A VISUAL FORMALISM FOR WEIGHTS SATISFYING REVERSE INEQUALITIES

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Abstract. In this expository article we introduce a diagrammatic scheme to represent reverse classes of weights and some of their properties.

1. Introduction

Let $B \subset \mathbb{R}^n$ be a Euclidean ball and let $|B|$ denote its Lebesgue measure. Given a function $w > 0$ defined on $B$, Hölder’s (or Jensen’s) inequality implies that for any $-\infty \leq p \leq q \leq \infty$ we have

\begin{equation}
\left( \frac{1}{|B|} \int_B w^p(x) \, dx \right)^{1/p} \leq \left( \frac{1}{|B|} \int_B w^q(x) \, dx \right)^{1/q},
\end{equation}

finite or infinite (see Definition 2 below for the meaning of these integrals when $p$ or $q$ equal 0, $-\infty$, or $\infty$). It is a remarkable fact that for large families of weights $w$ one or more of the natural inequalities in (1.1) can be reversed, up to a multiplicative constant, uniformly over vast collections of balls.

The purpose of this expository article is to formalize the study of such weights through a diagrammatic scheme. We found this visual scheme to be of pedagogical value when teaching or explaining Muckenhoupt weights as well as some topics on elliptic PDEs such as Harnack’s inequality and Moser’s iterations. It also serves as a mnemonic device for remembering various results relating weights in reverse classes and as a tool for posing questions and conjectures for such weights.

As part of the exposition we will review a number of results on the theory of Muckenhoupt weights; however, we will make no attempt at accounting for the history of weights satisfying reverse inequalities; readers are encouraged to consult, for instance, [14, Chapter 7], [16, Chapter IV], [18, Chapter 9], [27, Chapter 2], [38, Chapter 5], [39, Chapters I and II], [40, Chapter IX].

Since inequalities such as (1.1) require only the notions of ‘ball’ and ‘integral’, we set up the rest of the article in the context of spaces of homogeneous type, also known as doubling quasi-metric spaces.

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Definition 1. A quasi-metric space is a pair \((X,d)\) where \(X\) is a non-empty set and \(d\) is a quasi distance on \(X\), that is, \(d : X \times X \to [0, \infty)\) such that

(i) \(d(x,y) = d(y,x)\) for all \(x, y \in X\),
(ii) \(d(x,y) = 0\) if and only if \(x = y\), and
(iii) there exists \(K \geq 1\) (quasi-triangle constant), such that

\[
d(x,y) \leq K(d(x,z) + d(z,y)) \tag{1.2}
\]

\(\forall x, y, z \in X\).

Let \((X,d)\) be a quasi-metric space. The \(d\)-ball with center \(x \in X\) and radius \(r > 0\) in \((X,d)\) is defined by

\[
B_r(x) := \{y \in X : d(x, y) < r\}.
\]

For \(B := B_r(x)\) and \(\lambda > 0\), \(\lambda B\) denotes the ball \(B_{\lambda r}(x)\).

Let \(\mu\) be a measure defined on the balls of \(X\). The triple \((X,d,\mu)\) is said to be a space of homogeneous type (as introduced by Coifman and Weiss [9, Chapter III]) if \((X,d)\) is a quasi-metric space and \(\mu\) satisfies the doubling property, that is, if there exists a positive constant \(C_\mu > 1\) such that

\[
0 < \mu(B_{2r}(x)) \leq C_\mu \mu(B_r(x)) \quad \forall x \in X, r > 0. \tag{1.3}
\]

Any constant depending only on the quasi-triangle constant \(K\) in (1.2) and the doubling constant \(C_\mu\) in (1.3) will be called a geometric constant. For basic topological and measure theoretic results on spaces of homogeneous type, such as the existence of a constant \(\alpha \in (0, 1)\), depending only on \(K\), and a distance \(\rho\) in \(X\) with

\[
\frac{\rho(x,y)}{2} \leq d(x,y)^\alpha \leq 4\rho(x,y) \quad \forall x, y \in X,
\]

or the fact that the atoms for \((X,d,\mu)\) must be countable and isolated (where \(x \in X\) is an atom if \(\mu(\{x\}) > 0\)), see the pioneering work of Mac\’\i\’as and Segovia [32]. Another classical reference, which includes a long list of examples of spaces of homogeneous type, is the survey article by Coifman and Weiss [10]. For an exposition of some of the topics in this survey in the Euclidean context but without involving the doubling condition, see [35].

Definition 2. Given a function \(w > 0\), a ball \(B\), and \(-\infty \leq p \leq \infty\), set

\[
w(p,B) := \left(\frac{1}{\mu(B)} \int_B w^p \, d\mu\right)^{1/p},
\]

whenever finite, to be the \(p\)-mean of \(w\) over \(B\). Some special cases of (1.4) are noteworthy. The 1-mean of \(w\) over \(B\) is the arithmetic mean, or simply the average, of \(w\) over \(B\). When \(p = 0\), \(w(0,B)\) takes on the form

\[
w(0,B) := \lim_{p \to 0} w(p,B) = \exp\left(\frac{1}{\mu(B)} \int_B \ln w \, d\mu\right),
\]
which is the geometric mean of $w$ over $B$. With the exponent $p = -1$ we get $w(-1, B)$, the harmonic mean of $w$ over $B$, and the exponents $p = -\infty$ and $p = \infty$ yield the essential infimum and essential supremum of $w$ over $B$, respectively, that is,

$$w(-\infty, B) = \text{ess inf}_B w \quad \text{and} \quad w(\infty, B) = \text{ess sup}_B w.$$ 

2. Reverse Classes and their diagrams

Let $(X, d, \mu)$ be a space of homogeneous type and let $\Omega \subset X$ be an open set. Throughout the article we will consider balls in the collection

$$(2.5) \quad B_\Omega := \{ B \subset \Omega \mid 2B \subset \Omega \}.$$

**Definition 3.** Let $-\infty \leq r < s \leq \infty$ and $C \geq 1$. We write $w \in RC(r, s, C)$ and say that $w$ is in the reverse class with exponents $r$ and $s$ if $w^r \in L^1_{\text{loc}}(\Omega)$ and

$$(2.6) \quad w(r, B) \leq w(s, B) \leq C w(r, B) \quad \forall B \in B_\Omega.$$

The smallest constant $C \geq 1$ validating (2.6) will be called the reversal constant of the weight $w$ in the class $RC(r, s)$ and it will be denoted by $[w]_{RC(r, s)}$. Also, define

$$RC(r, s) := \bigcup_{C \geq 1} RC(r, s, C).$$

The fact that a weight $w$ belongs to a reverse class $RC(r, s)$ will be represented by the following diagram

$$w \in RC(r, s) \quad \Leftrightarrow \quad \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
-w \quad -r \quad -s \quad \infty
\end{array}
\end{array}$$

**Figure 1.** Illustration of the reverse class $RC(r, s)$, indicating the uniform comparability between $r$- and $s$-means.

In order to allow for more clear and less crowded pictures, we will make no distinctions between arrows drawn above or below the extended real line. That is,
In all rigor, we should write \( w \in RC(r, s, C, \Omega) \) to account for the open set \( \Omega \) where the balls are taken, but \( \Omega \) will be always understood from the context.

