Problem 1: A nilpotent element of a ring is an element $a$ such that $a^n = 0$ for some nonnegative integer $n$.

(a) Show that the set of nilpotent elements of a commutative ring form an ideal — called the nilradical.

(b) Let $R$ be a commutative ring with $1 \neq 0$. Prove that if $a \in R$ is nilpotent, $1 - ab$ is a unit for all $b \in R$.

Problem 2: Let $I$, $J$ and $K$ be ideals of a ring $R$. Prove:

(a) $I + J$ is the smallest ideal of $R$ containing both $I$ and $J$.

(b) $IJ \subset I \cap J$.

(c) $I(J + K) = IJ + IK$ and $(J + K)I = JI + KI$.

(d) If $J \subset I$ then $I \cap (J + K) = J + (I \cap K)$.

Problem 3: For an ideal $I$ of a ring $R$ denote with $M_n(I)$ the set of $n \times n$-matrices with entries in $I$. Prove that $I \mapsto M_n(I)$ defines a bijective map of the set of ideals of $R$ onto the set of ideals of the ring $M_n(R)$.

Problem 4: Show that every ring $R$ can be embedded into a ring $S$ with 1.

(Hint: Take $S = \mathbb{Z} \times R$ and define for $m, n \in \mathbb{Z}$ and $a, b \in R$ the addition by $(m, a) + (n, b) = (m + n, a + b)$ and the multiplication by $(m, a)(n, b) = (mn, mb + na + ab)$, where for a nonnegative integer $n$ one writes $na = \underbrace{a + \cdots + a}_{n-\text{times}}$.)