ON CERTAIN DEGENERATE AND SINGULAR ELLIPTIC PDES I: 
NONDIVergence FORM OPERATORS WITH UNBounded DRIFTS 
AND APPLICATIONS TO SubELLIPTIC EQUATIONS

DIEGO MALDONADO

Abstract. We prove a Harnack inequality for nonnegative strong solutions to degenerate and singular elliptic PDEs modeled after certain convex functions and in the presence of unbounded drifts. Our main theorem extends the Harnack inequality for the linearized Monge-Ampère equation due to Caffarelli and Gutiérrez and it is related, although under different hypotheses, to a recent work by N. Q. Le.

Since our results are shown to apply to the convex functions $|x|^p$ with $p \geq 2$ and their tensor sums, the degenerate elliptic operators that we can consider include subelliptic Grushin and Grushin-like operators as well as a recent example by A. Montanari of a nondivergence-form subelliptic operator arising from the geometric theory of several complex variables. In the light of these applications, it follows that the Monge-Ampère quasi-metric structure can be regarded as an alternative to the usual Carnot-Carathéodory metric in the study of certain subelliptic PDEs.

1. Introduction and main result

Fix an open bounded set $\Omega \subset \mathbb{R}^n$. The purpose of this article is to use the Monge-Ampère real-analysis and PDE techniques developed in [2, 3, 8, 9, 10, 16, 24, 25, 26, 27] to prove a Harnack inequality for nonnegative (strong) solutions $u \in C(\Omega) \cap W^{2,n}_{loc}(\Omega)$ to the degenerate/singular elliptic PDE

$$L^\varphi_A(u)(x) = f(x) \quad \text{a.e. } x \in \Omega,$$

with

$$(1.1)\quad L^\varphi_A(u)(x) := \text{trace}(A(x)D^2u(x)) + \langle b(x), \nabla \varphi u(x) \rangle + c(x)u(x),$$

where the first- and second-order terms of $L^\varphi_A$ are associated to the Hessian of a strictly convex function $\varphi \in C^1(\mathbb{R}^n)$. More precisely, given $0 < \lambda \leq \Lambda < \infty$, we will assume that, for (Lebesgue) a.e. $x \in \Omega$, the $n \times n$ symmetric matrix $A(x)$ in (1.1) satisfies

$$\lambda D^2\varphi(x)^{-1} \leq A(x) \leq \Lambda D^2\varphi(x)^{-1} \quad \text{a.e. } x \in \Omega$$

in the sense of non-negative-definite matrices. That is,

$$(1.2)\quad \lambda \langle D^2\varphi(x)^{-1}\xi, \xi \rangle \leq \langle A(x)\xi, \xi \rangle \leq \Lambda \langle D^2\varphi(x)^{-1}\xi, \xi \rangle,$$

for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^n$. We will write $A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)$ to indicate (1.2).

Date: September 14, 2017.

2000 Mathematics Subject Classification. Primary 35J70, 35J96; Secondary 35J75, 31E05.

Key words and phrases. Degenerate and singular elliptic PDEs, linearized Monge-Ampère operator, Grushin and subelliptic operators.

Author supported by NSF under grant DMS 1361754.
The first-order term of $L^2_\phi$ involves the Monge-Ampère gradient $\nabla \phi$ defined as
\begin{equation}
\nabla^\phi h(x) := D^2 \phi(x)^{-1/2} \nabla h(x).
\end{equation}

The strictly convex function $\phi \in C^1(\mathbb{R}^n)$ will also model the geometry shaping the Harnack inequality for $L^2_\phi$ by means of its Monge-Ampère sections. The Monge-Ampère section of $\phi$ centered at $x \in \mathbb{R}^n$ and with height $r > 0$ is defined as the open convex set
\begin{equation}
S_\phi(x, r) := \{ y \in \mathbb{R}^n : \phi(y) < \phi(x) + \langle \nabla \phi(x), y - x \rangle + r \}
\end{equation}
where
\begin{equation}
\delta_\phi(x, y) := \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle \quad \forall x, y \in \mathbb{R}^n.
\end{equation}

1.1. The hypotheses on $\phi$. Our hypotheses on $\phi$ are the following:

**H1.** $\phi \in C^1(\mathbb{R}^n)$ is strictly convex in the sense that its graph contains no line segments.

**H2.** The Monge-Ampère measure associated to $\phi$, denoted by $\mu_\phi$ and defined as
\begin{equation}
\mu_\phi(E) := |\nabla \phi(E)|
\end{equation}
for every Borel set $E \subset \mathbb{R}^n$ (where $|F|$ denotes the Lebesgue measure of $F \subset \mathbb{R}^n$), satisfies the following doubling property on the Monge-Ampère sections: There exist constants $C_d \geq 1$ and $\alpha \in (0, 1)$ such that
\begin{equation}
\mu_\phi(S_\phi(x, r)) \leq C_d \mu_\phi(\alpha S_\phi(x, r)) \quad \forall x \in \mathbb{R}^n, r > 0.
\end{equation}

Here $\alpha S_\phi(x, r)$ denotes the $\alpha$-contraction of $S_\phi(x, r)$ with respect to its center of mass (computed with respect to the Lebesgue measure), see [16, Sections 1.1 and 3.1]. We will sometimes write $\mu_\phi \in (DC)_\phi$ to indicate (1.6).

The hypothesis $\mu_\phi \in (DC)_\phi$ turns the triple $(\mathbb{R}^n, \delta_\phi, \mu_\phi)$ into a quasi-metric doubling space. In particular, there exists a constant $K \geq 1$, depending only on $C_d$, $\alpha$, and $n$, such that $\delta_\phi$ in (1.4) satisfies the following quasi-triangle inequality
\begin{equation}
\delta_\phi(x, y) \leq K \left( \min\{ \delta_\phi(z, x), \delta_\phi(x, z) \} + \min\{ \delta_\phi(z, y), \delta_\phi(y, z) \} \right),
\end{equation}
and the quasi-symmetry inequalities $K^{-1} \delta_\phi(x, y) \leq \delta_\phi(y, x) \leq K \delta_\phi(x, y)$, for every $x, y, z \in \mathbb{R}^n$, see [25] and references therein.

**H3.** There exists a non-decreasing function $\zeta : (0, 1) \to (0, \infty)$ with $\lim_{\epsilon \to 0^+} \zeta(\epsilon) = 0$ such that
\begin{equation}
\mu_\phi(S_\phi(x, (1 + \epsilon)R) \setminus S_\phi(x, R)) \leq \zeta(\epsilon) \mu_\phi(S_\phi(x, R)) \quad \forall \epsilon \in (0, 1).
\end{equation}

**H4.** $D^2 \phi(x) > 0$ for a.e. $x \in \Omega$. In particular, $D^2 \phi(x)^{-1}$ and $D^2 \phi(x)^{-1/2}$ exist and are positive-definite for a.e. $x \in \Omega$. Notice that, by A. D. Alexandrov’s theorem, $D^2 \phi(x)$ exists for a.e. $x \in \mathbb{R}^n$ due to the convexity of $\phi$.

**H5.** There exists $p > n$ such that
\begin{equation}
\|D^2 \phi\| \in L^p(\Omega).
\end{equation}
Notice that the convexity of $\phi$ implies that $\frac{1}{n} \Delta \phi(x) \leq \|D^2 \phi(x)\| \leq \Delta \phi(x)$ for a.e. $x \in \mathbb{R}^n$, so that (1.9) is equivalent to $\Delta \phi \in L^p(\Omega)$. From Remark 21 below we will see that **H5** can be weakened to $\|D^2 \phi\| \in L^p_{\text{loc}}(\Omega)$ when $b = 0$. Through this hypothesis the constant will only depend on $p$ and not on the $L^p$-norm of $\|D^2 \phi\|$. Also, if $b = 0$, the constants will not depend on the $L^p_{\text{loc}}$-norm of $\|D^2 \phi\|$.
There exists $\sigma \in (0,1)$ such that for every $x_0 \in \Omega$ and $r > 0$ with $S_\varphi(x_0, 2Kr) \subset\subset \Omega$, where $K \geq 1$ is the quasi-triangle constant in (1.7), we have
\begin{equation}
D_\varphi(x_0, x) \geq \sigma \delta_\varphi(x_0, x) \quad \text{for a.e. } x \in S_\varphi(x_0, 2r) \setminus S_\varphi(x_0, r)
\end{equation}
where
\begin{equation}
D_\varphi(x_0, x) := \langle D^2 \varphi(x)^{-1} (\nabla \varphi(x) - \nabla \varphi(x_0)), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle = |D^2 \varphi(x)^{-1} (\nabla \varphi(x) - \nabla \varphi(x_0))|^2.
\end{equation}

For the sake of convenience we have used the annulus $S_\varphi(x_0, 2r) \setminus S_\varphi(x_0, r)$ in (1.10), but it can be replaced by $S_\varphi(x_0, \Theta r) \setminus S_\varphi(x_0, \theta r)$ for any constants $0 < \theta \leq 1 < \Theta$.