**Definition 4.** Let \(-\infty \leq r < s \leq \infty\) and \( C > 0 \). We write \( w \in RC^{\text{weak}}(r, s, C) \), or simply, \( w \in RC^{\text{weak}}(r, s) \), and say that \( w \) is in the weak reverse class with exponents \( r \) and \( s \) if \( w^r \in L^1_{\text{loc}}(\Omega) \) and

\[
\left( \frac{1}{\mu(B)} \int_B w^s \, d\mu \right)^{1/s} \leq C \left( \frac{1}{\mu(B)} \int_{2B} w^r \, d\mu \right)^{1/r} \quad \forall B \in \mathcal{B}_\Omega.
\]

We visually represent this condition by joining the two exponents \( r \) and \( s \) by a dashed straight line with arrowheads pointing at them as illustrated in Figure 3.

**Remark 5.** Notice that if \( w \in RC^{\text{weak}}(r, s) \) and if \( w^r \) is a doubling weight, in the sense that there exists a constant \( C > 0 \) such that \( w^r(1, 2B) \leq C w^r(1, B) \) for every \( B \in \mathcal{B}_\Omega \), then \( w \in RC(r, s) \) and the dashed line in Figure 3 becomes a solid one.

### 3. The three axioms of the visual formalism

In this section we introduce the three main properties of the visual formalism: the shrinking property, the concatenation property, and the scaling property. These properties will be deduced from the definition of the reverse classes; however, they will be used as axioms during formal manipulations.
3.1. **The shrinking property.** This property says that the arrows representing reverse classes can always be shrunk without worsening reversal constants. More precisely,

**Lemma 6.** Fix any $-\infty \leq r \leq \tilde{r} < \tilde{s} \leq s \leq \infty$. If $w \in RC(r, s, C)$, then $w \in RC(\tilde{r}, \tilde{s}, C)$. Visually,

![Figure 4](image)

**Proof.** Since $\tilde{s} \leq s$ and $r \leq \tilde{r}$ we have natural inequalities

\begin{equation}
\text{(3.7)} \quad w(\tilde{s}, B) \leq w(s, B) \quad \text{and} \quad w(r, B) \leq w(\tilde{r}, B).
\end{equation}

Now, combining (3.7) with the defining inequality of $w \in RC(r, s, C)$, we obtain $w(\tilde{s}, B) \leq w(s, B) \leq Cw(r, B) \leq Cw(\tilde{r}, B)$. \qed

3.1.1. **The splitting property.** As a consequence of the shrinking property, it follows that: for every $-\infty \leq r < s < t \leq \infty$ we obtain that $RC(r, t, C) \subset RC(r, s, C) \cap RC(s, t, C)$; in particular, $\max\{[w]_{RC(r, s)} ; [w]_{RC(s, t)}\} \leq [w]_{RC(r, t)}$. We call this the splitting property and visually represent it with Figure 5.

![Figure 5](image)

**Figure 5.** The splitting property: $RC(r, t, C) \subset RC(r, s, C) \cap RC(s, t, C)$ and $\max\{[w]_{RC(r, s)} ; [w]_{RC(s, t)}\} \leq [w]_{RC(r, t)}$.

As shown, the arrows representing reverse classes can always be shrunk as much as desired. However, they cannot be stretched as much as we wish. As we will see in Section 5, stretching the arrows will describe self-improvement properties for reverse classes.
3.2. The concatenation property. This property provides the converse to the splitting property. It says that whenever an arrow begins where another ends, then they can be concatenated. Moreover, the reversal constant for the concatenation is controlled by the product of the reversal constants involved.

Lemma 7. Fix any $-\infty \leq r < s < t \leq \infty$. If $w \in RC(r, s, C_1) \cap RC(s, t, C_2)$, then $w \in RC(r, t, C_1 C_2)$. Consequently, $RC(r, s) \cap RC(s, t) = RC(r, t)$.

Bearing in mind that there is no distinction between arrows drawn above or below the extended real line, the statement of Lemma 7 can be visually recast as follows:

```
\[ \begin{array}{c}
\infty \\
| \\
| \\
| \\
| \\
| \\
\infty \\
\end{array} \quad \begin{array}{c}
\infty \\
| \\
| \\
| \\
| \\
\infty \\
\end{array} \]
\[ \begin{array}{c}
-\infty \\
| \\
| \\
| \\
\infty \\
\end{array} \quad \begin{array}{c}
-\infty \\
| \\
| \\
\infty \\
\end{array} \\
\]
\[ w \]
\[ \Rightarrow \]
\[ w \]
\[ w \]
\[ \begin{array}{c}
\infty \\
| \\
| \\
| \\
\infty \\
\end{array} \quad \begin{array}{c}
\infty \\
| \\
| \\
\infty \\
\end{array} \]
```

Notice that the concatenation property together with the splitting property implies that $RC(r, s) \cap RC(s, t) = RC(r, t)$.

Proof of Lemma 7. Since $w \in RC(r, s, C_1) \cap RC(s, t, C_2)$, we have $w(t, B) \leq C_2 w(s, B) \leq C_1 C_2 w(r, B)$.

3.3. The scaling property. This property dictates how reverse classes and reversal constants behave under the operation of taking powers. More precisely,

Theorem 8. Let $-\infty \leq r < s \leq \infty$ and fix $w \in RC(r, s, C)$. Then,

(i) for any $\theta > 0$, we have $w^\theta \in RC(\frac{r^\theta}{\theta}, \frac{s^\theta}{\theta}, C^\theta)$,
(ii) for any $\theta < 0$, we have $w^\theta \in RC(\frac{r^\theta}{\theta}, \frac{s^\theta}{\theta}, C^{\mid \theta \mid})$.

Visually, this is illustrated in Figure 7.

Figure 7. The scaling property. Quantitatively, $[w^\theta]_{RC(r/\theta,s/\theta)} = [w^\theta]_{RC(r,s)}$, for $\theta > 0$; and $[w^\theta]_{RC(s/\theta,r/\theta)} = |\theta| [w^\theta]_{RC(r,s)}$, for $\theta < 0$.

Proof. Let us prove (i). Since $w \in RC(r,s,C)$, given $B \in \mathcal{B}_\Omega$ we have

$$w(s,B) = \left(\frac{1}{\mu(B)} \int_B w^{\theta s} d\mu\right)^{1/s} \leq C w(r,B) = C \left(\frac{1}{\mu(B)} \int_B w^{\theta r} d\mu\right)^{1/r}.$$  

For $\theta > 0$ we get

$$\left(\frac{1}{\mu(B)} \int_B w^{\theta s} d\mu\right)^{\theta/s} \leq C^\theta \left(\frac{1}{\mu(B)} \int_B w^{\theta r} d\mu\right)^{\theta/r},$$

that is, $w^\theta \in RC(\frac{r}{\theta}, \frac{s}{\theta}, C^\theta)$. To prove (ii), we use (3.8) again, but now with $\theta < 0$

$$\left(\frac{1}{\mu(B)} \int_B w^{\theta s} d\mu\right)^{\theta/s} \geq C^\theta \left(\frac{1}{\mu(B)} \int_B w^{\theta r} d\mu\right)^{\theta/r},$$

that is, $w^\theta \in RC(\frac{s}{\theta}, \frac{r}{\theta}, C^\theta)$. \qed

Remark 9. It is convenient to point out that, during the formal manipulations, the axioms above provide qualitative as well as quantitative information about the reverse classes and their weights. Meaning that when a particular weight is subjected to a sequence of axioms of the visual formalism, there will always be an explicit control on the reversal constants.

