A CENTRAL SET OF DIMENSION 2

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ABSTRACT. The central set of a domain $D$ is the set of centers of maximal
discs in $D$. Fremlin proved that the central set of a planar domain has zero
area and asked whether it can have Hausdorff dimension strictly larger than
1. We construct a planar domain with central set of Hausdorff dimension 2.

1. INTRODUCTION

Let $D$ be a domain in $\mathbb{R}^2$. A subdisc of $D$ is maximal if it is not strictly contained
in any other subdisc of $D$. The central set of $D$ consists of the centers of maximal
discs, i.e.,

$$C(D) = \{ x \in D : D(x, d(x, \partial D)) \text{ is maximal in } D \},$$

where $D(x, r)$ denotes a disc of radius $r$ centered at $x$ and $d(A, B)$ denotes the
Euclidean distance between subsets $A, B \subset \mathbb{R}^2$. The skeleton or medial axis of $D$ is

$$M(D) = \{ x \in D : \exists \text{ distinct } y, y' \in \partial D \text{ s.t. } d(x, y) = d(x, y') = d(x, \partial D) \}.$$

It is easy to check that $M(D) \subset C(D)$, with equality for some domains (such as polygons),
but not in general. For example, a non-circular ellipse contains two maximal discs which are
each tangent to the boundary at only one point. Nevertheless, some sources in the literature mistakenly identify these sets, and one purpose of this
note is to emphasize how different they can be, even for quite reasonable domains.

In [4], Erdős proved that $M(D)$ has Hausdorff dimension 1 for planar domains.
In [6] Fremlin gives many interesting further results, including the fact that any
central set of a planar domain has zero area. He also gives an example of a domain
so that the closure of its medial axis covers a disc. Given the previous fact, we see
that the central set does not always contain the closure of the medial axis. Fremlin
asks whether the central set can have Hausdorff dimension strictly bigger than 1.
How big can the gap between the dimensions of the medial axis and central set be?
We answer this by proving

**Theorem 1.1.** There is a domain $D \subset \mathbb{R}^2$ with $\dim_H(C(D)) = 2$. 
Moreover, our domain is close to the unit disc in the following sense. For any \( \epsilon > 0 \) we can take \( D(0, 1) \subset D \subset D(0, 1 + \epsilon) \), and we can take \( \partial D \) to be an \( \epsilon \)-Lipschitz graph, i.e.,

\[
D = \{ re^{i\theta} : 0 \leq r < f(\theta) \},
\]

where \( f : [0, 2\pi] \to [1, 1 + \epsilon] \) satisfies \( |f(s) - f(t)| \leq \epsilon |s - t| \). The construction also gives something better than just dimension 2. We will show that we can take \( H_\varphi(C(D)) > 0 \) for measure functions \( \varphi \) so that \( \varphi(t)/t^2 \nearrow \infty \) as slowly as we wish, as \( t \to 0 \).

Recall the definitions of Hausdorff measures and Hausdorff dimension. Given a subset \( X \) of the plane and a continuous increasing function \( \varphi : [0, \infty) \to [0, \infty) \), one defines Hausdorff \( \varphi \)-measure of \( X \) as

\[
H_\varphi(X) = \lim_{\epsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \varphi(r_i) : X \subseteq \bigcup_{i=1}^{\infty} D(x_i, r_i), r_i < \epsilon \right\}.
\]

When \( \varphi(t) = t^s \), for some \( s > 0 \), this is called an \( s \)-dimensional measure and is denoted by \( H_s \). The Hausdorff dimension of \( X \) is

\[
\dim_H(X) = \inf\{s : H_s(X) = 0\} = \sup\{s : H_s(X) = \infty\}.
\]

The standard way to prove a lower bound on dimension is to use:

**The mass distribution principle.** If \( X \subset \mathbb{R}^2 \) supports a positive measure \( \mu \) such that

\[
\mu(D(x, r)) \leq C \varphi(r)
\]

for a fixed constant \( C > 0 \) and for all \( x \in \mathbb{R}^2 \) and \( r > 0 \), then \( H_\varphi(X) \geq 0 \) (see [7]).