Notice that $x \in S_\varphi(x_0, 2r) \setminus S_\varphi(x_0, r)$ if and only if $r \leq \delta_\varphi(x_0, x) < 2r$, so that a condition equivalent to (1.10) is the existence of $\tilde{\sigma} \in (0,1)$ such that
\begin{equation}
D_\varphi(x_0, x) \geq \tilde{\sigma} r \quad \text{for a.e. } x \in S_\varphi(x_0, 2r) \setminus S_\varphi(x_0, r),
\end{equation}
where $\sigma$ and $\tilde{\sigma}$ depend only on one another.

1.2. The hypotheses on the lower-order coefficients. Throughout the article we assume that $b(x)$ and $c(x)$ in (1.1) satisfy the following hypotheses
\begin{equation}
\|(D^2 \varphi)^{-1/2} b\| \in L^n(\Omega, d\mu_\varphi) \quad \text{and} \quad c \in L^n(\Omega, d\mu_\varphi).
\end{equation}
The hypothesis $(D^2 \varphi)^{-1/2} b \in L^n(\Omega)$ is crucial to the ABP maximum principle, which serves as the cornerstone to the whole approach towards the Harnack inequality for the nondivergence form elliptic operators $L^\varphi_A$. As discussed in the comments in Section 1.3, it cannot be improved to $(D^2 \varphi)^{-1/2} b \in L^{n-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

Our main result is the following (see Section 5.1 for the definitions of geometric and structural constants).

**Theorem 1.** Suppose that $\varphi$ satisfies H1–H6 and fix $A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)$. There exist structural constants $\varepsilon_H \in (0,1)$, $M_H > 1$, and a geometric constant $\tau \in (0,1)$ such that for every section $S_\varphi(z, r)$ with $S_{2Kr} := S_\varphi(z, 2Kr) \subset\subset \Omega$ and every $u \in C(S_{2Kr}) \cap W^{2,n}(S_{2Kr})$ satisfying $u \geq 0$ in $S_{2Kr}$ and
\begin{equation}
L^\varphi_A(u) := \operatorname{trace}(A(x) D^2 u(x)) + \langle b(x), \nabla^\varphi u(x) \rangle + c(x) u(x) = f \quad \text{a.e. in } S_{2Kr},
\end{equation}
the inequalities
\begin{equation}
\sup_{x \in S_\varphi(z, r)} \left( \rho^{-\frac{1}{2}} |S_\varphi(x, \rho)|^{-\frac{1}{n}} - \frac{1}{2} \|D^2 \varphi\|^{-1/2} b \|L^n(S_\varphi(x, \rho), d\mu_\varphi)\|D^2 \varphi\|^{-1/2} b \|L^n(S_\varphi(x, \rho), dx)\| \right) \leq \varepsilon_H
\end{equation}
and
\begin{equation}
|S_{2Kr}|^{\frac{1}{n}} \|c\| L^n(S_{2Kr}, d\mu_\varphi) \leq \varepsilon_H
\end{equation}

imply the Harnack inequality
\begin{equation}
\sup_{S_\varphi(z, 2r)} u \leq M_H \left( \inf_{S_\varphi(z, 2r)} u + |S_\varphi|^{\frac{1}{n}} \|f\| L^n(S_{2Kr}, d\mu_\varphi) \right).
\end{equation}

**Remark 2.** Notice that any first-order term of the form $\langle \tilde{b}, \nabla u \rangle$, involving the usual gradient $\nabla$ and a drift $\tilde{b}$, can be written as
\begin{equation}
\langle \tilde{b}, \nabla u \rangle = \langle b, \nabla^\varphi u \rangle,
\end{equation}
where \( b := (D^2 \varphi)^{1/2} B \), so that the condition \( |(D^2 \varphi)^{-1/2} b| \in L^n(\Omega, d\mu_\varphi) \) in (1.13) just means \( |\tilde{b}| \in L^n(\Omega, d\mu_\varphi) \). Along these lines, Theorem 1 can be equivalently stated with the equation (1.14) and the condition (1.15) replaced with
\[
L_A^\varphi(u) = \text{trace}(A(x)D^2u(x)) + \langle \tilde{b}, \nabla u \rangle + c(x)u(x) = f \quad \text{a.e. in} \ S_{2K_r}
\]
and
\[
\sup_{x \in S_{\varphi}(x, \rho) \atop \rho \leq r} \left( \rho^{-\frac{1}{2}} |\varphi\varphi(x, \rho)|^{\frac{1}{p}} \frac{1}{2} \left\| \tilde{b} \right\|_{L^n(\varphi(x, \rho), d\mu_\varphi)} \left\| D^2 \varphi \right\|_{L^p(\varphi(x, \rho), dx)} \right) \leq \varepsilon_H,
\]
respectively.

Let us briefly put Theorem 1 in the context of some related results.

1.3. A timeline on the interior Harnack inequality for \( L_A^\varphi \) with unbounded drift.

We first notice that when the function \( \varphi \) is taken as the \( \varphi_2(x) := \frac{1}{2} |x|^2 \), for \( x \in \mathbb{R}^n \), then its Monge-Ampère measure reduces to Lebesgue measure, its associated Monge-Ampère gradient \( \nabla \varphi^2 \) is just the regular gradient \( \nabla \), and \( D\varphi_2(x_0, x) = |x - x_0|^2 \) for every \( x, x_0 \in \mathbb{R}^n \). In particular, the sections \( S_{\varphi_2}(x, r) \) coincide with the Euclidean balls \( B(x, \sqrt{2r}) \) for every \( x \in \mathbb{R}^n \) and \( r > 0 \). Hence, all the hypotheses H1-H6 hold true for \( \varphi_2 \).

In addition, the condition \( A \in \mathcal{E}(\lambda, \Lambda, \varphi_2, \Omega) \) amounts to the uniform ellipticity of \( A \) on \( \Omega \). In this case, the Harnack inequality for positive solutions to \( L_A^\varphi(u) = f \) in \( \Omega \) with \( |b|, c \in L^\infty(\Omega) \) is due to the fundamental work of N. Krylov and M. Safonov [14, 30], see also [15, Sections 9.7-9.8].

More recently, and always in the uniformly elliptic case, Safonov in [31] established a Harnack inequality for nonnegative solutions to \( AD^2u + \langle b, \nabla u \rangle = 0 \) in the case of an unbounded drift \( b \) under the assumption \( |b| \in L^n(\Omega) \). The condition \( |b| \in L^n(\Omega) \) is critical in the following sense: if \( |b| \in L^p(\Omega) \) for some \( p > n \) a scaling argument lays a path towards the Harnack inequality along the lines of the Krylov-Safonov approach in [14, 30], see for instance [33]. On the other hand, if \( |b| \in L^p(\Omega) \) for some \( p < n \), the Harnack inequality is known not to hold. Indeed, Safonov’s example (see [31, Remark 1.3] and [29, Section 1]) shows that, in any dimension \( n \), the nonnegative function \( u(x) := \frac{1}{2} |x|^2 \) is a strong solution to \( \Delta u + \langle b, \nabla u \rangle = 0 \) with \( b(x) := -n |x|^{-2} x \). Notice that \( b \in L^n(\mathbb{R}^n, B(0, 1)) \backslash L^n(\mathbb{R}^n, B(0, 1)) \) and, in particular, that \( b \in L^{n - \frac{2}{p}}(\mathbb{R}^n, B(0, 1)) \) for every \( \varepsilon > 0 \). However, \( u(x) := \frac{1}{2} |x|^2 \) vanishes at \( x = 0 \) (and only at \( x = 0 \)) and therefore it cannot satisfy a Harnack inequality in, say, \( B(0, 1) \). Earlier versions of Hölder (as opposed to Harnack) estimates under the assumption \( |b| \in L^p(\Omega) \) had been proved by O. Ladyzhenskaya and N. Ural’tseva in [19] under a smallness condition on \( |b| \). See [31, Remark 1.4] for further details.

For later reference we mention a recent extension of the elliptic results in [31], under the assumption \( |b| \in L^n(\Omega) \), due to C. Mooney in [29], where a Harnack inequality is proved for nonnegative functions that are solutions only when their gradients are large. Mooney’s result also extends the work of C. Imbert and L. Silvestre in [17], who had assumed \( |b| \in L^\infty(\Omega) \), by implementing the method of “sliding paraboloids”.