4. Some well-known reverse classes

Definition 10. Let $1 < s \leq \infty$. A weight $w$ is said to belong to $RH_s$, the reverse Hölder class of order $s$, if the following inequality holds

$$[w]_{RH_s} := \sup_{B \in \mathcal{B}_\Omega} w(s,B)w(1,B)^{-1} < \infty.$$  

In other words, $RH_s = RC(1,s)$. Visually,
Definition 11. Let $1 < p < \infty$. A weight $w$ is said to belong to the Muckenhoupt class $A_p$ if

$$[w]_{A_p} := \sup_{B \in \mathcal{B}_1} w(1, B) w\left(\frac{1}{1 - p}, B\right)^{-1} < \infty.$$ 

In other words, $A_p = RC(1/(1 - p), 1)$. We write $w \in A_1$ if

$$[w]_{A_1} := \sup_{B \in \mathcal{B}_1} w(1, B) w(-\infty, B)^{-1} < \infty.$$ 

That is, $A_1 = RC(-\infty, 1)$. Finally, we write $w \in A_\infty$ if

$$(4.9) \quad [w]_{A_\infty} := \sup_{B \in \mathcal{B}_1} w(1, B) w(0, B)^{-1} < \infty.$$ 

That is, $A_\infty = RC(0, 1)$. Again, we have chosen to use the notation $A_p$ instead of $A_p(\Omega)$. Muckenhoupt classes can be visually represented by the following diagrams in Figure 9.
Again, we refer the reader to [14, Chapter 7], [16, Chapter IV], [18, Chapter 9], [27, Chapter 2], [34, Chapter 5], [38, Chapter 5], [39, Chapters I and II], [40, Chapter IX], as well as references therein, for all the basic material and history of reverse Hölder classes and Muckenhoupt weights.

Remark 12. The constant $[w]_{A_{\infty}}$ in (4.9) was introduced by Hruščev in [21]. Another constant, which has lately received a lot of attention, given by

\[(4.10) \quad [w]'_{A_{\infty}} := \sup_{B \in \mathcal{B}_{\Omega}} \frac{1}{\mu(B)} \int_B \mathcal{M}(w\chi_B) \, d\mu,\]

where $\mathcal{M}$ stands for the Hardy-Littlewood maximal operator, was introduced by Fujii [15] and Wilson [41]. The constant $[w]'_{A_{\infty}}$ also characterizes $A_{\infty}$. More precisely, in the Euclidean setting (with $\Omega = \mathbb{R}^n$), Hytönen and Pérez proved in...
that one has \( |w'|_{A_{\infty}} \leq c_n |w|_{A_{\infty}} \), for some dimensional constant \( c_n > 0 \). A corresponding inequality was proved in the context of spaces of homogeneous type by Hytönen, Pérez, and Rela in [23]. The study of \( A_{\infty} \) through the constant \( |w'|_{A_{\infty}} \) seems to be better suited for obtaining sharp reversal constants and reversal exponents, see [22, 23], and references therein.

4.1. Some practice. With the purpose of getting some practice with the visual formalism, next we go over some well-known properties of \( A_p \) weights (see, for instance, [25, Section 1] and [11]).

It will be useful to bear in mind Remark 9 as well as the facts that, for \( 1 < p < \infty \), \( A_p = RC((1 - p)^{-1}, 1) \), \( A_1 = RC(\infty, 1) \), and \( A_{\infty} = RC(0, 1) \).

Corollary 13. Fix \( 1 < p < \infty \) and let \( p' \) denote its Hölder conjugate, that is, \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then \( w \in A_p \) if and only if \( w^{1-p'} \in A_{p'} \). Moreover, \( [w^{1-p'}]_{A_{p'}} = [w]_{A_p}^p \).

Proof. This proof uses only the scaling property along with the fact that \( (1 - p)(1 - p') = 1 \).

\[
\begin{align*}
  w \in A_p & \quad \Downarrow \\
  w & \quad \text{Scaling} \\
  \frac{1}{1-p} & \quad 1 \\
\end{align*}
\]

\[
\begin{align*}
  w^{1-p'} \in A_{p'} & \quad \Downarrow \\
  w^{1-p'} & \quad \text{Scaling} \\
  \frac{1}{1-p'} & \quad 1
\end{align*}
\]

Figure 10. Visual proof of Corollary 13

Notice that, since scaling by a power \( \theta \) has the effect of raising reversal constants to the power \( |\theta| \) (see Theorem 8), for the sequence of steps, from left to right, in Figure 10 we have scaled by \( \theta = 1 - p' < 0 \), yielding

\[
[w^{1-p'}]_{A_{p'}} \leq [w]_{A_p}^{p-1}.
\]

Conversely, taking the steps in Figure 10 from right to left we have scaled by \( \theta = 1 - p < 0 \), giving

\[
[w]_{A_p} \leq [w^{1-p'}]_{A_{p'}}^{p-1}.
\]

Hence, \( [w^{1-p'}]_{A_{p'}} = [w]_{A_p}^{p-1} \).

\[\square\]

Corollary 14. Fix \( 1 < p < \infty \). Then, \( w \in A_1 \) implies \( w^{1-p} \in A_p \cap RH_{\infty} \), and we have \( \max\{[w^{1-p}]_{A_p}, [w^{1-p}]_{RH_{\infty}}\} \leq [w]_{A_1}^{p-1} \). Conversely, \( w^{1-p} \in A_p \cap RH_{\infty} \) implies \( w \in A_1 \), and we have \( [w]_{A_1} \leq ([w^{1-p}]_{A_p} [w^{1-p}]_{RH_{\infty}})^{p-1} \).
Proof. Here is the visual proof

Again, by Theorem 8 and since shrinkage does not worsen reversal constants, for the left-to-right sequence of steps in Figure 11 we have: scaling (with $\theta = 1 - p < 0$), which yields $[w^{1-p}]_{RC(1/(1-p),\infty)} = [w]_{A_1}$, followed by splitting, which then yields

$$\max\{[w^{1-p}]_{A_p}, [w^{1-p}]_{RH_\infty}\} \leq [w]_{A_1}^{p-1}.$$ 

On the other hand, now considering the right-to-left sequence of steps, we have: concatenation, which yields $[w^{1-p}]_{RC(1/(1-p),\infty)} \leq [w^{1-p}]_{A_p} [w^{1-p}]_{RH_\infty}$, followed by scaling (with $\theta = 1/(1-p) < 0$), which yields $[w]_{A_1} \leq ([w^{1-p}]_{A_p} [w^{1-p}]_{RH_\infty})^{\frac{1}{p-1}}$. \hfill $\Box$

Corollary 15. Fix $1 < s < \infty$. If $w \in A_1 \cap RH_s$, then $w^{1-p} \in A_p \cap RH_\infty$ for every $1 < p < \infty$ and $q > (p-1)/s + 1$. Moreover, $\max\{[w^{1-p}]_{A_q}, [w^{1-p}]_{RH_\infty}\} \leq ([w]_{A_1} [w]_{RH_s})^{p-1}$.

Proof. Let us start with the visual proof, which is immediate after realizing that the relation between the indices can be recast as $\frac{s}{1-p} < \frac{1}{1-q}$. We have
For the clockwise sequence of steps in Figure 12, we have: first concatenation, which yields \([w]_{RC(-∞, s)} \leq [w]_{A_1[w]_{RH_s}}\), followed by scaling (with \(θ = 1 - p\)), yielding \([w^{1-p}]_{RC(s/(1-p), ∞)} \leq ([w]_{A_1[w]_{RH_s}})^{p-1}\), followed by shrinking, which gives \([w^{1-p}]_{RC(1/(1-q), ∞)} \leq ([w]_{A_1[w]_{RH_s}})^{p-1}\), and finally, splitting gives

\[
\max\{[w^{1-p}]_{A_q}, [w^{1-p}]_{RH_∞}\} \leq ([w]_{A_1[w]_{RH_s}})^{p-1}.
\]

\[\square\]

**Corollary 16.** Fix \(1 < p < ∞\). Then, \(w ∈ A_p ∩ RH_∞\) implies that \(w^{1-p'} ∈ A_1\), with \([w^{1-p'}]_{A_1} \leq ([w]_{RH_∞}[w]_{A_p})^{p'-1}\). Conversely, \(w^{1-p'} ∈ A_1\) implies \(w ∈ A_p ∩ RH_∞\), with \(\max\{[w]_{RH_∞}, [w]_{A_p}\} \leq [w^{1-p'}]_{A_1}^{p-1}\).