Central sets and the medial axis arise naturally in various parts of analysis and computer science, e.g., [3], [2], [8] (see [1] for a connection to conformal mapping and its references for further applications of the medial axis).

The rest of the paper is organized as follows. In Section 2, we define a class of domains called “disc trees” for which the medial axes are trees and impose some conditions which imply that the closure of the medial axis is contained in the central set. In Section 4, we construct domains of this type so that \( M(D) \) has dimension 2.

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## 2. Disc trees

Our domains are unions of discs arranged with the structure of an infinite rooted tree. We will construct them inductively as an increasing union \( D_0 \subset D_1 \subset D_2 \ldots \) whose union is the desired domain \( D \). We start with \( D_0 \) being the unit disc. In this case the skeleton \( M_0 \) of \( D_0 \) is just one point, the origin.

Let \( G_1 = \{ D_{1,i} \}_{i=1}^{n_1} \) be the “children” of \( D_0 \). This is a collection of finitely many discs with centers in \( D_0 \) such that the corresponding crescents \( C_{1,i} = D_{1,i} \setminus D_0 \) are mutually disjoint. Let \( D_1 = D_0 \cup \bigcup_{i=1}^{n_1} D_{1,i} \). The skeleton \( M_1 \) of \( D_1 \) is obtained from \( M_0 \) by adding (radial) segments connecting \( M_0 \) to the centers of the discs \( D_{1,i} \).

The corresponding **bending points** are defined as \( \{ b^+_i, b^-_i \} = \partial D_0 \cap \partial D_{1,i} \). There is a 1-parameter family of subdiscs of \( D_1 \) which hit \( \partial D_1 \) exactly at these two points and whose centers sweep out the interval between the centers of \( D_0 \) and \( D_{1,i} \).
Figure 1. Here are $D_0$, $D_1$ and $D_2$. Black represents the medial axis, the union of the grey regions is $B$, the dark grey triangles are regions closest to bending points, light grey points have a unique closest boundary point and the white regions are the sectors $S$.

Since the crescents $C_{1,i}$ are mutually disjoint the maximal discs of $D_0 \cup D_{1,i}$ are still maximal in $D_1$. Hence $M_1$ is a tree with one vertex of degree $n_1$ and $n_1$ vertices of degree one. Let

$$B_1 = \{ x \in D_1 \mid \text{dist}(x, \partial D_1) = \text{dist}(x, \partial D_0 \cap \partial D_1) \}. $$

Then $D_1 \setminus B_1$ can be written as a disjoint union of circular sectors $S_{1,i}$ of $D_{1,i}$ corresponding to the crescents $C_{1,i}$ (see white regions in Figure 1). $B_1 \setminus M_1$ consists of two type of points: those for which the closest points on $\partial D_1$ are bending points (the dark grey triangular regions in Figure 1) and the rest (light grey circular sectors, which could possibly degenerate to a line segment if two successive sectors have a common bending point).

In general, suppose $D_k$ has been constructed from $D_{k-1}$ by adding discs $D_{k,i}$, $i = 1, \ldots, n_k$, and let $S_{k,i}$ denote the corresponding sectors. Let $G_{k+1} = \{ D_{k+1,i} \}_{i=1}^{n_{k+1}}$ be a collection of discs with centers in $\bigcup_{i=1}^{n_k} S_{k,i}$. Denote by $\tilde{D}_{k+1,j} \in G_k$ the ($k$th generation) disc which contains the center of $D_{k+1,j}$ (i.e., the “parent” of $D_{k+1,j}$). Assume that $k + 1$ generation discs satisfy the following conditions:
Let $D_{k+1} = \bigcup_{i=1}^{n_{k+1}} D_{k+1,i}$. The skeleton $M_{k+1}$ of $D_{k+1}$ is obtained from $M_k$ by adding edges connecting the centers of discs of $g_{k+1}$ to the corresponding degree one vertices of $M_k$. Let

$$B_{k+1} = \text{int}\{x \in D_{k+1} | \text{dist}(x, \partial D_{k+1}) = \text{dist}(x, \partial D_k \cup \partial D_{k+1})\}.$$ 

Just as before, $D_{k+1} \setminus B_{k+1} = \bigcup_{i=1}^{n_{k+1}} S_{k+1,i}$ is a union of disjoint sectors and $B_{k+1} \setminus M_{k+1}$ is a union of “bending” triangles and circular sectors. So for every edge $e$ of $M_k$ there are two triangles $T^+$ and $T^-$ which have $e$ as a common edge and the corresponding bending points $b^+$ and $b^-$ as vertices, respectively.