The results mentioned above are based on the choice \( \varphi_2(x) := \frac{1}{2} |x|^2 \). Let us now move on to the non-uniformly elliptic case and the pioneering work of L. Caffarelli and C. Gutiérrez. In [3], Caffarelli and Gutiérrez considered a function \( \varphi \in C^2(\Omega) \) with \( D^2 \varphi(x) > 0 \) for every \( x \in \Omega \) such that its Monge-Ampère measure \( \mu_\varphi = \text{det} \ D^2 \varphi \) satisfies the so-called \( \mu_\infty \)-condition: for every \( \delta_1 \in (0, 1) \) there exists \( \delta_2 \in (0, 1) \) such that for every section \( S := S_{\varphi}(x, r) \subseteq \Omega \) and
every measurable set $E \subset S$ the implication

$$|E| < \delta_2 |S| \Rightarrow \mu_\varphi(E) < \delta_1 \mu_\varphi(S)$$

holds true, where $|E|$ stands for the Lebesgue measure of the set $E$. Then, given a matrix $A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)$, they proved a Harnack inequality for classical solutions of $L_A^2(u) = 0$ in the case $|b| = c = 0$. The $\mu_\infty$-condition is a Muckenhoupt-type property of $\mu_\varphi$ with respect to Lebesgue measure which is strictly stronger than the doubling condition (1.6) from hypothesis H2, see [3, Sections 1 and 5] and [10, Section 3].

We point out that in [3] the hypothesis $\varphi \in C^2(\Omega)$ with $D^2 \varphi(x) > 0$ for every $x \in \Omega$ is crucial in the proof of the “passage to the double section” (that is, [3, Theorem 2], which, in turn, is essential for the weak-Harnack inequality in [3, Theorem 4]). This can be seen on [3, pp.439-445] where a Dirichlet problem having the Monge–Ampère measure as a nonvanishing factor of the right-hand side is solved for a function $w_\varepsilon$ required to be smooth (at least $C^2$).

In [25], under the assumptions $\varphi \in C^3(\Omega)$ with $D^2 \varphi(x) > 0$ for every $x \in \Omega$, but assuming (the weaker) H2 hypothesis instead of the $\mu_\infty$-condition, the author proved a Harnack inequality for nonnegative strong solutions to $L_A^2(u) = 0$ ([25, Theorem 1.4]) in the case $|b| = c = 0$ and $\lambda = \Lambda$. (Notice that, when $\lambda = \Lambda$, then $A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)$ just means $A = \lambda D^2 \varphi^{-1}$.) In particular, the doubling property (1.6) from hypothesis H2 makes the Harnack inequality intrinsic to the quasi-metric space $(\mathbb{R}^n, \mu_\varphi, \delta_\varphi)$, with no need of a priori comparisons between $\mu_\varphi$ and Lebesgue measure such as in the $\mu_\infty$-condition. In [25] the assumption $\varphi \in C^3(\Omega)$ was used to pass from the non-divergence form of trace($D^2 \varphi^{-1} D^2 u$) to its divergence form via the identity

$$\text{trace}(A_\varphi D^2 h) = \text{div}(A_\varphi \nabla h) \quad \forall h \in C^2(\Omega),$$

where $A_\varphi(x)$ is the matrix of cofactors of the Hessian $D^2 \varphi(x)$, that is,

$$A_\varphi(x) := D^2 \varphi(x)^{-1} \det D^2 \varphi(x) \quad \forall x \in \Omega.$$  

Later on, in [26], under the assumptions $\varphi \in C^2(\Omega)$ with $D^2 \varphi(x) > 0$ for every $x \in \Omega$ and H2, the author proved a Harnack inequality for nonnegative strong solutions to $L_A^2(u) = f$, also in the case $\lambda = \Lambda$, when the lower-order coefficients and source term satisfy

$$|(D^2 \varphi)^{-1/2} b| \in L^p(\Omega), \quad c \leq 0, \quad \text{and} \quad |b|, c, f \in L^\infty(\Omega),$$

see [26, Theorem 1]. The hypothesis $\varphi \in C^2(\Omega)$ with $D^2 \varphi(x) > 0$ for every $x \in \Omega$ allowed for an alternative proof of the critical density estimate ([26, Theorem 2]) to the original one by Caffarelli-Gutiérrez ([3, Theorem 1]). In order to prove a weak-Harnack inequality, the approach in [26] was based on the idea from [25] to use the variational side of the PDE. In this case a weaker form of (1.18) was used, for which the hypothesis $\varphi \in C^3(\Omega)$ (instead of $\varphi \in C^3(\Omega)$) sufficed, see [26, Remark 12].

More recently, under the hypothesis $\varphi \in C^2(\Omega)$ with $D^2 \varphi(x) > 0$ for every $x \in \Omega$ (which plays a crucial role) as well as

$$0 < \lambda_0 \leq \det D^2 \varphi(x) \leq \Lambda_0 \quad \forall x \in \Omega,$$

for some constants $\lambda_0, \Lambda_0 \in (0, \infty)$, N. Q. Le in [20] considered the PDE

$$L_A(u) := \text{trace}(A D^2 u) + (\bar{b}, \nabla u) + cu = f$$

(notice the gradient $\nabla$, as opposed to $\nabla \varphi$, on the first-order term and keep in mind Remark 2) with $A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)$ in the general case of $0 < \lambda \leq \Lambda$ when the lower-order coefficients and source term satisfy

$$|\bar{b}| \in L^p(\Omega) \quad \text{and} \quad c, f \in L^p(\Omega),$$

where $\text{trace}((\bar{b}, \nabla) u) = \sum_i (\bar{b}_i, \partial_i u)$.
for some $p > n(1 + \alpha_*)/(2\alpha_*)$, with $\alpha_* \in (0, 1]$ a structural constant, see [20, Theorem 1.1]. Le’s approach relies on his implementation of the “sliding Monge-Ampère paraboloids” technique, thus extending the approach by Mooney in [29]. Notice that the inequalities (1.21) make $\mu_\varphi$ uniformly comparable to Lebesgue measure, thus implying the $\mu_\infty$-condition for $\mu_\varphi$.

1.4. Our point of view and the reasons for the hypotheses H1–H6. In all of the results mentioned above, the hypothesis $\varphi \in C^2(\Omega)$ with $D^2\varphi(x) > 0$ for every $x \in \Omega$ has been essential. In particular, those results do not apply, for example, to the functions $\varphi_p(x) := \frac{1}{p}|x|^p$ for any $p > 1$ or their tensor sums. Moreover, the current lack of apriori estimates prevents an approximation argument to go from $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ to $\varphi_p$.

The role of the hypotheses H4–H6 is precisely to compensate for such lack of apriori estimates and to be able to apply Theorem 1 directly to ”rougher” (i.e. $\varphi \notin C^2(\Omega)$) and “flatter” (i.e. $D^2\varphi$ vanishing at some points) functions $\varphi$. We remark that the $L^p$-integrability condition in H5 is reminiscent of the one adopted by N. Trudinger in [32] in the context of degenerate divergence-form elliptic operators, see [32, Section 5].

In terms of the functions $\varphi_p$, in Sections 11 and 12 we prove that they, as well as their tensor sums, satisfy the hypotheses H1–H6 whenever $p \geq 2$. Such choices allow for a number of applications to singular and degenerate (including subelliptic) PDEs with unbounded drifts, as shown in Sections 2, 3, and 4.

Of course, other classes of general convex functions are expected to satisfy H1–H6 and to lead, in turn, to corresponding applications.

Finally, we mention that we have stated the “first-order” hypotheses H1–H3 on $\mathbb{R}^n$, while the “second-order” hypotheses H4–H6 are stated on $\Omega$. This is only a matter of convenience since stating H1–H3 on $\mathbb{R}^n$ allows to consider the triple $(\mathbb{R}^n, \delta_\varphi, \mu_\varphi)$ (as opposed to the triple $(\Omega, \delta_\varphi, \mu_\varphi)$) as a space of homogeneous type. However, at the expense of a number of technicalities that we have preferred to avoid, the hypotheses H1–H3 could also be stated on $\Omega$.

Regarding applications of Theorem 1. A central motivation for the Harnack inequality for nonnegative solutions of the linearized Monge-Ampère equation proved by Caffarelli and Gutiérrez in [3] stemmed from topics of fluid dynamics, see [3, Section 1]. Since then, other applications have appeared, for instance, in the contexts of optimal transport [23], [6, Section 2] and differential geometry; in particular, to affine geometry [34, 35, 36] and Abreu’s equation [21, 22] and references therein. Hence, Theorem 1 can be applied in all the contexts above to the case of “rougher” and “flatter” convex functions $\varphi$.

