7.2 Application to economics: Leontief Model

Wassily Leontief won the Nobel prize in economics in 1973.

The Leontief model is a model for the economics of a whole country or region. In the model there are \( n \) industries producing \( n \) different products such that the input equals the output or, in other words, consumption equals production. One distinguishes two models:

**open model:** some production consumed internally by industries, rest consumed by external bodies.

*Problem:* Find production level if external demand is given.

**closed model:** entire production consumed by industries.

*Problem:* Find relative price of each product.

**The open Leontief Model**

Let the \( n \) industries denoted by \( S_1, S_2, \ldots, S_n \). The exchange of products can be described by an input-output graph

Here, \( a_{ij} \) denotes the number of units produced by industry \( S_i \) necessary to produce one unit by industry \( S_j \) and \( b_i \) is the number of externally demanded units of industry \( S_i \).

**Example:** Primitive model of the economy of Kansas in the 19th century.
The following equations are satisfied:

<table>
<thead>
<tr>
<th>Production of</th>
<th>Total output</th>
<th>Internal consumption</th>
<th>+</th>
<th>External Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>farming industry (in tons):</td>
<td>$x = 0.05x + 0.5y + 8000$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>horse industry:</td>
<td>$y = 0.01x + 2000$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(in 1000km horse rides)

In general, let $x_1, x_2, \ldots, x_n$, be the total output of industry $S_1, S_2, \ldots, S_n$, respectively. Then

$$
\begin{align*}
    x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\
    x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\
    \vdots \\
    x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n
\end{align*}
$$

since $a_{ij}x_j$ is the number of units produced by industry $S_i$ and consumed by industry $S_j$. The total consumption equals the total production for the product of each industry $S_i$.

Let

$$
A = \begin{pmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{n1} & \cdots & a_{nn}
\end{pmatrix}, \quad
B = \begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}, \quad
X = \begin{pmatrix}
    x_1 \\
    \vdots \\
    x_n
\end{pmatrix}
$$

$A$ is called the input-output matrix, $B$ the external demand vector and $X$ the production level vector. The above system of linear equations is equivalent to the matrix equation

$$
X = AX + B.
$$

In the open Leontief model, $A$ and $B \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ are given and the problem is to determine $X$ from this matrix equation.

We can transform this equation as follows:

$$
\begin{align*}
    I_nX - AX &= B \\
    (I_n - A)X &= B \\
    X &= (I_n - A)^{-1}B
\end{align*}
$$

if the inverse of the matrix $I_n - A$ exists. ($(I_n - A)^{-1}$ is then called the Leontief inverse.) For a given realistic economy, a solution obviously must exist.

For our example we have:

$$
A = \begin{pmatrix}
    0.05 & 0.5 \\
    0.1 & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
    8,000 \\
    2,000
\end{pmatrix}, \quad
X = \begin{pmatrix}
    x \\
    y
\end{pmatrix}
$$
We obtain therefore the solution
\[ X = (I_2 - A)^{-1} B \]
\[ = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) - \left( \begin{array}{cc} 0.05 & 0.5 \\ 0.1 & 0 \end{array} \right)^{-1} \left( \begin{array}{c} 8,000 \\ 2,000 \end{array} \right) \]
\[ = \left( \begin{array}{cc} 0.95 & -0.5 \\ -0.1 & 1 \end{array} \right)^{-1} \left( \begin{array}{c} 8,000 \\ 2,000 \end{array} \right) \]
\[ = \frac{1}{9} \left( \begin{array}{cc} 10 & 5 \\ 1 & 9.5 \end{array} \right) \left( \begin{array}{c} 8,000 \\ 2,000 \end{array} \right) \]
\[ = \left( \begin{array}{c} 10,000 \\ 3,000 \end{array} \right), \]
i.e., \( x = 10,000 \) tons wheat and \( y = 3 \) Million km horse ride.

If the external demand changes, ex. \( B' = \left( \begin{array}{c} 7,300 \\ 2,500 \end{array} \right) \), we get
\[ \left( \begin{array}{c} x \\ y \end{array} \right)' = (I_2 - A)^{-1} B' = \frac{1}{9} \left( \begin{array}{cc} 10 & 5 \\ 1 & 9.5 \end{array} \right) \left( \begin{array}{c} 7,300 \\ 2,500 \end{array} \right) = \left( \begin{array}{c} 9,500 \\ 3,450 \end{array} \right), \]
i.e., one doesn’t need to recompute \((I_2 - A)^{-1}\).