In this way we obtain an increasing sequence of domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$. Let $D = \bigcup_{k=1}^{\infty} D_k$. We will call a domain a disc tree if it can be constructed as above. We will also impose three extra conditions for the remainder of this paper. First, we require that $\alpha_k$, the maximum angle of a sector in the $k$th generation, tends to zero. Second, we require that the medial axis of $D_k$ remains uniformly bounded away from $\partial D_k$ with an estimate independent of $k$. Thirdly, we assume that each sector is contained in the cone defined by its parent sector (this is satisfied if the $\alpha_k$ tend to zero fast enough).

With these assumptions it is fairly easy to see that the closure of the medial axis is contained in the central set, but we will give the details for completeness.

Let $D$ be a disc tree and let $B = \bigcup_{k=1}^{\infty} B_k$ and $L = D \setminus B$. Note that if $x \in B$, then $x \in B_k$ for some $k$ and so, by construction, the segment connecting $x$ to a nearest point of the boundary is also in $B$.

Given a sector $S_{k,i}$ from the construction, let $\tilde{S}_{k,i}$ be the “extended sector” $S_{k,i} \subset \tilde{S}_{k,i} \subset D$ so that $\partial S_{k,i} \cap \partial D = \partial S_{k,i} \cap D$, i.e., $\tilde{S}_{k,i}$ is $S_{k,i}$ plus the part of $D$ separated from the origin by $S_{k,i}$. By construction, $\tilde{S}_{k,i}$ is in the infinite cone obtained by extending the edges of $S_{k,i}$.

Note that $L$ is an intersection of finite unions of extended sectors (closed in $D$), $L = \bigcap_k \bigcup_{i=1}^{n_k} \tilde{S}_{k,i}$, and hence $L$ is closed in $D$. Moreover, every point $x \in L$ is contained in an infinite, decreasing sequence of closed extended sectors, whose angles decrease to zero. Therefore each connected component of $L$ is a line segment in $D$, touching $\partial D$ at one end and $\overline{M(D)}$ at the other. Moreover, since $\overline{M(D)} \setminus M(D)$ is contained in the union of $k$th generation extended sectors for each $k$, it must be contained in $L$. Since distinct sectors of the $k$th generation may only touch on $\partial D$ and since $M(D)$ is bounded away from $\partial D$, the only way for a sequence in $M(D)$ to approach a point of $\overline{M(D)} \setminus M(D)$ is through a sequence of nested extended sectors, and the only possible limit point is an endpoint of a connected component of $L$. Thus $\overline{M(D)} \setminus M(D)$ contains exactly one point in each connected component of $L$, and this point must be an endpoint of that component (which is a segment).

Finally, we want to show that every point of $\overline{M(D)} \setminus M(D)$ is in $C(D)$, the central set. Suppose not, i.e., suppose there is a point $x \in \overline{M(D)} \setminus (M(D) \cup C(D))$. Then $x \in L$ has a unique closest point $y$ on $\partial D$ but the disc $D(x, |x-y|)$ is not maximal in $D$. Therefore this disc is contained in a larger subdisc of $D$, which must be centered at a point $x'$ which lies on the line through $y$ and $x$. The point $x'$ must be in $L$, for otherwise it would be in $B$ and hence so would $x$ by our remark following the definition of $L$. This implies $x$ in an interior point of a component of $L$ and hence not in $\overline{M(D)}$, a contradiction.
3. Proof of Theorem 1.1

We will now construct a particular disc tree and prove $\overline{M(D) \setminus M(D)}$ has dimension 2. We actually describe how to build an infinite rooted tree in the plane. It is easy to place discs at the vertices of this tree so that the tree becomes the medial axis of the union of discs (this will be explained below).

