CANTOR SETS WHICH ARE MINIMAL FOR QUASISYMMETRIC MAPS

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Abstract. We show that middle interval Cantor sets of Hausdorff dimension 1 are minimal for quasisymmetric maps of a line. Combining this with a theorem of Wu we conclude that there are “rigid” subsets of a line whose every quasisymmetric image has zero length and Hausdorff dimension 1.

1. Introduction

Given $M \geq 1$ a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is said to be $M$-quasisymmetric if for every pair of adjacent intervals $I$ and $J$ of the same length

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M$$

(here and in sequel $| \cdot |$ stands for the 1-dimensional Lebesgue measure). A map is quasisymmetric if it is $M$-quasisymmetric for some $M \geq 1$. We denote by $QS$ and $QS(M)$ the set of all quasisymmetric and $M$-quasisymmetric homeomorphisms of $\mathbb{R}$ respectively. More generally a homeomorphism $f$ between metric spaces $(X,d_X)$ and $(Y,d_Y)$ is called $\eta$-quasisymmetric if there is a self-homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all $x, y, z \in X$ and $t > 0$

$$d_X(x, y) \leq td_X(y, z) \Rightarrow d_Y(f(x), f(y)) \leq \eta(t)d_Y(f(y), f(z)).$$

Given a compact set $E \subset \mathbb{R}$ we are interested in the distortion of the Hausdorff dimension of $E$ under quasisymmetric maps. If $\dim_H(E) = 0$ then $\dim_H(f(E)) = 0$ since quasisymmetric maps are Hölder continuous (see [1]). In [2] it is shown that if $\dim_H(E) > 0$ then for any $\varepsilon > 0$ one can find a quasisymmetric mapping such that $\dim_H(f(E)) > 1 - \varepsilon$. In the opposite direction Tukia [7] proved that for any $\varepsilon > 0$ there is a set $E \subset \mathbb{R}$ and $f \in QS$ such that $\dim_H(\mathbb{R} \setminus E) < \varepsilon$ and $\dim_H(f(E)) < \varepsilon$. In this article we prove the following theorem.

Theorem 1.1. Middle interval Cantor sets of dimension 1 are minimal for quasisymmetric maps.

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A set $E \subset \mathbb{R}$ is called minimal for quasisymmetric maps if $\dim_H (f(E)) \geq \dim_H (E), \forall f \in QS$. We say $E \subset \mathbb{R}$ is a middle interval Cantor set if it can be constructed as follows. Fix a sequence $c = \{c_i\}_{i=1}^{\infty}$ of numbers in $(0,1)$. From $[0,1]$, which we will denote by $E_{0,1}$, remove the middle interval of length $c_1$ centered at $1/2$. Call the removed interval $J_{1,1}$ and the components of the remaining set $E_{1,1}$ and $E_{1,2}$. From the middle of $E_{1,i}$ remove the interval $J_{2,j}$ of length $c_2|E_{1,i}|$, $i = 1, 2$. Continue in the same fashion. Let $E = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^n-1} E_{n,j}$ where $E_{n,j} = E_{n+1,2j-1} \cup J_{n+1,j} \cup E_{n+1,2j}$, $|J_{n,j}| = c_n|E_{n-1,j}|$ and $|E_{n+1,j}| = |E_{n+1,j'}|$ for any $j$ and $j'$. We will denote the set corresponding to a sequence $c = \{c_i\}_{i=1}^{\infty}$ by $E(c)$.

\begin{figure}[h]
\centering
\begin{tabular}{ccc}
$E_{0,1}$ & $E_{1,1}$ & $J_{1,1}$ & $E_{1,2}$ \\
$E_{2,1}$ & $J_{2,1}$ & $E_{2,2}$ & $J_{1,1}$ & $E_{2,3}$ & $J_{2,2}$ & $E_{2,4}$ \\
\end{tabular}
\caption{A middle interval Cantor set}
\end{figure}

The following definitions are by now standard. The conformal dimension of a metric space $X$ is the infimal Hausdorff dimension of quasisymmetric images of $X$

$$C \dim (X) = \inf \{ \dim_H (Y) \mid \exists f : X \to Y \text{ quasisymmetric} \}$$

whereas the quasiconformal or global conformal dimension of a set $E \subset \mathbb{R}^n$ is defined as the infimum over a smaller collection of maps

$$QC \dim (E) = \inf \{ \dim_H (f(E)) \mid f : \mathbb{R}^n \to \mathbb{R}^n \text{ quasisymmetric} \}.$$  