However, in Sections 2 and 3, we focus on applications of Theorem 1 and the Monge-Ampère quasi-metric structure to the study of regularity properties for solutions to certain subelliptic PDEs containing lower-order terms. These subelliptic PDEs will include Grushin PDEs as well as a recent subelliptic PDE studied by Montanari in [28] in the context of the geometric theory of several complex variables. In particular, we extend Montanari’s Harnack inequality for the subelliptic operator in [28] to a family of degenerate elliptic operators involving unbounded drifts. As a consequence of these applications, the Monge-Ampère quasi-metric structure can now be regarded as an alternative to the usual Carnot-Carathéodory metric in the study of such subelliptic PDEs.

In Section 4 we study other degenerate/singular PDEs which extend the Grushin PDEs. To the best of our knowledge, the Harnack inequalities in the presence of unbounded drifts in Theorems 3 and 5 below as well as the ones in Section 4 are all new contributions to subelliptic PDE literature.
Organization of the article. The article is organized as follows: In Sections 2 and 3 we explore applications of Theorem 1 to certain subelliptic operators. These operators include Grushin operators with unbounded drifts as well as a recent nondivergence form subelliptic operator introduced by A. Montanari in [28].

In Section 4 we apply Theorem 1 to classes of singular and degenerate elliptic PDEs with unbounded drifts. In particular, those degenerate classes will include the Grushin operators from Section 2, see Theorems 7 and 8.

In Section 5 we establish some notation and background material, including a Morrey-type estimate (Lemma 13) that is new to the Monge-Ampère quasi-metric structure.

In Section 6 we show that the Monge-Ampère quasi-metric space \((\mathbb{R}^n, \delta_\varphi)\) possesses the segment and segment-prolongation properties (Theorem 16). This then leads to the construction of geodesics on \((\mathbb{R}^n, \delta_\varphi)\) and to the fact that the Monge-Ampère sections are geodesically convex (Theorem 17). The material in Section 6 and the Morrey-type estimate from Section 5 are of independent interest.

In Section 7, under the hypotheses \(H_1, H_2, H_4, H_5, H_6\) on \(\varphi\), we prove the double-ball property for supersolutions (Theorem 20 and Corollary 22).

In Section 8, under the hypotheses \(H_1-H_6\) on \(\varphi\), we prove a critical-density estimate for supersolutions (Theorem 25). In Section 9, also under the hypotheses \(H_1-H_6\) on \(\varphi\), we prove mean-value inequalities for subsolutions (Theorem 27 and Corollaries 28 and 29).

In Section 10, the critical density property and the double-ball property combined with Vitali’s covering lemma are shown to imply the power-like decay of the distribution function of nonnegative supersolutions (Theorem 30) and their corresponding weak-Harnack inequalities (Corollary 31). Thus, with all the elements in place, the proof of Theorem 1 is then completed in Section 10.1.

In Section 11 we prove that (all positive multiples of) the functions \(\varphi_p(x) := \frac{1}{p}|x|^p\) satisfy all of the hypotheses \(H_1-H_6\) when \(p \geq 2\) (Theorem 33).

In Section 12 we prove that the hypotheses \(H_1, H_2, H_4, H_5, H_6\) are quantitatively preserved under tensor sums (Theorem 35) and that all the hypotheses \(H_1-H_6\) are preserved under tensor sums of the functions \(\varphi_p\) for \(p \geq 2\).

Finally, Section 13 corresponds to an Appendix where we include the proofs Lemma 13 (the Morrey-type estimate), Lemma 15 (the local Vitali covering lemma), and Theorem 27 (the mean-value inequality for nonnegative subsolutions).

In a forthcoming article we will address divergence-form operators with second-order terms of the form \(\text{div}(A \nabla u)\) with \(A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)\) and lower-order terms.

2. Applications to subelliptic PDEs I: Grushin operators

Consider \(n_1, n_2 \in \mathbb{N}, x := (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\), and for \(\gamma \geq 0\), let \(G_\gamma\) denote the degenerate elliptic Grushin operator

\[
G_\gamma(u)(x) := \Delta_1 u(x) + |x_1|^\gamma \Delta_2 u(x),
\]

where \(\Delta_j\) denotes the Laplacian on \(\mathbb{R}^{n_j}\) for \(j = 1, 2\). The literature on Grushin operators is vast, we will only mention the well-known works [11, 12] for related Harnack inequalities for nonnegative solutions to \(G_\gamma(u) = 0\). However, as mentioned in the introduction, the author is not aware of any systematic study of Grushin operators with first-order terms previously appeared in the literature.
By setting \( n := n_1 + n_2 \), the equation \( G_\gamma(u) = 0 \) a.e. in \( \Omega \subset \mathbb{R}^n \) can be recast as
\[
(2.24) \quad 0 = \frac{1}{|x_1|^\gamma} \Delta_1 u(x) + \Delta_2 u(x) = \text{trace}(A_\gamma(x)D^2 u(x)) \quad \text{a.e. } x \in \Omega,
\]
where \( A_\gamma(x) \) denotes the diagonal \( n \times n \) matrix
\[
(2.25) \quad A_\gamma(x) := \begin{bmatrix} \frac{1}{|x_1|^\gamma} I_{n_1 \times n_1} & 0 \\ 0 & I_{n_2 \times n_2} \end{bmatrix}.
\]
Hence, by introducing the convex functions \( \varphi^1_\gamma(x) := \frac{1}{\gamma + 2} |x_1|^{\gamma+2}, \varphi_2(x) := \frac{1}{2} |x_2|^2 \), and \( \varphi_\gamma(x) := \varphi^1_\gamma(x_1) + \varphi_2(x_2) \) we get
\[
(2.26) \quad D^2 \varphi_\gamma(x) = \begin{bmatrix} D^2 \varphi^1_\gamma(x_1) & 0 \\ 0 & I_{n_2 \times n_2} \end{bmatrix},
\]
where, for every \( x_1 \in \mathbb{R}^{n_1} \setminus \{0\} \), \( D^2 \varphi^1_\gamma(x_1) \) is the \( n_1 \times n_1 \) symmetric matrix
\[
(2.27) \quad D^2 \varphi^1_\gamma(x_1) = (\gamma + 1)|x_1|^\gamma \begin{pmatrix} \frac{x_1}{|x_1|} \otimes \frac{x_1}{|x_1|} \end{pmatrix} + |x_1|^\gamma \begin{pmatrix} f - \frac{x_1}{|x_1|} \otimes \frac{x_1}{|x_1|} \end{pmatrix},
\]
which has eigenvalues \( (\gamma + 1)|x_1|^\gamma \) with multiplicity 1 and \( |x_1|^\gamma \) with multiplicity \( n_1 - 1 \). That is,
\[
(2.28) \quad D^2 \varphi^1_\gamma(x_1) = P \begin{pmatrix} (\gamma + 1)|x_1|^\gamma & 0 & \cdots & 0 \\ 0 & |x_1|^\gamma & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & |x_1|^\gamma \end{pmatrix} P^t,
\]
for some orthogonal \( n_1 \times n_1 \) matrix \( P \). It then follows that
\[
(2.29) \quad \lambda_\gamma \langle D^2 \varphi_\gamma(x)^{-1}\xi, \xi \rangle \leq \langle A_\gamma(x)\xi, \xi \rangle \leq \Lambda_\gamma \langle D^2 \varphi_\gamma(x)^{-1}\xi, \xi \rangle,
\]
holds for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^n \), with \( \lambda_\gamma := 1 \) and \( \Lambda_\gamma := \gamma + 1 \). That is, \( A_\gamma \in \mathcal{E}(\lambda_\gamma, \Lambda_\gamma, \varphi_\gamma, \Omega) \).

Now, as proved in Sections 11 and 12, the convex function \( \varphi_\gamma \) satisfies all of the hypotheses \textbf{H1–H6} (moreover, \( \varphi_\gamma \) satisfies \textbf{H5} for any \( p > n \)) and Theorem 1 will apply. More precisely, given a PDE of the form
\[
G_\gamma(u) + \langle \bar{b}, \nabla u \rangle + \bar{c} u = \bar{f}
\]
we can recast it in the form of (1.14) after dividing by \( |x_1|^\gamma \), to get
\[
\frac{1}{|x_1|^\gamma} \Delta_1 u(x) + \Delta_2 u(x) + \frac{1}{|x_1|^\gamma} \left( \langle \bar{b}(x), \nabla u(x) \rangle + \frac{\bar{c}(x)}{|x_1|^\gamma} u(x) \right) = \frac{\bar{f}(x)}{|x_1|^\gamma},
\]
which can be written as
\[
\text{trace}(A_\gamma(x)D^2 u(x)) + \langle b(x), \nabla u(x) \rangle + c(x) u(x) = f(x),
\]
where \( A_\gamma(x) \) is given by (2.25), \( b(x) := |x_1|^{-\gamma}D^2 \varphi_\gamma(x)^{1/2} \bar{b}(x), c(x) := \nabla(x)|x_1|^{-\gamma}, \) and \( f(x) := \bar{f}(x)|x_1|^{-\gamma} \).