**Proof.** Figure 13 illustrates the visual proof.
Corollary 16. Let \( 1 < p, s < \infty \) and set \( q := s(p - 1) + 1 \). Then, \( w \in A_p \cap RH_s \) implies \( w^s \in A_q \), with \([w^s]_{A_q} \leq ([w]_{A_p}[w]_{RH_s})^s\). Conversely, \( w^s \in A_q \) implies \( w \in A_p \cap RH_s \), with \( \max\{[w]_{A_p}, [w]_{RH_s}\} \leq [w^s]_{A_q}^{1/s} \).

Proof. Figure 14 illustrates the visual proof

In Figure 14, from left to right, we have: concatenation, which gives \([w]_{RC(1/(1-p),s)} \leq [w]_{A_p}[w]_{RH_s}\), followed by scaling by \( s \), which gives...
\( [w^s]_{A_q} \leq ([w]_{A_p} [w]_{RH})^s \). From right to left we have: scaling by \(1/s\), yielding
\[ [w]_{RC(1/(1-p),s)} = [w^s]^{1/s}_{A_q} \], followed by splitting, which gives \( \max\{[w]_{A_p}, [w]_{RH_s}\} \leq [w^s]^{1/s}_{A_q}. \)

\[ \square \]

5. SELF-IMPROVING PROPERTIES

**Definition 18.** A reverse class \( RC(r, s, C_1) \) is said to have the right self-improving property if for every \( w \in RC(r, s, C_1) \) there exists \( \varepsilon, C_2 > 0 \) such that \( w \in RC(r, s + \varepsilon, C_2) \).

A reverse class \( RC(r, s, C_1) \) is said to have the left self-improving property if for every \( w \in RC(r, s, C_1) \) there exists \( \varepsilon, C_2 > 0 \) such that \( w \in RC(r - \varepsilon, s, C_2) \).

These self-improving properties are visually represented in Figure 15.

**Figure 15.** Left and right self-improving properties. The corresponding concatenations are tacitly understood. That is, the diagrams also indicate that \( w \in RC(r - \varepsilon, r) \cap RC(r, s) = RC(r - \varepsilon, s) \) and \( w \in RC(r, s) \cap RC(s, s + \varepsilon) = RC(r, s + \varepsilon) \), respectively.

In the next theorem we collect some well-known facts concerning self-improving properties for \( A_p \) and reverse-Hölder classes.

**Theorem 19.** Fix \( 1 < p, s < \infty \).

(i) For every \( w \in A_p \) there exists \( \delta > 0 \), depending on geometric constants, \( p \), and \( [w]_{A_p} \), such that \( w \in RH_{1+\delta} \).

(ii) For every \( w \in A_p \) there exists \( \varepsilon > 0 \), depending on geometric constants, \( p \), and \( [w]_{A_p} \), such that \( w \in A_{p-\varepsilon} \).

(iii) For every \( w \in RH_s \) there exists \( 1 < q < \infty \), depending on geometric constants, \( s \), and \( [w]_{RH_s} \), such that \( w \in A_q \).

(iv) For every \( w \in A_{\infty} \) there exists \( 1 < r < \infty \), depending on geometric constants and \( [w]_{A_{\infty}} \), such that \( w \in A_r \).

Theorem 19 illustrates the close connection between the reverse inequalities defining \( A_p \) and the reverse Hölder inequalities defining \( RH_s \). This connection was first proved by Coifman and Fefferman in [6]. For proofs of Theorem 19 see for instance, [6], [14] Chapter 7, [18] Chapter 9, and [38] Chapter 5. Precise estimates for \( \delta > 0 \) in (i) and \( \varepsilon > 0 \) in (ii), in the context of spaces of homogeneous type, can be found in [23]. Sharp reverse Hölder inequalities for \( A_{\infty} \) weights in \( \mathbb{R}^n \) can be found in [22, 32].
We can now illustrate the contents of Theorem 19 as follows

\[ w \]
\[ \frac{1}{1-p} 0 1 1 + \delta \]

(i)

\[ w \]
\[ \frac{1}{1-(p-\varepsilon)} \frac{1}{1-p} 0 1 \]

(ii)

\[ w \]
\[ \frac{1}{1-q} 0 1 s \]

(iii)

\[ w \]
\[ \frac{1}{1-r} \frac{1}{1-q} 0 1 \]

(iv)

**Figure 16.** The self-improving properties for the \( A_p \) classes as stated in Theorem 19. Notice that the “improvement leaps” in parts (i), (ii), and (iv) will typically be small. That is, \( \delta \) and \( \varepsilon \) will be small and \( r \) will be big. However, in the case of (iii), although the index \( q \) will also be typically large, the “improvement leap” is of length larger than one, crossing from 1 all the way back to the negative number \( \frac{1}{1-q} \).

In order to keep practicing our visual formalism, let us prove some well-known results using the diagrams for self-improving properties.

**Corollary 20.** Fix \( 1 < s < \infty \). If \( w \in RH_s \) then \( w^s \in A_\infty \).

**Proof.** The visual proof is illustrated in Figure 17. Notice how the first step uses the self-improving property from Theorem 19 (iii).
Corollary 21. Fix $1 < s < \infty$. If $w \in RH_s$, then there exists $t > s$ such that $w \in RC(0, t) \subset RH_t$.

Proof. The visual proof is illustrated in Figure 18. Again, the first step uses the self-improving property from Theorem 19 (iii).

Corollary 22. If $w \in A_\infty$, then there exists $\varepsilon > 0$ such that $w^\varepsilon \in A_2$.

Proof. The visual proof is illustrated in Figure 19. Notice how the first step uses the self-improving property from Theorem 19 (iv).
Weak reverse classes also enjoy self-improving properties. For instance, we have

**Theorem 23.** Fix $1 < s < \infty$ and $0 < p < q < r \leq \infty$. Then,

(i) for every $w \in RH^\text{weak}_s$ there exists $\varepsilon > 0$ such that $w \in RH^{\text{weak}}_s + \varepsilon$, and

(ii) $RC^{\text{weak}}(q, r) = RC^{\text{weak}}(p, r)$.

Visually, these properties are depicted in Figures 20 and 21.
The whole phenomenon of reverse inequalities and their self-improving properties stems from Gehring’s work [17] (see [2, Chapter 3] for an exposition of the so-called Gehring’s lemma in doubling metric spaces). As mentioned, the connection between reverse Hölder inequalities and $A_p$ weights was first explored in [6]. A proof for Theorem 23 (i) in spaces of homogeneous type can be found, for instance, in [28]. A proof for Theorem 23 (ii) in spaces of homogeneous type can be found, for instance, in [3, Lemma 1.4] and the case $r = \infty$ has been worked out, for instance, in [29, Remark 4.4].

6. Reverse classes, $BMO$, BLO, and BUO

Throughout this section we will consider $\Omega = X$.