Using this terminology Tukia’s theorem says that there are subsets of $\mathbb{R}$ of dimension 1 and (quasi)-conformal dimension $<1$ while our definition of minimality becomes a particular case of the following. $E \subset \mathbb{R}^n$ is minimal (for quasisymmetric maps) if

$$QC \dim (E) = \dim_H (E).$$

In [3] it has been shown that for every $\alpha \geq 1$ there are minimal Cantor sets in $\mathbb{R}^n$, $n \geq 2$, of Hausdorff dimension $\alpha$. All the previously known examples of minimal subsets of $\mathbb{R}$ had positive Lesbegue measure, these are quasisymmetrically thick sets. Recall from [8] that $E \subset \mathbb{R}$ is called a quasisymmetrically thick set if $|f(E)| > 0, \forall f \in QS$. In [4] it was shown that a middle interval Cantor set $E(c)$ with $c_1 < 1/2$ is quasisymmetrically thick if and only if $\sum_{i=1}^{\infty} c_i^p < \infty, \forall p > 0$. Our theorem gives a partial negative answer to the following question from [3] (see page 370): If $E$ is not a quasisymmetrically thick set, then is there a quasisymmetric image of $E$ of Hausdorff dimension $<1$? In other words: If a set is not quasisymmetrically thick then does it necessarily have conformal dimension $<1$? Theorem 1.1 shows that every middle interval Cantor set of zero length and dimension one is an example of a set which is not quasisymmetrically thick and has
quasiconformal dimension 1. Is it true in this case that $C\dim(E) = 1$? More generally: is there a set $E \subset \mathbb{R}$ such that $QC\dim(E) = 1$ but $C\dim(E) < 1$?

One could also ask: Is there a “rigid” set whose every $QS$-image has dimension 1 and length 0? In [8] a set is called \textit{quasisymmetrically null} if all its $QS$-images have zero length. In [9] Wu had shown that if $c = c_i \notin l^p, \forall p \geq 1$ then $E = E(c)$ is null. She noticed that in particular null sets can have dimension 1. Therefore combining theorem 1.1 with Wu’s theorem we get an affirmative answer to the question above.

**Corollary 1.2.** There are quasisymmetrically null sets which are minimal.

It’s enough to take the sequence

$$c_i = \begin{cases} (1/i)^{1/2} & \text{if } i = 2^m \\ (1/i)^{1/3} & \text{if } i \neq 2^m \end{cases}$$

and construct the corresponding Cantor set $E(c)$. We leave it to the reader to verify that the set has dimension 1 and that Wu’s condition is also satisfied.

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## 2. Background Material

Recall that the Hausdorff $t$-measure of a metric space $E$ is defined by

$$H^t(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}U_i)^t : E \subset \bigcup_{i=1}^{\infty} U_i, \text{diam}U_i < \varepsilon \right\},$$

where $\{U_i\}_{i=1}^{\infty}$ is an open cover of $E$. The Hausdorff dimension of $E$ is

$$\dim_H(E) = \inf \{ t : H^t(E) = 0 \}.$$  

One usually gives an upper bound on the Hausdorff dimension of a set by finding explicit covers for it. Lower bounds can be obtained from the \textit{mass distribution principle}: If $E \subset \mathbb{R}$ supports a positive Borel measure $\mu$ satisfying $\mu(E \cap I) \leq C|I|^d$, for some fixed constant $C > 0$ and every interval $I \subset \mathbb{R}$ then $\dim_H(E) \geq d$.

Let $N(E, \varepsilon)$ be the minimal number of $\varepsilon$ balls needed to cover $E$. \textit{Upper and lower Minkowski dimensions} of $E$ are defined as

$$\overline{\dim}_M(E) = \limsup_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{\log 1/\varepsilon}, \quad \underline{\dim}_M(E) = \liminf_{\varepsilon \to 0} \frac{\log N(E, \varepsilon)}{\log 1/\varepsilon}$$

respectively. When this two numbers are the same the common value is called the Minkowski dimension of $E$. Generally for a subset of a line one has $\dim_H(E) \leq \underline{\dim}_M(E) \leq \overline{\dim}_M(E) \leq 1$ (see [6]). And therefore when $\dim_H(E) = 1$ then the Minkowski dimension of $E$ exists and is equal 1.
Lemma 2.1. If \( E = E(c) \) and \( \dim_M(E) = 1 \) then

\[
\left( \prod_{i=1}^{n} (1 - c_i) \right)^{1/n} \to 1,
\]

\[
\frac{1}{n} \sum_{i=1}^{n} c_i^p \to 0, \quad 0 < p < 1.
\]

