > 0$ and $y \in Y$. Constant $C$ does not depend on $y$ and $r$.

To simplify the notation below we write $f(E_{n,j})$ for $f(E_{n,j} \cap E)$ (we don’t assume that $f$ extends to the real line). We will prove the lemma in several steps.

First we will show that there is a measure $\mu$ on $f(\bigcap_n \bigcup_{E_n} E_{n,j}) \subset f(E)$ such that

\[ \mu(f(E_{n,j})) \leq C \text{diam}f(E_{n,j}), \]

for some non-zero finite constant $C$ independent of $n$ and $j$. 


Proof of $\star$. Since $f$ is a homeomorphism $f(E)$ has a tree structure just like $E$. The notation of Definition 3.1 will also be used for $f(E)$. Namely, for an $I \subset Y$ of the form $I = I_{n,j} = f(E_{n,j})$ we will denote by $\tilde{I}_{n,j} = f(\tilde{E}_{n,j})$ and $I'_{n,j} = f(E'_{n,j})$ the parent and the sibling of $I$ respectively.

Construction of the measure. Now define $\mu$ as follows. Pick $E_0 \in \mathcal{E}_0$ and let

$$\mu(f(E_0)) = 1.$$ 

For any $I \subset Y$ of the form $I = f(E_{n,j})$, where $E_{n,j}$ is a “descendant” of $E_0$ let:

$$(4.4) \quad \mu(I) = \frac{\text{diam}^d I}{\text{diam}^d I + \text{diam}^d I'} \mu(I).$$

Given such an interval $I$ there is a unique sequence of nested subsets $I = I_n \subset I_{n-1} \subset I_{n-2} \subset \ldots \subset I_2 \subset I_1 \subset I_0 = Y$ containing it, so that $I_{k-1} = \tilde{I}_k$. By induction we have

$$\mu(I) \leq \prod_{i=1}^{n} \frac{(\text{diam}I_i + \text{dist}(I_i, I'_i) + \text{diam}I'_i)^d}{\text{diam}^d I_i + \text{diam}^d I'_i}.$$ 

(4.5)

Let

$$(4.6) \quad p_i = \frac{(\text{diam}I_i + \text{dist}(I_i, I'_i) + \text{diam}I'_i)^d}{\text{diam}^d I_i + \text{diam}^d I'_i}.$$ 

To prove $\star$ we need to show that $\prod_{i=1}^{n} p_i \to 0$ as $n \to \infty$. Indeed, if this is the case then $\exists C < \infty \text{ s.t. } \prod_{i=1}^{n} p_i < C, \forall n \in \mathbb{N}$. Now, to prove $\prod_{i=1}^{n} p_i \to 0$ we will need the following estimates.

Lemma 4.6 (Small gaps). $\exists a > 0, C_1 < 1 \text{ s.t } c_i < a \Rightarrow p_i < C_1 < 1.$

Lemma 4.7 (Large gaps). $\exists C_2 > 1 \text{ s.t. } p_i < \frac{C_2}{(1-c_i)^{d/\alpha}}, \forall i.$

Let us prove Theorem 3.2 assuming these two lemmas. First of all

$$\prod_{i=1}^{n} p_i \leq \prod_{\{i \leq n|c_i < a\}} C_1 \prod_{\{i \leq n|c_i \geq a\}} \frac{C_2}{(1-c_i)^{d/\alpha}} \quad \text{(by the two lemmas)}$$ 

$$\leq C_1^{s_n} \frac{C_2^{n-s_n}}{\prod_{i=1}^{n} (1-c_i)^{d/\alpha}} \quad \text{(where } s_n \text{ is like in Corollary 4.1).}$$
Now, if $C_1 < 1$ and $s_n/n \to 1$ then for every number $C_2 < \infty$ there is a $C_3 < 1$ and $N \in \mathbb{N}$ s.t. for $n > N$

$$C_1^{s_n} C_2^{n-s_n} \leq C_3^n.$$ 

Hence

$$\prod_{i=1}^n p_i \leq \left( \frac{C_3}{\sqrt[n]{\prod_{i=1}^n (1 - c_i)^{d/\alpha}}} \right)^n.$$ 

Since $\sqrt[n]{\prod_{i=1}^n (1 - c_i)} \to 1$ and $C_3 < 1$ it follows that $\prod_{i=1}^n p_i \to 0$.

Next we prove Lemmas 4.6 and 4.7.

*Proof of lemma 4.6.* Recall that for a given $a > 0$ we had

$$S_a = \{ i \in \mathbb{N} | c_i < a \}, S_n = S_a \cap \{ i \leq n \}, s_n = \text{card}(S_n).$$

Without loss of generality we can assume $a < 1/2$.

