Conformal dimension of self-affine sets.

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Outline

Preliminaries
definitions
Hausdorff dimension can increase
Hausdorff dimension can’t decrease

Results
Theorem A. conformal dimension of self-affine sets
Theorem B. conformal dimension and Fuglede modulus

Sketch of the proof of Thm. A
Step 1: definitions of $\mu, \mathcal{E}$ and $\mathcal{L}$.
Step 2: subsets of conformal dimension 1
Step 3: modulus is positive
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Definitions

- Let $F : X \to Y$ be an embedding of a metric space $X$ into $Y$. $F$ is $\eta$-quasisymmetric if for every triple of distinct points $x, y, z \in X$ we have
  $$\frac{|fx - fy|}{|fx - fz|} \leq \eta \left( \frac{|x - y|}{|x - z|} \right),$$
  (Here $\eta : [0, \infty) \to [0, \infty)$ is any homeomorphism).
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  (Here $\eta : [0, \infty) \to [0, \infty)$ is any homeomorphism).

- \textit{Conformal gauge} of $X$ is
  \[
  \mathcal{X} = \{ F(X) : F \in QS(X) \},
  \]
  where $QS(X)$ be the collection of all qs maps defined on $X$. 
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  \[
  \mathcal{X} = \{ F(X) : F \in QS(X) \},
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  where $QS(X)$ be the collection of all qs maps defined on $X$.

- **Conformal dimension** of $X$ (or $\mathcal{X}$) is
  \[
  C \dim X = \inf_{Y \in \mathcal{X}} \dim_H Y.
  \]
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Hausdorff dimension can increase

- In general $\dim_H F(X) \neq \dim_H X$ for $F \in QS(X)$. 
Hausdorff dimension can increase

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- The snowflake map

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id : (X, |x - y|) \to (X, |x - y|^{\varepsilon}),
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is quasisymmetric for every \( \varepsilon \in (0, 1) \) but changes \( \dim_H X \) to \( \frac{1}{\varepsilon} \dim_H X \).
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- The snowflake map
  \[ id : (X, |x - y|) \rightarrow (X, |x - y|^{\varepsilon}), \]
  is quasisymmetric for every $\varepsilon \in (0, 1)$ but changes $\dim_H X$ to $\frac{1}{\varepsilon} \dim_H X$.
- In particular for every $X$ with $\dim_H X > 0$ we have
  \[ \sup_{Y \in X} \dim_H Y = \infty. \]
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History

- Recall, $C \dim X = \inf_{Y \in X} \dim_H Y$. 

- Let $|z_1 - z_2|_{1/2} = |x_1 - x_2| + |y_1 - y_2|_{1/2}$. The Hausdorff dimension of $\mathbb{R}^2$ with this metric is 3 and $C \dim (\mathbb{R}^2, |z_1 - z_2|_{1/2}) = 3$.

- (Tyson, Bishop-Tyson) For every $\alpha \geq 1$ there is a space $X$ (could be a Cantor set) such that $C \dim X = \dim_H X = \alpha$.

- (Kovalev) If $\dim_H E < 1$ then $C \dim X = 0$.

- (H.) There are sets of length 0 and conformal dimension 1.

- What is the conformal dimension of the standard Sierpinski carpet?
History

- Recall, $C \dim X = \inf_{Y \in X} \dim_H Y$.
- $C \dim \mathbb{R}^n = n$
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Conformal Dimension of Self-Affine Sets

Hrant Hakobyan

Conformal Dimension of Self-Affine Sets
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**Preliminaries**

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**Sketch of the proof of Thm. A**

**Definitions**

**Hausdorff dimension can increase**

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Conformal dimension of self affine sets

- Let $m, n \in \mathbb{N}$ and $m < n$. Divide the unit square into $n$ equal rows and $m$ equal columns and consider a McMullen carpet $X$. 
Conformal dimension of self affine sets

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**Theorem (Binder, Hak.’08)**

Let $X$ be a self-affine McMullen carpet. Then

$$C \dim X \geq 1 + \frac{1}{m} \sum_{j=1}^{m} \log_n r(j),$$

where $r(j)$ is the number of selected rectangles in the $j$-th column. The second term on the right is the dimension of almost all vertical cross-sections of $X$. 
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**Theorem (Binder, Hak.’08)**

Let $X$ be a self-affine McMullen carpet. Then

$$\mathcal{C} \dim X \geq 1 + \frac{1}{m} \sum_{j=1}^{m} \log_{n} r(j),$$

where $r(j)$ is the number of selected rectangles in the $j$-th column. The second term on the right is the dimension of almost all vertical cross-sections of $X$.

- Proof uses the following theorem . . .
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**Theorem (Hak.’08)**

Suppose $p > q > 1$ and $(X, \mu)$ is a doubling measure space s.t.

If there is a system of measures $\mathcal{L} = \{\lambda_E\}_{E \in \mathcal{E}}$ s.t.

then

$$C \dim X \geq q.$$
Conformal dimension and Fuglede modulus

\textit{Theorem (Hak.’08)}

Suppose $p > q > 1$ and $(X, \mu)$ is a doubling measure space s.t.

