HARNACK’S INEQUALITY FOR SOLUTIONS TO THE
LINEARIZED MONGE-AMPÈRE OPERATOR WITH
LOWER-ORDER TERMS

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Abstract. Under minimal geometric assumptions, we prove a Harnack inequality for non-negative solutions to the non-homogeneous linearized Monge-Ampère equation with bounded lower-order terms compatible with the maximum principle.

1. Introduction and main result

Throughout this work, \( \varphi : \mathbb{R}^n \to \mathbb{R} \) will be a strictly convex, twice continuously differentiable function whose associated Monge-Ampère measure \( \mu_\varphi(x) := \det D^2 \varphi(x) \) satisfies \( \mu_\varphi(x) > 0 \) for every \( x \in \mathbb{R}^n \). The linearization of the Monge-Ampère operator \( v \mapsto \det D^2 v \) at \( \varphi \) renders the elliptic, typically degenerate, linear operator acting on, say, \( v \in W^{2,n}_{\text{loc}}(\mathbb{R}^n) \), as

\[
L_{\varphi}^{MA}(v)(x) := \text{trace}(A_{\varphi}(x)D^2v(x)) \quad \text{a.e. } x \in \mathbb{R}^n,
\]

where

\[
A_{\varphi}(x) := \mu_\varphi(x)D^2\varphi(x)^{-1} \quad \forall x \in \mathbb{R}^n
\]

is the matrix of cofactors of \( D^2\varphi(x) \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set. Given \( u \), a differentiable function in \( \Omega \), we define the gradient \( \nabla \varphi \) of \( u \) as

\[
\nabla \varphi u(x) := D^2\varphi(x)^{-\frac{1}{2}}\nabla u(x) \quad \forall x \in \Omega.
\]

We will consider the operator

\[
L_{\varphi}(u)(x) := \text{trace}(A_{\varphi}(x)D^2u(x)) + \langle b(x), \nabla \varphi u(x) \rangle \mu_\varphi(x) + c(x)u(x)\mu_\varphi(x),
\]

with the following assumptions on the coefficients for the lower-order terms

\[
|b|, c \in L^\infty(\Omega) \quad \text{and} \quad c \leq 0 \text{ in } \Omega.
\]

Our main result is the following (see Section 2 for the appropriate definitions)

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Theorem 1. (Harnack’s inequality). Assume $\mu_\varphi \in (DC)_\varphi$. There exists a geometric constant $0 < \rho < 1$ and a structural constant $C_H \geq 1$ and such that for every section $S_\varphi(z,t)$ with $S_\varphi(z,t) \subset \subset \Omega$ and every $u \in W^{2,n}_{\text{loc}}(S_\varphi(z,t)) \cap C(S_\varphi(z,t))$ non-negative solution to
\begin{equation}
\text{trace}(A_\varphi(x)D^2u(x)) + \langle b(x), \nabla \varphi u(x) \rangle \mu_\varphi(x) + c(x)u(x)\mu_\varphi(x) = f(x)\mu_\varphi(x),
\end{equation}
for a.e. $x \in S_\varphi(z,t)$, we have
\begin{equation}
\sup_{S_\varphi(z,\rho t)} u \leq C_H \left( \inf_{S_\varphi(z,\rho t)} u + \|f\|_{L^\infty(S_\varphi(z,t))} \right).
\end{equation}

Theorem 1 for non-negative solutions to (1.4) in the case $|b| \equiv c = f \equiv 0$ was proved by Caffarelli and Gutiérrez in [2] under the assumption that the Monge-Ampère measure $\mu_\varphi$ satisfies the so-called $(\mu_\infty)$-condition. This condition turns out to be significantly stronger than the $(DC)_\varphi$-doubling property, the gap being comparable to that between $A_\infty$ weights and doubling weights, see [5, Section 3] for a thorough discussion and examples. Under the hypothesis $\mu_\varphi \in (DC)_\varphi$ only, Theorem 1, always with $f \equiv |b| \equiv c \equiv 0$ was proved in [12]. As we will see in the development of this work, the arguments leading towards the proof of Theorem 1 in the presence of lower-order terms will require a novel approach to the so-called critical density estimate (Theorem 2), a precise quantification of the $C^\beta$-Hölder continuity of $\delta_\varphi(x,\cdot)$ with respect to $\delta_\varphi$ as in Lemma 4, a suitable first-order differential operator (namely, $\nabla \varphi$ above) for the first-order coefficient of $L_\varphi$, and a deeper understanding of the duality between the spaces of homogeneous type $(\mathbb{R}^n, \delta_\varphi, \mu_\varphi)$ and $(\mathbb{R}^n, \delta_\psi, |\cdot|)$ as well as their corresponding roles in the non-variational and variational sides of $L_\varphi$.

The article is organized as follows: in Section 2 we introduce notation as well as the main geometric and measure-theoretic objects associated to convex solutions to the Monge-Ampère equation. A brief comment on the version of the Alexandrov-Bakelman-Pucci maximum principle to be used is also included. In Section 3 we prove a critical density estimate (Theorem 2) involving positive sub-solutions to $L_\varphi^\alpha(u) = f$ (with $L_\alpha^\varphi$ as defined in (2.7)). Such critical-density estimates lie at the core of the proofs for Harnack’s inequality in the elliptic, non-variational setting. Theorem 2 extends Theorem 1 in Caffarelli-Gutiérrez [2] but its proof, however, somewhat deviates from the one for Theorem 1 in [2] in the construction of barrier functions and the avoidance of the normalization technique. In Sections 4 and 5 we use the critical-density estimate to obtain mean-value inequalities for positive and negative powers of sub- and super-solutions to $L_\alpha^\varphi(u) = f$. By resorting to the variational side of $L_\varphi$, in Section 6 we uniformly compare averages of some positive and negative powers of super-solutions to $L_\varphi(u) = f$ by means of a local-BMO estimate. In turn, the latter estimate follows from convex conjugation and a Poincaré-type inequality proved in [12] within the Monge-Ampère quasi-metric structure. Finally, in Section 7 we combine the results from previous sections, fine-tune their relevant constants, and render a proof of Theorem 1.
The reader is referred to [2, 5] and references therein for an introduction to the linearized Monge-Ampère operator as well as its connections with topics of fluid dynamics and differential geometry.

2. Notation and some background

Fix $0 < \Lambda_1 \leq \Lambda_2 < \infty$ and for each $x \in \Omega$ let $A(x)$ be an $n \times n$ real symmetric matrix. We write $A \in E(\Lambda_1, \Lambda_2, D^2 \varphi, \Omega)$ if the map $x \mapsto A(x)$ is continuous in $\Omega$ and for every $\xi \in \mathbb{S}^{n-1}$ (the Euclidean unit sphere in $\mathbb{R}^n$) we have

$$\Lambda_1 \leq \langle D^2 \varphi(x)^{1/2} A(x) D^2 \varphi(x)^{1/2} \xi, \xi \rangle \leq \Lambda_2 \quad \forall x \in \Omega. \tag{2.6}$$

That is, in the sense of positive-definite matrices, (2.6) means that

$$\Lambda_1 I \leq (D^2 \varphi)^{1/2} A (D^2 \varphi)^{1/2} \leq \Lambda_2 I \quad \text{in } \Omega.$$

For $A \in E(\Lambda_1, \Lambda_2, D^2 \varphi, \Omega)$, in Sections 3, 4, and 5 we will consider the elliptic, typically degenerate, non-divergence form operator $L_A^\varphi$ defined for $u \in W^{2,n}_{loc}(\Omega)$ by

$$L_A^\varphi(u)(x) := \text{trace}(A(x) D^2 u(x)) + \langle b(x), \nabla^2 u(x) \rangle + c(x)u(x) \quad \text{a.e. } x \in \Omega. \tag{2.7}$$

We say that $L_A^\varphi$ is adapted to $\varphi$ because the geometric and measure theoretic objects to study $L_A^\varphi$ are determined by the convex function $\varphi$. Indeed, define

$$\delta\varphi(x, y) := \varphi(y) - \varphi(x) - \langle \nabla \varphi(x), y - x \rangle \quad \forall x, y \in \mathbb{R}^n. \tag{2.8}$$

Given $x \in \mathbb{R}^n$ and $t > 0$, a section of $\varphi$ centered at $x$ with height $t$ is the open bounded convex set

$$S_{\varphi}(x, t) := \{ y \in \mathbb{R}^n : \delta\varphi(x, y) < t \}.$$

Let us also introduce the non-negative function

$$D_{\varphi}(x, y) := \langle D^2 \varphi(x)^{-1}(\nabla \varphi(x) - \nabla \varphi(y)), (\nabla \varphi(x) - \nabla \varphi(y)) \rangle \quad \forall x, y \in \mathbb{R}^n. \tag{2.9}$$

Associated to the functions $\delta\varphi$ and $D_{\varphi}$ we have the following notions for the diameter of $\Omega \subset \mathbb{R}^n$

$$\text{diam}_{\varphi}(\Omega) := \sup_{z, z' \in \Omega} \delta\varphi(z, z') \tag{2.10}$$

and

$$\text{diam}^2(\Omega) := \sup_{z, z' \in \Omega} D_{\varphi}(z, z'). \tag{2.11}$$

In particular, if for some section we have $S_{\varphi}(x_0, t) \subset \subset \Omega$, there is $y \in \Omega$ with $\delta\varphi(x_0, y) > t$ and therefore

$$\text{diam}_{\varphi}(\Omega) > t. \tag{2.12}$$

Notice that with the special choice $\varphi(x) := \varphi_2(x) := \frac{1}{2} |x|^2$ we recover the usual class of uniformly elliptic matrices $E(\Lambda_1, \Lambda_2, I, \Omega)$, $L_{\varphi_2} = \Delta + \langle b, \nabla \cdot \rangle + c$, as $\nabla^2 \varphi$ becomes the usual gradient. Also, the measure $d\mu_{\varphi_2}$ reduces to Lebesgue measure and

$$D_{\varphi_2}(x, y) = 2\delta_{\varphi_2}(x, y) = |x - y|^2 \quad \forall x, y \in \mathbb{R}^n,$$
so that, for every $x \in \mathbb{R}^n$ and $r > 0$ the Euclidean balls are given by

$$B(x, r) = S_{\varphi_2}(x, r^2/2).$$

2.1. The $(\text{DC})_{\varphi}$-doubling property. For the sake of introducing notation, we briefly recall some of conditions on $\mu_{\varphi}$ (references will be provided as needed). The following are equivalent (see, for instance, [2], Theorem 8 in [3], Theorem 4 in [4], [8], and [7, Chapter 3]):

(i) There exist constants $K_{\text{sym}}, K_{\text{trian}} \geq 1$ such that

$$\frac{\delta_{\varphi}(x, y)}{K_{\text{sym}}} \leq \delta_{\varphi}(y, x) \leq K_{\text{sym}} \delta_{\varphi}(x, y) \quad \forall x, y \in \mathbb{R}^n$$

and

$$\delta_{\varphi}(x, y) \leq K_{\text{trian}}(\delta_{\varphi}(z, x) + \delta_{\varphi}(z, y)) \quad \forall x, y, z \in \mathbb{R}^n.$$

(ii) $\mu_{\varphi}$ possesses the $(\text{DC})_{\varphi}$ doubling property, that is, there exist constants $B \geq 1$ and $0 < a_0 < 1$

$$\mu_{\varphi}(S_{\varphi}(x, t)) \leq B \mu_{\varphi}(a_0 S_{\varphi}(x, t)) \quad \forall x \in \mathbb{R}^n, \forall t > 0,$$

where, for a convex set $S$, $a_0 S$ denotes its $a_0$-contraction with respect to its (Euclidean) center of mass.

(iii) There exist constants $0 < K_1 < K_2$ such that

$$K_1^n t^n \leq |S_{\varphi}(x, t)| \mu_{\varphi}(S_{\varphi}(x, t)) \leq K_2^n t^n \quad \forall x \in \mathbb{R}^n, \forall t > 0,$$

where $|S|$ denotes the Lebesgue measure of a set $S \subset \mathbb{R}^n$.

The statements above are quantitative in the sense that all the constants depend only one another and dimension $n$.

Throughout this work, the relevant compatibility condition between the sections of $\varphi$ and its Monge-Ampère measure will be the $(\text{DC})_{\varphi}$ doubling property in (2.15) and we will indicate it by writing $\mu_{\varphi} \in (\text{DC})_{\varphi}$.

2.2. Geometric and structural constants. Constants depending only on the dimension $n$ and the parameters $B$ and $a_0$ in (2.15) will be called geometric constants. Notice that (2.13) and (2.14) imply that whenever $\mu_{\varphi} \in (\text{DC})_{\varphi}$, there exists a geometric constant $K \geq 1$ (for instance, one could choose $K := K_{\text{sym}} K_{\text{trian}}$) such that the following symmetrized $K$-quasi-triangle inequality holds true for every $x, y, z \in \mathbb{R}^n$

$$\delta_{\varphi}(x, y) \leq K \left( \min\{\delta_{\varphi}(z, x), \delta_{\varphi}(x, z)\} + \min\{\delta_{\varphi}(z, y), \delta_{\varphi}(y, z)\} \right).$$

Constants depending on geometric constants as well as on $\Lambda_1, \Lambda_2, \mu_{\varphi}(\Omega), \text{diam}_{\varphi}(\Omega), \text{diam}^1(\Omega), \|c\|_{L^\infty(\Omega)}, \|(D^2 \varphi)^{-\frac{1}{2}} b\|_{L^n(\Omega, d\mu_{\varphi})}$, and $\|b\|_{L^\infty(\Omega)}$ will be called structural constants.
2.3. The ABP maximum principle. Given \( A \in E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega) \) (which implies that \( \Lambda_1^2 \mu_\varphi^{-1} \leq \det A \leq \Lambda_2^2 \mu_\varphi^{-1} \)) and a section \( S := S_\varphi(x_0, t) \subset \Omega \), we will use the fact that whenever \( h \in W^{2, n}_{\text{loc}}(S) \cap C(\overline{S}) \) is a solution to \( L_A^\varphi(h) = H \) in \( S \) with \( h \leq 0 \) on \( \partial S \), then, by the Aleksandrov-Bakelman-Pucci (ABP) maximum principle (see Theorem 9.1 on page 220 of [6]), and since the first-order term equals \( \langle b, \nabla \varphi u \rangle = \langle (D^2 \varphi)^{-\frac{1}{2}} b, \nabla u \rangle \), we have

\[
\sup_S |h| \leq C_{\text{abp}} |S|^{1/n} \left( \int_S |H|^n \, d\mu_\varphi \right)^{1/n},
\]

where \( C_{\text{abp}} > 0 \) depends only on \( \Lambda_1, \Lambda_2, \) dimension \( n \) and \( \int_S |(D^2 \varphi)^{-\frac{1}{2}} b|^n \mu_\varphi \) (see inequality (9.14) on page 224 of [6]) and, in particular, it does not depend on the zeroth-order term \( c \). Now, since \( S \subset \Omega \), the constant \( C_{\text{abp}} > 0 \) can be regarded as a structural constant depending only on \( \Lambda_1, \Lambda_2, \) dimension \( n \), \( \mu_\varphi(\Omega) \), and \( \|(D^2 \varphi)^{-\frac{1}{2}} b\|_{L^n(\Omega, d\mu_\varphi)} \).

