Quasisymmetric Geometry of Slit Carpets

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Outline

1. Uniformization and Rigidity
   - Conformal Mappings
   - Quasisymmetric Mappings

2. Modulus
Circle domains

- A domain \( D \subset \mathbb{C} \) is a **circle domain** if every component of \( \mathbb{C} \setminus D \) is a point or a (round) disc.
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**Theorem (Uniformization & Rigidity - Koebe' 1918)**

Every finitely connected domain $D \subset \mathbb{C}$ is conformal to a circle domain. Finitely connected circle domains are conformally rigid.

**Conjecture. (Koebe' 1908)**

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**Theorem (He, Schramm' 1993)**

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To prove Koebe’s conjecture it is enough to prove it for slit domains.
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It is also important to understand if there is a (sub-)limit when the number of boundary components tends to infinity.
What if $D_1 \supset D_2 \supset \ldots$ is a decreasing sequence of domains, such that $\bigcap_{i=1}^{\infty} D_i$ has no interior?

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**Theorem (Whyburn)**

Let $D$ be a Jordan domain and $E = D \setminus \bigcup_{i=1}^{\infty} D_i$ such that

- $\overline{D_i} \cap \overline{D_j} = \emptyset$, if $i \neq j$,
- $\text{diam} D_i \to 0$, as $i \to \infty$,
- $E$ has no interior points.

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**Definition**

A carpet is a metric space homeomorphic to the standard Sierpinski carpet.
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   - Quasisymmetric Mappings

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Definition

A homeomorphism $F : X \rightarrow Y$ between metric spaces is (weakly) Quasisymmetric (QS) if $\exists H \geq 1$ such that for every triple of distinct points $x, y, z \in X$ the following holds

$$\frac{|x - y|}{|x - z|} \leq 1 \implies \frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq H.$$
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If $E \subseteq \mathbb{C}$ is a carpet whose peripheral circles are uniformly relatively separated uniform quasicircles then $E$ is quasisymmetric to a round carpet.
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If $E \subset \mathbb{C}$ is a carpet whose peripheral circles are uniformly relatively separated uniform quasicircles then $E$ is quasisymmetric to a round carpet.

- Relative distance $\Delta(E, F) = \frac{\text{dist}(E,F)}{\min(\text{diam}E, \text{diam}F)}$,

- Uniformly relatively separated means $\Delta(C_i, C_j) \geq c > 0$, for every pair of distinct peripheral circles $C_i, C_j$.

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**Theorem (Rigidity - Bonk, Kleiner, Merenkov’2009)**

Round carpets are quasisymmetrically rigid if and only if they have zero measure.
Example from Kleinian groups
Uniformizing square carpets
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Given a subset of $E \subset \mathbb{C}$ define the inner path metric on $E$ as follows. For $x, y \in E$

$$d(x, y) := \inf\{\ell(\gamma) : \gamma(0) = x, \gamma(1) = y\},$$

where $\ell(\gamma)$ denotes the length $\gamma$. 

\begin{itemize}
  \item Given a slit domain $S$ we will denote by $\bar{S}$ the completion of $S$ with respect to the path metric.
  \item Given a sequence of reals $r = \{r_i\}_{i=1}^{\infty}$, $r_i \in (0, 1)$, $\forall i \in \mathbb{N}$ can construct the corresponding slit carpet $S(r)$ as follows: Remove from every diadic square $Q \in \Delta$ a vertical slit of the same length $l_i = \frac{1}{2^i}r_i$.
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**Standard Slit domains and Path metric**

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Ahlfors 2-regular space $(X, d, \mu)$ is 2 Loewner if there is a decreasing function $\psi(0, \infty) \to (0, \infty)$ such that

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**Theorem (Rigidity - H.’2012)**

Let $r_1, r_2 \notin \ell^2$. If $r_1 \neq r_2$ then $S(r_1)$ and $S(r_2)$ are not quasisymmetric to each other.
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Question.