Consider an increasing function $\varphi$ on $(0, 1)$ so that $\lim_{t \to 0} \varphi(t)/t^2 = \infty$. Let $\phi(t) = \varphi(t)/t^2$. For example, if we show the central set has positive measure for the measure function $\varphi(t) = t^2 \log 1/t$, then it certainly has Hausdorff dimension 2.

The construction is by induction. Suppose we have two strictly increasing integer sequences $p = \{p_i\}$ and $n = \{n_i\}$ such that $p_i$ divides $n_i$. Let $q_i = p_i + 1$ and set

$$N_k = \prod_{i=1}^{k} n_i, \quad P_k = \prod_{i=1}^{k} p_i, \quad Q_k = \prod_{i=1}^{k} q_i.$$

Let $r_k = 1/Q_k N_k$ and assume these sequences satisfy

$$(3.1) \quad Q_{k+1} \geq Q_k N_k = r_k^{-1},$$

$$(3.2) \quad \phi(r_k) = \phi(Q_k^{-1} N_k^{-1}) \geq q_k^2 Q_k^2.$$

Note that since $\phi(t) \nearrow \infty$ as $t \to 0$ we can make the left hand side of (3.2) as large as we want by taking $N_k \to \infty$, while keeping $q_k, p_k, Q_k$ fixed. Taken together, these conditions imply $Q_{k+1} \geq (\phi^{-1}(q_k^2 Q_k^2))^{-1}$, so that $\{Q_k\}$ grows very quickly if $\phi$ grows slowly. For example, if $\phi(t) = \log 1/t$, then $Q_{k+1} \geq \exp(q_k^2 Q_k^2)$.

**Initial step:** Let 0 be the root of the tree. Divide the plane into $n_1$ disjoint sectors with vertex at 0 and all with angle $\alpha_1 := 2\pi/n_1$. On the bisector of the $j$th sector place segments of the form $[0, z]$, where $|z| = (1 + (j \mod p_1))/Q_1$ for $j = 0, \ldots, n_1 - 1$. Thus these segments increase in length by $1/Q_1$ at each step until they reach length $p_1/Q_1$ and then start at $1/Q_1$ again; see Figure 2. We let $V_1$ denote the non-zero endpoints of these segments. Note that $V_1$ can be thought of as consisting of $p_1$ rows, or annular shells, in which the vertices which are equidistant from 0, and each row contains $n_1/p_1$ vertices.

![Figure 2. The first generation segments. Here $p_1 = 4$, $n_1 = 36$ and $\alpha_1 = 10^\circ$.](image-url)
General step: Suppose the $k$th generation edges and vertices $V_k$ have been constructed. For each $v \in V_k$ let $r(v)$ be the ray starting at $v$, the parent of $v$, and passing through $v$. Also let $r^\pm(v)$ be the rays starting at $v$ and making an angle $\pm \alpha_k/2$ with $r(v)$, where $\alpha_k = \alpha_{k-1}/n_k = 2\pi/N_k$. Let $C(v)$ be the cone with sides $r^\pm(v)$ and angle $\alpha_k$ at $v$.

Divide $C(v)$ into $n_{k+1}$ congruent cones of opening $\alpha_{k+1} = 2\pi/N_{k+1}$. On the bisectors of these cones place the $(k+1)$st generation segments of lengths $(1 + (j \mod p_{k+1}))/Q_{k+1}$ for $j = 0, \ldots, n_{k+1} - 1$. The new endpoints can be divided into $p_{k+1}$ rows with $n_{k+1}/p_{k+1}$ vertices per row. The collection $V_{k+1}$ is given by doing this construction for every vertex in $V_k$. Continue by induction. Denote the resulting tree by $\Gamma = \Gamma(p,n)$. See Figure 3.

To construct a domain for which $\Gamma$ is the medial axis one needs to start with a disc $D_0$ of radius $R$ strictly larger than 2. For each first generation vertex $v \in V_1$ consider the disc $D(v)$ centered at $v$ such that $\partial D(v) \cap \partial D_0 = (r^+(v) \cup r^-(v)) \cap \partial D_0$. Define $D_1 = D_0 \cup_{v \in V_1} D(v)$. For a $v \in V_2$ denote by $D(v)$ the disc centered at $v$ such that $\partial D(v) \cap \partial D_1 = (r^+(v) \cup r^-(v)) \cap \partial D_1$. Then $D_2 = D_1 \cup_{v \in V_2} D(v)$. Continuing by induction we get a sequence of domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$ and get $D = \bigcup_{k=0}^\infty D_k$. By construction, all sector angles tend to zero uniformly. According to the previous section, $\Gamma$ is the medial axis of $D$ and $C(D)$ contains $\Gamma$.