On the other hand, notice that (2.26) implies that the Monge-Ampère measure of \( \varphi_\gamma \) is given by \( \mu_{\varphi_\gamma}(x) = \det D^2 \varphi_\gamma(x) = (\gamma + 1)|x_1|^{\gamma m_1} \). Consequently, given a set \( S \subset \Omega \), we have
\[
\int_S |(D^2 \varphi)^{-1/2} b(x)|^n d\mu_{\varphi_\gamma}(x) = \int_S |x_1|^{-n \gamma} |\bar{b}(x)|^n (\gamma + 1)|x_1|^{\gamma m_1} dx
\]
\[
= (\gamma + 1) \int_S |\bar{b}(x)|^n |x_1|^{\gamma(n_1 - n)} dx = (\gamma + 1) \int_S |\bar{b}(x)|^n |x_1|^{-\gamma n_2} dx.
\]
Similarly, \( c \in L^n(S, d\mu_{\phi_n}) \) and \( f \in L^n(S, d\mu_{\phi_n}) \) translate into \( \tau \in L^n(S, |x_1|^{-\gamma_2} dx) \) and \( \Upsilon \in L^n(S, |x_1|^{-\gamma_2} dx) \), respectively.

By bringing all this together, we can now state a regularity result for solutions to Grushin operators with lower-order coefficients and unbounded drifts. Namely,

**Theorem 3.** Fix an open bounded set \( \Omega \subset \mathbb{R}^n \), \( \gamma \geq 0 \), and consider the subelliptic PDE

\[
G_{\gamma}(u) + \langle \mathbf{b}, \nabla u \rangle + cu = \Upsilon
\]

where \( G_{\gamma} \) denotes the Grushin operator in (2.23). Then, there exist constants \( 0 < \tau < 1 < K_H \) (depending only on \( \gamma \) and \( n \)) as well as structural constants \( 0 < \varepsilon_H < 1 < M_H \), such that for every section \( S_{\phi_n}(z, R) \) with \( S_{K_H R} := S_{\phi_n}(z, K_H R) \subset \subset \Omega \) and every \( u \in C(S_{K_H R} \cap W^{2,1}(S_{K_H R})) \) satisfying \( u \geq 0 \) in \( S_{K_H R} \) and solving the subelliptic PDE (2.30) a.e. in \( S_{K_H R} \), we have that the inequalities

\[
\sup_{y \in S_{\phi_n}(z, R)} \left( \rho^{-\frac{1}{2}} |S_{\phi_n}(y, \rho)|^{\frac{1}{\gamma^2}} \| \mathbf{b} \|_{L^n(S_{\phi_n}(y, \rho), |x_1|^{-\gamma_2} dx)} \| D^2 \phi_n \|_{L^p(S_{\phi_n}(y, \rho), dx)} \right) \leq \varepsilon_H
\]

and

\[
|S_{K_H R}|^{\frac{1}{2}} \| \Upsilon \|_{L^n(S_{K_H R}, |x_1|^{-\gamma_2} dx)} \leq \varepsilon_H
\]

imply the Harnack inequality

\[
\sup_{S_{\phi_n}(z, \tau R)} u \leq M_H \left( \inf_{S_{\phi_n}(z, \tau R)} u + |S_{K_H R}|^{\frac{1}{2}} \| \Upsilon \|_{L^n(S_{K_H R}, |x_1|^{-\gamma_2} dx)} \right).
\]

Here \( p \), the exponent from hypothesis \( H5 \), can be taken as any number bigger than \( n \).

**Remark 4.** In order to illustrate how Theorem 3 can be used, let us break down the inequality (2.31). A discussion on the sections of convex functions given as tensor sums of other convex functions (such as \( \varphi_n \) above) is included in Sections 11 and 12. In particular, we have that, given \( y = (y_1, y_2) \in S_{\varphi_n}(z, R) \), the inclusions (11.119) and (12.135) yield

\[
S_{\varphi_n}(y, \rho) \subset S_{\varphi_1}(y_1, \rho) \times S_{\varphi_2}(y_2, \rho) \subset B_1(y_1, C_\gamma \rho^{\frac{1}{\gamma+2}}) \times B_2(y_2, (2\rho)^{1/2})
\]

and

\[
B_1(y_1, c_\gamma \rho^{\frac{1}{\gamma+2}}) \times B_2(y_2, 2\rho^{1/2}) \subset S_{\varphi_1}(y_1, \rho/2) \times S_{\varphi_2}(y_2, \rho/2) \subset S_{\varphi_n}(y, \rho),
\]

where \( B_1 \) and \( B_2 \) denote Euclidean balls in \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_2} \), respectively, and \( c_\gamma, C_\gamma > 0 \) depend only on \( \gamma \) and \( n_1 \). For the sake of conciseness, since the degeneracy of the PDE (2.30) effectively occurs on \( |z_1| = 0 \), let us assume that \( z = (z_1, z_2) \) with \( |z_1| < 1 \). Also, we can assume that \( 0 < R < 1 \). Hence, given \( y = (y_1, y_2) \in S_{\varphi_n}(z, R) \), \( \rho \in (0, R) \), and \( x = (x_1, x_2) \in S_{\varphi_n}(y, \rho) \) it follows from the inclusions above that \( |x_1| \leq |x_1 - y_1| + |y_1 - z_1| + |z_1| \leq \rho^{\frac{1}{\gamma+2}} + R^{\frac{1}{\gamma+2}} + 1 \leq 1 \), where the implied constants depend only on \( n_1 \) and \( \gamma \). Now, \( |x_1| \leq 1 \) along with (2.26) and (2.28) yields \( \| D^2 \phi_n \| \approx 1 \) on \( S_{\varphi_n}(y, \rho) \). Consequently, for any given \( p > n \), we get

\[
|S_{\varphi_n}(y, \rho)|^{-\frac{1}{p}} \| D^2 \phi_n \|_{L^p(S_{\varphi_n}(y, \rho), dx)} \approx 1.
\]

On the other hand, by taking Lebesgue measure on the inclusions above,

\[
\rho^{-\frac{1}{2}} |S_{\varphi_n}(y, \rho)|^{\frac{1}{\gamma^2}} \| \mathbf{b} \|_{L^n(S_{\varphi_n}(y, \rho), |x_1|^{-\gamma_2} dx)} \approx \rho^{-\frac{1}{2}} \rho^{\frac{1}{\gamma^2}} \left( \frac{n_1+2}{\gamma+2} \right) \| \mathbf{b} \|_{L^n(S_{\varphi_n}(y, \rho), |x_1|^{-\gamma_2} dx)}.
\]
so that the condition (2.31) can be recast as

\[
(2.33) \quad \sup_{y \in \mathcal{S}_{\varphi}(z,R)} \left( \rho^{\frac{1}{2}} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) \right) - \frac{1}{2} \left\| \mathbf{b} \right\|_{L^n(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)} \leq \tilde{\varepsilon}_H,
\]

which would place the drift \( \mathbf{b} \) in a Morrey space with respect to the weight \(|x_1|^{-\gamma n_2} dx\). Alternatively, the condition (2.33) can be realized by asking for higher integrability of \( \mathbf{b} \) with respect to a weight. More precisely, given \( \beta \geq 1 \), by Hölder’s inequality with exponents \( \beta \) and \( \beta' \), we have

\[
\left\| \mathbf{b} \right\|_{L^n(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)} \leq \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\beta n_2} dx)} |S_{\varphi}(y,\rho)| \frac{1}{n^\beta'}
\]

\[
\simeq \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\beta n_2} dx)} \rho \frac{1}{n^\beta'} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right),
\]

and then, by choosing \( \beta \) so that \( \frac{1}{n} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) - \frac{1}{2} + \frac{1}{n^\beta'} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) = 0 \), we get

\[
\rho \frac{1}{n} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) - \frac{1}{2} \left\| \mathbf{b} \right\|_{L^n(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)}
\]

\[
\leq \rho \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) - \frac{1}{2} \left( \frac{n_1}{\gamma+2} + \frac{n_2}{2} \right) \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)} = \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\beta n_2} dx)},
\]

and (2.33) is then implied by the condition \( \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)} \leq \tilde{\varepsilon}_H \) with

\[
(2.34) \quad \beta := \frac{n_1}{\gamma+2} + \frac{n_2}{2}.
\]

Notice that \( \beta \geq 1 \) for every \( \gamma \geq 0 \) with

\[
(2.35) \quad \frac{1}{\gamma+2} \geq \frac{1}{4} \left( 1 - \frac{n_2}{n_1} \right).
\]

Therefore, the condition (2.31) is equivalent to the weighted Morrey-space estimate (2.33) and it is weaker than the integrability condition \( \left\| \mathbf{b} \right\|_{L^{n\beta}(S_{\varphi}(y,\rho),|x_1|^{-\gamma n_2} dx)} \leq \tilde{\varepsilon}_H \) with \( \beta \) and \( \gamma \) as in (2.34) and (2.35).

