Problem 1: Let \( x_1, \ldots, x_n \) be elements of a field \( K \). Prove the following formula for the Vandermonde determinant:

\[
\begin{vmatrix}
1 & x_1 & \cdots & x_1^{n-1} \\
1 & x_2 & \cdots & x_2^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \cdots & x_n^{n-1}
\end{vmatrix} = \prod_{i<j}(x_j - x_i).
\]

Problem 2: Prove the following theorem about the rank of a matrix: Let \( A \in \text{Mat}(n \times m, K) \) and let \( r \) be the largest non-negative integer such that there exist an \( r \times r \) submatrix \( A' \) of \( A \) with \( \det A' \neq 0 \). Then \( \text{rank } A = r \). Here, a submatrix of \( A \) is a matrix obtained by removing an arbitrary number of rows and columns from \( A \).

Problem 3: Suppose \( A \) is an \( n \times n \) matrix with real entries such that the diagonal elements are all positive, the off-diagonal elements are all negative and the row sums are all positive. Prove that \( \det A \neq 0 \).

Problem 4: Compute the determinant of the matrix

\[
A = \begin{pmatrix} B_1 & C \\ 0 & B_2 \end{pmatrix}
\]

in terms of the three submatrices \( B_1, B_2 \) and \( C \).

Problem 5*: Let \( A \) be an \( n \times n \) matrix. Let \( I = \{i_1, \ldots, i_p\} \) be a subset of \( \{1, \ldots, n\} \) and let \( J = \{j_1, \ldots, j_q\} \) be the complement of \( I \) in \( \{1, \ldots, n\} \). Denote by \( \Gamma \) the set of all permutations \( \sigma \in S_n \) such that the restrictions of the map \( \sigma \) to \( I \) and \( J \) are monotone increasing functions. For \( \gamma \in \Gamma \) let \( A_\gamma \) be the submatrix of \( A \) obtained by removing the rows \( j_1, \ldots, j_q \) and columns \( \gamma(j_1), \ldots, \gamma(j_q) \) from \( A \) and let \( A_\gamma^* \) be the submatrix of \( A \) obtained by removing the rows \( i_1, \ldots, i_p \) and columns \( \gamma(i_1), \ldots, \gamma(i_p) \) from \( A \). Prove the following expansion theorem of Laplace:

\[
\det A = \sum_{\gamma \in \Gamma} \epsilon(\gamma) \cdot \det A_\gamma \cdot \det A_\gamma^*.
\]