We shall see below that $\Gamma \subset D(0,2) \subset D_0$ which implies $\Gamma$ is bounded away from $\partial D_0$, as desired. Note that if $R$ is large, then the boundary of the domain lies in a thin annulus between radii $R$ and $R + \epsilon$ and is Lipschitz with a small constant. Thus rescaling gives the claim following the statement of Theorem 1.1.

Let $\tilde{\Gamma} = \Gamma \setminus \Gamma$. To estimate the dimension of $\tilde{\Gamma}$ we consider a special covering of it by circular sectors. To do that first note that if $v$ is a vertex of generation $k$, then all its descendants are contained in a subsector of the cone $C(v)$. We are interested in the radius of the smallest such subsector. To find it, note that the children of $v$ are at most $p_{k+1}/Q_{k+1}$ away from $v$, the grandchildren are at most $p_{k+1}/Q_{k+1}^2$ away and so on. Hence, we see that any descendant of $v$ is at most $I_k := \sum_{i=k+1}^{\infty} p_i/Q_i$ away. Let us denote by $C(v)$ the circular sector centered at $v$.
of angle $\alpha_k$ and radius $\sum_{i=k+1}^{\infty} p_i/Q_i$. Then

$$\Gamma = \bigcup_{i=1}^{\infty} C(v).$$

**Lemma 3.1.** With notation as above $l_k = \frac{1}{q_k}$.

**Proof.** First, note that

$$l_k = \lim_{n \to \infty} \sum_{i=k+1}^{n} \frac{p_i}{Q_i} = \frac{L_{k+1}}{Q_k},$$

where $L_{k+1} := \lim_{n \to \infty} \left[ \frac{p_{k+1}}{q_{k+1}} + \ldots + \frac{p_n}{q_{k+1} \ldots q_n} \right]$. We claim $L_{k} = 1$, for every $k$. Indeed, the general term of the sequence can be rewritten using the fact that $q_i = p_i + 1$ as follows:

$$\begin{align*}
\frac{p_k}{q_k} + \frac{1}{q_k} \frac{p_{k+1}}{q_{k+1}} + \ldots + \left( \frac{1}{q_k} \frac{1}{q_{k+1}} \ldots \frac{1}{q_{n-1}} \right) \frac{p_n}{q_n} \\
= \frac{p_k}{q_k} + \left( 1 - \frac{p_k}{q_k} \right) \frac{p_{k+1}}{q_{k+1}} + \ldots + \prod_{i=k}^{n-1} \left( 1 - \frac{p_i}{q_i} \right) \frac{p_n}{q_n}.
\end{align*}$$

Now, given a sequence of numbers $c_i < 1$, induction on $n$ implies

$$c_k + (1 - c_k)c_{k+1} + \ldots + \prod_{i=k}^{n-1} (1 - c_i) c_n = 1 - \prod_{i=k}^{n} (1 - c_i).$$

Applying this in our case we get

$$L_k = \lim_{n \to \infty} \left[ 1 - \frac{1}{q_k \ldots q_n} \right] = 1,$$

since $q_i = p_i + 1 > 2, \forall i$. \qed

Now we are ready to calculate the Hausdorff dimension of $\Gamma$. We will use the mass distribution principle. Define a probability measure $\mu$ on $\Gamma$ by distributing it evenly among all the sectors of the same generation, i.e.,

$$\mu(C(v)) = \frac{1}{N_i}, \quad \forall v \in V_i.$$

For a ball $B \subset \mathbb{R}^2$ and $i \in \mathbb{N}$ let $\nu_i(B)$ be the number of $i$th generation sectors which have positive $\mu$-mass when intersected with $B$, or

