Calculus I - Lecture 6
Limits D & Intermediate Value Theorem

Lecture Notes:  
http://www.math.ksu.edu/~gerald/math220d/

Course Syllabus:  

Gerald Hoehn (based on notes by T. Cochran)

February 10, 2014
Section 2.7 – Limits at Infinity

Recall: 1) A **vertical asymptote** is a guideline that the graph of $f(x)$ approaches at points where $\lim_{x \to a^+} f(x) = \pm \infty$ or $\lim_{x \to a^-} f(x) = \pm \infty$.

2) A **horizontal asymptote** is a guideline that the graph of $f(x)$ approaches at points where $x \to \pm \infty$.
Limits at Infinity

**Definition**
The limits of a function \( f(x) \) at **infinity** are the values \( f(x) \) approaches as \( x \to \infty \), written \( \lim_{x \to \infty} f(x) \), or \( x \to -\infty \), written \( \lim_{x \to -\infty} f(x) \). They may or may not exist.

**Example:** In the previous example we find:

\[
\lim_{x \to \infty} f(x) = -3 \quad \text{(horizontal asymptote to the right)} \\
\lim_{x \to -\infty} f(x) = 2 \quad \text{(horizontal asymptote to the left)}
\]

**Note:**

\[
\lim_{x \to \infty} f(x) = L \iff y = L \text{ horizontal asymptote for } x \to \infty \\
\lim_{x \to -\infty} f(x) = L \iff y = L \text{ horizontal asymptote for } x \to -\infty
\]
\( \infty \)-type limits \hspace{1em} \text{(Example:} \lim_{x \to \infty} \frac{x + 2}{x - 3} \text{)}

**Basic Trick** for evaluating \( \infty \)-type limits (without doing any graphing!):

Divide top and bottom by the largest power of \( x \) occurring in the denominator.

**Example:** a) Evaluate \( \lim_{x \to \infty} \frac{2x + 3}{2 - x} \).

**Solution:** \( x \) appears with first degree in denominator.

\[
\lim_{x \to \infty} \frac{2x + 3}{2 - x} = \lim_{x \to \infty} \frac{(2x + 3) \cdot \frac{1}{x}}{(2 - x) \cdot \frac{1}{x}}
\]

\[
= \lim_{x \to \infty} \frac{2x \cdot \frac{1}{x} + 3 \cdot \frac{1}{x}}{2 \cdot \frac{1}{x} - x \cdot \frac{1}{x}}
\]

\[
= \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{\frac{2}{x} - 1}
\]

\[
= \frac{\lim_{x \to \infty} (2 + \frac{3}{x})}{\lim_{x \to \infty} (\frac{2}{x} - 1)} = \frac{2 + 0}{0 - 1} = -2
\]
b) What information does the limit in part a) provide about the graph of \( f(x) = \frac{2x + 3}{2 - x} \)?

**Solution:**

Horizontal asymptote: \( y = -2 \) (for \( x \to \infty \))

\[
\lim_{x \to -\infty} \frac{2x + 3}{2 - x} = -2 \quad \text{(by same method as in part a))}
\]

Horizontal asymptote: \( y = -2 \) (for \( x \to -\infty \))
Rational Functions

Example: Evaluate \( \lim_{x \to \infty} \frac{2x - 5x^3}{x^3 - x + 1} \).

Solution: \( x \) appears with 3\(^{rd}\) degree in denominator.

\[
\lim_{x \to \infty} \frac{2x - 5x^3}{x^3 - x + 1} = \lim_{x \to \infty} \frac{(2x - 5x^3) \frac{1}{x^3}}{(x^3 - x + 1) \frac{1}{x^3}}
\]

\[
= \lim_{x \to \infty} \frac{\frac{2x}{x^3} - \frac{5x^3}{x^3}}{\frac{x^3}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}}
\]

\[
= \lim_{x \to \infty} \frac{\frac{2}{x^2} - \frac{5}{1}}{1 - \frac{1}{x^2} + \frac{1}{x^3}} = \frac{0 - 5}{1 - 0 + 0} = -5
\]
Alternate way:

This works for any rational function (quotient of polynomials)

\[
\lim_{x \to \infty} \frac{a x^n + \text{lower degree terms}}{b x^m + \text{lower degree terms}} = \lim_{x \to \infty} \frac{a x^n}{b x^m}.
\]

Drop all lower degree terms.