The fact that \( |S|^{1/n} \), instead of diam(\( S \)) appears in (2.18), is due to \( S \) being a bounded, convex set (see, for instance, [5, Lemma 8]). Also, notice that the assumption \( c \leq 0 \) allows us to use the following \textit{weak maximum principle}: If \( L_A^\varphi(h) \leq 0 \) in an open set \( \mathcal{O} \subset \Omega \) with \( h \geq 0 \) on \( \partial \mathcal{O} \), then \( h \geq 0 \) in \( \mathcal{O} \) (see Theorem 9.6 on page 225 of [6]).

3. A critical density estimate when \( L_A^\varphi(u) \leq f \) and \( u > 0 \)

The next theorem extends, by different methods, Theorem 1 in [2], where the case \( f \equiv |b| \equiv c \equiv 0 \) was addressed.

**Theorem 2.** (Critical density) Assume \( \mu_\varphi \in (\text{DC})_\varphi \). Let \( L_A^\varphi \) be as in (2.7) with \( A \in E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega) \). Let \( \delta > 0 \) and \( M_0 > 1 \) be the structural constants defined by

\[
\delta := \frac{1}{n\Lambda_2 + \text{diam}_\varphi(\Omega)\|c\|_{L^\infty(\Omega)} + \text{diam}^\varphi(\Omega)^\frac{1}{2}\|b\|_{L^\infty(\Omega)}},
\]

and

\[
M_0 := 2C_{\text{abp}}K_2 > 1,
\]

where \( C_{\text{abp}} \) is as in (2.18) and \( K_2 \) is as in (2.16). Given any \( \sigma, \tau \in (0, 1) \) let \( \varepsilon_0, \varepsilon_1 \in (0, 1) \) be structural constants \( (\varepsilon_0 \text{ close to } 1 \text{ and } \varepsilon_1 \text{ close to } 0) \), depending only on \( \delta, \sigma, \tau, C_{\text{abp}}, K_2, \) and \( n \), such that

\[
C_{\text{abp}}K_2 \left( \varepsilon_1 + (1 - \varepsilon_0)^{1/n} \right) := \delta(1 - \tau)(1 - \sigma).
\]

Then, given any section \( S_\varphi(x_0, t) \subset \subset \Omega \), any \( f \in L^n(S_\varphi(x_0, t), d\mu_\varphi) \) and any \( u \in W^{2, n}_{\text{loc}}(S_\varphi(x_0, t)) \cap C(\overline{S_\varphi(x_0, t)}) \) satisfying \( u \geq 0 \) and \( L_A^\varphi(u) \leq f \) in \( S_\varphi(x_0, t) \), we have that for every \( \lambda \geq 1 \) and \( \sigma, \tau \in (0, 1) \), the inequalities

\[
\inf_{S_\varphi(x_0, t)} u \leq \sigma \delta(1 - \tau)\lambda
\]
and

\[(3.23) \quad \frac{t}{\mu(S_\varphi(x_0, t))^{1/n}} \|f\|_{L^n(S_\varphi(x_0, t), d\mu_\varphi)} \leq \varepsilon_1 \]

imply that

\[(3.24) \quad \frac{\mu_\varphi(\{ y \in S_\varphi(x_0, t) : u(y) \geq \lambda M_0 \})}{\mu_\varphi(S_\varphi(x_0, t))} \leq \varepsilon_0, \]

where \(\varepsilon_0, \varepsilon_1 \in (0, 1)\) are as in \((3.21)\).

\textbf{Proof.} First assume that \(x_0 = 0, \varphi(0) = 0,\) and \(\nabla \varphi(0) = 0.\) For \(\tau \in (0, 1),\) set \(S_\tau := S_\varphi(0, \tau t)\) so that \(x \in S_\tau\) if and only if \(\varphi(x) < \tau t.\) Also, put \(S := S_1 := S_\varphi(0, t) \subset \subset \Omega.\) Notice that the problem

\[(3.25) \quad \left\{ \begin{array}{ll} L^\varphi_A(h) = H & \text{in } S \\ h = 0 & \text{on } \partial S, \end{array} \right. \]

is always solvable for \(H \in L^n(S, d\mu_\varphi)\) with a unique \(h \in W^{2,n}_{loc}(S) \cap C(\overline{S})\) because, by assumption, the elliptic matrix \(A(x)\) in \(L^\varphi_A\) has continuous coefficients in \(\Omega\) and the lower-order terms satisfy (1.3) (see [6, Section 9.6]). Here we make use of the fact that the eigenvalues of \(A(x)\) are comparable to those of \((D^2 \varphi(x))^{-1}\) and they will be bounded and bounded away from zero on compact subsets of \(\Omega.\) Notice that we use this fact just to deduce the existence of solutions, however we are allowed to use apriori estimates for the solutions, such as (2.18), only when they depend on the eigenvalues of \(A(x)\) through \(\det A(x).\) Throughout the article, the only admissible gauge for the ellipticity of \(A(x)\) is provided by the \((DC)_{\varphi}\)-doubling property or, equivalently, inequalities (2.16).

Now let \(f \in L^n(S, d\mu_\varphi)\) and \(u \in C(S) \cap W^{2,n}_{loc}(S)\) with \(L^\varphi_A(u) \leq f\) in \(S\) be as in the statement of Theorem 2. Put

\[Q(f) := \frac{t}{\mu_\varphi(S)^{1/n}} \|f\|_{L^n(S, d\mu_\varphi)}.\]

By hypothesis, \(Q(f) < \varepsilon_1 < 1.\) By contrapositive, let us suppose that \((3.24)\) does not hold, that is, suppose that

\[(3.26) \quad \frac{\mu_\varphi(\{ y \in S : u(y) \geq \lambda M_0 \})}{\mu_\varphi(S)} > \varepsilon_0 \]

and prove

\[(3.27) \quad \inf_{S_\tau} u > \sigma \delta (1 - \tau) \lambda.\]

From \((3.26)\) and the left continuity of the function \(\epsilon \rightarrow \mu_\varphi(S_\epsilon)\) at \(\epsilon = 1,\) there exists \(\epsilon^* \in (0, 1)\) (close to 1 and certainly not a structural constant, but this will be irrelevant) such that the closed set

\[\Gamma := \{ y \in \overline{S_{\epsilon^*}} : u(y) \geq \lambda M_0 \} \subset S\]
satisfies
\[(3.28)\quad \mu_\varphi(\Gamma) > \varepsilon_0 \mu_\varphi(S)\).

Letting \(\chi_E\) denote the characteristic function of a set \(E \subset \mathbb{R}^n\), let \(w\) solve
\[
\begin{cases}
L_\varphi^\tau(w) = t|f| - \chi_\Gamma & \text{in } S \\
w = 0 & \text{on } \partial S.
\end{cases}
\]

Now, by the ABP maximum principle (2.18) and (2.16), for every \(x \in S\) we have
\[
|w(x)| \leq C_{abp}|S|^{1/n} \left(t\|f\|_{L^n(S, \mu_\varphi)} + \mu_\varphi(\Gamma)^{1/n}\right) \\
= C_{abp}|S|^{1/n} \mu_\varphi(S)^{1/n} \left(Q(f) + \left(\frac{\mu_\varphi(\Gamma)}{\mu_\varphi(S)}\right)^{1/n}\right) \\
\leq C_{abp}K_2 t(Q(f) + 1).
\]

In particular, for \(x \in \Gamma \subset S\), from the definition of \(M_0\) in (3.20) we have
\[
\frac{1}{t} w(x) \leq C_{abp}K_2(Q(f) + 1) \leq 2C_{abp}K_2 = M_0 \leq \frac{1}{\lambda} u(x).
\]

Thus,
\[(3.29)\quad \frac{u(x)}{\lambda} - \frac{w(x)}{t} \geq 0 \quad \forall x \in \Gamma,
\]
and, consequently, \(u(x)/\lambda - w(x)/t \geq 0\) on \(\partial \Gamma\). Also, for every \(x \in \partial S\) we have \(u(x)/\lambda - w(x)/t = u(x)/\lambda \geq 0\). In addition, for every \(x \in S \setminus \Gamma\),
\[
L_\varphi^\tau(u/\lambda - w/t)(x) = L_\varphi^\tau(u/\lambda)(x) - L_\varphi^\tau(w/t)(x) \\
\leq \frac{1}{\lambda} f(x) - (|f(x)| - \frac{1}{t} \chi_\Gamma(x)) \\
\leq \left(\frac{1}{\lambda} - 1\right) |f(x)| + \frac{1}{t} \chi_\Gamma(x) \leq \frac{1}{t} \chi_\Gamma(x) = 0.
\]

That is, on the open set \(S \setminus \Gamma\) we have
\[
\begin{cases}
L_\varphi^\tau(u/\lambda - w/t) \leq 0 & \text{in } S \setminus \Gamma \\
u/\lambda - w/t \geq 0 & \text{on } \partial(S \setminus \Gamma).
\end{cases}
\]

Thus, from the weak maximum principle (see Section 2.3), it follows that \(u/\lambda - w/t \geq 0\) in \(S \setminus \Gamma\) and, along with (3.29), this yields
\[(3.30)\quad u(x)/\lambda - w(x)/t \geq 0 \quad \forall x \in S.
\]

Next we want to establish a lower bound of the form
\[
\frac{w(x)}{t} \geq \gamma \quad \forall x \in S_\tau,
\]
for a structural constant \(\gamma \in (0, 1)\). Let \(v\) solve
\[
\begin{cases}
L_\varphi^\tau(v) = -t|f| - \chi_{S \setminus \Gamma} & \text{in } S \\
v = 0 & \text{on } \partial S.
\end{cases}
\]
As before, by the ABP maximum principle (2.18) and (2.16), but now using (3.28), for every \( x \in S \) we have
\[
|v(x)| \leq C_{\text{abp}} |S|^{1/n} (t \|f\|_{L^n(S, \mu_\phi)} + \mu_\phi(S \setminus \Gamma)^{1/n})
\]
\[
= C_{\text{abp}} |S|^{1/n} \mu_\phi(S)^{1/n} \left( Q(f) + \left( \frac{\mu_\phi(S)}{\mu_\phi(S)} - \frac{\mu_\phi(\Gamma)}{\mu_\phi(S)} \right)^{1/n} \right)
\]
\[
< C_{\text{abp}} |S|^{1/n} \mu_\phi(S)^{1/n} \left( Q(f) + (1 - \varepsilon_0)^{1/n} \right)
\]
\[
\leq C_{\text{abp}} K_2 t \left( Q(f) + (1 - \varepsilon_0)^{1/n} \right).
\]

Consequently,
\[
(3.31) \quad \frac{|v(x)|}{t} < C_{\text{abp}} K_2 \left( Q(f) + (1 - \varepsilon_0)^{1/n} \right) \quad \forall x \in S.
\]

For \( \delta > 0 \) as in (3.19), define the barrier function
\[
(3.32) \quad q(x) := w(x) + v(x) + \delta (\varphi(x) - t) \quad \forall x \in \overline{S}.
\]

For every \( x \in S \) we have
\[
L_A^\varphi(q)(x) = L_A^\varphi(w)(x) + L_A^\varphi(v)(x) + \delta L_A^\varphi(\varphi - t)(x)
\]
\[
= t |f(x)| - \chi_{\Gamma}(x) - t |f(x)| - \chi_S \setminus \Gamma(x)
\]
\[
+ \delta \text{trace}(A(x) D^2 \varphi(x)) + \delta \langle b(x), \nabla^2 \varphi(x) \rangle + \delta (\varphi(x) - t) c(x)
\]
\[
= -1 + \delta (\varphi(x) - t) c(x) + \delta \langle b(x), D^2 \varphi(x)^{-1/2} (\nabla \varphi(x) - \nabla \varphi(0)) \rangle
\]
\[
+ \delta \text{trace}(A(x) D^2 \varphi(x))
\]
\[
\leq -1 + \delta t \|c\|_{L^\infty(S)} + \delta \|b\|_{L^\infty(S)} D_x \varphi(x, 0)^{1/2} + \delta n \Lambda_2
\]
\[
\leq -1 + \delta \left( n \Lambda_2 + \text{diam}_\varphi(\Omega) \|c\|_{L^\infty(\Omega)} + \text{diam}_\varphi(\Omega)^{1/2} \|b\|_{L^\infty(\Omega)} \right) = 0,
\]

where the last equality is due to the definition of \( \delta \) in (3.19). That is, we obtained \( L_A^\varphi(q) \leq 0 \) in \( S \). On the other hand, from (3.32) we see that \( q = 0 \) on \( \partial S \). The weak maximum principle then implies that \( q \geq 0 \) in \( S \), which means
\[
w(x) + v(x) \geq \delta (t - \varphi(x)) \quad \forall x \in S.
\]

In particular, (recall that \( x \in S_\tau \) if and only if \( \varphi(x) < \tau t \)),
\[
(3.33) \quad w(x) + v(x) \geq \delta (1 - \tau) t \quad \forall x \in S_\tau.
\]
Now, using (3.30), (3.33) and (3.31), for every $x \in S_\tau$ we obtain
\[
\frac{u(x)}{\lambda} \geq \frac{w(x) - v(x)}{t} = \frac{w(x) + v(x) - v(x)}{t} \geq \delta(1 - \tau) - \frac{v(x)}{t}
\]
\[
> \delta(1 - \tau) - C_{\text{abs}}K_2 \left( Q(f) + (1 - \varepsilon_0)^{1/n} \right)
\]
\[
\geq \delta(1 - \tau) - C_{\text{abs}}K_2 \left( \varepsilon_1 + (1 - \varepsilon_0)^{1/n} \right) =: \gamma.
\]

Then, the choice of $\varepsilon_0$ and $\varepsilon_1$ as in (3.21) yields $\gamma = \sigma \delta(1 - \tau)$ and (3.27) follows.