Proof. From the definition of Minkowski dimension we get

\[
\dim_M(E) = \lim_{n \to \infty} \frac{\log 2^n}{\log \prod_{i=1}^{n} (1 - c_i)} = \lim_{n \to \infty} \frac{1}{\log 2} \frac{1}{\sqrt[1/n]{\prod_{i=1}^{n} (1 - c_i)}} = 1.
\]

Therefore (2.1) holds. Now, from the usual inequality between geometric and arithmetic means

\[
\left( \prod_{i=1}^{n} (1 - c_i) \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} (1 - c_i) \leq 1
\]

we get

\[
\frac{1}{n} \sum_{i=1}^{n} c_i^p \to 0. \quad \text{Combined with Jensen’s inequality (} \frac{1}{n} \sum_{i=1}^{n} c_i^p \leq \frac{1}{n} \sum_{i=1}^{n} c_i \text{ for } p < 1 \text{ this gives (2.2).} \]

Our main tool for proving theorem 1.1 will be the following lemma from [9]. Here we reformulate it in a rescaled manner.

Lemma 2.2. If \( f \) is an \( M \)-quasisymmetric function then for any two intervals \( J \subset I \)

\[
\frac{1}{(1 + M)^2} \left( \frac{|J|}{|I|} \right)^q \leq \frac{|f(J)|}{|f(I)|} \leq 4 \left( \frac{|J|}{|I|} \right)^p
\]

where \( p = p(M) = \log_2(1 + 1/M), q = q(M) = \log_2(1 + M). \)

3. PROOF OF THEOREM 1.1

Proof. Fix \( d < 1, M \geq 1 \) and take any \( f \in QS(M) \). We will construct a measure \( \mu \) on \( f(E) \) satisfying \( \mu(I) \leq C|I|^d \) for some constant \( C > 0 \) and therefore will conclude by the mass distribution principle that \( \dim_H(f(E)) \geq d \). Since \( d \) is arbitrary it will follow that \( \dim_H(f(E)) = 1 \). First note that \( E \) has a tree structure where each parent interval has exactly two children and the same is true for \( f(E) \).

Denote \( I_{n,j} = f(E_{n,j}) \) and define the measure \( \mu \) inductively as follows:

\[
\mu(I_{0,1}) = 1
\]

\[
\mu(I_{n,j}) = \frac{|I_{n,j}|^d}{|I_{n,j}|^d + |I'_{n,j}|^d} \mu(I_{n-1,k}), \quad n \geq 1
\]

where \( I_{n-1,k} \) is the parent of \( I_{n,j} \) and \( I'_{n,j} \) is the other interval with the same parent as \( I_{n,j} \). Now pick an interval \( I_{n,j} \) and compute \( \mu(I_{n,j})/|I_{n,j}|^d \). There is a unique sequence of nested intervals \( I_{n,j_n} \subset I_{n-1,j_{n-1}} \subset \ldots \subset I_{2,j_2} \subset I_{1,j_1} \subset I_0 = [0,1] \) and to simplify the notation we denote \( I_{k,j_k} \) by \( I_k \). Also
letting $G_n = I_{n-1} \setminus (I_n \cup I_n')$ for $n = 1, 2, \ldots$ from Wu’s inequalities it follows that

$$\frac{c_i^q}{1 + M^2} \leq \frac{|G_i|}{|I_{i-1}|} \leq 4c_i^p,$$

and hence by induction we get

$$\frac{\mu(I_n)}{|I_n|^d} = \frac{1}{|I_n|^d + |I_n'|^d} \cdot \frac{|I_n-1|^d + |I_n'|^d}{|I_n|^d + |I_n'|^d} \cdot \ldots \cdot \frac{|I_1|^d + |I_1'|^d}{|I_1|^d + |I_1'|^d} \cdot |I_0|$$

$$= \frac{(|I_1| + |G_n| + |I_n'|)^d}{|I_1|^d + |I_1'|^d} \cdot \frac{(|I_{n-1}| + |G_{n-1}| + |I_n'|)^d}{|I_{n-1}|^d + |I_{n-1}'|^d} \cdot \ldots \cdot \frac{(|I_1| + |G_1| + |I_1'|)^d}{|I_1|^d + |I_1'|^d}$$

$$= \left( \prod_{i=1}^n \frac{|I_i|^d + |I_i'|^d}{|I_i|^d + |I_i'|^d} \right)^{-1}$$

(3.1)

The second equality above uses the fact that $|I_{k-1}| = |I_k| + |G_k| + |I_k'|$, $\forall k$.