In Section 4 we will extend Theorem 3 to a large class of degenerate PDEs that will include the Grushin operators in (2.23).

### 3. Applications to Subelliptic PDEs II: Extensions of Montanari’s example

#### 3.1. Montanari’s example

Let \( x = (x_1, x_2) \in \mathbb{R}^2 \) and consider the vector fields \( X_1 := \partial x_1 \) and \( X_2 := x_1 \partial x_2 \), in particular, notice that \([X_1, X_2] = \partial x_2\). In [28] and motivated by topics on the geometric theory of several complex variables, A. Montanari introduced the subelliptic operator

\[
(3.36) \quad L = a_{11} X_1^2 + 2a_{12} X_2 X_1 + a_{22} X_2^2,
\]

where the coefficient matrix \( a := (a_{ij})_{i,j=1} \) satisfies, for some constants \( 0 < \lambda_0 \leq \Lambda_0 \), the uniform ellipticity condition

\[
(3.37) \quad \lambda_0 |\xi|^2 \leq \langle a(x) \xi, \xi \rangle \leq \Lambda_0 |\xi|^2 \quad \forall x, \xi \in \mathbb{R}^2.
\]

Then, by means of a weighted version of the ABP maximum principle, she established a Harnack inequality for nonnegative classical \( C^2 \)-solutions to \( Lu = 0 \) with respect to balls of the Carnot-Carathéodory metric generated by \( X_1 \) and \( X_2 \).
In our next application, by means of Theorem 1 and the Monge-Ampère quasi-metric structure, we extend Montanari’s Harnack inequality to a family of subelliptic operators that include \( L \) in (3.36).

### 3.2. Extensions of Montanari’s example.

Fix an open bounded set \( \Omega \subset \mathbb{R}^2 \) and \( \nu \in \mathbb{N}_0 \).

Define the vector fields \( X_1 := \partial_{x_1} \) and \( X_2 := x_1^\nu \partial_{x_2} \), and introduce the subelliptic operator

\[
L_{\nu} = a_{11}X_1^2 + 2a_{12}X_2X_1 + a_{22}X_2^2,
\]

where the coefficient matrix \( a := (a_{ij})_{i,j=1}^2 \) satisfies (3.37) for a.e. \( x \in \Omega \).

Notice that in this case we have \([X_1, X_2] = \nu x_1^{-1} \partial_{x_2} \) and this commutator will vanish on \( x_1 = 0 \) whenever \( \nu > 1 \). However, the vector fields \( X_1 \) and \( X_2 \) will still satisfy Hörmander’s condition, but commutators of up to order \( \nu \) will be required.

Also, notice that, when \( \nu = 1 \), \( L_1 \) recovers \( L \) from (3.36).

Let us write \( L_{\nu} \) as \( L_{\nu}(u) = x_1^{2\nu} \text{trace}(A_{\nu}D^2u) \) where

\[
A_{\nu}(x) = A_{\nu}(x_1, x_2) := \begin{bmatrix} a_{11}(x)/x_1^{2\nu} & a_{12}(x)/x_1^{\nu} \\ a_{12}(x)/x_1^{\nu} & a_{22}(x) \end{bmatrix}
\]

and introduce the convex function \( \varphi_{\nu} : \mathbb{R}^2 \to \mathbb{R} \) as \( \varphi_{\nu}(x) := \frac{1}{(2\nu+1)(2\nu+2)} x_1^{2\nu+2} + \frac{1}{2} x_2^2 \) so that

\[
D^2\varphi_{\nu}(x) = \begin{bmatrix} x_1^{2\nu} & 0 \\ 0 & 1 \end{bmatrix}
\]

and

\[
D^2\varphi_{\nu}(x)^{1/2}A_{\nu}(x)D^2\varphi_{\nu}(x)^{1/2} = \begin{bmatrix} a_{11}(x)/x_1^{2\nu} & a_{12}(x)/x_1^{\nu} \\ a_{12}(x)/x_1^{\nu} & a_{22}(x) \end{bmatrix}.
\]

Then, the ellipticity condition (3.37) implies that

\[
\lambda_0 |\xi|^2 \leq \langle D^2\varphi_{\nu}(x)^{1/2}A_{\nu}(x)D^2\varphi_{\nu}(x)^{1/2}\xi, \xi \rangle \leq \Lambda_0 |\xi|^2
\]

holds for a.e. \( x \in \Omega \) and every \( \xi \in \mathbb{R}^n \), that is, \( A_{\nu} \in \mathcal{E}(\lambda_0, \Lambda_0, \varphi_{\nu}, \Omega) \). Next, let us write any subelliptic PDE of the form

\[
L_{\nu}(u) + \langle \bar{b}, \nabla u \rangle + \bar{c}u = \bar{f}
\]

as

\[
\text{trace}(A_{\nu}D^2u) + \langle b, \nabla^\varphi u \rangle + cu = f,
\]

where \( b(x) := x_1^{-2\nu}D^2\varphi_{\nu}(x)^{1/2}\bar{b}(x) \), \( \bar{c}(x) := \bar{c}(x) x_1^{-2\nu} \), and \( f(x) := \bar{f}(x) x_1^{-2\nu} \), and notice that, since the Monge-Ampère measure of \( \varphi_{\nu} \) equals \( \mu_{\varphi_{\nu}}(x) = \det D^2\varphi_{\nu}(x) = x_1^{2\nu} \), given \( S \subset \Omega \) the condition \( |(D^2\varphi_{\nu})^{-1/2}b| \in L^2(S, d\mu_{\varphi_{\nu}}) \) from (1.13) translates into

\[
\int_S |(D^2\varphi_{\nu})^{-1/2}b(x)|^2 d\mu_{\varphi_{\nu}}(x) = \int_S x_1^{-4\nu}|\bar{b}(x)|^2 x_1^{2\nu} dx = \int_S |\bar{b}(x)|^2 x_1^{-2\nu} dx < \infty,
\]

and, similarly, \( c \in L^2(S, d\mu_{\varphi_{\nu}}) \) and \( f \in L^2(S, d\mu_{\varphi_{\nu}}) \) translate into \( \bar{c} \in L^2(S, x_1^{-2\nu}) \) and \( \bar{f} \in L^2(S, x_1^{-2\nu}) \), respectively. Thus, by bringing things together, Theorem 1 yields the following Harnack inequality for nonnegative solutions to the subelliptic PDEs (3.40) with unbounded drifts.

**Theorem 5.** Fix an open bounded set \( \Omega \subset \mathbb{R}^2 \), \( \nu \in \mathbb{N}_0 \), and consider the subelliptic PDE

\[
L_{\nu}(u) + \langle \bar{b}, \nabla u \rangle + \bar{c}u = \bar{f}
\]

where \( L_{\nu} \) denotes the subelliptic operator in (3.38). Then, there exist constants \( 0 < \tau < 1 < K_H \) (depending only on \( \nu \)) as well as structural constants \( 0 < \varepsilon_H < 1 < M_H \), such that
for every section \( S_{\varphi}(z, R) \) with \( S_{K_H R} := S_{\varphi}(z, K_H R) \subset \Omega \) and every \( u \in C(S_{K_H R}) \cap W^{2,2}(S_{K_H R}) \) satisfying \( u \geq 0 \) in \( S_{K_H R} \) and solving the subelliptic PDE (3.41) a.e. in \( S_{K_H R} \), we have that the inequalities

\[
(3.42) \quad \sup_{\rho \leq r} \left( \rho^{-\frac{1}{2}} \|S_{\varphi}(y, \rho)\|^{\frac{1}{2} - \frac{1}{2p}} \|D^2 \varphi\|_{L^p(S_{\varphi}(y, \rho), dx)} \right) \leq \varepsilon_H
\]

and

\[
(3.43) \quad |S_{K_H R}|^{\frac{1}{2}} \|\overline{\tau}\|_{L^2(S_{K_H R}, |x_1|^{-2\nu} dx)} \leq \varepsilon_H
\]

imply the Harnack inequality

\[
\sup_{S_{\varphi}(z, \tau R)} u \leq M_H \left( \inf_{S_{\varphi}(z, \tau R)} u + |S_{K_H R}|^{\frac{1}{2}} \|\overline{\tau}\|_{L^2(S_{K_H R}, |x_1|^{-2\nu} dx)} \right).
\]

Here \( p \), the exponent from hypothesis \( H5 \), can be taken as any number bigger than 2.