For arbitrary $x_0 \in \Omega$ and section $S_{\varphi}(x_0, t) \subset \subset \Omega$, define
\[
\varphi_{x_0}(x) := \varphi(x_0 - x) - \varphi(x_0) + \langle \nabla \varphi(x_0), x \rangle \quad \forall x \in \mathbb{R}^n.
\]

Then $\mu_{\varphi_{x_0}} \in (\text{DC})_{\varphi_{x_0}}$ with the same constants as $\mu_{\varphi}$ does (uniformly in $x_0$). Also
\[
\nabla \varphi_{x_0}(0) = 0, \quad \varphi_{x_0}(0) = 0, \quad \text{and}
\]
\[
S_{\varphi_{x_0}}(0, t) = x_0 - S_{\varphi}(x_0, t).
\]

Thus, $\mu_{\varphi_{x_0}}(S_{\varphi_{x_0}}(0, t)) = \mu_{\varphi}(S_{\varphi}(x_0, t))$ and if we now apply the obtained result to $\varphi_{x_0}$ in the open set $x_0 - \Omega$, we obtain the general case just by changing variables $x_0 - x \mapsto x$. \hfill \square

4. **Mean-value inequalities when $L_A^2(u) \geq f$ and $u > 0$**

The main result in this section is the following mean-value inequality for subsolutions $L_A^2(u) \geq f$.

**Theorem 3.** Assume $\mu_{\varphi} \in (\text{DC})_{\varphi}$. Let $A \in E(\Lambda_1, \Lambda_2, D^2 \varphi, \Omega)$. There exist structural constants $C_7, C_8 > 0$ such that for every section $S_{\varphi}(z, t)$ with $S_K := S_{\varphi}(z, 2Kt) \subset \subset \Omega$ whenever $u \in W_{\text{loc}}^{2,n}(S_K) \cap C(S_K)$ and $f \in L^n(S_K, d\mu_{\varphi})$ satisfy $L_A^2(u) \geq f$ and $u \geq 0$ in $S_K$, we have

\[
(4.34) \quad \sup_{S_{\varphi}(z, t/2)} u \leq \frac{C_7}{\mu_{\varphi}(S_{\varphi}(z, 2Kt))} \int_{S_{\varphi}(z, 2Kt)} u \, d\mu_{\varphi} + \frac{C_8 t}{\mu_{\varphi}(S_{\varphi}(z, t))^{1/n}} \|f\|_{L^n(S_{\varphi}(z, 2Kt), d\mu_{\varphi})}.
\]

The proof of Theorem 3 will be based on Lemma 5 below, which is modeled after Lemma 15 in [5] where the cases $f \equiv |b| \equiv c \equiv 0$ were considered (notice the misprints on lines 16 and 19 on page 264 of [5] where the inequalities should be reversed). In turn, Lemma 15 in [5] stems from Lemma 4.1 in [2] where infima, instead of averages, are used. Both Lemma 15 in [5] and Lemma 4.1 in [2] are based on the normalization technique (see Section 1 in [2]) and the so-called Property (B) (as stated on page 446 of [2]) for normalized sections. However, due to the presence of the first-order term in $L_A^2$, it will be necessary to prove Theorem 3 without resorting to normalization. This is so because after using the normalization technique to switch from $u(x)$ to $\bar{u}(y) := u(T^{-1}y)$, where $T$ is an affine transformation that normalizes a given section.
S, the conditions on the coefficients of the “normalized” version of $L^r_{\lambda}$ will no longer be structural as the term $\langle b(x), \nabla^r u(x) \rangle$ in (2.7) will turn (following the notation in [2, Section 1]) into $\langle \bar{b}(y), \nabla^r \bar{u}(y) \rangle$ with $\bar{b}(y) := D^2 \varphi^*(y) \frac{1}{2} T(\lambda T \varphi(y))^{-\frac{1}{2}} b(Ty)$. Consequently, we lose control of $\| \bar{b} \|_{L^\infty(S^*)}$ as a structural constant. Indeed, to the best of our knowledge, no pointwise estimates for $\| D^2 \varphi^*(y) \frac{1}{2} T(\lambda T \varphi(y))^{-\frac{1}{2}} \|$, in terms of $\lambda$ and geometric constants, are known under just $\mu_{\varphi} \in (DC)_\varphi$. For this reason we put aside the normalization technique and provide a detailed account on how to prove Theorem 3 by means of real analysis techniques in spaces of homogeneous type. Our substitute for Property (B) will be the Lemma 4 below which, for each fixed $x \in \mathbb{R}^n$, quantifies the Hölder continuity of the function $\delta_{\varphi}(x, \cdot)$ with respect to $\delta_{\varphi}$.

**Lemma 4.** Assume $\mu_{\varphi} \in (DC)_\varphi$. There exist geometric constants $C_2 \geq 1$ and $\beta \in (0, 1]$ such that for every $x \in \mathbb{R}^n$ the function $\delta_{\varphi}(x, \cdot) : \mathbb{R}^n \to [0, \infty)$, with $\delta_{\varphi}$ as in (3.19), is locally $\beta$-Hölder continuous with the following estimate

$$
|\delta_{\varphi}(x, z) - \delta_{\varphi}(x, y)| \leq C_2 \max_{\beta} (\delta_{\varphi}(x, y) + \delta_{\varphi}(x, z))^{1-\beta} \quad \forall x, y, z \in \mathbb{R}^n.
$$

**Proof.** Let us start by mentioning that, from Theorem 2.4(i) in [8], there exist geometric constants $c_0 > 0$ and $p_1 \geq 1$ such that if $0 < r_0 < s_0 \leq 1, t > 0$, and $x_1 \in S_{\varphi}(x_0, r_0 t)$ (for any $x_0, x_1 \in \mathbb{R}^n$) then

$$
S_{\varphi}(x_1, c_0 (s_0 - r_0)^{p_1} t) \subset S_{\varphi}(x_0, s_0 t).
$$

Now, given $x, y, z \in \mathbb{R}^n$ suppose $0 < \delta_{\varphi}(x, y) < \delta_{\varphi}(x, z)$ and then set $t := \delta_{\varphi}(x, z) > 0$ and $r := \delta_{\varphi}(x, y)/t \in (0, 1)$. Let $\varepsilon > 0$ small enough so that $r < r + \varepsilon < 1$. By definition of $r$ we have that $y \in S_{\varphi}(x, (r + \varepsilon)t)$ (since this only means $\delta_{\varphi}(x, y) < (r + \varepsilon)t = \delta_{\varphi}(x, y) + \varepsilon t)$. We now claim that

$$
c_0 (1 - (r + \varepsilon))^{p_1} t < \delta_{\varphi}(y, z) + \varepsilon.
$$

Indeed, if we had $\delta_{\varphi}(y, z) + \varepsilon \leq c_0 (1 - (r + \varepsilon))^{p_1} t$ instead, then by (3.36) applied with $x_1 := y, x_0 := y$ and $0 < r_0 := r + \varepsilon < s_0 := 1$, we would have

$$
z \in S_{\varphi}(y, \delta(y, z) + \varepsilon) \subset S_{\varphi}(y, c_0 (1 - (r + \varepsilon))^{p_1} t) \subset S_{\varphi}(x, t),
$$

which would imply $z \in S_{\varphi}(x, t)$, that is, $\delta_{\varphi}(x, z) < t$; but $t = \delta_{\varphi}(x, z)$, a contradiction. That is, (4.37) holds for every $\varepsilon > 0$ and by taking limits as $\varepsilon \to 0^+$, we obtain

$$
c_0 (1 - r)^{p_1} t \leq \delta_{\varphi}(y, z).
$$

Next, define $\beta$ as the geometric constant

$$
\beta := \frac{1}{p_1} \in (0, 1)
$$

and put $\gamma := c_0 (1 - r)^{p_1} > 0$ to write

$$
\delta_{\varphi}(x, z) - \delta_{\varphi}(x, y) = t - rt = t(1 - r) = \frac{1}{c_0} t^{1-\beta}(t \gamma)^{\beta}.
$$
From (4.38) and (2.17) we have
\[(t\gamma)^\beta \leq \delta_\varphi(y, z)^\beta \leq K^\beta (\delta(x, y) + \delta_\varphi(x, z))^\beta.\]
Thus, by taking convex combinations of exponents, for every \(\alpha \in (0, 1)\) we have
\[(t\gamma)^\beta \leq \delta_\varphi(y, z)^{\alpha\beta} K^{(1-\alpha)\beta} (\delta(x, y) + \delta_\varphi(x, z))^{(1-\alpha)\beta} = \delta_\varphi(y, z)^{\alpha\beta} K^{(1-\alpha)\beta} (rt + t)^{(1-\alpha)\beta} = K^{(1-\alpha)\beta} \delta_\varphi(y, z)^{\alpha\beta} t^{(1-\alpha)\beta} (r + 1)^{1-\alpha\beta} \leq K^{(1-\alpha)\beta} \delta_\varphi(y, z)^{\alpha\beta} t^{(1-\alpha)\beta} (r + 1)^{1-\alpha\beta}.

Therefore, for every \(\alpha \in (0, 1)\), we can write
\[
\delta_\varphi(x, z) - \delta_\varphi(x, y) \leq \frac{K^{(1-\alpha)\beta}}{c_0^\beta} t^{1-\alpha\beta} \delta_\varphi(y, z)^{\alpha\beta} t^{(1-\alpha)\beta} (r + 1)^{1-\alpha\beta} = \frac{K^{(1-\alpha)\beta}}{c_0^\beta} \delta_\varphi(y, z)^{\alpha\beta} t^{1-\alpha\beta} (r + 1)^{1-\alpha\beta} = \frac{K^{(1-\alpha)\beta}}{c_0^\beta} \delta_\varphi(y, z)^{\alpha\beta} (\delta_\varphi(x, y) + \delta_\varphi(x, z))^{1-\alpha\beta},
\]
and taking limits as \(\alpha \to 1^-\) yields (4.35) with \(C_2 := 1/c_0^\beta\), a geometric constant. \(\Box\)

At this point we mention (see \([7, \text{Corollary 3.3.2}]\)) that whenever \(\mu_\varphi \in (DC)_\varphi\), there exists a geometric constant \(C_1 > 1\) such that
\[(4.40) \quad \mu_\varphi(S_\varphi(x, t)) \leq C_1 \mu_\varphi(S_\varphi(x, t/2)) \quad \forall x \in \mathbb{R}^n, \forall t > 0.
\]
Define the geometric constant \(\zeta > 0\) as
\[(4.41) \quad \zeta := \log_2(C_1) > 0.
\]
Then, (4.40) implies that for every \(x_0 \in \mathbb{R}^n\) and \(0 < R_1 < R_2\) we have
\[(4.42) \quad \mu_\varphi(S_\varphi(x_0, R_2)) \leq C_1 \left(\frac{R_2}{R_1}\right)^\zeta \mu_\varphi(S_\varphi(x_0, R_1)).\]
In particular, whenever we have \(S_\varphi(z, t) \subset S_\varphi(x_0, 2Kt)\) (with \(K\) always as in (2.17)) and \(0 < s < t\) it follows that
\[(4.43) \quad \mu_\varphi(S_\varphi(x_0, s)) \geq \frac{1}{(2K)^\zeta C_1} \left(\frac{s}{t}\right)^\zeta \mu_\varphi(S_\varphi(z, t)).\]
The setup is now ready. After proving Lemma 5 below, the proof of Theorem 3 will follow along the lines of Lemma 1 and Theorem 6 in [9] and a fine tuning of the critical density estimate in Theorem 2.

**Lemma 5.** Assume that \(\mu_\varphi \in (DC)_\varphi\). Let \(A \in E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega)\) and let \(M_0 > 1\) be the structural constant in (3.20). Let \(\varepsilon_0, \varepsilon_1 \in (0, 1)\) be the structural constants in (3.21) corresponding to the values \(\sigma = \tau = 1/2\) and let \(\lambda\) (structural as well) be defined
by \( \lambda := 4/\delta \) with \( \delta \in (0, 1) \) as in (3.19). (Notice that, with these choices, the product \( \sigma \delta (1 - \tau) \lambda \) in (3.22) equals 1). Also structural are the constants

\[
M_1 := \lambda M_0 > 1,
\]

\[
\nu := \frac{2M_1}{2M_1 - 1} > 1,
\]

and

\[
c_3 := \frac{(1 - \varepsilon_0)^{1/n}}{(2C_1(2K)^{\xi})^{1/n}}.
\]

Fix a section \( S := S_\varphi(z, t) \subset \Omega \) with \( S_K := S_\varphi(z, 2Kt) \subset \subset \Omega \) and let \( u \in W_{\text{loc}}^2(S_K) \cap C(S_K) \) and \( f \in L^n(S_K, d\mu_\varphi) \) such that \( u \geq 0 \) and \( L_\varphi A(u) \geq f \) in \( S_K \) with

\[
\frac{t}{\mu(S_\varphi(z, t))^{1/n}} \| f \|_{L^n(S_\varphi(z, 2Kt), d\mu_\varphi)} \leq \frac{\varepsilon_1}{2c_3M_1}
\]

and

\[
\frac{1}{\mu_\varphi(S_\varphi(z, 2Kt))} \int_{S_\varphi(z, 2Kt)} u \, d\mu_\varphi < M_1.
\]

Suppose that there exist \( x_0 \in S_\varphi(z, t) \) and \( j \in \mathbb{N} \) verifying

\[
u j^{-1} M_1
\]

and

\[
\mu_\varphi(S_\varphi(x_0, \rho)) \geq \left( \frac{2C_1(2K)^{\xi}}{(1 - \varepsilon_0)\nu^j} \right) \mu_\varphi(S_\varphi(z, t)),
\]

for some \( \rho < t \). Then,

\[
\sup_{S_\varphi(x_0, \rho)} u > \nu^j M_1.
\]

**Proof.** Let us assume that

\[
\sup_{S_\varphi(x_0, \rho)} u \leq \nu^j M_1
\]

and reach a contradiction. Since \( x_0 \in S_\varphi(z, t) \), given \( y \in S_\varphi(x_0, \rho) \) from (2.17) and the fact that \( \rho < t \), we get \( \delta_\varphi(z, y) \leq K(\delta_\varphi(z, x_0) + \delta_\varphi(x_0, y)) \leq K(t + \rho) < 2Kt \). Thus,

\[
S_\varphi(x_0, \rho) \subset S_\varphi(z, 2Kt).
\]