**Example:** Redo the limit \(\lim_{x \to \infty} \frac{2x - 5x^3}{x^3 - x + 1}\) using the alternate way.

**Solution:**

\[
\lim_{x \to \infty} \frac{2x - 5x^3}{x^3 - x + 1} = \lim_{x \to \infty} \frac{-5x^3 + 2x}{x^3 - x + 1}
\]

\[
= \lim_{x \to \infty} \frac{-5x^3}{x^3} = \lim_{x \to \infty} \frac{-5}{1} = -5 \quad \text{(Books method)}
\]
Example: Evaluate $\lim_{x \to \infty} \frac{5x^4 - 2x}{6x^3 + 7x^5 - 3}$.

Solution:

First method: $x$ appears with $5^{th}$ degree in denominator.

$$= \lim_{x \to \infty} \frac{(5x^4 - 2x) \frac{1}{x^5}}{(6x^3 + 7x^5 - 3) \frac{1}{x^5}}$$

$$= \lim_{x \to \infty} \frac{5 \frac{x}{x^5} - 2 \frac{1}{x^4}}{6 \frac{1}{x^2} + 7 - 3 \frac{3}{x^5}}$$

$$= \frac{0 - 0}{0 + 7 - 0} = \frac{0}{7} = 0$$

Second method:

$$= \lim_{x \to \infty} \frac{5x^4}{7x^5}$$

$$= \lim_{x \to \infty} \frac{5}{7x} = 0$$
Example: Compute \( \lim_{x \to -\infty} \frac{\sqrt{4x^2 - 2}}{x + 3} \).

Solution:
\[
\lim_{x \to -\infty} \frac{\sqrt{4x^2 - 2}}{x + 3} = \lim_{x \to -\infty} \frac{\sqrt{4x^2 - 2} \cdot \frac{1}{x}}{(x + 3) \cdot \frac{1}{x}} = \lim_{x \to -\infty} \frac{\sqrt{x^2(4 - \frac{2}{x^2})} \cdot \frac{1}{x}}{1 + \frac{3}{x}}
\]
\[
\sqrt{x^2} = x \quad \text{if} \quad x > 0 \quad \text{and} \quad \sqrt{x^2} = -x \quad \text{if} \quad x < 0.
\]
One has: \( \sqrt{x^2} = x \) if \( x > 0 \) and \( \sqrt{x^2} = -x \) if \( x < 0 \).

because \( x < 0 \)
\[
= \lim_{x \to -\infty} \frac{-\sqrt{4 - \frac{2}{x^2}} \cdot \frac{1}{x}}{1 + \frac{3}{x}}
\]
\[
= \lim_{x \to -\infty} \frac{-\sqrt{4 - 0}}{1 + 0} = \frac{-2}{1} = -2
\]
Section 2.8 – Intermediate Value Theorem

Theorem (Intermediate Value Theorem (IVT))

Let $f(x)$ be continuous on the interval $[a, b]$ with $f(a) = A$ and $f(b) = B$.

Given any value $C$ between $A$ and $B$, there is at least one point $c \in [a, b]$ with $f(c) = C$.

Example: Show that $f(x) = x^2$ takes on the value 8 for some $x$ between 2 and 3.

Solution: One has $f(2) = 4$ and $f(3) = 9$. Also $4 < 8 < 9$. Since $f(x)$ is continuous, by the IVT there is a point $c$, with $2 < c < 3$ with $f(c) = 8$. 
**Note:** The IVT fails if $f(x)$ is not continuous on $[a, b]$.

**Example:**

There is no $c \in [a, b]$ with $f(c) = C$. 
Important special case of the IVT:
Suppose that $f(x)$ is \textbf{continuous} on the interval $[a, b]$ with $f(a) < 0$ and $f(b) > 0$.

Then there is a point $c \in [a, b]$ where $f(c) = 0$.

\textbf{Example:} Show that the equation

$$x^3 - 3x^2 + 1 = 0$$

has a solution on the interval $(0, 1)$.

\textbf{Solution:}

1) $f(x) = x^3 - 3x^2 + 1$ is continuous on $[0, 1]$ (polynomial).
2) $f(0) = 1 > 0$.
3) $f(1) = 1 - 3 + 1 = -1 < 0$.

Thus by the IVT, there is a $c \in (0, 1)$ with $f(c) = 0$. 