$$\nu_i(B) = \# \{ C(v) : \mu(C(v) \cap B) > 0, v \in V_i \}.$$

Recall that $r_k = l_k/N_k = Q_k^{-1} N_k^{-1}$. This is approximately the length of the circular arc edge of $C(v)$ in the $k$th generation. The sectors of the next generation which are contained in $C(v)$ are actually contained in a truncated sector obtained by removing all points within $l_k/q_k$ of $v$. The remaining region has two long radial edges (with respect to $v$) and two circular arc edges, one of length about $r_k$ (the one farther from $v$) and one of length about $r_k/q_k$. Thus, if $B$ is a ball of diameter $\leq r_k$, it can hit at most $O(q_k)$ $k$th generation sectors in positive measure. (With a little more work one can show only $O(1)$ sectors can hit $B$, but the weaker estimate is easier and sufficient for us.)
Clearly $r_k = l_k/N_k < l_k/q_k < l_k$ (since $N_k > n_k \geq q_k$). Also note that by (3.1), we have $l_{k+1} = Q_{k+1}^{-1} \leq Q_k^{-1}N_k^{-1} = r_k$ so
\[ \cdots < r_{k+1} < l_{k+1} < r_k < l_k/q_k < l_k < \ldots. \]

Fix a ball $B$, let $|B|$ denote its diameter and choose an index $k$ so that $r_{k+1} < |B| \leq r_k$. As noted above, $B$ hits at most $O(q_k)$ sectors in the $k$th generation and it is enough to estimate the mass coming from one of them. Fix such a sector, $C(v)$, hitting $B$ and consider the $(k+1)$st generation subsectors of $C(v)$. They are arranged into $p_k$ levels, according to their distance from the point $v$ (which are multiples of $l_k/q_k$). Since $|B| < l_k/q_k$, $B$ can hit at most two of these levels. Inside each level, there are $n_{k+1}$ subsectors $(k+1)$st generation sectors equidistributed in a row which is $l_k/q_k$ “high” and $w$ “wide” where $w$ is at most $r_k$ (for the row farthest from $v$) and at least $r_k/q_k$ (for the row closest to $v$). Thus the number of $(k+1)$st generation sectors that hit $B$ is approximately $n_{k+1}/p_k$ times $|B|/w$, and so is at most $|B|q_kn_{k+1}/rp_k$.

Next, each row of $(k+1)$st generation sectors is divided into $q_{k+2}$ bands of $(k+2)$nd generation sectors. (We use the term “bands” instead of “rows” since the situation is slightly different than before; the $(k+2)$nd generation subsectors of the fixed $(k+1)$st generation sector do lie in rows equidistant from the vertex of the sector, exactly as before, but the union of the subsectors over different $(k+1)$st sectors are not all equidistant from a single point. However, they are arranged in obvious bands which are close to being equidistant from the vertex of the $kth$ generation sector containing them.)

The height of each of these bands is approximately $u = l_k/(q_kq_{k+1})$ and so at most $1 + O(|B|/u)$ rows can hit $B$ (and this is less than $O(|B|/u)$).

Each row contains $n_{k+2}/p_{k+2}$ sectors. Thus the total number of $(k+2)$nd generation sectors that hit $B$ is less than a bounded multiple of
\[ |B|^2 q_k^2 q_{k+1} Q_k^2 N_k n_{k+2} n_{k+1} p_{k+2} p_{k+1} \leq |B|^2 q_k Q_k^2 N_{k+2}. \]

(Recall that $p_{k+1} \geq p_k + 1 = q_k$ since the sequence is strictly increasing.)

Every $(k+2)$nd generation sector has mass $N_{k+2}^{-1}$, and $B$ hits at most $O(q_k)$ $k$th generation sectors in positive mass, so the total mass of the $(k+2)$nd generation sectors hitting $B$ is bounded by a constant times
\[ \frac{q_k}{N_{k+2}} \cdot (|B|^2 q_k Q_k^2 N_{k+2}) \leq |B|^2 \phi(|B|) Q_k^2 \frac{\phi(|B|)}{\phi(|B|)} \leq |B|^2 \phi(|B|) = \phi(|B|), \]

by (3.2). Thus $H_\phi(\bar{\Gamma}) > 0$ by the mass distribution principle, which proves Theorem 1.1.

References


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