One difficulty with the model: How to determine the matrix \( A \) from a given economy? Typically, \( X = \left( \begin{array}{c} x_1 \\ \vdots \\ x_n \end{array} \right) \) is known, \( B = \left( \begin{array}{c} b_1 \\ \vdots \\ b_n \end{array} \right) \) is known and \((a_{ij}x_j)_{i,j=1,\ldots,n}\) is known. One takes therefore the matrix \((a_{ij}x_j)_{i,j=1,\ldots,n}\) and divides the \( j \)-th column by \( x_j \) for \( j = 1, \ldots, n \) to get \( A \).

**Example**: An economy has the two industries \( R \) and \( S \). The current consumption is given by the table

<table>
<thead>
<tr>
<th>Industry</th>
<th>( R ) production</th>
<th>( S ) production</th>
<th>( \text{external} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Industry ( R )</td>
<td>50</td>
<td>50</td>
<td>20</td>
</tr>
<tr>
<td>Industry ( S )</td>
<td>60</td>
<td>40</td>
<td>100</td>
</tr>
</tbody>
</table>

Assume the new external demand is 100 units of \( R \) and 100 units of \( S \). Determine the new production levels.

**Solution**: The total production is 120 units for \( R \) and 200 units for \( S \). We obtain
\[ X = \left( \begin{array}{c} 120 \\ 200 \end{array} \right), \quad B = \left( \begin{array}{c} 20 \\ 100 \end{array} \right), \quad A = \left( \begin{array}{cc} 50 & 50 \\ 60 & 40 \end{array} \right), \quad \text{and} \quad B' = \left( \begin{array}{c} 100 \\ 100 \end{array} \right). \]
The solution is
\[ X' = (I_2 - A)^{-1} B' = \frac{1}{41} \left( \begin{array}{cc} 96 & 30 \\ 60 & 70 \end{array} \right) \left( \begin{array}{c} 100 \\ 100 \end{array} \right) = \left( \begin{array}{c} 307.3 \\ 317.0 \end{array} \right). \]
The new production levels are 307.3 and 317.0 for \( R \) and \( S \), respectively.
The closed Leontief Model

The closed Leontief model can be described by the matrix equation

\[ X = AX, \]

i.e., there is no external demand. The matrix \( I_n - A \) is usually not invertible.

(Otherwise, the only solution would be \( X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \).

The input-output graph looks now as follows:

There is only internal consumption.

Example: Extended model of the economy of Kansas in the 19th century including labor.

The corresponding matrix equation is:

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = \begin{pmatrix}
  0.05 & 0.5 & 0.5 \\
  0.1 & 0 & 0.1 \\
  0.4 & 0.1 & \frac{1331}{1800}
\end{pmatrix} \begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}.
\]
If $X$ is a solution, also $t \cdot X$ for every $t > 0$ is a solution. (Usually, one gets a one parameter family of solutions.) If $x \neq 0$, we can assume $x = 1,000$ by choosing the appropriate parameter $t$. One obtains then the solution

$$x = 1,000, \quad y = \frac{2900}{11} \approx 263.63, \quad z = \frac{18000}{11} \approx 1636.36.$$  

For this computation, it is important to use rational numbers (i.e., fractions) as matrix entries since otherwise the approximation to the matrix $I_n - A$ usually will be invertible and only the trivial uninteresting solution $x = 0$, $y = 0$, and $z = 0$ will exist. This is also the reason, why the entry $a_{33}$ has large numerator and denominator.

In a closed economy, the absolute units of output are less interesting. More important is the relative consumption of a product.

We can normalize therefore the matrix $A$ such that the sum of every row is 1. This is a matrix $\hat{A}$, such that

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \hat{A} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \quad \text{The recipe is: Divide the } i\text{-th row of } A \text{ by the } i\text{-th component of } A \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{(that is the sum of the } i\text{-th row}).$$

For our example, we have

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{21}{20} \\ \frac{1}{6} \\ \frac{2231}{1800} \end{pmatrix},$$

leading to the matrix

$$\hat{A} = \begin{pmatrix} \frac{1}{11} & \frac{10}{2231} & \frac{10}{2231} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{720}{2231} & \frac{180}{2231} & \frac{1331}{2231} \end{pmatrix}, \quad \hat{A} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$  

The entries of the matrix $\hat{A} = (\hat{a}_{i,j})_{i,j=1,...,n}$ have the following meaning: $\hat{a}_{ij}$ is the relative consumption of the product of industry $S_i$ by industry $S_j$.