Now, for \( x \in S_\varphi(x_0, \rho) \) and \( j \in \mathbb{N} \), define

\[
w_j(x) := \frac{\nu^j M_1 - u(x)}{\nu^j - (\nu - 1)M_1}.
\]
It follows from (4.51) that \( w_j \geq 0 \) in \( S_\varphi(x_0, \rho) \) and, since \( L^r_A(u) \geq f \) and \( c \leq 0 \),

\[
L^r_A(w_j) = \frac{-L^r_A(u)}{\nu^j-1(\nu - 1)M_1} + \frac{cv^jM_1}{\nu^j-1(\nu - 1)M_1} \leq \frac{-f}{\nu^j-1(\nu - 1)M_1} \leq \frac{|f|}{\nu^j-1(\nu - 1)M_1} =: f_j.
\]

By (4.50), we have (recall that \( \nu, M_1 > 1 \) and \( \nu/(\nu - 1) = 2M_1 \))

\[
\frac{\rho}{\mu_\varphi(S_\varphi(x_0, \rho))^{1/n}} \|f_j\|_{L^n(S_\varphi(x_0, \rho), d\mu_\varphi)} \leq \frac{\rho((1 - \varepsilon_0)\nu^jM_1)^{1/n}}{(2C_1/(2K)^\zeta)^{1/n}\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f_j\|_{L^n(S_\varphi(z, t), d\mu_\varphi)}
\]

\[
= \frac{\rho((1 - \varepsilon_0)\nu^jM_1)^{1/n}}{(2C_1/(2K)^\zeta)^{1/n}(\nu^j-1(\nu - 1)M_1)\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f\|_{L^n(S_\varphi(x_0, \rho), d\mu_\varphi)}.
\]

Now, bringing up the definition of \( c_3 \) in (4.46), we can keep bounding

\[
\frac{\rho}{\mu_\varphi(S_\varphi(x_0, \rho))^{1/n}} \|f_j\|_{L^n(S_\varphi(x_0, \rho), d\mu_\varphi)} \leq \frac{\rho((1 - \varepsilon_0)\nu^jM_1)^{1/n}}{(2C_1/(2K)^\zeta)^{1/n}(\nu^j-1(\nu - 1)M_1)\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f\|_{L^n(S_\varphi(z, t), d\mu_\varphi)}
\]

\[
= \frac{c_3\rho(\nu^jM_1)^{1/n}}{(\nu^j-1(\nu - 1)M_1)\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f\|_{L^n(S_\varphi(x_0, \rho))}
\]

\[
\leq \frac{c_3\rho}{(\nu^j-1(\nu - 1)M_1)\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f\|_{L^n(S_\varphi(z, 2Kt))}
\]

\[
\leq \frac{2c_3M_1\rho}{\mu_\varphi(S_\varphi(z, t))^{1/n}}\|f\|_{L^n(S_\varphi(z, 2Kt))} \leq \varepsilon_1,
\]

where we have used that the definition of \( \nu \) in (4.45) yields \((\nu - 1)/\nu = 1/(2M_1)\), the inclusion (4.52), and the hypothesis (4.47). Also, by (4.49),

\[
w_j(x_0) = \frac{(\nu^jM_1 - u(x_0))}{\nu^j-1(\nu - 1)M_1} \leq \frac{(\nu^jM_1 - (\nu^j-1)M_1)}{\nu^j-1(\nu - 1)M_1} = 1.
\]

Consequently,

\[
\inf_{S_\varphi(x_0, \rho/2)} w_j \leq 1,
\]

and Theorem 2 (applied to \( w_j \) in \( S_\varphi(x_0, \rho) \) with \( M_1 \) as in (4.44)) yields

\[
\mu_\varphi\{x \in S_\varphi(x_0, \rho) : w_j(x) \geq M_1\} \leq \varepsilon_0\mu_\varphi(S_\varphi(x_0, \rho)).
\]

Next, introduce the following subsets of \( S_\varphi(z, 2Kt) \) (recall (4.52))

\[
A_1 := \{x \in S_\varphi(z, 2Kt) : u(x) \geq \nu^jM_1/2\}
\]
and
\[ A_2 := \{ x \in S_\varphi(x_0, \rho) : w_j(x) > M_1 \} \subset S_\varphi(z, 2Kt). \]

We claim that
\[ S_\varphi(x_0, \rho) \subset A_1 \cup A_2. \] (4.54)

Indeed, given \( x_1 \in S_\varphi(x_0, \rho) \setminus A_1 \), that is, \( x_1 \in S_\varphi(x_0, \rho) \) and \( u(x_1) < \nu^j M_1/2 \), it follows that \( x_1 \in A_2 \) because
\[ w_j(x_1) = \frac{\nu^j M_1 - u(x_1)}{\nu^{j-1}(\nu - 1)M_1} > \frac{\nu^j M_1 - \nu^j M_1/2}{\nu^{j-1}(\nu - 1)M_1} = \frac{\nu}{2(\nu - 1)} = M_1. \]

From inclusion (4.54), Chebyshev’s inequality applied to (4.48) (to control \( \mu(A_1) \)), and inequality (4.53) (to control \( \mu(A_2) \)), we get
\[ \mu_\varphi(S_\varphi(x_0, \rho)) \leq \mu_\varphi(A_1) + \mu_\varphi(A_2) \]
(4.55)
\[ \leq \frac{2}{\nu^j M_1 \int_{S_\varphi(z, 2Kt)} u \, d\mu_\varphi} + \varepsilon_0 \mu_\varphi(S_\varphi(x_0, \rho)) \]
\[ < \frac{2\mu_\varphi(S_\varphi(z, 2Kt))}{\nu^j} + \varepsilon_0 \mu_\varphi(S_\varphi(x_0, \rho)). \]

Now, the doubling property (4.42) implies
\[ \mu_\varphi(S_\varphi(z, 2Kt)) \leq C_1(2K)\varepsilon \mu_\varphi(S_\varphi(z, t)) \]
which, along with (4.55), yields
\[ \mu_\varphi(S_\varphi(x_0, \rho)) < \frac{2C_1(2K)\varepsilon}{(1 - \varepsilon_0)\nu^j} \mu_\varphi(S_\varphi(z, t)), \]
contradicting (4.50). \( \square \)

Lemma 6. Assume that \( \mu_\varphi \in (DC)_\varphi \). Let \( A \in E(\Lambda_1, \Lambda_2, D^2 \varphi, \Omega) \). There exists a structural constant \( C_6 > 0 \) such that for every section \( S := S_\varphi(z, t) \subset \Omega \) with \( S_K := S_\varphi(z, 2Kt) \subset \subset \Omega \) and every \( u \in W^{2,n}_{loc}(S_K) \cap C(\overline{S_K}) \) and \( f \in L^n(S_K, d\mu_\varphi) \) with \( u \geq 0 \) and \( L_\varphi^A(u) \geq f \) in \( S_K \) satisfying
\[ \frac{t}{\mu(S_\varphi(z, t))^{1/n}} \| f \|_{L^n(S_\varphi(z, 2Kt), d\mu_\varphi)} \leq \frac{\varepsilon_1}{2c_3 M_1} \]
(4.56)
and
\[ \frac{1}{\mu_\varphi(S_\varphi(z, 2Kt))} \int_{S_\varphi(z, 2Kt)} u \, d\mu_\varphi < M_1, \]
(4.57)
we have
\[ \sup_{S_\varphi(z, t/2)} u \leq C_6. \]
(4.58)
Proof. Let \( u, f \) and \( S \) be as in the statement of Lemma 6. Let \( C_4 > 1 \) be the structural constant defined by

\[
C_4 := (2K)^2 \left( \frac{2C_1}{1 - \varepsilon_0} \right)^{\frac{1}{\zeta}}.
\]

Let \( m \in \mathbb{N} \) be a structural constant large enough so that

\[
C_2^{1/\beta} C_4 \nu^{-m/\zeta} < 1,
\]

(4.60)

(where \( C_2 > 1 \) and \( \beta \in (0, 1] \) are as in (4.35)) and put

\[
C_5(m) := \frac{2}{1 - (C_2^{1/\beta} C_4 \nu^{-m/\zeta})}.
\]

In addition to (4.60) choose \( m \in \mathbb{N} \), always structural, big enough so that

\[
\exp \left[ \frac{C_5(m) C_2 C_1^{\beta/\zeta}}{\nu^{m/\zeta} (\nu^{\beta/\zeta} - 1)} \right] < 2.
\]

(4.62)

We claim that, for the structural \( m \) chosen above, we have

\[
\sup_{S_\varphi(z,t/2)} u \leq \nu^{m-1} M_1 =: C_6.
\]

(4.63)

Indeed, we will see that if (4.63) fails to be true, then it is possible to construct a sequence \( \{x_{m+j}\}_{j \in \mathbb{N}_0} \subset S_\varphi(z,t) \) such that

\[
u(x_{m+j}) > \nu^{m+j-1} M_1 \quad \forall j \in \mathbb{N}_0,
\]

(4.64)

which contradicts the local boundedness (and therefore the continuity) of \( u \) in \( S_\varphi(z,t) \).

Let us then suppose that (4.63) is false and use Lemma 5 to construct such sequence.

For \( k \in \mathbb{N} \) with \( k \geq m \), let \( \{\rho_k\} \) be the decreasing sequence defined by

\[
\rho_k := C_4 \nu^{-k/\zeta} t/2.
\]

(4.65)

Notice that (4.60) implies

\[
\rho_k < t/2 \quad \forall k \in \mathbb{N}, k \geq m.
\]

(4.66)

Also, notice that for every \( y \in S_\varphi(z,t) \) the quasi-triangle inequality (2.17) implies \( S_\varphi(z,t) \subset S_\varphi(y,2Kt) \). Then, using (4.43) with \( S_\varphi(z,t) \subset S_\varphi(y,2Kt) \) and \( \rho_{m+j} < t/2 \), and from the definitions of \( \rho_{m+j} \) and \( C_4 \) (in (4.65) and (4.59), respectively), for every \( j \in \mathbb{N} \) we have

\[
\mu_\varphi(S_\varphi(y,\rho_{m+j})) \geq \frac{1}{C_1(2K)^{\zeta}} \left( \frac{2\rho_{m+j}}{t} \right)^{\zeta} \mu_\varphi(S_\varphi(z,t))
\]

\[
= \frac{C_4^{\zeta}}{C_1(2K)^{\zeta}} \nu^{-(m+j)} \mu_\varphi(S_\varphi(z,t))
\]

\[
= \frac{2C_1(2K)^{\zeta}}{(1 - \varepsilon_0)} \nu^{-(m+j)} \mu_\varphi(S_\varphi(z,t)).
\]

(4.67)
That is, we have shown that, with the choice of \(\rho_{m+j}\) as in (4.65), for every \(j \in \mathbb{N}_0\) and every \(y \in S_\varphi(z, t)\), the sections \(S_\varphi(y, \rho_{m+j})\) and \(S_\varphi(z, t)\) verify the inequality (4.50) in Lemma 5. What comes next is the inductive selection of the appropriate \(y\)'s in \(S_\varphi(z, t)\) that will form the sequence \(\{x_{m+j}\}_{j \in \mathbb{N}_0}\). Let us then start by taking \(j = 0\) and selecting \(x_m\). Since we are assuming that (4.63) is false, there exists \(x_m \in S_\varphi(z, t/2)\) (that is, \(x_m\) satisfies \(d_0 := \delta_\varphi(z, x_m) < t/2\)) such that

\[
u^2 M_1.
\]

Now, by Lemma 5 applied to \(x_m\), we obtain

\[
(4.68) \quad \sup_{S_\varphi(x_m, \rho_m)} u > \nu^2 M_1.
\]

Therefore, there exists \(x_{m+1} \in S_\varphi(x_m, \rho_m)\) (that is, \(\delta_\varphi(x_m, x_{m+1}) < \rho_m\)) such that

\[
u^2 M_1.
\]

And it is clear that we intend to use Lemma 5 again, now applied to \(x_{m+1}\), to get

\[
(4.69) \quad d_j := \delta_\varphi(z, x_{m+j}) \quad \text{and} \quad \theta_j := C_2^{1/3} \rho_{m+j} \quad \forall j \in \mathbb{N}_0.
\]

From the inductive construction above, at each step \(j \in \mathbb{N}_0\), (4.35) yields

\[
d_{j+1} - d_j \leq C_2 \delta_\varphi(x_{m+j}, x_{m+j+1})^{1/3} (d_j + d_{j+1})^{1-\beta}
\]

\[
\leq C_2 \rho_{m+j}^{\beta} (d_j + d_{j+1})^{1-\beta} = \theta_j^{\beta} (d_j + d_{j+1})^{1-\beta},
\]

with \(d_0 = \delta_\varphi(z, x_m) < t/2\), and (from (4.60)) \(\theta_{j+1} < \theta_j < t/2\) for every \(j \in \mathbb{N}_0\). By Lemma 1 in [9] (with "\(R := t/2\" in the notation in there) it follows that

\[
(4.70) \quad d_{j+1} \leq \frac{t}{2} \exp \left[ C_5(m) \sum_{k=0}^{\infty} \left( \frac{\theta_k}{t} \right)^{\beta} \right] \quad \forall j \in \mathbb{N}_0.
\]

Finally, the definitions of \(d_j\) and \(\theta_j\) in (4.69), the one for \(\rho_{m+j}\) in (4.65), and the choice of \(m\) in (4.62) imply that

\[
\exp \left[ C_5(m) \sum_{k=0}^{\infty} \left( \frac{\theta_k}{t} \right)^{\beta} \right] < 2.
\]
Therefore, from (4.70) we get that \( x_{m+j} \in S_{\varphi}(z,t) \) at each step \( j \in \mathbb{N}_0 \) and the construction of \( \{x_{m+j}\}_{j \in \mathbb{N}_0} \subset S_{\varphi}(z,t) \) satisfying (4.64) is complete. \( \square \)