Definition 24. We recall the definitions of $BMO$, BLO and BUO; namely, the spaces of functions of bounded mean oscillation, bounded lower oscillation, and bounded upper oscillation, respectively. $w \in BMO$ if and only if
$$
\|w\|_{BMO} := \sup_{B \in \mathcal{B}_1} \left( \frac{1}{\mu(B)} \int_B |w - w_B| \, d\mu \right) < \infty,
$$
where, as usual, $w_B := \left( \frac{1}{\mu(B)} \int_B w \, d\mu \right)$. $w \in BLO$ if and only if
$$
\|w\|_{BLO} := \sup_{B \in \mathcal{B}_1} \left( \frac{1}{\mu(B)} \int_B (w - \text{ess inf}_B w) \, d\mu \right) < \infty.
$$
$w \in BUO$ if and only if
$$
\|w\|_{BUO} := \sup_{B \in \mathcal{B}_1} \left( \frac{1}{\mu(B)} \int_B (\text{ess sup}_B w - w) \, d\mu \right) < \infty.
$$

Proposition 25 below provides characterizations of $BMO$ and $BLO$ in terms of $A_p$ classes.

Proposition 25. The following characterizations hold true:

(i) $\log w \in BMO$ if and only if $w^\alpha \in A_p$ for some $1 < p < \infty$ and some $\alpha \in \mathbb{R}$.
(ii) $\log w \in BLO$ if and only if $w^{\epsilon} \in A_1$ for some $\epsilon > 0$.
(iii) $\log w \in BUO$ if and only if $w^{\epsilon} \in RH_\infty$ for some $\epsilon > 0$.

For a proof of (i) see, for instance, [14, p.151], [18, p.300], and [38, p.218]. The proof of (ii) can be found in [8, Lemma 1]. The proof of (iii) follows, for instance, from the characterization $RH_\infty = e^{BUO}$ as in [38, Theorem 3.2]. In view of Proposition 25, we have

Corollary 26. The following characterizations hold true:

(i) $\log w \in BMO$ if and only if $w \in RC(r, s)$ for some $-\infty \leq r < s \leq \infty$.
(ii) $\log w \in BLO$ if and only if $w \in RC(-\infty, r)$ for some $-\infty < r \leq \infty$.
(iii) $\log w \in BUO$ if and only if $w \in RC(r, \infty)$ for some $-\infty \leq r < \infty$. 

Visually,

\[ \log w \in BMO \iff \begin{array}{c}
\downarrow \\
\downarrow \\
-\infty & r & s & \infty
\end{array} \]

**Figure 22.** A characterization of \( BMO: \ BMO = \log \left( \bigcup_{-\infty \leq r < s \leq \infty} RC(r, s) \right) \).

\[ \log w \in BLO \iff \begin{array}{c}
\downarrow \\
\downarrow \\
-\infty & r & \infty
\end{array} \]

**Figure 23.** A characterization of \( BLO: \ BLO = \log \left( \bigcup_{-\infty < r \leq \infty} RC(-\infty, r) \right) \).

\[ \log w \in BUO \iff \begin{array}{c}
\downarrow \\
\downarrow \\
-\infty & r & \infty
\end{array} \]

**Figure 24.** A characterization of \( BUO: \ BUO = \log \left( \bigcup_{-\infty \leq r < \infty} RC(r, \infty) \right) \).

**Proof.** Here is the visual proof of Corollary 26 by means of Proposition 25.

\[ \log w \in BMO \iff w^r \in A_q \iff \begin{array}{c}
-\frac{1}{1-q} & 0 & 1
\end{array} \]

\[ \log w \in BLO \iff \begin{array}{c}
-\frac{1}{r-1} & 0 & 1
\end{array} \]

**Figure 25.** Proof of the “only if” part of Corollary 26 (i), for \( 0 < r \), using Proposition 25 (i). In the case of \( r < 0 \), only the positions of the points \( 1/r \) and \( 1/(r(1-q)) \) will be switched in the figure.
Figure 26. Proof of the “if” part of Corollary 26 (i), for $0 < r < s$, by using Proposition 25 (i). The case $r < s < 0$ reduces to this case after scaling by $-1$. By the shrinking property, the general case $r < s$ reduces to one of the previous two cases (see Exercise 5 in Section 10).

Figure 27. Proof of Corollary 26 (ii) for the case $r > 0$, by using Proposition 25 (ii).

Figure 28. Proof of Corollary 26 (iii) for the case $r < 0$. It is shown how after scaling and self-improving, this case reduces to the case $r > 0$.

The proof of Corollary 26 (iii) is left as an exercise (see Exercise 6 in Section 10).

7. Interpolation

Theorem 27. Let $r_1 \leq r_2 < 0 < s_1 \leq s_2$, and $u \in RC(r_1, s_1, C_1)$ and $v \in RC(r_2, s_2, C_2)$. Then, for every $\theta \in [0, 1]$, we have $u^\theta v^{1-\theta} \in RC(r_\theta, s_\theta, C_\theta)$, where $C_\theta := C_1^\theta C_2^{1-\theta}$ and

\[
\frac{1}{r_\theta} := \frac{\theta}{r_1} + \frac{1-\theta}{r_2} < 0, \quad \frac{1}{s_\theta} := \frac{\theta}{s_1} + \frac{1-\theta}{s_2} > 0.
\]
Visually, this is illustrated in Figure 29.

![Diagram](image)

**Figure 29.** Interpolation of two reverse classes crossing zero.

**Proof.** Assume \( \theta \in (0, 1) \) (the cases \( \theta = 0 \) or \( \theta = 1 \) being trivial). We need to prove that

\[
\left( \frac{1}{\mu(B)} \int_B u^{s_\theta} v^{s_\theta (1-\theta)} d\mu \right)^{1/s_\theta} \left( \frac{1}{\mu(B)} \int_B u^{r_\theta} v^{(1-\theta)r_\theta} d\mu \right)^{-1/r_\theta} \leq C_\theta.
\]

By Hölder’s inequality with the exponents \( q := \frac{s_\theta}{s_\theta r_\theta} \) and \( q' := \frac{s_\theta}{(1-\theta)r_\theta} \), one gets

\[
\left( \frac{1}{\mu(B)} \int_B u^{r_\theta} v^{(1-\theta)r_\theta} d\mu \right)^{-1/r_\theta} \leq u(r_1, B)^{-\theta} v(r_2, B)^{-(1-\theta)}.
\]

Again, applying Hölder’s inequality with the exponents \( r := \frac{s_\theta}{s_\theta r_\theta} \) and \( r' := \frac{s_\theta}{(1-\theta)r_\theta} \), and using the assumptions \( u \in RC(r_1, s_1, C_1) \) and \( v \in RC(r_2, s_2, C_2) \),

\[
\left( \frac{1}{\mu(B)} \int_B u^{s_\theta} v^{s_\theta (1-\theta)} d\mu \right)^{1/s_\theta} \leq w(s_1, B)^{\theta} w(s_2, B)^{1-\theta}
\]

\[
\leq C_1^{\theta} C_2^{1-\theta} u(r_1, B)^{\theta} v(r_2, B)^{1-\theta}.
\]

Therefore, (7.11) follows from (7.12) and (7.13).

**Corollary 28.** Let \( w_j \in A_{p_j}, \ j = 1, 2, \) where \( 1 \leq p_1 < p_2 < \infty \), then \( w_\theta w_1^{1-\theta} \in A_{p_3} \) where \( p_3 = \theta p_1 + (1 - \theta)p_2 \) and \( \theta \in (0, 1) \).

**Proof.** The proof follows immediately from Theorem 27 after the statement is recast in terms of reverse classes as follows: if \( w_j \in RC(1/p_j, 1, C_j) \), where \( j = 1, 2, \) and \( 1 \leq p_1 < p_2 < \infty \), then \( w_\theta w_1^{1-\theta} \in RC(\frac{1}{\theta p_1 + (1-\theta)p_2}, 1, C_\theta) \), \( \theta \in (0, 1) \).