Suppose now $i \in S_n$. We find it easier to estimate $p_i^{-1}$ from below.

$$p_i^{-1} = \frac{\text{diam}^d I_i + \text{diam}^d I_i'}{\text{diam} I_i + \text{diam} I_i'} \geq \frac{\text{diam}^d I_i + \text{diam}^d I_i'}{\text{diam} I_i + \text{diam} I_i'} \left( 1 - \frac{\text{dist}(I_i, I_i')}{\text{diam} I_{i-1}} \right)^d$$

(by (4.3))

$$\geq \frac{\text{diam}^d I_i + \text{diam}^d I_i'}{\text{diam} I_i + \text{diam} I_i'} \left( 1 + \frac{\text{diam} I_i'}{\text{diam} I_i} \right)^d \left( 1 - \eta(2c_i) \right)^d.$$

We will show that the first term in this product is bounded below by a constant strictly greater than 1. To do that, first note that there is a constant $1 < D(\eta, M) < \infty$ so that $D^{-1} < \text{diam} I_i / \text{diam} I_i' < D$. Indeed,

(by (4.3))

$$\frac{\text{diam} I_i}{\text{diam} I_i'} \geq \frac{\text{diam} I_i}{\text{diam} I_{i-1}} \geq \frac{1}{2\eta \left( \frac{\text{diam} E_{i-1}}{\text{diam} E_i} \right)}.$$ 

Since $c_{i-1} < 1/2$ it follows from (3.2) that

$$\text{dist}(E_i, E_i') \leq \text{diam} E_i + \text{diam} E_i' \leq (1 + M)\text{diam} E_i.$$ 

Therefore

$$\frac{\text{diam} E_{i-1}}{\text{diam} E_i} \leq \frac{\text{diam} E_i + \text{dist}(E_i, E_i') + \text{diam} E_i'}{\text{diam} E_i} \leq 2(1 + M),$$

and hence

$$\frac{\text{diam} I_i}{\text{diam} I_i'} \geq \frac{1}{2\eta(2(1 + M))} > 0.$$ 

The second inequality follows by symmetry.
Considering the function \( x \mapsto \frac{1+x^d}{(1+x)^d} \) for \( d < 1 \) one can easily see that on an interval \([D^{-1}, D]\) its smallest value is attained at \( D \) and is strictly larger than 1. We will denote this value by \( C_4 = C_4(\eta, d) > 1 \). Therefore

\[
p_i^{-1} \geq C_4(1 - \eta(2c_i))^d \geq C_4(1 - \eta(2a))^d.
\]

Since \( \eta \) is increasing and \( c_i < a \). Now, \( \eta(t) \to 0 \) as \( t \to 0 \). Therefore we can always choose a small enough so that \( C_4(1 - \eta(2a))^d > 1 \). So finally we conclude that there is an \( a \) so that for \( i \in S_a \) one has \( p_i^{-1} \geq C_5 > 1 \).

Equivalently \( p_i \) is bounded from above by a constant strictly less than 1.  

Note that we haven’t yet used the fact that \( f \) is power quasisymmetric.

Proof of lemma 4.7. Since \( \text{diam} I_i, \text{diam} I'_i, \text{dist}(I_i, I'_i) < \text{diam} I_{i-1} \), we have

\[
p_i = \frac{(\text{diam} I_i + \text{dist}(I_i, I'_i) + \text{diam} I'_i)^d}{\text{diam}^d I_i + \text{diam}^d I'_i} \leq \frac{3^d \text{diam}^d I_{i-1}}{\text{diam}^d I_i + \text{diam}^d I'_i} = 3^d \left[ \left( \frac{\text{diam} I_i}{\text{diam} I_{i-1}} \right)^d + \left( \frac{\text{diam} I'_i}{\text{diam} I_{i-1}} \right)^d \right]^{-1}
\]

(by (4.3)) \( \leq 3^d \frac{\eta d}{2} \frac{\text{diam} E_{i-1}}{\text{diam} E_i} \).

From (3.2) we have

\[
\frac{\text{diam} E_i}{\text{diam} E_{i-1}} \geq 1 - c_i - \frac{\text{diam} E'_i}{\text{diam} E_{i-1}} \geq 1 - c_i - M \frac{\text{diam} E_i}{\text{diam} E_{i-1}}
\]

and therefore

\[
\frac{\text{diam} E_{i-1}}{\text{diam} E_i} \leq 1 + M \frac{1}{1 - c_i}.
\]

It follows that

\[
p_i \leq \frac{3^d}{2} \eta d \frac{1 + M}{1 - c_i}.
\]

Now, since \( \eta(t) \leq C \max\{t^{1/\alpha}, t^\alpha\} \), the last inequality yields

\[
p_i \leq \frac{C_2(d, \alpha)}{(1 - c_i)^{d/\alpha}}.
\]

As shown before this completes the proof of inequality (⋆).  