We estimate from below the product in the parentheses using (2.3). The idea is to use the left and right inequalities of (2.3) for "small" and "large" $c_i$-s respectively. First note that $1 - 4x \geq (1 - x)^5$ for $x < 1/10$ and let

$$S = \left\{ i \in \mathbb{N} : c_i < \min \left( \sqrt[5]{\frac{1}{10}}, \frac{1}{3} \right) \right\},$$

$$S_n = S \cap \{ i \leq n \}$$

$$s_n = \text{card}(S_n).$$

From (2.1) it follows that $s_n/n \to 1$ as $n \to \infty$.

For $i \in S$ the second inequality in (2.3) gives us the following estimate

$$p_i = \frac{|I_i|^d + |I_i'|^d}{(|I_i| + |I_i'|)^d} \cdot \frac{(|I_i| + |I_i'|)^d}{(G_i + |I_i'|)^d} = \frac{|I_i|^d + |I_i'|^d}{(|I_i|^d + |I_i'|^d)} \cdot \left( 1 - \frac{|G_i|}{|I_{i-1}|} \right)^d$$

$$\geq \frac{|I_i|^d + |I_i'|^d}{(|I_i| + |I_i'|)^d} \cdot (1 - 4c_i^p)^d = \frac{1 + \left( \frac{|I_i'|}{|I_i|} \right)^d}{\left( 1 + \frac{|I_i'|}{|I_i|} \right)^d} \cdot (1 - 4c_i^p)^d$$

(3.2)

Now since $c_i < 1/3$, $I_i$ and $I_i'$ are images of two intervals $E_i$ and $E_i'$ of the same length which are at most $|E_i'|$ away from each other. Therefore, twice applying the definition of quasisymmetry we immediately get that

$$\frac{1}{M} \left( 1 + \frac{1}{M} \right) \leq \frac{|I_i'|/|I_i|} = \frac{|f(E_i')|/|f(E_i)|} \leq M(M + 1).$$

Considering the function $x \mapsto \frac{1+x^d}{(1+x)^d}$ for $d < 1$ one can easily see that on an interval $[(M + 1)/M^2, M(M + 1)]$ its smallest value is attained at $M(M + 1)$ and is equal to $\frac{1+(M(M+1))^d}{(1+M(M+1))^d} > 1$. We will denote this value by $C_1(M, d)$. The important
thing is that it is larger than 1. Therefore (3.2) finally gives us the following estimate

\[ p_i \geq C_1(M, d)(1 - c_i^p)^{5d} \]

for \( i \in S \).

For \( i \notin S \) we use the first inequality of (2.3) to get

\[ p_i = \frac{|I_i|^d + |I'_i|^d}{(|I_i| + |G_i| + |I'_i|)^d} = \left( \frac{|I_i|}{|I_{i-1}|} \right)^d + \left( \frac{|I'_i|}{|I_{i-1}|} \right)^d \]

\[ \geq \frac{2}{(1 + M)^{2d}} \left( \frac{|E_i|}{|E_{i-1}|} \right)^{dq} = \frac{2}{(1 + M)^{2d}} \left( \frac{1 - c_i}{2} \right)^{dq} \]

\[ = \frac{1}{C_2(M, d)} (1 - c_i)^{dq} \]

Combining (3.1),(3.2) and (3.4) we get

\[ \prod_{i=1}^{n} p_i \geq \prod_{i \in S_n} C_1(1 - c_i^p)^{5d} \cdot \prod_{\{i \leq n\} \backslash S_n} C_2^{-1}(1 - c_i)^{dq} \]

\[ \geq \frac{C_1^{s_n}}{C_2^{n-s_n}} \prod_{i \in S_n} (1 - c_i^p)^{5d} \prod_{i=1}^{n} (1 - c_i)^{dq} \]

where \( C_1, C_2 > 1 \). Now (3.5) together with \( s_n/n \to 1 \) imply that there is a constant \( C(M, d) > 1 \) such that

\[ \prod_{i=1}^{n} p_i \geq C^n \prod_{i \in S_n} (1 - c_i^p)^{5d} \cdot C^n \prod_{i=1}^{n} (1 - c_i)^{dq} \]