**8. Factorization**

The basic factorization theorem for \( A_p \) weights reads

**Theorem 29.** \( w \in A_p \) if and only if \( w = w_1 w_2^{1-p} \), for some \( w_1, w_2 \in A_1 \).
Theorem 29 was first proved, in the Euclidean setting, by P. Jones in [26] and other proofs then appeared in [7, 37]. For a proof in spaces of homogeneous type, see [39, Chapter II]. A thorough discussion of this topic can be found, for instance, in [12, Sections 1.1 and 6.2], [14, Section 5.5], and [40, Chap IX].

The visual rendition of the factorization theorem yields a certain intersection property for the arrows, see Figure 30.

![Figure 30. The factorization of $A_p$ weights as an intersection property for the arrows.](image)

Theorem 29 can be extended as follows:

**Theorem 30.** Let $r < 0 < s$. Then $w \in RC(r, s)$ if and only if $w = uv$ for some $u \in RC(r, \infty)$ and $v \in RC(-\infty, s)$. Visually,

![Figure 31. The factorization of a weight $w$ belonging to a reverse class crossing zero as an intersection property for the arrows.](image)

**Proof.** For the ‘if’ part, we take $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$ and we will prove $uv \in RC(r, s, C_1 C_2)$, that is,

$$\text{(8.14)} \quad (uv)(s, B) \leq C_1 C_2 (uv)(r, B) \quad \forall B \in \mathcal{B}_\Omega.$$

Using the assumptions $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$, given $B \in \mathcal{B}_\Omega$ we have

$$\text{(8.15)} \quad (uv)(s, B) \leq (\text{ess sup}_B u)v(s, B) \leq C_2 u(r, B)v(s, B)$$

and

$$\text{(8.16)} \quad (uv)(r, B)^{-1} \leq (\text{ess inf}_B v)^{-1}u(r, B)^{-1} \leq C_1 v(s, B)^{-1}u(r, B)^{-1}.$$
Hence, the estimate (8.14) follows by multiplying (8.15) and (8.16).

For the ‘only if’ part, we take $w \in RC(r, s)$ and need to show that there exist $u \in RC(r, \infty)$ and $v \in RC(-\infty, s)$ such that $w = uv$. We know that $w \in RC(r, s)$, then, by Theorem 8 (scaling), we have $w^s \in A_p$, with $p := (1 - \frac{s}{r}) > 1$. Then, by Theorem 29 (factorization), there exist $\tilde{u}, \tilde{v} \in A_1$ such that $w^s = \tilde{u} \tilde{v}^{1-p}$. Defining $u := \tilde{u}^{1-s}$ and $v := \tilde{v}^{1-s}$, we get $w^s = uv$. Now, since $\tilde{u}, \tilde{v} \in A_1 = RC(-\infty, 1)$ and $s > 0$, by Theorem 8 (scaling), we have that $u := \tilde{u}^{1-s} \in RC(-\infty, s)$. On the other hand, since $r = \frac{s}{1-p}$, from the definition of $p$, by scaling we have that $v := \tilde{v}^{1-s} \in RC(\frac{s}{1-p}, \infty) = RC(r, \infty)$. □

\textbf{Corollary 31.} (See [8, Section III]) Every BMO function is the difference of two BLO functions, that is,

$$BMO = BLO - BLO.$$  

\textit{Proof.} In light of Corollary 26 (i) and (ii), we only need to show that $w \in RC(r, s)$, for some $r < 0 < s$, if and only if $w = w_1 w_2^{-1}$ for some $w_1 \in RC(-\infty, r_1)$ and $w_2 \in RC(-\infty, -r)$, with $r_1, r_2 > 0$. Let $r < 0 < s$ and $w \in RC(r, s)$. Then, take $w_1 \in RC(-\infty, s)$ and $w_2 \in RC(-\infty, -r)$ provided by Theorem 30 to get $w = w_1 w_2^{-1}$. Conversely, let $w_1 \in RC(-\infty, r_1)$ and $w_2 \in RC(-\infty, -r)$, for some $r_1, r_2 > 0$. Finally, since $-r_2 < 0 < r_1$, by Theorem 30 we have $w_1 w_2^{-1} \in RC(-r_2, r_1)$.

\section*{8.1. The visual formalism as a tool for generating conjectures.} Figure 31 motivates the following question:

\begin{center}
\begin{tikzpicture}
  \node (x1) at (0,0) {$-\infty$};
  \node (x2) at (1,0) {$r_1$};
  \node (x3) at (2,0) {$r_2$};
  \node (x4) at (3,0) {$0$};
  \node (x5) at (4,0) {$s_1$};
  \node (x6) at (5,0) {$s_2$};
  \node (x7) at (6,0) {$\infty$};
  \path (x1) edge (x2)
  (x2) edge (x3)
  (x3) edge (x4)
  (x4) edge (x5)
  (x5) edge (x6)
  (x6) edge (x7);
  \node (u) at (2.5,1) {$u$};
  \node (v) at (2.5,-1) {$v$};
  \node (uv) at (2.5,2) {$uw$};
  \node (w) at (2.5,3) {$w$};
  \node (Q) at (2.5,0) {$?$};
  \path (u) edge (x4)
  (v) edge (x4)
  (w) edge (x4)
  (uv) edge (x4);
\end{tikzpicture}
\end{center}

\textbf{Figure 32.} Is the implication true? Does the product honor the reversal property in the intersection of the reversal indices of the factors? It turns out that the answer is no, unless $r_1 = -\infty$ and $s_2 = \infty$ (and then Theorem 30 applies).

Next, we will see that in order for the implication in Figure 32 to hold true it is necessary that both factors touch opposite infinities. This fact will be substantiated by Examples 32 and 33 below. These examples are constructed with the help of the following fact: $|x|^a \in A_p(\mathbb{R}^n)$ if and only if $-n < a < n(p - 1)$, see [18, p.286]; visually, this fact is represented in Figure 33.
Example 32. Take $\Omega = \mathbb{R}$. In this example the reverse classes of both the factors do not touch an infinity and we will see that the product does not lie in the intersecting reverse class. Take $u := |x| \in A_3 = RC(-\frac{1}{2}, 1)$ and $v := |x|^3 \in A_5 = RC(-\frac{1}{4}, 1)$. Clearly, we have

$$RC\left(-\frac{1}{2}, 1\right) \cap RC\left(-\frac{1}{4}, 1\right) = RC\left(-\frac{1}{4}, 1\right).$$

But, $uv = |x|^4 \notin A_5$. This example is illustrated in Figure 34.