To complete the proof of Lemma 4.5 and Theorem 3.2 we need to show that the upper bound similar to the (⋆) holds for every ball \( B = B(y, r) \) with \( y \in Y \). First we show that (⋆) implies the following lemma.

Lemma 4.8. There is a constant \( C \) such that for any interval \( J \subset \mathbb{R} \) we have

\[
\mu(f(J \cap E)) \leq C [\text{diam} f(J \cap E)]^d
\]
Proof. Note first that for every \( J \subset \mathbb{R} \) there are two (or one) intervals \( E_1, E_2 \in \bigcup_n \mathcal{E}_n \) such that 
\[
E_1, E_2 \subset J \quad \text{and} \quad J \cap E \subset \tilde{E}_1 \cup \tilde{E}_2.
\]
Indeed, consider the collection 
\[
\mathcal{E}_J = \left\{ E_{n,j} \in \bigcup_n \mathcal{E}_n : E_{n,j} \subseteq J, \text{ but } \tilde{E}_{n,j} \not\subseteq J \right\},
\]
in other words, the collection of intervals \( E_{n,j} \) which are contained in \( J \) with parents \( \tilde{E}_{n,j} \) that are not. Since every interval \( E_{n,j} \subset J \) has an “ancestor” in \( \mathcal{E}_J \) it follows that 
\[
J \subset \bigcup_{E_{n,j}} E_{n,j}.
\]
Now, choose \( E_1 \in \mathcal{E}_J \) so that \( \text{diam} \tilde{E}_1 \geq \text{diam} \tilde{E}_{n,j} \), for any \( E_{n,j} \in \mathcal{E}_J \). If \( \tilde{E}_1 \supset J \) then we are done \((E_2 = \emptyset)\). If not, consider \( \mathcal{E}_J \setminus \tilde{E}_1 \) and choose \( E_2 \) from this collection in a similar fashion, i.e. \( \text{diam} \tilde{E}_2 \geq \text{diam} \tilde{E}_{n,j} \), for any \( E_{n,j} \in \mathcal{E}_J \setminus \tilde{E}_1 \). Since for every \( E_{n,j} \in \mathcal{E}_J \) its parent \( \tilde{E}_{n,j} \) contains at least one of the end points of \( J \) it means it intersects either \( \tilde{E}_1 \) or \( \tilde{E}_2 \) and therefore must be contained in one of them (since every two elements of \( \mathcal{E} \) are either disjoint or one of them contains the other one). Therefore \( J \cap E \subset \tilde{E}_1 \cup \tilde{E}_2 \).

Just as before let \( E_1' \) and \( E_2' \) be the siblings of \( E_1 \) and \( E_2 \) respectively. Note that if \( J \cap E_1' = \emptyset \) then \( J \cap \tilde{E}_i = E_i \). Therefore we need to consider the contribution of \( E_i' \) only if \( J \cap E_i' \neq \emptyset \) in which case, since \( \text{diam} E_i' \leq M \text{diam} E_i \leq M \text{diam} J \), we obtain 
\[
E_i' \subset 2M J,
\]
where \( 2M J \) is just the dilation of \( J \) by \( 2M \). Now from \( \star \) it follows that 
\[
\mu(f(J \cap E)) = \sum_{i=1,2} \mu(f(E_i)) + \mu(f(E_i')) \leq C \sum_{i=1,2} [\text{diam} f(E_i)]^d + [\text{diam} f(E_i')]^d
\]
Since 
\[
\text{diam} f(E_i), \text{diam} f(E_i') \leq \text{diam} f(2M J \cap E),
\]
and by \( \square \) we have 
\[
\text{diam} f(2M J \cap E) \leq 2\eta(2M) \text{diam} f(J \cap E)
\]
it follows that 
\[
\mu(f(J \cap E)) \leq C[\text{diam} f(J \cap E)]^d
\]
for some constant \( C \) and any interval \( J \subset R \). \( \blacklozenge \)

Proof of Lemma 4.5. By quasisymmetry there is a number \( 1 \leq H < \infty \) such that for every \( y \in \bar{Y} \) and \( r > 0 \) 
\[
B(x, R) \subset f^{-1}(B(y, r)) \subset B(x, HR),
\]
where \( x = f^{-1}(y) \). Therefore
\[
\mu(B(y, r)) \leq \mu(f(B(x, HR)))
\]
(by Lemma (4.8))
\[
\leq C[\text{diam}(B(x, HR))]^d
\]
(by (4.2))
\[
(f(B(x, R)) \subseteq B(y, r))
\]
\[
\leq Cr^d.
\]
As we noted before it follows that \( \dim_H(f(E)) \geq 1 \) since \( d \) could be chosen as close to 1 as one would like. \( \square \)

**Remark 4.9.** Note that \( E_c \) is uniformly perfect for every \( c \in (0, 1] \) and therefore Theorem 1.3 is contained in the proof of Theorem 3.2 if we disregard Lemma 4.7.