**Proof of Theorem 3.** Given \( u \) and \( f \) as in the statement of Theorem 3, we define

\[
Q(u, f) := \frac{1}{M_1 \mu_\varphi(S_{\varphi}(z,2Kt))} \int_{S_{\varphi}(z,2Kt)} u \, d\mu_\varphi + \left( \frac{2c_3 M_1}{\varepsilon_1} \right) \frac{t}{\mu(S_{\varphi}(z,t))^{1/n}} \|f\|_{L^n(S_{\varphi}(z,2Kt),d\mu_\varphi)}
\]

and, for every \( x \in S_K \), put \( w(x) := u(x)/Q(u,f) \). Then we have \( L^q_{\varphi}(w) \geq \bar{f} := f/Q(u,f) \) in \( S_K \) with \( w \) and \( \bar{f} \) verifying the hypotheses of Lemma 6. Hence, from (4.58), we have

\[
\sup_{S_{\varphi}(z,t/2)} w \leq C_6,
\]

which, from the definition of \( Q(u, f) \) in (4.71), translates into (4.34) with structural constants

\[
C_7 := \frac{C_6}{M_1} \quad \text{and} \quad C_8 := \frac{2c_3 C_6 M_1}{\varepsilon_1}.
\]

\( \square \)

Given a Borel subset \( E \subset \mathbb{R}^n \) with \( \mu_\varphi(E) > 0 \) let us denote the average of a function \( w \) over \( E \) with respect to \( d\mu_\varphi \) by

\[
\frac{1}{\mu_\varphi(E)} \int_E w \, d\mu_\varphi := \int_E w(y) \, d\mu_\varphi(y).
\]

**Theorem 7.** Assume \( \mu_\varphi \in (\text{DC})_\varphi \). Let \( A \in E(\Lambda_1, \Lambda_2, \Lambda^2 \varphi, \Omega) \). There are geometric constants \( K_3, K_5 \geq K \) such that for every \( q > 0 \) there exist constants \( C_9, C_{10} > 0 \), depending only on structural constants and \( q \), such that for every section \( S_{\varphi}(z,t) \) with \( S_{K_3} := S_{\varphi}(z,K_3t) \subset \subset \Omega \) whenever \( u \in W^{2,n}_{\text{loc}}(S_{K_3}) \cap C(S_{K_3}) \) and \( f \in L^n(S_{K_3}, d\mu_\varphi) \) satisfy \( F_A^\varphi(u) \geq f \) and \( u \geq 0 \) in \( S_{K_3} \), we have

\[
\sup_{S_{\varphi}(z,t/K_3)} u \leq C_9 \left( \int_{S_{\varphi}(z,2t)} w^q \, d\mu_\varphi \right)^{1/q} + \frac{C_{10} t}{\mu_\varphi(S_{\varphi}(z,t))^{1/n}} \|f\|_{L^n(S_{\varphi}(z,K_3t),d\mu_\varphi)}.
\]

**Proof.** The following detailed proof of Theorem 7 will allow for the close tracing of the relevant geometric and structural constants. Notice that, by Hölder’s inequality, it is enough to prove (4.73) for \( 0 < q < 1 \). Let us symmetrize \( \delta_\varphi \) by defining

\[
\rho_\varphi(x,y) := \max\{\delta_\varphi(x,y), \delta_\varphi(y,x)\} \quad \forall x,y \in \mathbb{R}^n.
\]

Then, \( \rho_\varphi \) is symmetric, \( \rho_\varphi(x,y) = 0 \) if and only if \( x = y \) (due to the strict convexity of \( \varphi \), in the sense that its graph does not contain line segments) and \( \rho_\varphi \) satisfies the \( K \)-quasi-triangle inequality (2.17), all of which turn \( \rho_\varphi \) into a quasi-distance on \( \mathbb{R}^n \). Now, a classical result on quasi-metric spaces (see [10, Theorem 2]) establishes the
existence of a distance $d_\varphi$ defined on $\mathbb{R}^n$ such that, setting $\alpha := 1/\log_2(3K^2) \in (0,1)$ (where $K$ is as in the quasi-triangle inequality (2.17)), $d_\varphi^n$ is equivalent to $\rho_\varphi$. More precisely,

\begin{equation}
\frac{1}{2} \rho_\varphi(x,y) \leq d_\varphi(x,y)^\alpha \leq 4 \rho_\varphi(x,y) \quad \forall x,y \in \mathbb{R}^n.
\end{equation}

For $x \in \mathbb{R}^n$ and $R > 0$, let us write

\begin{equation}
B_\varphi(x,R) := \{ y \in \mathbb{R}^n : d_\varphi(x,y) < R \}.
\end{equation}

Let us also introduce the geometric constants

\begin{equation}
K_3 := 16K^2, \quad K_4 := 4^{-1/\alpha}, \quad K_5 := (8K)^{1/\alpha}, \quad \text{and} \quad K_6 := K_5K^{1/\alpha}.
\end{equation}

From (4.74), for every $x_0 \in \mathbb{R}^n$ and $t > 0$ we have

\begin{equation}
B_\varphi(x_0,R) \subset S_\varphi(x_0,2t) \subset B_\varphi(x_0,K_5R) \quad \text{with} \quad R := t^{1/\alpha},
\end{equation}

where in the second inclusion we have used the fact that whenever $\delta_\varphi(x,y) < 2t$ then (2.13) implies that $\delta_\varphi(y,x) < 2K_{sym}t$ and we can assume $K_{sym} \leq K$ (where $K$ is always as in (2.17)). From (4.74) and (4.76), for every $x_0 \in \mathbb{R}^n$ and $t > 0$ we have (always for $R := t^{1/\alpha}$)

\begin{equation}
B_\varphi(x_0,K_4R) \subset S_\varphi(x_0,t/2) \subset S_\varphi(x_0,2Kt) \subset B_\varphi(x_0,K_6R) \subset S_\varphi(x_0,K_3t).
\end{equation}

Also, through (4.77), the doubling property (4.42) implies that for every $x_0 \in \mathbb{R}^n$ and $0 < R_1 < R_2$ we have

\begin{equation}
\mu_\varphi(B_\varphi(x_0,R_2)) \leq C_1K_5^{\alpha\zeta} \left( \frac{R_2}{R_1} \right)^{\alpha\zeta} \mu_\varphi(B_\varphi(x_0,R_1)).
\end{equation}

Throughout this proof the positive numbers $R$ and $t$ will be related by $t = R^{\alpha}$. Setting

\begin{equation}
K_7 := \frac{1}{C_1K_5^{\alpha\zeta}} \quad \text{and} \quad K_8^n := \frac{1}{C_1K_5^{2\alpha\zeta}(2K)^{\zeta}},
\end{equation}

for every $x_0 \in \mathbb{R}^n$ with $S_\varphi(x_0,K_3t) \subset \subset \Omega$ we obtain

\begin{equation}
\mu_\varphi(S_\varphi(x_0,2Kt)) \geq K_7\mu_\varphi(B_\varphi(x_0,K_6R))
\end{equation}

and

\begin{equation}
\mu_\varphi(S_\varphi(x_0,t))^{1/n} \geq K_8\mu_\varphi(B_\varphi(x_0,K_6R))^{1/n}.
\end{equation}

Now, given $u$ as in the statement of Theorem 7, Theorem 3 applied to $u$ on any section $S_\varphi(x_0,t_0)$ with $S_\varphi(x_0,K_3t_0) \subset \subset \Omega$, together with the inclusions in (4.78) and thinking of $R'_0 := t_0$, gives

\begin{equation}
\sup_{B_\varphi(x_0,K_3R_0)} u \leq \frac{C_7}{K_7} \int_{B_\varphi(x_0,K_6R_0)} u \, d\mu_\varphi + \frac{C_8R_0^n \| f \|_{L^n(S_\varphi(x_0,2Kt_0),d\mu_\varphi)}}{K_8\mu_\varphi(B_\varphi(x_0,K_6R_0))^{1/n}},
\end{equation}

for every $R_0 > 0$ with $B_\varphi(x_0,K_6R_0) \subset \subset \Omega$. 
Let $S_\varphi(z,t)$ be a section with $S_\varphi(z,K_3t) \subset \subset \Omega$. Fix $0 < \rho < r < K_6R$, always with $R := t^{1/2}$. For $\varepsilon > 0$ pick $y \in B_\varphi(z,\rho)$ with $u(y) \geq \sup_{B_\varphi(z,\rho)} u - \varepsilon$. That is,

$$\sup_{B_\varphi(z,\rho)} u \leq u(y) + \varepsilon.$$ 

We will now apply (4.81) with $x_0 := y$ and $R_0 := (r - \rho)/(2K_6)$. Notice that this can be done since, using the fact that $y \in B_\varphi(z,\rho)$, we have

$$(4.82) \quad B_\varphi(y,R_0) = B_\varphi(y,(r - \rho)/2) \subset B_\varphi(z,r) \subset B_\varphi(z,K_6R) \subset \subset \Omega.$$ 

Then, we can write

$$\sup_{B_\varphi(z,\rho)} u \leq u(y) + \varepsilon \leq \frac{C_7}{K_7} \int_{B_\varphi(y,(r - \rho)/2)} u \, d\mu_\varphi + \frac{C_8R_0\|f\|_{L^n(S_\varphi(y,2K_0), d\mu_\varphi)}}{K_8\mu_\varphi(B_\varphi(y,K_6R_0))^{1/n}} + \varepsilon.$$ 

The fact that $B_\varphi(z,r) \subset B_\varphi(y,2r)$ (because $y \in B_\varphi(z,\rho)$) and the doubling property (4.79) yield

$$\mu_\varphi(B_\varphi(z,r)) \leq \mu_\varphi(B_\varphi(y,2r)) \leq C_1(4K_5)^{\alpha_\zeta} \left( \frac{r}{r - \rho} \right)^{\alpha_\zeta} \mu_\varphi(B_\varphi(y,(r - \rho)/2),$$

which together with the inclusions in (4.82) and the inequalities $R_0 < r/(2K_6) < R$ (so that, in particular, $t_0 < t$), gives

$$(4.83) \quad \sup_{B_\varphi(z,\rho)} u \leq \frac{C_7C_1(4K_5)^{\alpha_\zeta}}{K_7} \int_{B_\varphi(z,r)} u \, d\mu_\varphi + \frac{C_8C_1^{1/n}(4K_5)^{\alpha_\zeta/n}}{2^{\alpha_\zeta}K_8K_6^{\alpha_\zeta} \mu_\varphi(B_\varphi(z,r))^{1/n}} + \varepsilon,$$

where we have used the inclusion

$$(4.84) \quad S_\varphi(y,2Kt_0) \subset S_\varphi(z,K_3t),$$

which follows from the fact that $y \in B_\varphi(z,\rho)$ (by construction) and the definition of $R_0 := (r - \rho)/(2K_6)$. Indeed, since $y \in B_\varphi(z,\rho)$ and since $d_\varphi$ is an actual distance, we immediately get

$$(4.85) \quad B_\varphi(y,r - \rho) \subset B_\varphi(z,r).$$

Now, by recalling the definitions of $K_3, K_5$ and $K_6$ in (4.76), from (4.74) (or (4.77)) and (4.85) we get

$$S_\varphi(y,2Kt_0) \subset S_\varphi(y,2^{1+\alpha}Kt_0) = S_\varphi(y,2(2K_6R_0/K_5)^{\alpha}) \subset B_\varphi(y,2K_6R_0)$$

$$= B_\varphi(y,r - \rho) \subset B_\varphi(z,r) \subset S_\varphi(z,2r^\alpha) \subset S_\varphi(z,2(K_6R)^{\alpha})$$

$$= S_\varphi(z,16K^2t) = S_\varphi(z,K_3,t),$$

and (4.84) follows.
Next, letting \( \varepsilon \to 0^+ \) in (4.83), introducing the structural constants

\[
C_{13} := \frac{C_7 C_1 (4K_5)^{\alpha \zeta}}{K_7} \quad \text{and} \quad C_{14} := \frac{C_8 C_1^{1/n} (4K_5)^{\alpha \zeta/n}}{2^n K_8 K_6^n},
\]

and using the fact that \((r/(r - \rho))^{\alpha \zeta/n} < (r/(r - \rho))^{\alpha \zeta}\), we have then obtained that for every \( 0 < \rho < r < K_6 R \)

\[
(4.86) \quad \sup_{B_\varphi(z, \rho)} u \leq \left( \frac{r}{r - \rho} \right)^{\alpha \zeta} \left( C_{13} \int_{B_\varphi(z,r)} u \, d\mu_\varphi + \frac{C_{14} r^\alpha \|f\|_{L^n(S_\varphi(z,K_3 t), d\mu_\varphi)}}{\mu_\varphi(B_\varphi(z, r))^{1/n}} \right).
\]

Now, given \( q \in (0, 1) \), we apply Young’s inequality:

\[
ab \leq \frac{a^s}{s} + \frac{b^{s'}}{s'} \quad \forall a, b > 0 \text{ and } 1 < s, s' < \infty \text{ with } s' + s = ss',
\]

with \( s := 1/(1 - q) \), to obtain

\[
C_{13} \left( \frac{r}{r - \rho} \right)^{\alpha \zeta} \int_{B_\varphi(z,r)} u \, d\mu_\varphi \leq C_{13} \left( \frac{r}{r - \rho} \right)^{\alpha \zeta} \left( \int_{B_\varphi(z,r)} u^q \, d\mu_\varphi \right)^{1/q} \left( \frac{r}{r - \rho} \right)^{\alpha \zeta/q} \left( \int_{B_\varphi(z,r)} u^{\alpha \zeta/q} \, d\mu_\varphi \right)^{1 - 1/q}.
\]

Thus, for every \( 0 < \rho < r < K_6 R \), we have

\[
(4.87) \quad \sup_{B_\varphi(z, \rho)} u \leq (1 - q) \sup_{B_\varphi(z,r)} u + (r - \rho)^{-\alpha \zeta/q} A
\]

with

\[
A := q C_{13}^{1/q} r^{\alpha \zeta/q} \left( \int_{B_\varphi(z,r)} u^q \, d\mu_\varphi \right)^{1/q} + \frac{C_{14} r^\alpha \|f\|_{L^n(S_\varphi(z,K_3 t), d\mu_\varphi)}}{\mu_\varphi(B_\varphi(z, r))^{1/n}}.
\]