- $\mu(B(x, r))/r^p \to 0$, $\forall x \in X$,

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Theorem (Hak.’08)

Suppose $p > q > 1$ and $(X, \mu)$ is a doubling measure space s.t.

- $\mu(B(x, r))/r^p \to 0$, $\forall x \in X$,
- $\mathcal{E} = \{E\} \subset X$ is a family of subsets s.t. $C \dim E \geq 1$, $\forall E \in \mathcal{E}$.

If there is a system of measures $\mathcal{L} = \{\lambda_E\}_{E \subset \mathcal{E}}$ s.t.

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If there is a system of measures $\mathcal{L} = \{\lambda_E\}_{E \subseteq \mathcal{E}}$ s.t.
- for every $s > 1$
  $$\lambda_E(B(x, r)) \geq Cr^s, \ \forall x \in E \text{ and } r < r_E.$$  

then
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Suppose $p > q > 1$ and $(X, \mu)$ is a doubling measure space s.t.

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- for every $s > 1$
  $$\lambda_E(B(x, r)) \geq Cr^s, \ \forall x \in E \text{ and } r < r_E.$$
- $\text{mod } q\mathcal{L} > 0$

then

$$C \dim X \geq q.$$
Conformal dimension and Fuglede modulus

Theorem (Hak.’08)

Suppose \( p > q > 1 \) and \((X, \mu)\) is a doubling measure space s.t.

- \( \mu(B(x, r))/r^p \rightarrow 0, \forall x \in X \),
- \( E = \{E\} \subset X \) is a family of subsets s.t. \( \mathcal{C} \dim E \geq 1, \forall E \in \mathcal{E} \).

If there is a system of measures \( \mathcal{L} = \{\lambda_E\}_{E \subset \mathcal{E}} \) s.t.

- for every \( s > 1 \)

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\lambda_E(B(x, r)) \geq Cr^s, \ \forall x \in E \text{ and } r < r_E.
\]

- \( \text{mod } q \mathcal{L} > 0 \)

then

\[ \mathcal{C} \dim X \geq q. \]

- Here \( \text{mod } q \) is the modulus of a family of measures, defined as follows...
**Definition of modulus**

- The \( p \)-modulus of \( \mathcal{L} \) is

\[
\text{mod} \ p(\mathcal{L}, \mu) = \inf_{\rho} \int_{\mathcal{X}} \rho^p d\mu,
\]

where \( \inf \) is over all \( \rho \geq 0 \) Borel functions \( \rho \) s.t.

\[
\int_{\mathcal{E}} \rho d\lambda_E \geq 1, \quad \forall \lambda_E \in \mathcal{L}
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Definition of modulus

- The $p$-modulus of $\mathcal{L}$ is

$$\text{mod}_p(\mathcal{L}, \mu) = \inf_{\rho} \int_X \rho^p d\mu,$$

where $\inf$ is over all $\rho \geq 0$ Borel functions $\rho$ s.t.

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- Here
  $\mathcal{E} = \{E\}$ be a family of subsets in $X$,
  $\mathcal{L} = \{\lambda_E\}_{E \in \mathcal{E}}$ be a family of measures supported on $\mathcal{E}$. 

Theorem A. Conformal dimension of self-affine sets

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Step 1: definitions of \( \mu, \mathcal{E} \) and \( \mathcal{L} \).
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Definition of $\mu$

- Consider the probability measure $\mu_p$ on $X$ corresponding to the probability vector $p$ with

$$p_{ij} = \mu(R_{ij}) = \frac{1}{mr(j)},$$

here $R_{ij}$ is the rectangle in the $i$-th raw and $j$-th column.
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- Note, $\dim \mu = D = 1 + \frac{1}{m} \sum_{j=1}^{m} \log_n r(j)$. 

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- Consider

  \[ X_0 = \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B_r(x))}{r^D} = 0 \right\} \]
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- $\dim_H X_0 = D$. 
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- Consider

$$X_0 = \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B_r(x))}{r^D} = 0 \right\}$$

- $\dim_H X_0 = D$.

- Want to show that $X_0$ is minimal.
Definition of $\mathcal{E}$ and $\mathcal{L}$

- Need to find a family of sets $\mathcal{E} = \{E\}$ and measures $\{\lambda_E\}_{E \in \mathcal{E}}$ s.t.
  1. $\mathcal{C} \dim E = 1, \forall E \in \mathcal{E}$,
  2. $\text{mod}_1(\{\lambda_E\}, \mu) > 0$. 
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- Every $E \in \mathcal{E}$ is the intersection of a “piecewise horizontal Cantor set" in $X$ with $X_0$. 
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- Every $E \in \mathcal{E}$ is the intersection of a “piecewise horizontal Cantor set” in $X$ with $X_0$.
- $\lambda_E$ is the pullback of the Lesbegue measure under the projection, so clearly satisfies the condition of the theorem.
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Showing that every $E \in \mathcal{E}$ is minimal amounts to showing that

$$X_j = \left\{ \sum_{i=1}^{\infty} \frac{x_i}{m^i} : \frac{\#\{i \leq n : x_i = j\}}{n} \to \frac{1}{m} \right\} \subset [0, 1],$$

has conformal dimension 1 for every $j \in \{0, 1, \ldots, m-1\}$. This follows from the law of large numbers combined with the following...
$C\dim E = 1$