Example 33. Take $\Omega = \mathbb{R}$. In the following example, illustrated in Figure 35 only one of the factors touches an infinity and we will see that this is not sufficient for the product to lie in the intersecting reverse class. Take $\varepsilon_0, \varepsilon_1 > 0$ so that $0 < (\varepsilon_0 + \varepsilon_1) \ll 1$. Then, since $-1 < -1 + \varepsilon_0 < 0$, $|x|^{-1+\varepsilon_0} \in A_1 = RC(-\infty, 1)$. Equivalently, $u := |x|^{1-\varepsilon_0} \in RC(-1, \infty)$. Again, since $-1 < \varepsilon_0 + \frac{\varepsilon_1}{2} < \varepsilon_0 + \varepsilon_1$, $v := |x|^ {\varepsilon_0 + \frac{\varepsilon_1}{2}} \in A_1 = RC(-\infty, 1)$.

$$RC\left(-1, \infty\right) \cap RC\left(-1, 1\right) = RC\left(-1, 1\right).$$

But, $uv = |x|^{1+\frac{\varepsilon_1}{2}} \notin A_2$ since $1 + \frac{\varepsilon_1}{2} > 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure33}
\caption{A necessary and sufficient condition for the power weight $|x|^a$ to be in $A_p(\mathbb{R}^n)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure34}
\caption{The product will not be in the intersecting reverse class if the reverse classes for both factors do not touch an infinity.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure35}
\caption{The product will not be in the intersecting reverse class if the reverse classes for both factors do not touch an infinity.}
\end{figure}
9. Using the visual formalism to illustrate proofs of Harnack’s inequality

Fix an open set $\Omega \subset X$. A weight $w$ is said to satisfy Harnack’s inequality, with constant $C_H \geq 1$, in $\Omega$ if $w \in RC(-\infty, \infty, C_H)$, that is,

$$\operatorname{ess sup}_B w \leq C_H \operatorname{ess inf}_B w, \quad \forall B \in \mathcal{B}_\Omega.$$ 

That is, $w$ satisfies the most extreme reversal inequality. Visually,

![Figure 36. The Harnack class $RC(-\infty, \infty)$.](image)

In this section we use the visual formalism to provide a brief description of Moser’s and Krylov-Safonov’s proofs of Harnack’s inequality for positive solutions to elliptic PDEs. While Moser’s method is most notable for his ingenious iteration scheme [33] in the context of divergence-form operators, the Krylov-Safonov’s method, based on innovative probabilistic tools [30, 31], was developed in the context of non-divergence-form operators. Both of these models stand as cornerstones in the study of regularity properties of solutions to PDEs and are flexible enough to be carried out in more general types of doubling quasimetric spaces possessing suitable additional structure (e.g., carrying Sobolev or Poincaré-type inequalities). In what follows the underlying space of homogeneous type is Euclidean space $\mathbb{R}^n$ with Lebesgue measure (the latter indicated by $|\cdot|$).

9.1. Moser’s iterations and Harnack’s inequality. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and for each $x \in \Omega$ let $A(x)$ be an $n \times n$ symmetric matrix verifying
the uniform ellipticity condition

\[ \Lambda_1 |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \Lambda_2 |\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \]

for some constants \(0 < \Lambda_1 \leq \Lambda_2\). Harnack’s inequality for positive solutions to the divergence-form elliptic equation

\[ Lu := \sum_{i,j=1}^{n} (a_{ij}(x)u_i)_j = \text{div}(A(x)\nabla u) = 0 \text{ in } \Omega \subset \mathbb{R}^n \]

was established by J. Moser in [33]. Moreover, Moser showed that the Harnack constant \(C_H\) depends only on dimension \(n\) and the ratio \(\Lambda_2/\Lambda_1\).

The first step in his approach is based on an interaction between a Sobolev inequality and an energy estimate (Caccioppoli’s inequality) to show that any positive subsolution \(u\) (i.e., \(Lu \geq 0\) in \(\Omega\)) satisfies \(u \in R^{\text{weak}}_{\rho} (2, \infty)\), where \(\rho := 2n/(n-2) > 1\). More precisely,

\[ \left( \frac{1}{|B|} \int_B u^{2\rho} \, dx \right)^{\frac{1}{2\rho}} \leq C(n, \Lambda_2/\Lambda_1) \left( \frac{1}{|2B|} \int_{2B} u^2 \, dx \right)^{\frac{1}{2}}, \quad \forall B \in \mathcal{B}_\Omega. \]

Then, by means of a finely tuned iterative procedure, illustrated in Figure 37,

\[ \begin{array}{cccccc}
-\infty & 0 & 2 & 2\rho & 2\rho^2 & 2\rho^3 & \infty \\
\end{array} \]

**Figure 37.** Moser iterations for a positive subsolution \(u\).

Moser improved (9.18) to obtain \(u \in R^{\text{weak}}_{\rho} (2, \infty)\) as in Figure 38.

\[ \begin{array}{cccccc}
-\infty & 0 & 2 & \infty \\
\end{array} \]

**Figure 38.** Concatenation and Moser’s iterations imply that every positive subsolution \(u\) belongs to \(R^{\text{weak}}_{\rho} (2, \infty)\).

Consequently, by the self-improving properties for the reverse weak classes (see Figure 21), we have \(u \in R^{\text{weak}}_{\rho} (p, \infty)\), for every \(p > 0\), that is,

\[ \text{ess sup}_B u \leq C(p, n, \Lambda_2/\Lambda_1) \left( \frac{1}{|B|} \int_{2B} u^p \, dx \right)^{\frac{1}{p}}, \quad \forall B \in \mathcal{B}_\Omega. \]
as illustrated in Figure 39.

Figure 39. Every positive subsolution $u$ belongs to $RC^{weak}(p, \infty)$ for all $p > 0$.

Inequality (9.19) is usually referred to as the *local boundedness property* for $u$ and it can also be proved by means of De Giorgi’s truncations. For both methods, Moser’s iterations and De Giorgi’s truncations, see for instance [20, Section 4.2].

Now, if $u$ is a positive solution, then both $u$ and $u^{-1}$ are positive subsolutions and (9.19) applied to them yields $u \in RC^{weak}(p, \infty)$ and $u^{-1} \in RC^{weak}(p, \infty)$. But, by the scaling property with $\theta = -1$, the latter means $u \in RC^{weak}(-\infty, -p)$, that is, for all $p > 0$,

$$ (9.20) \quad \left( \frac{1}{|B|} \int_{2B} u^{-p} \, dx \right)^{-\frac{1}{p}} \leq C(p, n, \Lambda_{2}/\Lambda_{1}) \text{ess inf}_B u \; \forall B \in B_\Omega. $$

Hence, positive solutions satisfy both (9.19) and (9.20), this is illustrated in Figure 40.

Figure 40. Every positive solution $u$ belongs to $RC^{weak}(-\infty, -p) \cap RC^{weak}(p, \infty)$ for all $p > 0$.

The next step in Moser’s Harnack inequality consists in establishing the existence of $p_0 > 0$ such that if $u$ is a positive supersolution (i.e., $\mathcal{L}u \leq 0$ in $\Omega$), then $u^{p_0} \in A_2(\Omega)$. This is illustrated in Figure 41.
There exists a structural $p_0 > 0$ such that every positive supersolution $u$ (i.e., $Lu \leq 0$) satisfies $u^{p_0} \in A_2$. By the scaling property, this is equivalent to $u \in RC(-p_0, p_0)$.

The step in Figure 41 is typically accomplished by using a Poincaré inequality and an energy estimate for $\log u$ to obtain that $\log u \in BMO(\Omega)$. Then, by Corollary 22 and the techniques in the proof of Corollary 26 (see also Exercise 5 in Section 10), it follows that there is a $p_0 > 0$ such that $u^{p_0} \in A_2$.

Finally, by choosing the $p$ in Figure 40 equal to the $p_0$ in Figure 41 and using the fact that $u^{p_0}$ is doubling, since it is $A_2$, so that dashed lines in Figure 40 turn into solid ones (see Remark 5), the concatenation property yields $u \in RC(-\infty, -p_0) \cap RC(-p_0, p_0) \cap RC(p_0, \infty) = RC(-\infty, \infty)$.