**Remark 6.** A comment on the condition (3.42) follows along the lines of Remark 4. For instance, assuming \( z := (z_1, z_2) \) with \( |z_1| < 1 \) and \( 0 < R < 1 \) we get that \( \|D^2 \varphi\| \simeq 1 \) on \( S_{\varphi}(y, \rho) \). In this case the condition (3.42) turns out to be equivalent to the Morrey-space estimate

\[
(4.44) \quad \sup_{\rho \leq r} \left( \rho^{-\frac{1}{1+2\nu}} \|\overline{\tau}\|_{L^2(S_{\varphi}(y, \rho), |x_1|^{-2\nu} dx)} \right) \leq \tilde{\varepsilon}_H,
\]

as well as weaker than the integrability \( \|\overline{\tau}\|_{L^{2\beta}(S_{\varphi}(y, \rho), |x_1|^{-2\nu} dx)} \leq \tilde{\varepsilon}_H \) with \( \beta := \frac{2\nu + 2}{2} \).

We close this application by mentioning that the weight in the weighted ABP maximum principle from [28, Theorem 2.5] is precisely the Monge-Ampère measure \( \rho \) and that the Monge-Ampère quasi-distance \( \delta_{\varphi} \) provides an equivalent gauge to the Carnot-Carathéodory and Grushin metrics. In the case of \( \varphi \) as above, by [9, Theorem 4(iii)] and (12.136) from Section 12 we have

\[
\delta_{\varphi}(x_1, x_2, (y_1, y_2)) \simeq (x_1^{2\nu+1} - y_1^{2\nu+1})(x_1 - y_1) + (x_2 - y_2)^2, \quad \forall (x_1, x_2, (y_1, y_2)) \in \mathbb{R}^2,
\]

where the implied constants depend only on \( \nu \). Several equivalent distances and quasi-distances were considered in [28, Section 2].

### 4. Applications to Other Singular and Degenerate Elliptic Operators

In this section we extend the class of subelliptic Grushin operators from Section 2, introduce a related singular elliptic operator, and establish a Harnack inequality for their corresponding PDEs including unbounded drifts.

Fix \( m \in \mathbb{N}, n_1, \ldots, n_m \in \mathbb{N} \), and set \( n := n_1 + \cdots + n_m \). Fix \( \Omega \subset \mathbb{R}^n \) and for each \( j = 1, \ldots, m \), let \( \Omega_j \) denote the projection of \( \Omega \) over \( \mathbb{R}^{n_j} \) and let the functions \( \Gamma_j : \Omega_j \rightarrow \mathbb{R} \) satisfy

\[
(4.45) \quad \lambda_j |x_j|^\gamma_j \leq \Gamma_j(x_j) \leq \Lambda_j |x_j|^\gamma_j \quad \text{a.e. } x_j \in \Omega_j,
\]

for some constants \( 0 < \lambda_j \leq \Lambda_j \) and \( \gamma := (\gamma_1, \ldots, \gamma_m) \in [0, \infty)^m \). Notice that only measurability is required from the \( \Gamma_j \)'s.

Introduce the convex function \( \varphi_\gamma : \mathbb{R}^n \rightarrow \mathbb{R} \) as the tensor sum

\[
\varphi_\gamma(x) := \frac{1}{(1 + \gamma_1)(2 + \gamma_1)} |x_1|^{2+\gamma_1} + \cdots + \frac{1}{(1 + \gamma_m)(2 + \gamma_m)} |x_m|^{2+\gamma_m},
\]
for $x = (x_1, \ldots, x_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} = \mathbb{R}^n$, so that $D^2 \varphi_\gamma$ is a direct sum, for $j = 1, \ldots, m$, of $n_j \times n_j$ matrices of the form

$$
P_j \begin{bmatrix} (1 + \gamma_j)x_j^{\gamma_j} & 0 & \cdots & 0 \\
0 & |x_j|^{\gamma_j} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & |x_j|^{\gamma_j} \end{bmatrix} P_j^t,
$$

for some $n_j \times n_j$ orthogonal matrix $P_j$. In particular, $D^2 \varphi_\gamma(x) = \prod_{j=1}^m (1 + \gamma_j)|x_j|^{n_j \gamma_j}$.

### 4.1. The degenerate case: Grushin-like operators.

Let us consider the following degenerate elliptic operator

$$(4.46) \quad G_\Gamma(u) := \Pi_1(x)\Delta_1 u(x) + \cdots + \Pi_m(x)\Delta_m u(x),$$

where $\Delta_j$ is the Laplacian on $\mathbb{R}^{n_j}$ and, for $j = 1, \ldots, m$,

$$\Pi_j(x) := \prod_{k \neq j} \Gamma_k(x_k).$$

For instance, if $m = 2$ and $\lambda_j = \Lambda_j = 1$ for $j = 1, 2$, then we have

$$G_\Gamma(u) := |x_2|^{\gamma_2} \Delta_1 u(x) + |x_1|^{\gamma_1} \Delta_2 u(x),$$

so that the choice $\gamma_2 = 0$ recovers the Grushin operator (2.23), and, for general $0 < \lambda_j \leq \Lambda_j$, we have

$$G_\Gamma(u) := \Gamma_2(x_2)\Delta_1 u(x) + \Gamma_1(x_1)\Delta_2 u(x),$$

with $\lambda_j|x_j|^{\gamma_j} \leq \Gamma_j \leq \Lambda_j|x_j|^{\gamma_j}$ and $j = 1, 2$. If $m = 3$ we have

$$G_\Gamma(u) := \Gamma_2(x_2)\Gamma_3(x_3)\Delta_1 u(x) + \Gamma_1(x_1)\Gamma_3(x_3)\Delta_2 u(x) + \Gamma_1(x_1)\Gamma_2(x_2)\Delta_3 u(x),$$

e tc. Now, given a PDE of the form

$$G_\Gamma(u) + \langle \vec{b}, \nabla u \rangle + \bar{c} u = \bar{f},$$

after dividing it by the product $\Pi_\Gamma(x) := \prod_{j=1}^m \Gamma_j(x_j)$, it turns into

$$\text{trace}(A_\Gamma(x)D^2 u(x)) + \langle b(x), \nabla u(x) \rangle + c(x)u(x) = f(x),$$

where $A_\Gamma$ is the direct sum

$$(4.47) \quad A_\Gamma(x) := \begin{bmatrix} \frac{1}{\Gamma(x_1)} I_{n_1 \times n_1} & 0 & \cdots & 0 \\
0 & \frac{1}{\Gamma(x_2)} I_{n_2 \times n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \frac{1}{\Gamma(x_m)} I_{n_m \times n_m} \end{bmatrix},$$

and $b(x) := \Pi_\Gamma(x)^{-1}D^2 \varphi_\gamma(x)^{1/2}\bar{b}(x)$, $c(x) := \Pi_\Gamma(x)^{-1}\bar{c}(x)$, and $f(x) := \Pi_\Gamma(x)^{-1}\bar{f}(x)$. In particular, it follows that $A_\Gamma \in \mathcal{E}(\lambda_\Gamma, \Lambda_\Gamma, \varphi_\gamma, \Omega)$, for some constants $0 < \lambda_\Gamma \leq \Lambda_\Gamma$ depending only on the $\lambda_j$’s and $\Lambda_j$’s in (4.45).
Notice that, given \( S \subset \Omega \), the condition \( |(D^2 \varphi_\gamma)^{-1/2} b| \in L^n(S, d \mu_{\varphi_\gamma}) \) from (1.13) means
\[
\int_S |(D^2 \varphi_\gamma)^{-1/2} b(x)|^n d \mu_{\varphi_\gamma}(x) = \int_S \Pi_\Gamma(x)^{-\frac{n}{m}} |\bar{b}(x)|^n \prod_{j=1}^m (1 + \gamma_j |x_j|)^{n_{\gamma_j}} \, dx \\
\simeq \int_S |\bar{b}(x)|^n \prod_{j=1}^m |x_j|^{-(n-n_j)\gamma_j} \, dx < \infty,
\]
where the implicit constants depend only on the \( \gamma_j \)'s and the \( \lambda_j \)'s from (4.45). Similarly with \( c, f \in L^n(S, d \mu_{\varphi_\gamma}) \). Thus, Theorem 1 yields the following Harnack inequality.