### 5. Modulus of measures & Conformal dimension

It was shown by Tyson in [19] that one obstruction for lowering the Hausdorff dimension of a space by quasisymmetric maps is the existence of a family of curves of positive modulus. Theorem 5.5 states that a large family of regular enough, but possibly totally disconnected minimal sets also gives a lower bound for the conformal dimension. For the sake of completeness before stating Theorem 5.5 we will recall the definition of a modulus of a family of curves and formulate Tyson’s theorem.

Given a family of curves \( \Gamma \) in a metric measure space \((X, \mu)\) and a real number \( d \geq 1 \) the \( d \)-modulus of \( \Gamma \) is defined as
\[
\text{mod}_d \Gamma = \inf \int_X \rho^d d\mu,
\]
where the infimum is taken over all \( \Gamma \)-admissible positive Borel functions \( \rho \). Here a function \( \rho : X \to [0, \infty) \) is \( \Gamma \) admissible if \( \int_\gamma \rho ds \geq 1 \) for every locally rectifiable curve \( \gamma \in \Gamma \), where \( ds \) denotes the arclength element.

Recall also that a metric measure space \((X, \mu)\) is doubling if there is a constant \( C \) such that \( \mu(B_{2r}) \leq C\mu(B_r) \) for every ball \( B_r \subset X \). For further details on moduli of curve families, definition of the arclength in general metric spaces and a short and elegant proof (due to Bonk and Tyson) of the following Theorem we refer the reader to [9].

**Theorem 5.1** *(Tyson [19]).* Suppose \((X, \mu)\) is a doubling metric measure space such that \( \mu(B_r) \lesssim r^d \) for every ball \( B_r \subset X \) of radius \( 0 < r < \text{diam}X \). If there is a curve family \( \Gamma \) in \( X \) such that \( \text{mod}_d \Gamma > 0 \) then \( C \dim X \geq d \).

To state Theorem 5.5 we need the following definition of a modulus of a system of measures due to Fuglede, see [7].

**Definition 5.2.** Let \((X, \mu)\) be a measure space. Let \( E \) be a collection of measures on \( X \) the domains of which contain the domain of \( \mu \). A positive
Borel function $\rho : X \to \mathbb{R}$ is said to be admissible for the system of measures $\mathbf{E}$ if $\int_X \rho \, d\lambda \geq 1$ for every $\lambda \in \mathbf{E}$. Next we define the $p$-modulus of $\mathbf{E}$ as

$$\text{mod}_p(\mathbf{E}) = \inf \int_X \rho^p \, d\mu,$$

where $\inf$ is taken over all $\mathbf{E}$-admissible functions $\rho$.

Just like the usual modulus of a family of curves the modulus of a system of measures is monotone and sub-additive, see [7].

**Lemma 5.3.** The $p$-modulus is monotone and countably subadditive:

\begin{align}
\text{mod}_p \mathbf{E} &\leq \text{mod}_p \mathbf{E}', \quad \text{if } \mathbf{E} \subseteq \mathbf{E}', \\
\text{mod}_p \mathbf{E} &\leq \sum_{i=1}^{\infty} \text{mod}_p \mathbf{E}_i, \quad \text{if } \mathbf{E} = \bigcup_{i=1}^{\infty} \mathbf{E}_i.
\end{align}

We will also need the following property of modulus.

**Lemma 5.4.** Suppose $\mu(X) < \infty$ and $t > t' \geq 1$. If $\text{mod}_{t'} \mathbf{E} > 0$ then $\text{mod}_t \mathbf{E} > 0$.

**Proof.** From Hölder’s inequality we obtain $\int_X \rho^t \, d\mu \leq \left( \int_X \rho^{t'} \right)^{\frac{t}{t'}} \mu(X)^{1-\frac{t}{t'}}$. Since $\mu(X) < \infty$ the lemma follows.

**Theorem 5.5.** Let $p > 1$ and $(X, \mu)$ be a compact, doubling metric measure space. Suppose there is a constant $0 < C < \infty$ such that for every ball $B_r \subset X$

$$\mu(B_r) \leq Cr^p,$$

and there is a system of measures $\mathbf{E} = \{\lambda_E\}_{E \in \mathcal{E}}$ associated to $\mathcal{E}$ ($\lambda_E$ is supported on $E$) such that for every $s > 1$ there are constants $C_1 = C_1(s)$ and $C_2 = C_2(s)$ such that $\forall E \in \mathcal{E}$

(5.5) \hspace{1cm} \lambda_E(B_r \cap E) \geq C_1 r^s, \quad \text{if} \quad \frac{1}{C_2} B_r \cap E \neq \emptyset.

If $\text{mod}_p \mathbf{E} > 0$ for some $1 \leq p' < p$ then $C \dim X \geq p$.