Inequality (4.87) and a classical real analysis lemma (see Remark 10 in Section 7) imply that

\[
(4.88) \quad \sup_{B_\varphi(z, \rho)} u \leq K(q)(r - \rho)^{-\alpha \zeta/q} A \quad \forall 0 < \rho < r < K_6 R,
\]

where \( K(q) \), depending only on geometric constants and \( q \), is defined by

\[
K(q) := (1 - \tau)^{-\alpha \zeta/q} (1 - (1 - q)\tau^{-\alpha \zeta/q})^{-1},
\]
and $\tau \in (0,1)$ is any number (close to 1) satisfying $(1 - q)\tau^{-\frac{\alpha q}{q}} < 1$. Hence, for every $0 < \rho < r < K_6 R = K_6 t^\frac{1}{\alpha}$ we get

$$
\sup_{B_\varphi(z,\rho)} u \leq qK(q)C_{13} \left( \frac{r}{r - \rho} \right)^{\frac{\alpha q}{q}} \left( \int_{B_\varphi(z,r)} u^q d\mu_\varphi \right)^{\frac{1}{q}} + C_{14} K(q) \left( \frac{r}{r - \rho} \right)^{\frac{\alpha q}{q}} r^\alpha \|f\|_{L^n(S_\varphi(z,K_3 t), d\mu_\varphi)}.
$$

At this point we choose $r := t^\frac{1}{\alpha} = R$ and $\rho := r/2$ so that the inclusions in (4.77) render

$$
S_\varphi(z,t/K_5) \subset B_\varphi(z,\rho) \subset B_\varphi(z,r) \subset S_\varphi(z,2t),
$$

which, together with the doubling property (4.42) we have

$$
\frac{1}{\mu_\varphi(B_\varphi(z,r))} \leq \max \left\{ \frac{C_{1} K_5^\zeta}{\mu_\varphi(S_\varphi(z,t))}, \frac{C_{1} (2K_5)^\zeta}{\mu_\varphi(S_\varphi(2z,t))} \right\}.
$$

Finally, setting

$$
C_9 := qK(q)(2^{\alpha q} C_{13})^{\frac{1}{q}} C_{17}^{\frac{1}{q}} (2K_5)^{\frac{\eta}{q}}
$$

and

$$
C_{10} := C_{14} K(q) 2^{\alpha q} (C_{1} K_5^\zeta)^{\frac{1}{q}}
$$

the $q$-mean value inequality (4.73) follows. □

5. A MEAN-VALUE INEQUALITY WHEN $L_A^q(u) \leq f$ AND $u > 0$

**Theorem 8.** Assume $\mu_\varphi \in (DC)_\varphi$. Let $A \in E(\Lambda_1, \Lambda_2, D^2 \varphi, \Omega)$. For every $p \in (n, \infty]$ and $q > 0$ there exist constants $C_{11}, C_{12} > 0$, depending only on structural constants as well as on $p$ and $q$, such that for every section $S := S_\varphi(z,t)$ with $S_K := S_\varphi(z,2Kt) \subset \subset \Omega$ whenever $u \in W^{2,n}_{\text{loc}}(S_K) \cap C(\overline{S_K})$ and $f \in L^p(S_K, d\mu_\varphi)$ satisfy $L_A^q(u) \leq f$ and $u \geq 0$ in $S_K$ and whenever $k > 0$ satisfies

$$
(5.89) \quad k \geq \left( \int_{S_K} |f|^p d\mu_\varphi \right)^{\frac{1}{p}},
$$

then we have

$$
(5.90) \quad \left( \int_{S_\varphi(z,2Kt)} (u + k)^{-q} d\mu_\varphi \right)^{-\frac{1}{q}} \leq (C_{11} + C_{12} t^{\frac{1}{\beta}}) \inf_{S_\varphi(z,t/2)} (u + k),
$$

where

$$
(5.91) \quad \beta := q \left( \frac{1}{n} - \frac{1}{p} \right) > 0.
$$
Proof. Given \( k > 0 \) as in the statement and \( \beta > 0 \) as in (5.91), set
\[
    w_k(x) := (u(x) + k)^{-\beta} \quad \forall x \in S_\varphi(z, 2Kt).
\]
Then, in \( S_\varphi(z, 2Kt) \) we have
\[
    L_A^\varphi (w_k) = \frac{\beta (\beta + 1)}{(u+k)^{\beta+2}} \text{trace}(A \nabla u \otimes \nabla u) - \frac{\beta L_A^\varphi (u)}{(u+k)^{\beta+1}} + \frac{\beta cu}{(u+k)^{\beta}} + \frac{c}{(u+k)^{\beta}}
\]
that is,
\[
    (5.92) \quad L_A^\varphi (w_k) \geq -\beta f + \beta cu + c(u+k) \quad \text{in } S_\varphi(z, 2Kt).
\]
By Theorem 3 we have
\[
    (5.93) \quad \sup_{S_\varphi(z,t/2)} w_k \leq C_7 \int_{S_\varphi(z,2Kt)} w_k d\mu_\varphi + \frac{C_8 t}{\mu_\varphi (S_\varphi(z,t))^{1/n}} \| f_k \|_{L^n (S_\varphi(z,2Kt), d\mu_\varphi)}.
\]
The definition of \( f_k \) in (5.92) yields (recall that \( S_K := S_\varphi(z, 2Kt) \))
\[
    \frac{1}{\mu_\varphi (S_K)^{1/n}} \| f_k \|_{L^n (S_K, d\mu_\varphi)} \leq \left( \int_{S_K} \beta^n |f|^n d\mu_\varphi \right)^{\frac{1}{n}} + \left( \int_{S_K} \frac{\beta^n |u^n d\mu_\varphi|}{(u+k)^{\beta n+n}} \right)^{\frac{1}{n}} + \left( \int_{S_K} |c^n (u+k)^n d\mu_\varphi| \right)^{\frac{1}{n}}
\]
\[
    =: J_1 + J_2 + J_3.
\]
Now, given \( p \) as in the statement set \( r := p/n \in (1, \infty] \). By Hölder’s inequality with \( r \) and \( r' \) and by the condition on \( k \) in (5.89) we get
\[
    J_1 \leq \frac{\beta}{k} \left( \int_{S_K} \frac{|f|^n d\mu_\varphi}{(u+k)^{\beta n}} \right)^{\frac{1}{n}} \leq \frac{\beta}{k} \left( \int_{S_K} |f|^p d\mu_\varphi \right)^{\frac{1}{p}} \left( \int_{S_K} \frac{d\mu_\varphi}{(u+k)^{\beta n r'}} \right)^{\frac{1}{n r'}}
\]
\[
    \leq \beta \left( \int_{S_K} \frac{d\mu_\varphi}{(u+k)^{\beta n r'}} \right)^{\frac{1}{n r'}}.
\]
Also,
\[
    J_2 + J_3 \leq (\beta + 1) \left( \int_{S_K} \frac{|c^n d\mu_\varphi|}{(u+k)^{\beta n}} \right)^{\frac{1}{n}} \leq (\beta + 1) \| c \|_{L^n(S_K)} \left( \int_{S_K} \frac{d\mu_\varphi}{(u+k)^{\beta n}} \right)^{\frac{1}{n}}.
\]
so that (5.93) implies
\[
\frac{1}{\inf_{S_\varphi(z,t/2)} (u+k)} \leq (2C_7)^{\frac{1}{\beta}} \left( \frac{\int_{S_\varphi(z,2Kt)} d\mu_\varphi}{(u+k)^\beta} \right)^{\frac{1}{\beta}} + \frac{(2C_8)^{\frac{1}{\beta}} \|f_k\|_{L^n(S_\varphi(z,2Kt))}^{\frac{1}{\beta}}}{\mu_\varphi(S_\varphi(z,t))^{\frac{1}{3n}}}.
\]

Setting \( C_{15} := C_1^{1/n\beta} (2K)^{\zeta/(n\beta)} \), we get
\[
\left( \frac{\|f_k\|_{L^n(S_\varphi(z,2Kt))}}{\mu_\varphi(S_\varphi(z,t))^{1/n}} \right)^{\frac{1}{\beta}} \leq C_{15} \left( \frac{\|f_k\|_{L^n(S_\varphi(z,2Kt))}}{\mu_\varphi(S_K^{1/n})} \right)^{\frac{1}{\beta}} \leq C_{15} (J_1 + J_2 + J_3)^{\frac{1}{\beta}}
\]
\[
\leq C_{15} \left[ (2\beta)^{\frac{1}{\beta}} \left( \int_{S_K} d\mu_\varphi \right)^{\frac{1}{3n\tau}} + (2(\beta + 1)\|c\|_{L^\infty(S)}^{\frac{1}{\beta}} \left( \int_{S_K} d\mu_\varphi \left( \frac{d\mu_\varphi}{(u+k)^{3n\tau}} \right)^{\frac{1}{3n\tau}} \right) \right]
\]
\[
\leq C_{15} \left( (2\beta)^{\frac{1}{\beta}} + (2(\beta + 1)\|c\|_{L^\infty(S)}^{\frac{1}{\beta}} \right) \left( \int_{S_K} \frac{d\mu_\varphi}{(u+k)^{3n\tau}} \right)^{\frac{1}{3n\tau}}.
\]

where for the last two inequalities we used the fact that (from Hölder’s inequality) the function \( \tau \to (\int_S u^r d\mu_\varphi)^{1/r} \) is non-decreasing, and the doubling property (4.42). Notice that, from the assumption \( p > n \), we have \( r' < \infty \) and the choice of \( \beta \) in (5.91) yields \( q = \beta n r' > \beta \). Then,
\[
\frac{1}{\inf_{S_\varphi(z,t/2)} (u+k)} \leq \left( (2C_7)^{\frac{1}{\beta}} + (2C_8)^{\frac{1}{\beta}} C_{15} \right) \left( \int_{S_\varphi(z,2Kt)} \frac{d\mu_\varphi}{(u+k)^q} \right)^{\frac{1}{q}}
\]
and (5.90) follows with \( C_{11} := (2C_7)^{\frac{1}{\beta}} \) and \( C_{12} := (2C_8)^{\frac{1}{\beta}} C_{15} \). \( \Box \)

6. Exploiting the variational side: A local \( BMO \) estimate

In this section we look at the inequality \( L_\varphi(u) \leq f \) from the conjugate side of \( \varphi \) which will be amenable to a suitable Poincaré-type inequality for the Monge-Ampère quasi-metric structure.

6.1. The convex conjugate. Given a strictly convex, twice continuously differentiable \( \varphi : \mathbb{R}^n \to \mathbb{R} \), its Legendre transform or convex conjugate, will be denoted by \( \psi \). Under the hypothesis \( \mu_\varphi \in (DC)_\varphi \) we have that \( \mu_\psi := \det D^2 \psi \in (DC)_\psi \), where the \( \psi \)-doubling constants for \( \mu_\psi \) (with respect to the sections of \( \psi \)) depend only on the ones for \( \mu_\varphi \) and dimension \( n \). Also, \( \psi \) is a strictly convex twice continuously differentiable function whose domain is \( \mathbb{R}^n \) and
\[
(6.94) \quad \nabla \varphi(\nabla \psi(x)) = \nabla \psi(\nabla \varphi(x)) = x \quad \forall x \in \mathbb{R}^n,
\]
Consequently,
\begin{equation}
D^2 \varphi(\nabla \psi(y)) D^2 \psi(y) = D^2 \psi(\nabla \varphi(x)) D^2 \varphi(x) = I \quad \forall x, y \in \mathbb{R}^n
\end{equation}
and, for every Borel set \(E \subset \mathbb{R}^n\),
\begin{equation}
|E| = |\nabla \varphi(\nabla \psi(E))| = \mu_\varphi(\nabla \psi(E)) = \mu_\psi(\nabla \varphi(E))
\end{equation}
Now, the fact that \(\mu_\psi \in (\text{DC})_\psi\) implies (and, as discussed in Section (2.1), it is actually equivalent to) the existence of a geometric constant \(K_\psi \geq 1\) such that
\begin{equation}
\delta_\psi(x, y) \leq K_\psi (\min \left\{ \delta_\psi(z, x), \delta_\psi(x, z) \right\} + \min \left\{ \delta_\psi(z, y), \delta_\psi(y, z) \right\}) \quad \forall x, y, z \in \mathbb{R}^n.
\end{equation}
Recall that
\begin{equation}
\delta_\psi(x, y) := \psi(y) - \psi(x) - \langle \nabla \psi(x), y - x \rangle \quad \forall x, y \in \mathbb{R}^n.
\end{equation}
On the other hand, by Lemma 5(a) in [2], Lebesgue measure is doubling with respect to the sections of any convex function with doubling constant \(2^n\), in particular, we have
\begin{equation}
|S_\psi(x, t)| \leq 2^n |S_\psi(x, t/2)| \quad \forall x \in \mathbb{R}^n, t > 0.
\end{equation}
Hence, the triple \((\mathbb{R}^n, \delta_\psi, |\cdot|)\) possesses the structure of a space of homogeneous type. Moreover, from Theorem 12 in [4], there exists \(K_0 \geq 1\), depending only on the \(\text{(DC)}_\varphi\) constants for \(\mu_\varphi\) and dimension \(n\) such that
\begin{equation}
\nabla \varphi(S_\varphi(x, t/K_0)) \subset S_\varphi(\nabla \varphi(x), t) \subset \nabla \varphi(S_\varphi(x, K_0 t)) \quad \forall x \in \mathbb{R}^n, \forall t > 0.
\end{equation}
In fact, one can take
\begin{equation}
K_0 := 2K,
\end{equation}
always with \(K\) as in (2.17). It is useful to bear in mind that, as a consequence of (6.100), the mapping
\[ \nabla \varphi : (\mathbb{R}^n, \delta_\varphi) \to (\mathbb{R}^n, \delta_\psi) \]
is a biLipschitz homeomorphism (with inverse \(\nabla \psi\)).

### 6.2. Poincaré inequalities.

Recall that for a differentiable function \(g\) in \(\Omega\) we have defined
\[ \nabla^\varphi g(x) := D^2 \varphi(x)^{-1/2} \nabla g(x) \quad \forall x \in \Omega. \]
There are at least two compelling reasons as to why the gradient \(\nabla^\varphi\), adapted to \(\varphi\), should be considered as the natural one when developing a first-order calculus in the Monge-Ampère quasi-metric setting. Firstly, \(\nabla^\varphi\) rules the Sobolev and Poincaré-type inequalities recently obtained in [11, 12] for the spaces of homogeneous type \((\mathbb{R}^n, \rho_\varphi, \mu_\varphi)\) and \((\mathbb{R}^n, \rho_\varphi, |\cdot|)\). Indeed, by Theorem 1.3 in [12], the Monge-Ampère quasi-metric structure admits a \((1, 2)\)-Poincaré inequality; more precisely, whenever
\(\mu_\varphi \in (\mathrm{DC})_\varphi\) there exists a geometric constant \(C_P > 0\) such that for every section \(S := S_\varphi(x_0, t)\) and every \(g \in C^1(S)\) we have

\[
\frac{1}{|S|} \int_S |g(x) - g_S| \, dx \leq C_P t^{\frac{1}{2}} \left( \frac{1}{|S|} \int_S |\nabla^\varphi g(x)|^2 \, dx \right)^{\frac{1}{2}},
\]

where \(g_S := \frac{1}{|S|} \int_S g(x) \, dx\).