Are standard slit carpets rigid if $r \in \ell^2$?
Given a curve family $\Gamma \subset \mathbb{R}^n$ we say a Borel function $\rho : \mathbb{R}^n \to \mathbb{R}_{[0,\infty)}$ is admissible for $\Gamma$, denoted by $\rho \wedge \Gamma$, if

$$\int_\gamma \rho ds \geq 1, \forall \gamma \in \Gamma.$$
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$\mathcal{L}^n$ is $n$-regular, i.e. there is a constant $C \geq 1$ s.t. for every $x \in \mathbb{R}$ and $R > 0$ holds

$$C^{-1} R^n \leq \mathcal{L}^n B(x, R) \leq CR^n.$$
Modulus in $\mathbb{R}^n$

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Uniformization and Rigidity Modulus

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$\Phi = U + iU^*$. 
There is a unique conformal mapping which maps $W$ onto a rectangle so that $E, F$ are mapped onto the vertical sides. No analogs in higher dimensions.
There is a unique conformal mapping which maps $W$ onto a rectangle with horizontal slits removed and so that $E, F$ are mapped onto the vertical sides.
Horizontal path families in slit domains

Figure: Path families $\Gamma_1, \Gamma_2, \Gamma_3$
Theorem (H.’2012)

Let $S(r)$ be the slit carpet corresponding to $r$. Then

- $\text{mod}_p \Gamma_n \to 0$, for some $p > 1$, if and only if
- $\text{mod}_p \Gamma_n \to 0$, for all $p > 1$, if and only if
- $r \not\in \ell^2$, if and only if
- $S(r)$ does not quasisymmetrically embed into $\mathbb{C}$.
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**Theorem (H.'2012)**

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QS geometry of slit carpets
Suppose $W$ is a finitely connected domain with $\partial W = C_1 \cup \ldots \cup C_N$. Given a curve family $\Gamma \subset W$ we say a Borel function $\rho : \hat{W} \to [0, \infty)$ is admissible for $\Gamma$, denoted by $\rho \wedge \Gamma$, if

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O. Schramm’s transboundary modulus

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$$\text{mod}_p \Gamma = \inf_{\rho \land \Gamma} \int_{\mathbb{R}^n} \rho^p d\mathcal{L}^n.$$ 

$\mathcal{L}^n$ is $n$-regular, i.e. there is a constant $C \geq 1$ s.t. for every $x \in \mathbb{R}$ and $R > 0$ holds

$$C^{-1} R^n \leq \mathcal{L}^n B(x, R) \leq CR^n.$$ 

In $\mathbb{R}^n$ the $n$-modulus $\text{mod}_n \Gamma$ is called conformal modulus.
Non self-similar square carpets

Given a sequence $a = (\frac{1}{n_1}, \frac{1}{n_2}, \ldots)$, where $n_i$-s are odd integers can construct a square carpet.
Non self-similar square carpets

Given a sequence $a = (a_1, a_2, \ldots)$, where $a_i$-s are reciprocals of odd integers can construct a square carpet.
Non self-similar square carpets

Given a sequence $\mathbf{a} = (\frac{1}{n_1}, \frac{1}{n_2}, \ldots)$, where $n_i$-s are odd integers can construct a square carpet.

**Theorem (MacKay, Tyson, Wildrick - 2011)**

Let $S$ be the square carpet corresponding to $\mathbf{a}$. Then

- $\text{mod}_p \Gamma_n \to 0$, for some $p > 1$, if and only if
- $\text{mod}_p \Gamma_n \to 0$, for all $p > 1$, if and only if
- $\mathbf{a} \notin \ell^2$.

**Theorem (MacKay, Tyson, Wildrick - 2011)**

Let $S$ be the slit carpet corresponding to $\mathbf{a}$. Then

- $\text{mod}_1 \Gamma_n \to 0$, if and only if
- $\mathbf{a} \notin \ell^1$. 

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QUASISYMMETRIC GEOMETRY OF SLIT CARPETS