Theorem 5.5 is proved in the next section. Here we show how Theorem 1.2 follows from Theorem 5.5.

**Corollary 5.6.** Suppose $E \subseteq \mathbb{R}$ is a set of conformal dimension 1 which supports a measure $\lambda_E$ such that for every $\varepsilon > 0$ there is a constant $C$ so that whenever $x \in E$ and $R < \text{diam} E$

$$\frac{1}{C} R^{1+\varepsilon} \leq \lambda_E(B(x, R) \cap E) \leq CR^{1-\varepsilon}.$$

Then for every Borel set $Y \subseteq \mathbb{R}^n$

$$C \dim(E \times Y) \geq \dim_H E \times Y.$$
Proof. Let $d < \dim_H Y$. By Frostman’s lemma for every $\varepsilon$ such that $2\varepsilon \in (0, \dim_H Y - d)$ there is a measure $\nu$ on $Y$ such that $\nu(Y) > 0$ and $\nu(B_R) \lesssim R^{d+2\varepsilon}$ for every ball $B_R \subset Y$. Let $\mu = \lambda_E \times \nu$. Then there is a constant $0 < C < \infty$ such that

$$\mu(B_R) \leq CR^{1-\varepsilon}R^{d+2\varepsilon} = R^{1+d+\varepsilon}$$

for every $B_R \subset E \times Y$.

Let $E = \{E \times \{y\} : y \in Y\}$ and $\lambda_{E \times \{y\}}(U \times \{y\}) = \lambda_E(U)$ for every $\lambda_E$ measurable $U \subset E$. Define

$$E = \{\lambda_{E \times \{y\}} : y \in Y\}.$$

The proof would be complete if we could show that $\text{mod}_{1+d} E > 0$. Indeed, Theorem 5.5 would imply then that $C \dim(E \times Y) \geq 1 + d$ for every $d < \dim_H Y$ and therefore $C \dim(E \times Y) \geq 1 + \dim_H Y$.

The argument for $\text{mod}_{1+d} E > 0$ is standard and we include it only for completeness. Take $\rho : E \times Y \to [0, \infty)$ s.t. $\int_E \rho(x, y)d\lambda_E \geq 1, \forall y \in Y$. By Hölder’s inequality we get that

$$\int_E \rho^{1+d}(x, y)dx \geq 1, \forall y \in Y.$$

Integrating both sides of the inequality with respect to $\nu$ we obtain

$$\int_{E \times Y} \rho^{1+d}(x, y)d\lambda_E \times d\nu \geq \nu(Y) > 0.$$

Therefore $\text{mod}_{1+d} E \geq \nu(Y) > 0$. \hfill \Box

Remark 5.7. It is not hard to see that uniformly perfect middle interval Cantor sets satisfy the conditions of the previous corollary. In fact the measure which gives equal mass to every interval of the same length is an example of a measure which satisfies the required inequalities.

6. **Proof of Theorem 5.5**

In the course of the proof we will need a notion of discrete modulus of a family of subsets of $X$ which in essence is due to Heinonen and Koskela, see [10], even though in [10] it was only defined and used for families of curves.

**Definition 6.1.** Let $E = \{E\}$ be a collection of subsets of $X$. Let $B = \{B\}$ be a cover of $X$ by balls and $v : B \to [0, \infty)$ a function. The pair $(v, B)$ is **admissible for $E$** if

$$\frac{1}{5}B \cap \frac{1}{5}B' = \emptyset,$$

whenever $B \neq B'$, and

$$\sum_{\frac{1}{5}B \cap E \neq \emptyset} v(B) \geq 1$$

for every $E \in E$. 
For \( \delta > 0 \) set

\[
\text{d-mod}_p(\mathcal{E}) = \inf \sum_{B \in \mathcal{B}} v(B)^p,
\]

where the infimum is over all pairs \((v, \mathcal{B})\) which are admissible for \( \mathcal{E} \) and such that \( \text{diam} B \leq \delta \) for every \( B \in \mathcal{B} \). The discrete \( p \)-modulus of \( \mathcal{E} \) is

\[
\text{d-mod}_p(\mathcal{E}) = \lim_{\delta \to 0} \text{d-mod}_p^\delta(\mathcal{E}).
\]

The need for the disjointness property in the definition of admissibility comes from the following covering lemma, see for instance [13] Theorem 2.3.

**Lemma 6.2 (Covering Lemma).** Every family \( \mathcal{B} \) of balls of bounded diameter in a compact metric space \( X \) contains a countable subfamily of disjoint balls \( B_i \subset \mathcal{B} \) such that

\[
\bigcup_{B \in \mathcal{B}} B \subset \bigcup_{i} 5B_i.
\]

**Remark 6.3.** Even though the monotonicity of the discrete modulus is easy to see we do not know if it is (countably-) subadditive.