9.2. Krylov-Safanov’s approach to Harnack’s inequality. As before, let $A(x)$ be a uniformly elliptic matrix satisfying (9.17). During the 1980’s, N. Krylov and M. Safonov took recourse to completely new measure-theoretic tools [30, 31] in order to establish Harnack’s inequality for positive solutions to the non-divergence-form elliptic equation

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = \text{tr}(A(x)D^2 u) = 0 \text{ in } \Omega \subset \mathbb{R}^n.$$  

Later on, L. Caffarelli’s seminal work [4] on fully non-linear elliptic equations (see also [5, Section 4.2]) greatly enriched and simplified Krylov-Safonov’s theory rendering it quite flexible and still manageable. This, in fact, paved the way for the axiomatizations of Krylov-Safonov’s theory in doubling quasi-metric spaces carried out in [1, 13, 24]. All these axiomatic approaches involve, implicitly or explicitly, the so-called critical density and power-like decay properties.

**Definition 34.** Let $\mathbb{K}_\Omega$ denote a family of non-negative measurable functions with domain contained in $\Omega$. If $u \in \mathbb{K}_\Omega$ and $A \subset \text{dom}(u)$ then we write $u \in \mathbb{K}_\Omega(A)$, where $\text{dom}(u)$ stands for the domain of the function $u$. Assume that $\mathbb{K}_\Omega$ is closed under multiplication by positive scalars.

Let $0 < \varepsilon < 1 \leq M$. $\mathbb{K}_\Omega$ is said to satisfy the **critical density property** with constants $\varepsilon$ and $M$ if for every $B_{2R}(x_0) \in B_\Omega$ and $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with

$$\text{ess inf}_{B_R(x_0)} u \leq 1,$$
it follows that

\[ \mu(\{x \in B_{2R}(x_0) : u(x) > M\}) \leq \epsilon \mu(B_{2R}(x_0)). \]

Let \(0 < \varrho < 1 < N \) . \( \mathbb{K}_\Omega \) is said to satisfy the power-like decay property with constants \(N\) and \(\varrho\) if for every \(B_{2R}(x_0) \in B_\Omega\) and every \(u \in \mathbb{K}_\Omega(B_{2R}(x_0))\) with

\[ \text{ess inf}_{B_{R}(x_0)} u \leq 1, \]

it follows that

\[ \mu(\{x \in B_{R/2}(x_0) : u(x) > N^k\}) \leq \varrho^k \mu(B_{R/2}(x_0)) \quad \forall k \in \mathbb{N}. \]

In the context of the non-divergence form elliptic operators (9.21), the critical density property plays the role analogous to the Moser iterations in the divergence-form setting. Indeed, let \( \mathbb{K}_\Omega \) denote the class of supersolutions of (9.21) (that is, \( u \in \mathbb{K}_\Omega \) if and only if \( Lu \leq 0 \) in \( \Omega \)), as a first step it is proved that \( \mathbb{K}_\Omega \) possesses the critical density property; see, for instance, Theorem 2.1.1 on page 31 of [19]. Then, it is proved that if for any given ball \( B \in B_\Omega \) and any scalar \( \lambda \) with \( \lambda - u > 0 \) in \( B \) we have \( \lambda - u \in \mathbb{K}_\Omega \), then \( u \in RC^{\text{weak}}(p, \infty) \) for every \( p > 0 \). See, for instance [24, Section 3]. Therefore, putting these two results together, if \( u \) is a positive subsolution \((Lu \geq 0)\), then \( L(\lambda - u) = -Lu \leq 0 \), so that \( \lambda - u \in \mathbb{K}_\Omega \) and, consequently, \( u \in RC^{\text{weak}}(p, \infty) \) for every \( p > 0 \). This is illustrated in Figure 42.

\[ \text{Figure 42. The critical density property implies that every positive subsolution } u \text{ (i.e. } Lu \geq 0) \text{ belongs to } RC^{\text{weak}}(p, \infty) \text{ for all } p > 0. \]

The next key step in Krylov-Safonov’s approach consists in showing that the class of positive supersolutions possesses the power-like decay property, see for instance Theorem 2.1.3 on page 36 of [19] and Lemma 4.6 on page 33 of [5]. Now, the fact that the class of supersolutions has the power-like decay property (and since it is closed under multiplication by positive constants) amounts to the existence of constants \(0 < \delta < 1 \leq C\), depending only on \(\varrho\), \(N\), \(\Lambda_2/\Lambda_1\), and dimension \(n\), such that every supersolution \(u\) satisfies the reverse inequality \(RC(-\infty, \delta, C)\) (see [24, Remark 2]); namely,

\[ (9.22) \quad \frac{1}{|B|} \int_B u^\delta \, dx \leq C^\delta \text{ ess inf}_{B} u^\delta \quad \forall B \in B_\Omega. \]
In other words, \( u^\delta \in A_1(\Omega) \). Inequality (9.22) is usually referred to as the \textit{weak Harnack inequality} for \( u \) and it is here illustrated in Figure 43.

![Figure 43](image)

There exists a structural \( \delta > 0 \) such that every positive supersolution \( u \) (i.e., \( Lu \leq 0 \)) satisfies \( u^\delta \in A_1 \). By the scaling property, this is equivalent to \( u \in RC(-\infty, \delta) \).

Finally, if \( u \) is a positive solution in \( \Omega \), then it is both a subsolution and a supersolution and by choosing the \( p \) in Figure 42 equal to the \( \delta \) in Figure 43 (and using the fact that \( u^\delta \) is doubling, since it is \( A_1 \), turns the dashed arrow in Figure 42 into a solid one), the concatenation property yields \( u \in RC(-\infty, \delta) \cap RC(\delta, \infty) = RC(-\infty, \infty) \).

10. \textbf{Further practice}

\textbf{Exercise 1.} Prove that the three axioms of the visual formalism for the reverse classes \( RC(r, s) \) also apply to the weak reverse classes \( RC^{\text{weak}}(r, s) \) and study the behavior of the weak-reversal constants.

\textbf{Exercise 2.} Use the visual formalism to prove that if \( w \in A_\infty \) and \( w^r \in A_1 \) for some \( 0 < r < \infty \), then \( w \in A_1 \). Estimate \( [w]_{A_1} \) in terms of \( [w]_{A_\infty} \) and \( [w^r]_{A_1} \) by considering the cases \( r > 1 \) and \( 0 < r < 1 \).

\textbf{Exercise 3.} Fix \( 1 < s < \infty \). Use the visual formalism to prove that \( w^s \in A_\infty \) implies \( w \in RH_s \) with \( [w]_{RH_s} \leq [w^s]_{A_\infty}^{1/s} \).

\textbf{Exercise 4.} Prove Theorem 30 using the visual formalism.

\textbf{Exercise 5.} Use the techniques depicted in Figures 26 and 28 to prove that every weight satisfying a reverse inequality “must cross the exponent zero”. That is, if \( w \in RC(r, s) \) for some \( -\infty \leq r < s < 0 \), then \( w \in RC(r, \varepsilon) \) for some \( \varepsilon > 0 \). Similarly, if \( w \in RC(r, s) \) for some \( 0 < r < s \leq \infty \), then \( w \in RC(-\varepsilon, s) \) for some \( \varepsilon > 0 \). Consequently, every reverse class \( RC(r, s) \) self-improves “to touch zero”. Meaning that whenever \( -\infty \leq r < s < 0 \), then \( RC(r, s) \subset RC(r, 0) \). Similarly, if \( 0 < r < s \leq \infty \), then \( RC(r, s) \subset RC(0, s) \).

\textbf{Exercise 6.} Adapt the arguments in Figures 27 and 28 to visually prove Corollary 26 (iii).

\textbf{Exercise 7.} Use Figures 24 and 31 to visually prove Corollary 31. In the same vain, use Figures 24 and 31 to visually prove that \( BMO = BUO - BUO \).
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