Secondly, \(\nabla^\varphi\) is invariant under conjugate change of variables, that is, if \(v(y) := u(x)\), with \(x = \nabla^\psi(y)\), then \((6.95)\) implies that

\[
(6.103) \quad \nabla^\psi v(y) = \nabla^\varphi u(x) \quad \forall x \in \mathbb{R}^n.
\]

**Theorem 9.** Assume \(\mu_\varphi \in (\mathrm{DC})_\varphi\). Then, there exist geometric constants \(0 < \tau_0 < 1 < K_{10}\) and a structural constant \(q_0 > 0\) such that for every section \(S := S_\varphi(z, t)\) with \(S_\varphi(z, t) \subset \subset \Omega\) and every non-negative \(u \in W^{2,n}_{\mathrm{loc}}(S) \cap C(\overline{S})\) with

\[
\text{trace}(A_\varphi(x) D^2 u(x)) + \langle b(x), \nabla^\varphi u(x) \rangle \mu_\varphi(x) + c(x) u(x) \mu_\varphi(x) \leq f(x) \mu_\varphi(x),
\]

for a.e. \(x \in S\) and for every \(k > 0\) with

\[
(6.104) \quad k \geq \|f\|_{L^\infty(S)},
\]

we have

\[
(6.105) \quad \int_{S_\varphi(z, \tau_0 t)} (u + k)^{q_0} \, d\mu_\varphi \leq K_{10}^2 \left( \int_{S_\varphi(z, \tau_0 t)} (u + k)^{-q_0} \, d\mu_\varphi \right)^{-1}.
\]

**Proof.** Set \(S := S_\varphi(z, t)\) with \(S_\varphi(z, t) \subset \subset \Omega\) and let \(u \in W^{2,n}_{\mathrm{loc}}(S) \cap C(\overline{S})\) be non-negative with

\[
\text{trace}(A_\varphi(x) D^2 u(x)) + \langle b(x), \nabla^\varphi u(x) \rangle \mu_\varphi(x) + c(x) u(x) \mu_\varphi(x) \leq f(x) \mu_\varphi(x),
\]

for a.e. \(x \in S\). Now, multiplication of the above inequality by an arbitrary \(\eta_0 \in C_c^1(S), \eta_0 \geq 0\), and integration by parts, using the fact that the columns of \(A_\varphi(x)\) are divergence free, that is,

\[
(6.106) \quad \text{trace}(A_\varphi(x) D^2 g(x)) = \text{div}(A_\varphi(x) \nabla g(x)) \quad \text{a.e.} \; x \in S, \forall g \in W^{2,n}_{\mathrm{loc}}(S),
\]

yields

\[
(6.107) \quad -\int_S \langle (D^2 \varphi)^{-1} \nabla u, \nabla \eta_0 \rangle \, d\mu_\varphi + \int_S \langle b, \nabla^\varphi u \rangle \eta_0 \, d\mu_\varphi + \int_S c u \eta_0 \, d\mu_\varphi \leq \int_S f \eta_0 \, d\mu_\varphi.
\]

Next, set \(S^\psi := \nabla^\psi(S)\) so that \(S = \nabla^\psi(S^\psi)\). Notice that \(S^\psi\) may fail to be a section of \(\psi\), but it is “almost” a section of \(\psi\), in the sense of the inclusions in \((6.100)\). For every \(y \in S^\psi\), define

\[
v(y) := u(\nabla^\psi(y)), \quad \eta(y) := \eta_0(\nabla^\psi(y)),
\]
and \( b^\psi (y) := b(\nabla \psi (y)) \), \( c^\psi (y) := c(\nabla \psi (y)) \), \( f^\psi (y) := f(\nabla \psi (y)) \), so that by changing variables \( x := \nabla \psi (y) \) in (6.107), for every \( \eta \in C^1_c(S^\psi) \), \( \eta \geq 0 \), we obtain
\[
(6.108) - \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla \eta \rangle \, dy + \int_{S^\psi} \langle b^\psi, \nabla \psi \rangle \eta \, dy + \int_{S^\psi} c^\psi v \eta \, dy \leq \int_{S^\psi} f^\psi \eta \, dy.
\]
For \( h \in C^1_c(S^\psi) \) and \( k > 0 \) as in (6.104) set
\[
\eta(y) := \frac{h(y)^2}{v(y) + k} \quad \forall y \in S^\psi,
\]
so that \( \nabla \eta = \frac{2h}{v + k} \nabla h - \frac{h^2}{v + k} \nabla v \) and (6.108) implies
\[
\int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla \eta \rangle \frac{h^2 \, dy}{(v + k)^2} \leq \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla h \rangle \frac{2h \, dy}{(v + k)} - \int_{S^\psi} \langle b^\psi, \nabla \psi \rangle \frac{h^2 \, dy}{(v + k)} - \int_{S^\psi} c^\psi h^2 \, dy \leq \int_{S^\psi} f^\psi h^2 \, dy = I_1 + I_2 + I_3 + I_4.
\]
By Cauchy-Schwarz (twice), we estimate \( I_1 \) as follows
\[
|I_1| \leq 2 \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{1}{2} \langle (D^2 \psi)^{-1} \nabla h, \nabla h \rangle \frac{1}{2} \frac{h \, dy}{(v + k)} \leq 2 \left( \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)^2} \right)^{\frac{1}{2}} \left( \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla h, \nabla h \rangle \, dy \right)^{\frac{1}{2}} \leq \frac{1}{4} \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)^2} + 4 \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla h, \nabla h \rangle \, dy.
\]
By using the Cauchy-Schwarz twice again, for \( I_2 \) we have
\[
|I_2| \leq \int_{S^\psi} |\langle b^\psi, \nabla \psi \rangle| \frac{h^2 \, dy}{(v + k)} = \int_{S^\psi} |\langle b^\psi, (D^2 \psi)^{-1} \nabla v \rangle| \frac{h^2 \, dy}{(v + k)} \leq \int_{S^\psi} |b^\psi| \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)} \leq \left( \int_{S^\psi} |b^\psi|^2 h^2 \, dy \right)^{\frac{1}{2}} \left( \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)^2} \right)^{\frac{1}{2}} \leq \frac{1}{4} \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)^2} + \int_{S^\psi} |b^\psi|^2 h^2 \, dy.
\]
Hence,
\[
|I_2| \leq \frac{1}{4} \int_{S^\psi} \frac{h^2}{(v + k)^2} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \, dy + \|b^\psi\|^2_{L^\infty(S^\psi)} \|h\|^2_{L^\infty(S^\psi)} |S^\psi|.
\]
For $I_3$ and $I_4$ we do
\[
|I_3| + |I_4| \leq (\|c^\psi\|_{L^\infty(S^\psi)} + \frac{1}{k}\|f^\psi\|_{L^\infty(S^\psi)})\|h\|_{L^\infty(S^\psi)}^2 |S^\psi|,
\]
and putting all together we get
\[
(6.109) \quad \frac{1}{2} \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2}{(v + k)^2} \leq 4 \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla h, \nabla h \rangle dy
\]
\[
+ \left( \|b^\psi\|_L^\infty(S^\psi) + \|c^\psi\|_L^\infty(S^\psi) + \frac{1}{k}\|f^\psi\|_L^\infty(S^\psi) \right) \|h\|_{L^\infty(S^\psi)}^2 |S^\psi|.
\]
For the first term on the right-hand side above, changing variables back, we have
\[
\int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla h, \nabla h \rangle dy = \int_S \langle A_\varphi(x) \nabla H(x), \nabla H(x) \rangle dx,
\]
where $H \in C^1_c(S)$ is related to $h \in C^1_c(S^\psi)$ by
\[
H(x) := h(\nabla \varphi(x)) \quad \forall x \in S.
\]
Next, let $\gamma : \mathbb{R} \to [0, 1]$ be a smooth function supported in $[0, 1]$ with $\gamma \equiv 1$ on $[0, 1/2]$. Considering $z$ fixed, for every $x \in S$, set
\[
(6.110) \quad \delta_z(x) := \delta_\varphi(z, t) \quad \text{and} \quad H(x) := \gamma(\delta_z(x)/t)
\]
so that
\[
H \in C^1_0(S) \quad \text{and} \quad \nabla H(x) = \frac{1}{t} \gamma'(\delta_z(x)/t) \nabla \delta_z(x)
\]
with
\[
(6.111) \quad \|\gamma'\|_{L^\infty} \leq 2 \quad \text{and} \quad h(y) = 1 \quad \forall y \in \nabla \varphi(S_\varphi(z, t/2)).
\]
Integrating by parts again (recall that $\delta_z(x) = t$ for $x \in \partial S$), and using the null-Lagrangian property (6.106), we write
\[
\frac{t^2}{\|\gamma'\|^2_{L^\infty}} \int_S \langle A_\varphi(x) \nabla H(x), \nabla H(x) \rangle dx
\]
\[
= \frac{1}{\|\gamma'\|^2_{L^\infty}} \int_S \gamma'(\delta_z(x)/t)^2 \langle A_\varphi(x) \nabla \delta_z(x), \nabla \delta_z(x) \rangle dx
\]
\[
\leq \int_S \langle A_\varphi(x) \nabla \delta_z(x), \nabla \delta_z(x) \rangle dx = \int_S \langle A_\varphi(x) \nabla (t - \delta_z(x)), \nabla (t - \delta_z(x)) \rangle dx
\]
\[
= \int_S \langle A_\varphi(x) \nabla \delta_z(x)(t - \delta_z(x)) dx \rangle \nabla t \nabla \delta_z(x) \rangle dx
\]
\[
= \int_S \langle A_\varphi(x) D^2 \varphi(x) \rangle (t - \delta_z(x)) dx
\]
\[
= \int_S \langle A_\varphi(x) D^2 \varphi(x) \rangle (t - \delta_z(x)) dx \leq nt \mu_\varphi(S) = nt |S^\psi|.
\]
Notice that (6.104) implies

\[ k \geq \| f \|_{L^\infty(S)} = \| f^\psi \|_{L^\infty(S^\psi)}. \]

From (6.109), and using (2.12), we get

\[
\int_{S^\psi} |\nabla \psi \log(v + k)|^2 h^2 \, dy = \int_{S^\psi} \langle (D^2 \psi)^{-1} \nabla v, \nabla v \rangle \frac{h^2 \, dy}{(v + k)^2} \leq \frac{8n\gamma'^2 \| S^\psi \|}{t} + 2 \left( \| b \|_{L^\infty(S^\psi)}^2 + \| c \|_{L^\infty(S^\psi)} + \frac{1}{k} \| f^\psi \|_{L^\infty(S^\psi)} \right) \| h \|_{L^\infty(S^\psi)} |S^\psi|.
\]

where the constant $C_{16}$ is a structural constant. From the inclusions (6.100) we obtain

\[ S^\psi := S^\psi(\nabla \varphi(z), t/(2K_0)) \subset \nabla \varphi(S^\varphi(z, t/2)) \subset \nabla \varphi(S^\varphi(z, t)) = S^\psi. \]

Now, by the Poincaré inequality (6.102) applied to $\psi$ (since $\mu_{\psi} \in (\text{DC})_\psi$) and $V := \log(v + k) \in W^{1,2}(S^\psi)$, and by the inclusions (6.113) and the fact that, from (6.111), $h \equiv 1$ on $\nabla \varphi(S^\varphi(z, t/2))$ (and therefore on $S^\psi$), we can do

\[
\frac{1}{|S^\psi|} \int_{S^\psi} |V(y) - V_{S^\psi}| \, dy \leq C_P \left( \frac{t}{K_0} \right)^{\frac{1}{2}} \left( \int_{S^\psi} |\nabla \psi V(y)|^2 \, dy \right)^{\frac{1}{2}} \leq C_P \left( \frac{C_{16} |S^\psi|}{K_0 |S^\psi|} \right)^{\frac{1}{2}}.
\]

From the inclusions in (6.100) again and the doubling property (4.42), it follows that

\[
|S^\psi| = |\nabla \varphi(S^\varphi(z, t))| = \mu_{\varphi}(S^\varphi(z, t)) \leq C_1(2K_0^2)^{\zeta} c_{\varphi}(S^\varphi(z, t/(2K_0^2)))
\]

\[
= C_1(2K_0^2)^{\zeta} \left| \nabla \varphi(S^\varphi(z, t/(2K_0^2))) \right| \leq C_1(2K_0^2)^{\zeta} \left| (S^\psi(\nabla \varphi(z), t/(2K_0))) \right|
\]

\[
= C_1(2K_0^2)^{\zeta} |S^\psi|.
\]

Therefore,

\[ \frac{1}{|S^\psi|} \int_{S^\psi} |V(y) - V_{S^\psi}| \, dy \leq C_P \left( C_{16} C_1 2^{\zeta} K_0^{2^\zeta - 1} \right)^{\frac{1}{2}}, \]

for every section $S^\psi := S^\psi(\nabla \varphi(z), t/(2K_0))$ such that $z \in \mathbb{R}^n$ and $t > 0$ satisfy $S^\varphi(z, 2t) \subset \subset \Omega$. Now, using that (6.100) implies

\[
\nabla \varphi(S^\varphi(z, t/(2K_0^2))) \subset S^\psi = S^\psi(\nabla \varphi(z), t/(2K_0)),
\]

setting $\tau \in (0, 1)$ as the geometric constant

\[ \tau := \frac{1}{2K_0^2} = \frac{1}{8K^2}, \]
Given $S$ a section $k$ same value of $k$ (6.117) imply that $K$ and we can denote (6.121) by writing (6.121)

$$
(6.121)\text{ where } U \text{ and } \mu(x) \text{ is as in (6.104). In other words, (6.116) says that } U \text{ belongs to a local version of } BMO(\Omega) \text{ with respect to the space of homogeneous type } (\mathbb{R}^n, \delta, \mu). \text{ However, in order to obtain an } A_2 \text{ type condition for the weight } u + k \text{ with no assumptions on } \partial \Omega, \text{ we will restrict the } BMO \text{ estimate to a fixed section.}
$$

Fix $S \subset \subset \Omega$ and consider any section $S' := S(\delta, t')$ such that (6.117)

$$
S' := S(\delta, t') \subset S(\delta, \tau t).
$$

We claim that (6.118)

$$
S(\delta, t'/\tau) \subset \subset \Omega.
$$

In order to prove (6.118), let us first see that (6.119)

$$
t' \leq 2K^2 t.
$$

Given $\varepsilon > 0$ pick $z'' \subset S(\delta, t')$ with $t' - \varepsilon \leq \delta(z, z'')$ and, recalling (6.117), do

$$
t' - \varepsilon \leq \delta(z, z') = K(\delta(z, z') + \delta(z, z'')) \leq 2K^2 t,
$$

and (6.119) follows after $\varepsilon \to 0^+$. Now, to prove (6.118) it suffices to check that (6.120)

$$
S(\delta, t'/\tau) \subset S(\tau, t'),
$$

(since $S(\delta, t) \subset \subset \Omega$). For $z'' \subset S(\delta, t'/\tau)$ we write

$$
\delta(z, z'') \leq K(\delta(z, z') + \delta(z, z')) \leq K(t'/\tau + \tau t)
$$

$$
\leq K(2K^2 + \tau^2) t = \frac{K^2}{4K^2} t < t,
$$

where in the last line we used (6.119), the definition of $\tau$ in (6.115), and the value of $K_0$ in (6.101). Then, (6.120) follows and so does (6.118).

