Problem 1: Prove Cauchy’s Theorem: Let $p$ be a prime divisor of the order of a finite group $G$. Then there exists an element $a \in G$ of order $p$.

Problem 2: Prove that there is no simple group of order $pqr$, where $p$, $q$ and $r$ are distinct primes.

Problem 3: Prove that there is no simple group of order 2907.

Problem 4: Let $A$ be a finite abelian group and let $p$ be a prime. The $p^{th}$-power map is the homomorphism defined by

$$\varphi : A \rightarrow A, \quad a \mapsto \varphi(a) = a^p.$$ 

Let $A^p$ be the image and let $A_p$ be the kernel of $\varphi$, respectively. Prove that $A/A^p \cong A_p$. (Hint: Show that both groups are elementary abelian $p$-groups of the same order. An elementary abelian $p$-group is an abelian group where each element besides the identity has order $p$.)

Problem 5*: Prove that there is no simple group of order 1004913. (Hint: Find all possible values for the numbers $n_p$ of $p$-Sylow subgroups and exclude them one by one. Finally, use a permutation representation of degree 819 to exclude the last case.)