Theorem 5.5 would follow from the following two lemmas.

**Lemma 6.4.** Let \( t > 0 \) and suppose \( \mathcal{E} \) is a collection of subsets in \( X \) such that for some \( \delta > 0 \)

\[
H_1^\delta t E \geq c > 0, \forall E \in \mathcal{E}.
\]

Then \( \text{d-mod}_q \mathcal{E} = 0 \) for every \( q > \frac{1}{t} \text{dim}_H X \).

**Lemma 6.5.** If conditions (5.3) and (5.5) of Theorem 5.5 are satisfied. Then for every \( q < p \) there is a constant \( C < \infty \) such that

\[
\text{mod}_q \mathcal{E} \leq C \text{d-mod}_q f(\mathcal{E}).
\]

Before proving the lemmas let us prove the theorem assuming they are true.

**Proof of Theorem 5.5.** Suppose \( f \) is a quasisymmetric map such that \( \text{dim}_H f(X) < p \). Assume that \( q < p \) is chosen so that \( q > \max(p', \text{dim}_H f(X)) \) and then choose \( t < 1 \) in such a way that \( q > \frac{1}{t} \text{dim}_H f(X) \). Since \( \text{dim}_H f(E) \geq 1 \) for every \( E \in \mathcal{E} \) it follows that \( H_1^{1/k} f(E) \to \infty \) as \( k \to \infty \). Let \( k_E \in \mathbb{N} \) denote the smallest integer such that \( H_1^{1/k} f(E) \geq 1 \). Let

\[
\mathcal{E}_j = \{ E \in \mathcal{E} : k_E \geq j \}, \quad \text{and} \quad \mathcal{E}_j = \{ \lambda_E \in \mathcal{E} : E \in \mathcal{E}_j \}.
\]

Then \( \mathcal{E} = \bigcup_{j=1}^\infty \mathcal{E}_j \) and therefore from (5.2) and from Lemma 6.5 it follows that

\[
\text{mod}_q \mathcal{E} \leq \sum_{i=1}^\infty \text{mod}_q \mathcal{E}_i \leq C \sum_{i=1}^\infty \text{d-mod}_q f(\mathcal{E}_i)
\]

where \( f(\mathcal{E}_i) \) is the image of the family \( \mathcal{E}_i \). Lemma 6.4 implies that \( \text{d-mod}_q f(\mathcal{E}_j) = 0 \) and it follows that \( \text{mod}_q \mathcal{E} = 0 \). Hence Lemma 6.4 implies that \( \text{mod}_p \mathcal{E} = 0 \), which contradicts our assumption. \( \square \)
Next we prove Lemmas 6.4 and 6.5.

Proof of Lemma 6.4. Let $q'$ be such that $\dim_H X < q' < tq$. For every $\delta > \varepsilon > 0$ there is a covering $\mathcal{B}$ of $X$ by balls $B_1, B_2, \ldots$ with radii $r_1, r_2, \ldots$ such that $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$, for $i \neq j$ and
\[
\sum_i r_i^{q'} < \varepsilon.
\]

Let $v(B_i) = r_i^t$. Since $r_i < \delta$ it follows that for every $E \in \mathcal{E}$ we have
\[
\sum_{B \cap E \neq \emptyset} v(B) \geq H^t_i(E) \geq 1
\]
and so $(v, \mathcal{B})$ is admissible for $\mathcal{E}$. Now,
\[
\sum_i v(B_i)^q = \sum_i r_i^{qt} \leq \sum_i r_i^{q'} < \varepsilon.
\]
Therefore $d_{\text{mod}} q \mathcal{E} < \varepsilon$. \hfill \Box

Proof of Lemma 6.5. Below we will need the following well known inequality, see [9] or [4] Lemma 4.2 in the case of $\mathbb{R}^n$, which is a consequence of the boundedness of the Hardy-Littlewood maximal operator.

Lemma 6.6. Suppose $\mathcal{B} = \{B_1, B_2, \ldots\}$ is a countable collection of balls in a doubling metric measure space $(X, \mu)$ and $a_i \geq 0$ are real numbers. Then there is a positive constant $C$ such that
\[
\left( \int_X \left( \sum_B a_i \chi_{AB_i}(x) \right)^p d\mu \right)^{1/p} \leq C(A, p, \mu) \left( \int_X \left( \sum_B a_i \chi_{B_i}(x) \right)^p d\mu \right)^{1/p}
\]
for every $1 < p < \infty$ and $A > 1$.