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$$X_j = \left\{ \sum_{i=1}^{\infty} \frac{x_i}{m_i} : \#\{i \leq n : x_i = j\} \to \frac{1}{m} \right\} \subset [0, 1],$$

has conformal dimension 1 for every $j \in \{0, 1, \ldots, m-1\}$.

This follows from the law of large numbers combined with the following

**Lemma (Binder, Hak.)**

*Construct a Cantor set $E \subset [0, 1]$ by dividing every component left from the $(i-1)$-st step into $n_i$ equal parts and removing $m_i$. If

$$\sum_{i=1}^{\infty} \left( \frac{m_i}{n_i} \right)^t < \infty, \forall t > 0$$

then $C\dim E = 1$.***
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mod $\,_{1}(\mathcal{L}, \mu) = 1$

Lemma ($\mu \ll \mathcal{L}$)
\[\text{mod}_1(\mathcal{L}, \mu) = 1\]

<table>
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<tr>
<th>Lemma ((\mu \ll \mathcal{L}))</th>
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<tbody>
<tr>
<td><strong>AC_1</strong> If (\lambda_E(A) = 0), (\forall E \in \mathcal{E}) then (\mu(A) = 0).</td>
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Lemma (\mu \ll L)

AC\_1 \text{ If } \lambda_E(A) = 0, \forall E \in \mathcal{E} \text{ then } \mu(A) = 0.

AC\_2 \text{ For every } A \subset \mathbb{R}^2

\mu(A) = \int_0^1 \nu_t(A \cap \{x = t\}) dt,

where \nu_t is the conditional measure of \mu given t.
mod_1(\mathcal{L}, \mu) = 1

**Lemma (\mu \ll \mathcal{L})**

\begin{itemize}
  \item **AC_1** If \( \lambda_E(A) = 0, \forall E \in \mathcal{E} \) then \( \mu(A) = 0 \).
  \item **AC_2** For every \( A \subset \mathbb{R}^2 \)
    \[ \mu(A) = \int_0^1 \nu_t(A \cap \{x = t\}) dt, \]
    where \( \nu_t \) is the conditional measure of \( \mu \) given \( t \).
\end{itemize}

- Our choice of \( \mathcal{E} \) and \( \mathcal{L} \) implies that if \( \rho \) is extremal then there is a function \( \varrho(x) \) s.t. \( \forall E \in \mathcal{E} \)
  \[ \rho(x, y) = \varrho(x), \text{ for a.e. } x \in [0, 1]. \]
mod\(_1(\mathcal{L}, \mu) = 1\)

**Lemma (\(\mu \ll \mathcal{L}\))**

\(\text{AC}_1\) If \(\lambda_E(A) = 0, \forall E \in \mathcal{E}\) then \(\mu(A) = 0\).

\(\text{AC}_2\) For every \(A \subset \mathbb{R}^2\)

\[
\mu(A) = \int_0^1 \nu_t(A \cap \{x = t\}) dt,
\]

where \(\nu_t\) is the conditional measure of \(\mu\) given \(t\).

- Our choice of \(\mathcal{E}\) and \(\mathcal{L}\) implies that if \(\rho\) is extremal then there is a function \(\varrho(x)\) s.t. \(\forall E \in \mathcal{E}\)

\[
\rho(x, y) = \varrho(x), \text{ for a.e. } x \in [0, 1].
\]

- Lemma (\(\mu \ll \mathcal{L}\)) implies

\[
\int_X \rho(z) d\mu = \int_0^1 \varrho(x) dx \geq 1.
\]
mod \(1(\mathcal{L}, \mu) = 1\)

Indeed,

- Let

\[ A = \{(x, y) \in X : \rho(x, y) \neq \varrho(\pi(x))\} \]
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Indeed,

- Let
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- our choice of \( \lambda_E \) implies \( \lambda_E(A) = 0, \forall E \in \mathcal{E} \)

- \( AC_1 \quad \Rightarrow \quad \mu(A) = 0. \)
Indeed,

- Let
  \[ A = \{(x, y) \in X : \rho(x, y) \neq \varrho(\pi(x))\} \]
- our choice of \( \lambda_E \) implies \( \lambda_E(A) = 0, \forall E \in \mathcal{E} \)
- \( AC_1 \Rightarrow \mu(A) = 0. \)
- \( AC_2 \Rightarrow \]

\[ \int_X \rho(x, y) d\mu = \int_A \rho(x, y) d\mu = \int_0^1 \varrho(x) dx \geq 1. \]

Q.E.D.