Theorem will be proved if we show that the right hand side above tends to infinity as \( n \to \infty \). Indeed, from (3.6) we will get that \( \prod_{i=1}^{n} p_n \to \infty \) as \( n \to \infty \) and therefore (3.1) would imply that there is a constant \( C \) such that

\[ \mu(I_n) \leq C |I_n|^d \]

for every interval \( I_n \).

For the second term in (3.6) one has

\[ \log \left( C^n \prod_{i=1}^{n} (1 - c_i)^{dq} \right) \geq n \left( \log C + dq \log \prod_{i=1}^{n} (1 - c_i) \right) . \]

By (2.1) \( \sqrt[n]{\prod_{i=1}^{n} (1 - c_i)} \to 0 \) and therefore \( C^n \prod_{i=1}^{n} (1 - c_i)^{dq} \to \infty \). 

On the other hand the first term in (3.6) can be bounded from below in the following manner:

\[ C^n \prod_{i \in S_n} (1 - c_i^p)^{5d} \geq C^n \prod_{i=1}^{n} (1 - c_i^p)^{5d} \]
where $\tilde{c}_i = \min(c_i, \sqrt{0.1}) \leq \sqrt{0.1} < 1$. To show that the right hand side in (3.7) tends to infinity it’s enough to show that

$$\log \left( \prod_{i=1}^{n} (1 - \tilde{c}_i^p) \right) = \frac{1}{n} \sum_{i=1}^{n} \log(1 - \tilde{c}_i^p) \to 0.$$  

To see the later first note that $\log(1 - x) > -2x$ for $0 < x \leq 0.1$. Since $\tilde{c}_i^p \leq 0.1$ we have

$$0 > \frac{1}{n} \sum_{i=1}^{n} \log(1 - \tilde{c}_i^p) > \frac{2}{n} \sum_{i=1}^{n} \tilde{c}_i^p > -2 \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{c}_i^p + \frac{n - s_n}{n} \cdot \frac{1}{10} \right) \to 0$$

by (2.1) and by the fact that $s_n/n \to 1$. So we get that

$$C^n \prod_{i \in S_n} (1 - c_i^p)^a \to \infty, \ n \to \infty$$

and hence have proved that the growth condition of the mass distribution principle holds for all intervals of the form $f(E_{i,j})$.

To complete the proof we need to show that it holds for any interval $I \subset \mathbb{R}$. So fix an interval $I$. Let $l_i = |E_{i,j}|, \forall i,j$. Clearly $l_i \searrow 0$ so there is an $i$ such that $l_{i+1} \leq |f^{-1}(I)| < l_i$. It follows then that there are at most 2 intervals of generation $i$ which intersect $|f^{-1}(I)|$ and therefore at most 4 such intervals of generation $i + 1$. Denoting the latter ones by $E_1, \ldots, E_4$ (some of these may be empty) we get

$$\mu(I) \leq \mu(f(E_1)) + \ldots + \mu(f(E_4)) \leq C(|f(E_1)|^d + \ldots + |f(E_4)|^d).$$

Now since $|E_i| \leq |f^{-1}(I)|$ it follows that $E_1 \cup \ldots \cup E_4 \subset 3f^{-1}(I)$ ($3f^{-1}(I)$ is just the dilation of $f^{-1}(I)$) and hence

$$|f(E_1)|^d + \ldots + |f(E_4)|^d \leq 4|f(3f^{-1}(I))|^d.$$  

From the definition of quasisymmetry it follows that

$$|f(3f^{-1}(I))|^d \leq (1 + 2M)^d|f(f^{-1}(I))|^d = C(M, d)|I|^d$$

and hence combining the last three inequalities we conclude the desired growth

$$\mu(I) \leq C|I|^d$$

for some constant $C$ and any interval $I$. As we noted in the beginning if follows that $\dim_H(f(E)) = 1$ since $d$ could be chosen as close to 1 as one would like. \hfill $\Box$

References


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