Our interest in (6.118) comes from the need of being able to use (6.116) for every section $S'$ satisfying (6.117). Notice that the choice of $k$ in (6.104) and the inclusion (6.117) imply that $k \geq \|f\|_{L^\infty(S')}$ and therefore (6.116) holds true for $S'$ with the same value of $k$ satisfying (6.112). With all this we have obtained the following:

Given $S(\delta, t)$ with $S(\delta, t) \subset \subset \Omega$, for every section $S' := S(\delta, \tau t) \subset S(\delta, \tau^2 t)$ we have

$$
\int_{S'} |U(x) - U| d\mu(x) \leq C_17,
$$

and we can denote (6.121) by writing $U \in BMO(S(\delta, \tau^2 t), \delta, \mu)$. Consequently, just as in the proof of Theorem 3.15 in [1], there exist geometric (and always explicit)
constants $0 < \eta < 1 < K_9, K_{10}$ such that the following John-Nirenberg inequality holds true

\[ \mu_\varphi(\{x \in S_\varphi(z, \eta^2 t) : |U(x) - U_{S_\varphi(z, \eta^2 t)}| > \lambda\}) \leq K_{10}\mu_\varphi(S_\varphi(z, \eta^2 t))e^{-\frac{K_9\lambda}{\tau^{\eta}}}, \]

which, in turn, implies (see Corollary 3.21 in [1]) the existence of a structural constant $q_0 \in (0, 1)$ such that

\[ \int_{S_\varphi(z, \eta^2 t)} e^{q_0|U(x) - U_{S_\varphi(z, \eta^2 t)}|} d\mu_\varphi(x) \leq K_{10}. \]

Hence, setting $\tau_0 := \eta^2$, we have

\[ (6.122) \left( \int_{S_\varphi(z, \tau_0 t)} (u(x) + k)^{q_0} d\mu_\varphi(x) \right) \left( \int_{S_\varphi(z, \tau_0 t)} (u(x) + k)^{-q_0} d\mu_\varphi(x) \right) \leq K_{10}^2, \]

and (6.105) follows.

7. Proof of Theorem 1 and closing remarks

Proof. Let $S_\varphi(z, t)$ be a section of $\varphi$ with $S_\varphi(z, t) \subset \subset \Omega$. Let $\tau_0$ and $q_0$ be as in Theorem 9. By applying Theorem 7 with $q := q_0$ to the section $S_\varphi(z, \tau_0 t/2)$ (notice that, from (4.76) and (6.115), $\tau_0 K_3/2 = \eta^{\tau} K_3/2 < \tau K_3/2 = 1$, so that $S_\varphi(z, \tau_0 K_3 t/2) \subset S_\varphi(z, t) \subset \subset \Omega$), we get

\[ \sup_{S_\varphi(z, \tau_0 t/(2K_3))} u \leq C_9 \left( \int_{S_\varphi(z, \tau_0 t)} u^{q_0} d\mu_\varphi \right)^{\frac{1}{q_0}} + \frac{C_{10}}{\mu_\varphi(S_\varphi(z, \tau_0 t/2))^{1/n}} \|f\|_{L^n(S_\varphi(z, t), d\mu_\varphi)}, \]

and, by the doubling property (4.42),

\[ \frac{C_{10}\|f\|_{L^n(S_\varphi(z, t), d\mu_\varphi)}}{\mu_\varphi(S_\varphi(z, \tau_0 t/2))^{1/n}} \leq C_{10} \left( \frac{\mu_\varphi(S_\varphi(z, t))}{\mu_\varphi(S_\varphi(z, \tau_0 t/2))} \right)^{1/n} \|f\|_{L^n(S_\varphi(z, t), d\mu_\varphi)} \]

\[ \leq C_{10} C_1^{1/n} \left( \frac{2}{\tau_0} \right) \zeta/n \|f\|_{L^\infty(S_\varphi(z, t), d\mu_\varphi)}. \]

Hence, setting $C_{18} := C_9 + C_{10} C_1^{1/n} \left( \frac{2}{\tau_0} \right) \zeta/n$, we get

\[ (7.123) \sup_{S_\varphi(z, \tau_0 t/(2K_3))} u \leq C_{18} \left( \int_{S_\varphi(z, \tau_0 t)} (u + \|f\|_{L^\infty(S_\varphi(z, t), d\mu_\varphi)})^{\frac{q_0}{q_0}} d\mu_\varphi \right)^{\frac{1}{q_0}}. \]
By applying Theorem 9 to \( S_\varphi(z,t) \subset \Omega \) with \( k:=\|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)} \) we get

\[
(7.124) \quad \left( \int_{S_\varphi(z,\tau_0 t)} (u + \|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)})^{q_0} \, d\mu_\varphi \right)^{\frac{1}{q_0}} \leq K_{10}^{2/q_0} \left( \int_{S_\varphi(z,\tau_0 t)} (u + \|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)})^{-q_0} \, d\mu_\varphi \right)^{-\frac{1}{q_0}}.
\]

By applying Theorem 8 with \( p = \infty \) and \( q = q_0 \) to \( S_\varphi(z,\tau_0 t/(2K)) \) (notice that \( 2K(\tau_0/(2K)) = \tau_0 < 1 \) so that \( S_\varphi(z,2K(\tau_0 t/(2K))) \subset S_\varphi(z,t) \subset \Omega \) with \( k:=\|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)} \geq \|f\|_{L^\infty(S_\varphi(z,\tau_0 t),d\mu_\varphi)} \)) it follows that

\[
(7.125) \quad \left( \int_{S_\varphi(z,\tau_0 t)} (u + \|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)})^{-q_0} \, d\mu_\varphi \right)^{\frac{1}{q_0}} \leq (C_{11} + C_{12} t^{\frac{1}{2}}) \inf_{S_\varphi(z,\tau_0 t/(4K))} (u + \|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)}).
\]

Now, by (2.12),

\[
(C_{11} + C_{12} t^{\frac{1}{2}}) \leq (C_{11} + C_{12} \text{diam}(\Omega)^{\frac{1}{2}}) =: C_{19}.
\]

Setting \( C_H > 0 \) as the structural constant

\[
C_H := C_{18} C_{19} K_{10}^{2/q_0},
\]

from (7.123), (7.124), and (7.125), we get

\[
\sup_{S_\varphi(z,\tau_0 t/(2K))} u \leq C_H \left( \inf_{S_\varphi(z,\tau_0 t/(4K))} (u + \|f\|_{L^\infty(S_\varphi(z,t),d\mu_\varphi)}) \right).
\]

Finally, from the definition of \( K_3 \) in (4.76), we have that \( 4K < 2K_3 \); therefore, (1.5) follows with \( \rho := \tau_0/(2K_3) \).

\[ \square \]

Remark 10. The classical real analysis lemma alluded to in the proof of Theorem 7 says that whenever \( f : [0,\tau] \rightarrow [0,M] \), for some \( 0 < M < \infty \), satisfies

\[
(7.126) \quad f(t) \leq \theta f(s) + A(s-t)^{-\alpha} \quad \forall 0 \leq t < s \leq \tau,
\]

for some \( \theta \in (0,1) \) and \( A > 0 \). Then

\[
(7.127) \quad f(t) \leq C(\alpha,\theta) A(s-t)^{-\alpha} \quad \forall 0 \leq t < s \leq \tau.
\]

The simple proof of the lemma starts by fixing \( 0 \leq t < s \leq \tau \) and, for \( \eta \) with \( \theta < \eta^{\alpha} < 1 \), setting \( t_0 := t \) and \( t_{j+1} = t_j + \eta^{j}(1-\eta)(s-t) \) for \( j \in \mathbb{N} \) to obtain

\[
f(t) = f(t_0) \leq \theta^k f(t_k) + A(s-t)^{-\alpha}(1-\eta)^{-\alpha} \sum_{j=0}^{k-1} \theta^j \eta^{-\alpha j} \quad \forall k \in \mathbb{N}.
\]
In the proof of Theorem 7 we used the lemma above with $f(t) := \sup_{B_\varphi(z,t)} u$. The reason we switched from the quasi-metric $\rho_\varphi$ to the metric $d_\varphi$ is that, instead of a setup as in (7.126), the quasi-metric $\rho_\varphi$ would lead to

$$f(t) \leq \theta f(Ks) + A(s - t)^{-\alpha} \quad \forall 0 \leq t < s \leq \tau,$$

for some $\theta \in (0,1)$ and $A > 0$, where $K$ is as in (2.17). It is not clear to me how to obtain the conclusion (7.127) from (7.128), with some $C(\alpha, \theta, K)$. Since $d_\varphi$ satisfies a triangle inequality (rather than a $K$-quasi triangle inequality), we worked with $d_\varphi$, applied the lemma, and then switched back to $\rho_\varphi$ and $\delta_\varphi$.

In any case, when doing real analysis within the Monge-Ampère quasi-metric structure, it is useful to bear in mind that there is a genuine distance $d_\varphi$ behind $\rho_\varphi$ and $\delta_\varphi$. However, when utilizing tools from elliptic PDEs (such as the construction of barrier functions, the ABP maximum principle, integration by parts, etc.), the geometric features of the sections, that is, of the “$\delta_\varphi$-balls” themselves (as well as their boundaries) played an instrumental role.

**Remark 11.** During the proof of Theorem 9 we saw that whenever $u$ is a (strong) super solution to $L_\varphi(u) \leq f$ in $S$, then, after the conjugate change of variables $v(y) := u(\nabla \psi(y))$, $v$ turns out to be a (variational) super solution in the sense of (6.108). Moving the discussion to the “conjugate side” allowed for the use of the Poincaré-type inequality (6.102). Accordingly, the mean-value inequalities in Sections 4 and 5 were obtained on the $\varphi$-side by using the non-divergence form structure of $L_\varphi$, through the ABP maximum principle, whereas the local $BMO$ and $A_2$ estimates in Section 6 came from the $\psi$-side by exploiting the divergence-form structure in (6.108) through the Poincaré-type inequality (6.102).

**Remark 12.** Notice that we have resorted to the null-Lagrangian property (6.106) only in a weak, integral sense. That is, we have actually used that for every $g \in W^{2,n}_{loc}(S)$ and $h \in C^1_0(S)$ we have

$$\int_S \text{trace}(A_\varphi(x)D^2g(x))h(x) \, dx = -\int_S \langle A_\varphi(x)\nabla g(x), \nabla h(x) \rangle \, dx.$$

The point being that (7.129) requires $\varphi$ to be only twice continuously differentiable, whereas the pointwise (6.106) requires $\varphi \in C^3(\mathbb{R}^n)$.

**Remark 13.** For the past 20 years or so there has been an intense systematic development, by a large number of authors, of Harnack inequalities in abstract spaces of homogeneous type that admit Sobolev-type and Poincaré-type inequalities. Such Harnack inequalities have been derived from abstract implementations of Moser’s iterations, DeGiorgi’s truncations, or the so-called double-ball property as originated in the work of Krylov and Safonov. In any case, as a common and essential feature in all these approaches, a pointwise estimate of the form

$$|\nabla H(z)|^2 \leq C(R - r)^{-2} \quad \forall z \in B(x, R)$$
is used, where $0 < r < R$ and $H$ is supported in some ball $B(x, R)$ with $H \equiv 1$ in $B(x, r)$ (and $|\nabla H|$ is often replaced with a suitable notion of upper-gradient of $H$). In the Monge-Ampère quasi-metric structure (always under the hypothesis $\mu_\phi \in (DC)_\phi$ only) and taking, say, a section $S_\phi(0, t)$, along the lines of the definition of $H$ in (6.110) the appropriate version of the pointwise inequality (7.130) hinges upon the pointwise inequality

$$
|\nabla^\phi \varphi(z)|^2 \leq C t \quad \forall z \in S_\phi(0, t), \forall t > 0,
$$

for some constant $C > 0$. Indeed, fixing $0 < s < t$ and $\gamma \in C^1(\mathbb{R})$ supported in $[-t, t]$ with $\gamma \equiv 1$ in $[0, s]$ setting $H(x) := \gamma(\varphi(x))$ (and assuming $\varphi(0) = 0$ and $\nabla \varphi(0) = 0$) inequality (7.131) would yield

$$
|\nabla^\phi H(z)|^2 \leq C(t + s)^{-2} \quad \forall z \in S_\phi(0, t),
$$

which, in turn, would play the role of (7.130) and much from the above mentioned approaches towards Harnack inequalities in spaces of homogeneous type could be implemented in the Monge-Ampère quasi-metric structure. However, an inequality such as (7.131), with $C > 0$ being a geometric constant, is unknown to me under the hypothesis $\mu_\phi \in (DC)_\phi$ alone. As it can be seen from the proof of Theorem 9, the Legendre transform, integration by parts, and the use of the null-Lagrangian property for $A_\phi$ in (6.106) compensated for the lack of (7.132).

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REFERENCES


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