It is clear that all we need to show is
\[
\text{mod}_q \mathcal{E} \leq C_{\text{d-mod}}\delta f(\mathcal{E})
\]
for some $\delta \in (0, 1)$. For that suppose $(v, \mathcal{B})$ is an $f(\mathcal{E})$-admissible pair, where $\mathcal{B}' = \{B_i'\}_{i=1}^\infty$ is a cover of $f(X)$ by balls $B_i'$ of radii $r_i' < \delta$ . Choose $B_i \subset X$ with radius $r_i$ so that
\[
\frac{1}{H} B_i \subset f^{-1} \left( \frac{1}{5} B_i' \right) \subset f^{-1}(B_i') \subset B_i,
\]
where $H$ is constant depending on $f$ (there is such a constant since $f$ is quasisymmetric). Note that since $\mathcal{B}'$ is admissible it follows that $\frac{1}{H} B_i \cap \frac{1}{H} B_j = \emptyset$ whenever $i \neq j$.

We want to construct an $\mathcal{E}$-admissible function $\rho$ such that
\[
\int_X \rho^q d\mu \leq C \sum_{B \in \mathcal{B}'} v(B')^q.
\]
Define

\[ \rho(x) = \sum_i \frac{v(B'_i)}{[\text{diam}B_i]^s} \chi_{C_2B_i}(x), \]

where \( C_2 \) is as in the formulation of Theorem 5.5. Then for every \( E \in \mathcal{E} \) the following holds

\[
\int_E \rho d\lambda_E = \int_E \sum_i \frac{v(B'_i)}{[\text{diam}B_i]^s} \chi_{C_2B_i}(x) d\lambda_E \geq \int_E \sum_i \frac{v(B'_i)}{[\text{diam}B_i]^s} \chi_{C_2B_i}(x) d\lambda_E
\]

\[
= \sum_{i: B_i \cap E \neq \emptyset} \frac{v(B'_i)}{[\text{diam}B_i]^s} \int_{E \cap C_2B_i} d\lambda_E = \sum_{i: B_i \cap E \neq \emptyset} \frac{v(B'_i)}{[\text{diam}B_i]^s} \lambda(E \cap C_2B_i)
\]

\[
\geq \frac{1}{C_1} \sum_{i: f(E) \cap B'_i \neq \emptyset} v(B'_i) \geq \frac{1}{C_1}
\]

It follows that

\[ \mod_q(E) \leq C_1^q \int_X \rho^q d\mu. \]

Next, take \( s > 1 \) so that \( qs < p \). Then we have

\[
\int_X \rho^q d\mu = \int_X \left( \sum_i \frac{v(B'_i)}{[\text{diam}B_i]^s} \chi_{C_2B_i}(x) \right)^q d\mu
\]

(by (†))

\[
\leq C(5H, q, \mu) \int_X \left( \sum_i \frac{v(B'_i)}{[\text{diam}B_i]^s} \chi_{C_2B_i}(x) \right)^q d\mu
\]

(6.1)

\[
= C(5H, q, \mu) \sum_i \left( \frac{v(B'_i)}{[\text{diam}B_i]^s} \right)^q \mu\left( \frac{1}{\mu(B_i)} \right)
\]

(by (5.3))

\[
\lesssim \sum_i v(B'_i)^q r_i^{p-qs}
\]

( \( r_i < \delta < 1 \))

\[
\lesssim \sum_i v(B'_i)^q.
\]

Taking infimum over all \( f(\mathcal{E}) \)-admissible pairs \((v', B')\) we obtain \( \mod_q E \leq C d \cdot \mod_q f(\mathcal{E}) \) for some \( C \) independent of \( \delta \) and hence

\[ \mod_q E \leq C d \cdot \mod_q f(\mathcal{E}) \]

therefore completing the proof. \( \square \)

7. Remarks and Problems

In [1] Beurling and Ahlfors showed that there are full measure sets in \([0, 1] \) (or \( \mathbb{R} \)) which can be mapped to a zero measure set by a quasisymmetry. Examples of sets which could not be mapped to zero length sets by a quasisymmetries of a line were given in [17] and [9]. The problem of characterization of such quasisymmetrically thick sets is well known but has not yet been solved completely. Tukia gave the first examples of subsets of the
line of full measure and conformal dimension 0 in [18]. See [2] and [16] for
further such examples and the relation to conformal welding. The following
seems to be a natural problem.

**Problem 1.** Characterize (compact) subsets of the line of conformal dimen-
sion 1.

One motivation for understanding the spaces of conformal dimension 1
comes from Theorem 5.5, which shows that one may bound the conformal
dimension of a general metric space $X$ if there are sufficiently many subsets
of $X$ of conformal dimension 1. This approach will be used in a subsequent
paper to obtain lower bounds for the conformal dimension of certain self-
affine sets.

