Problem: Prove that if $S$ has more than two elements, then the only element $f_0$ in $A(S)$ such that $ff_0 = f_0f$ for all $f \in A(S)$ must satisfy $f_0 = i$.

Solution: Assume $f_0 \neq i$. Then there exists an element $a \in S$ with $f(a) \neq a$. Let $b = f(a)$ and $c \in S$ be an element different from $a$ and $b$. Such an element $c$ exists because $S$ contains at least three elements. Let $f : S \rightarrow S$ be the function defined by $f(b) = c$, $f(c) = b$ and $f(x) = x$ for $x \in S \setminus \{b, c\}$. The function $f$ is a bijection since obviously $ff = i$. One has $(ff_0)(a) = f(f_0(a)) = f(b) = c$ but $(f_0f)(a) = f_0(f(a)) = f(a) = b$. Thus $ff_0 \neq f_0f$. The only function $f_0$ with $ff_0 = f_0f$ could therefore be the identity function $i$, which indeed has this property.