Problem 1: Let $G_1, \ldots, G_n$ be finite simple groups. Show that $G_1, \ldots, G_n$ are the composition factors of the direct product $G_1 \times \cdots \times G_n$.

Problem 2: Show that if the automorphism group of the finite cyclic group of order $n$ is itself cyclic then $n$ must be contained in the set $S = \{2, 4, p^k, 2p^k\}$ with $p$ an odd prime and $k$ a positive integer. (Hint: Show $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_i^{a_i}\mathbb{Z}$ where $p_1^{a_1} \cdots p_i^{a_i}$ is the prime factorization of $n$ and use this to investigate the $\varphi(n) = \varphi(p_1^{a_1}) \cdots \varphi(p_i^{a_i}) = (p_1^{a_1-1}(p_1-1)) \cdots (p_i^{a_i-1}(p_i-1))$ generators of $\mathbb{Z}/n\mathbb{Z}$. Identify the automorphism group of $\mathbb{Z}/n\mathbb{Z}$ with the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ of residue classes modulo $n$ of the $\varphi(n)$ non-negative integers $m < n$ which are relatively prime to $n$.)

Problem 3: If $K$ is a normal subgroup of the group $G$ and $K$ is cyclic, prove that the commutator $G'$ is a subgroup of the centralizer $C(K)$ of $K$. (Hint: Use from Problem 2 that the automorphism group of a cyclic group is abelian.)

Problem 4: Let $G$ be a finite solvable group. Prove that there exists a chain $G = N_0 > N_1 > \cdots > N_n = \{e\}$ of subgroups of $G$ such that $N_i$ is normal in $G$ and $N_i/N_{i+1}$ is abelian for $i = 0, 1, \ldots, n - 1$. (Hint: Prove that a minimal nontrivial normal subgroup $M$ of $G$ is necessarily abelian and use induction. To see that $M$ is abelian, let $N$ be a normal subgroup of $M$ of prime index and show that $[x, y] \in N$ for all $x, y \in M$. Apply the same argument to $gNg^{-1}, g \in G$, to show that $[x, y]$ lies in the intersection of all $G$-conjugates of $N$, and use minimality of $M$ to conclude that $[x, y] = e$.)

Problem 5*: Show that the automorphism group of the finite cyclic group of order $n$ is itself cyclic if $n$ is contained in the set $S = \{2, 4, p^k, 2p^k\}$ with $p$ an odd prime and $k$ a positive integer.