Recall from [11] that given a sequence of integers $n_i \geq 2$, and a sequence
of real number $\gamma_i \in (0, 1)$ a uniform Cantor set $E$ corresponding to these se-
quences is constructed as follows. Divide $E_0 = [0, 1]$ into $n_0$ intervals of equal
length so that the spacing between adjacent “children” of $E_0$ is $\gamma_1 \text{diam} E_0$.
In the $i$-th step divide every component $E_{i,j}$ remaining from the previous
step into $n_i$ equal length intervals so that the distance between every two
adjacent ones is $\gamma_i \text{diam} E_{i,j}$. It is not hard to see that if $n_i \leq N, \forall i$ and
$\dim_H E = 1$ then $E$ satisfies the conditions of Theorem 3.2. Furthermore,
under these conditions $E$ is uniformly perfect, which means $\gamma_i < C < 1$, 
then $C \dim E = 1$. Also, even if $E$ is not uniformly perfect, $\dim_H f(E) \geq 1$
if $f$ extends to a quasisymmetry of a uniformly perfect space (for instance
a quasiconformal map of a Euclidean space as in [11]). We summarize the
previous discussion as follows.

**Corollary 7.1.** Suppose $E = E(\{n_i\}, \{\gamma_i\})$ is a uniform Cantor sets of
Hausdorff dimension 1.

- If $\{n_i\}$ is bounded then $E$ is minimal for every map which extend to
  a quasisymmetry of a uniformly perfect space.
- If $\{n_i\}$ is bounded and $\gamma_i < C < 1, \forall i$ then $C \dim E = 1$.

The condition that $\{n_i\}$ is a bounded sequence is crucial in this corollary
since otherwise one can easily construct a uniform Cantor set of Hausdorff
dimension 1 which does not satisfy the condition (3.1) of Theorem 3.2 (take
$n_i = 3^i$ and $\gamma_i = 1/3^{i-1}$). The authors of [11] asked if uniform Cantor sets
are minimal without the assumption of boundedness of the sequence $\{n_i\}$.
One may construct many examples of uniform Cantor sets with unbounded
$\{n_i\}$ which still satisfy the conditions of Theorem 3.2 however the answers
to the following questions are not known to us.

**Problem 2.** Suppose $E = E(\{n_i\}, \{\gamma_i\})$ is a uniform Cantor set.

- Obtain necessary and sufficient conditions (in terms of $\{n_i\}$ and
  $\{\gamma_i\}$) for $E$ to be minimal.
- Suppose $\gamma_i n_i$ is non-increasing. Is it true that
  $C \dim E = 1 \iff \gamma_i n_i \to 0$ as $i \to \infty$?
Here is a different construction of compact subsets of a line. Consider two sequences \( \{p_i\} \) and \( \{q_i\} \) of integers with \( 1 \leq q_i < p_i, \forall i \in \mathbb{N} \). Construct the compact set \( F = F(\{p_i\}, \{q_i\}) \), corresponding to \( \{p_i\} \) and \( \{q_i\} \) as before but in the \( i \)-th stage divide each interval remaining from the previous stage into \( p_i \) equal parts and remove any \( q_i \) out. The results of Staples and Ward [17] and Theorem 3.2 suggest the following questions.

**Problem 3.** Let \( F = F(\{p_i\}, \{q_i\}) \) be constructed as above.
- Suppose \( \sum_i (q_i/p_i)^t < \infty \), for every \( t > 0 \) and let \( F = F(\{p_i\}, \{q_i\}) \).
- Does \( F \) have conformal dimension 1? Is it quasisymmetrically thick?
- Is there an example of \( F = F(\{p_i\}, \{q_i\}) \) of conformal dimension 0 and \( q_i/p_i \to 0 \) as \( i \to \infty \)?

In [3] Bishop and Tyson asked for a characterization of subsets \( E \) of the line such that the products \( E \times Y \) are minimal for every compact \( Y \). Clearly \( E \) would have to be minimal itself to have that property.

**Problem 4.** Let \( E \subset \mathbb{R} \). Is \( E \times Y \) minimal for every compact \( Y \) if (and only if) \( E \subset \mathbb{R} \) is minimal?

Theorem 1.2 does not answer this question since a lower bound on the Hausdorff dimension does not in general imply that there is a measure \( \lambda \) satisfying the lower estimate of (1.2). One may try to answer the last question by proving a result similar to Theorem 5.5 but assuming positivity of the discrete modulus for a family of sets. In fact, positive answers to the following two questions would imply a positive answer to Problem 4.

**Problem 5.** Let \( t > 1 \).
- Suppose \( \mathcal{E} = \{E\} \) is a collection of minimal subsets of dimension 1 in a metric space \( X \). Is it true that if \( d\text{-mod}_t(\mathcal{E}) > 0 \) then \( \mathcal{C}\dim X \geq t \)?
- Let \( E \subset \mathbb{R} \) be minimal, \( Y \) be any metric space of positive Hausdorff dimension and \( \mathcal{E} = \{E \times \{y\} : y \in Y\} \). Is it true that \( d\text{-mod}_t(\mathcal{E}) > 0 \) for any \( 1 < t < \dim_H(E \times Y) \)?

**References**


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