**Market prices**

The consumption of products is regulated by prices. All income of an industry is used for buying other (or the own) products, i.e., income equals expenditure.

Let $P = (p_1, \ldots, p_n)$ the price vector; $p_i$ is the relative price of the product of industry $S_i$. We can draw the flow of money into the input-output graph, the money flows in exchange for the products:
One has

\[
\begin{align*}
p_1 &= a_{11}p_1 + a_{21}p_2 + \cdots + a_{n1}p_n \\
p_2 &= a_{12}p_1 + a_{22}p_2 + \cdots + a_{n2}p_2 \\
&\quad \vdots \\
p_n &= a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{nn}p_n,
\end{align*}
\]

since \( a_{ij} p_i \) is the amount paid by industry \( S_j \) for products produced by industry \( S_i \). The total income of industry \( S_j \) equals the total price \( S_j \) has to pay to all other industries.

Again, one can write this as a matrix equation:

\[ PA = P. \]

This equation can be transformed in the following way

\[ P \cdot I_n = P \cdot \tilde{A} \\
P \cdot (I_n - \tilde{A}) = (0, \ldots, 0). \]

The matrix \( I_n - \tilde{A} \) is (similar as \( I_n - A \)) not invertible, since \( (I_n - \tilde{A}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \).

One can show that this implies that there is also a solution \( P \neq \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \). Since with \( P \) also \( t \cdot P \) for \( t > 0 \) is a solution, only the relative price between the different products has a well-defined meaning.

**Example (continued):** Assume \( p_1 = $1,000 \). One gets \( p_2 = \frac{40000}{63} \approx $634.92 \) and \( p_3 = \frac{1115500}{907} \approx $1967.37 \). We can compare these relative prices with the production levels measured by the original units and obtain the following relative prices per unit: \( p_1/x = \frac{1000}{1000} = 1 \) for one ton of wheat, \( p_2/y \approx \frac{634.92}{2.4} \approx 2.4 \) for 1000km horse ride, and \( p_3/z \approx \frac{1967.37}{1.2} \approx 1.636 \) for one man-year.

Since the above matrix equation for \( P \) is not of the usual form which we have studied so far, we make a final modification. We define

\[ \tilde{\tilde{A}} = (\tilde{a}_{i,j})_{i,j=1,\ldots,n}, \quad \text{where} \quad \tilde{a}_{i,j} = \tilde{a}_{j,i}. \]
This gives us (just by switching the rôle of rows and columns) the price equation

\[ \tilde{P} = \tilde{A}\tilde{P}, \]

where \( \tilde{a}_{i,j} \) is now the relative consumption of industry \( S_j \) by industry \( S_i \), so that the sum of each column is 1, and \( \tilde{P} = \left( \begin{array}{c} p_1 \\ \vdots \\ p_n \end{array} \right) \) is the price column vector.

In the textbook, our matrix \( \tilde{A} \) is again denoted by \( A \) and our \( \tilde{P} \) is denoted by \( X \). The price equation is therefore \( X = A \cdot X \). However, one has to keep in mind that this matrix \( A \) is different from the input-output matrix \( A \) we used in the open Leontief model!

**Example:** Let

\[ A = \left( \begin{array}{ccc} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{7} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{7} & \frac{1}{2} \end{array} \right). \]

Compute all wages, given that the wages for the 3rd product is $30,000.

**Solution:** Let \( X = \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \) be the different wages with \( z = 30,000 \). We have to solve

\[ X = AX, \]

\[ (I_3 - A)X = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right). \]

\[ \left( \begin{array}{ccc} \frac{1}{2} & -\frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & -\frac{2}{3} & \frac{1}{2} \end{array} \right) \left( \begin{array}{c} x \\ y \\ 30,000 \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right). \]

This system of linear equations for \( x \) and \( y \) has the solution \( x = 30,000 \) and \( y = 22,500 \). The wages for the first and second product are therefore $,30,000 and $22,500, respectively.