**Theorem 7.** Fix an open bounded set \( \Omega \subset \mathbb{R}^n, \gamma = (\gamma_1, \ldots, \gamma_m) \in [0, \infty)^m \), and consider the degenerate PDE
\[
(4.48) \quad G_\Gamma(u) + \langle \bar{b}, \nabla u \rangle + \varpi u = \bar{f}
\]
where \( G_\Gamma \) denotes the degenerate operator in (4.46). Then, there exist constants \( 0 < \tau < 1 < K_H \) (depending only on \( \gamma \) and \( n \)) as well as structural constants \( 0 < \varepsilon_H < 1 < M_H \), such that for every section \( S_{\varphi_\gamma}(z, R) \) with \( S_{K_H R} := S_{\varphi_\gamma}(z, K_H R) \subset \Omega \) and every \( u \in C(\overline{S_{K_H R}}) \cap W^{2,n}(S_{K_H R}) \) satisfying \( u \geq 0 \) in \( S_{K_H R} \) and solving the degenerate PDE (4.48) a.e. in \( S_{K_H R} \), we have that the inequalities
\[
(4.49) \quad \sup_{y \in S_{\varphi_\gamma}(z, \tau R)} \left( \rho^{-\frac{1}{2}} |S_{\varphi_\gamma}(y, \rho)|^{\frac{1}{2}} \| \bar{b} \|_{L^n(S_{\varphi_\gamma}(y, \rho), \prod_{j=1}^m |x_j|^{-(n-n_j)\gamma_j} \, dx} \| D^2 \varphi_\gamma \|_{L^p(S_{\varphi_\gamma}(y, \rho), dx)} \right) \leq \varepsilon_H
\]
and
\[
|S_{K_H R}|^{\frac{1}{2}} \| \varpi \|_{L^n(S_{K_H R}, \prod_{j=1}^m |x_j|^{-(n-n_j)\gamma_j} \, dx)} \leq \varepsilon_H
\]
imply the Harnack inequality
\[
\sup_{S_{\varphi_\gamma}(z, \tau R)} u \leq M_H \left( \inf_{S_{\varphi_\gamma}(z, \tau R)} u + |S_{K_H R}|^{\frac{1}{2}} \| \bar{f} \|_{L^n(S_{K_H R}, \prod_{j=1}^m |x_j|^{-(n-n_j)\gamma_j} \, dx)} \right).
\]

Again, here \( p \) is any number bigger than \( n \) and, regarding the Lebesgue measure of the sections \( S_{\varphi_\gamma}(z, R) \), we have
\[
(4.50) \quad |S_{\varphi_\gamma}(z, R)| \simeq R^{\sum_{j=1}^m \frac{n_j}{2+\gamma_j}} \quad \forall z \in \mathbb{R}^n, R > 0,
\]
where the implicit constants depend only on \( m \) and the \( n_j \)'s for \( j = 1, \ldots, m \).

### 4.2. The singular case.
In this case let us consider the singular elliptic operator
\[
(4.51) \quad H_\Gamma(u) = \frac{1}{\Gamma_m(x_1)} \Delta_1 u(x) + \cdots + \frac{1}{\Gamma_m(x_m)} \Delta_m u(x)
\]
which can be written as \( H_\Gamma(u) = \text{trace}(A_\Gamma D^2 u) \) where \( A_\Gamma \) is the \( n \times n \) matrix in (4.47). Reasoning along the lines of the previous example, we obtain the following Harnack inequality for the singular operator \( H_\Gamma \) with unbounded drifts.
Theorem 8. Fix an open bounded set $\Omega \subset \mathbb{R}^n$, $\gamma = (\gamma_1, \ldots, \gamma_m) \in [0, \infty)^m$, and consider the singular PDE

\[ H_\Gamma(u) + \langle b, \nabla u \rangle + cu = f \]

where $H_\Gamma$ denotes the singular operator in (4.51). Then, there exist constants $0 < \tau < 1 < K_H$ (depending only on $\gamma$ and $n$) as well as structural constants $0 < \varepsilon_H < 1 < M_H$, such that for every section $S_{\varphi_\gamma}(z, R)$ with $S_{K_H R} := S_{\varphi_\gamma}(z, K_H R) \subset \subset \Omega$ and every $u \in C(S_{K_H R}) \cap W^{2,n}(S_{K_H R})$ satisfying $u \geq 0$ in $S_{K_H R}$ and solving the singular PDE (4.52) a.e. in $S_{K_H R}$, we have that the inequalities

\[
\sup_{y \in S_{\varphi_\gamma}(z, R), \rho \leq r} \left( \rho^{-\frac{1}{2}} |S_{\varphi_\gamma}(y, \rho)|^{\frac{1}{n}} - \frac{1}{2} \|B\| L^n(S_{\varphi_\gamma}(y, \rho), \prod_{j=1}^m |x_j|^{\gamma_j} \, dx) \right) \leq \varepsilon_H
\]

and

\[
|S_{K_H R}|^{\frac{1}{2}} \|\mathbf{f}\| L^n(S_{K_H R}, \prod_{j=1}^m |x_j|^{\gamma_j} \, dx) \leq \varepsilon_H
\]

imply the Harnack inequality

\[
\sup_{S_{\varphi_\gamma}(z, R)} u \leq M_H \left( \inf_{S_{\varphi_\gamma}(z, R)} u + |S_{K_H R}|^{\frac{1}{n}} \|\mathbf{f}\| L^n(S_{K_H R}, \prod_{j=1}^m |x_j|^{\gamma_j} \, dx) \right).
\]

Once more, here $p$ is any number bigger than $n$ and the Lebesgue measure of a section $S_{\varphi_\gamma}(z, R)$ behaves as in (4.50).

Remark 9. The conditions (4.49) and (4.53) can be commented upon as done in Remarks 4 and 6 and we leave the details to the interested reader.

5. Preliminaries

5.1. Geometric and structural constants. Constants depending only on dimension $n$, on the constants $C_d$ and $\alpha$ from the doubling condition $\mu_\varphi \in (DC)_\varphi$ in (1.6), and on the function $\zeta$ from hypothesis H3 will be called geometric constants.

Constants depending only on $\lambda$ and $\Lambda$ in (1.2), on the exponent $p > n$ from hypothesis H5, on the constant $\sigma > 0$ from hypothesis H6, on $\|(D^2 \varphi)^{-1/2} b\|_{L^n(\Omega, d\mu_\varphi)}$, as well as on geometric constants, will be called structural constants.

5.2. Doubling properties. By [3, Lemma 5.2], and this requires only the hypothesis H1, we have the following doubling property for the Lebesgue measure on Monge-Ampère sections

\[
|S_\varphi(x, 2r)| \leq 2^n |S_\varphi(x, r)| \quad \forall x \in \mathbb{R}^n, r > 0,
\]

which in turn implies

\[
|S_\varphi(x, s)| \leq 2^n \left( \frac{s}{r} \right)^n |S_\varphi(x, r)| \quad \forall x \in \mathbb{R}^n, 0 < r < s.
\]

By [27, Lemma 2.1], under hypotheses H1 and H2 there exists a geometric constant $K_D > 1$ such that

\[
\mu_\varphi(S_\varphi(x, s)) \leq K_D \left( \frac{s}{r} \right)^n \mu_\varphi(S_\varphi(x, r)) \quad \forall x \in \mathbb{R}^n, 0 < r < s.
\]

From [8, Theorem 8], there exist geometric constants $0 < \kappa \leq 1 \leq K_2$ such that

\[
r^n r^n \leq |S_\varphi(x, r)| \mu_\varphi(S_\varphi(x, r)) \leq K_2^n r^n \quad \forall x \in \mathbb{R}^n, r > 0.
\]
5.3. **On the hypothesis H3.** In the literature on real analysis on spaces of homogeneous type, the inequality (1.8) is sometimes referred to as the *annular decay property* or *ring condition* and it quantifies the relative smallness if thin annuli with respect to \( \mu_\varphi \). Typical choices for the function \( \zeta \) include \( \zeta(\epsilon) = C\epsilon^{\alpha_0} \) or \( \zeta(\epsilon) = C|\log(\epsilon)|^{-\alpha_0} \) for some constants \( C_0, \alpha_0 > 0 \).

In this section we point out that (1.8) is implied by the following Coifman-Fefferman condition for \( \mu_\varphi \) (see [3, Section 5]): There exist constants \( C_0, \theta_0 > 1 \) such that for every section \( S := S_\varphi(x,r) \) and every Borel set \( E \subset S \) it holds true that

\[
\frac{\mu_\varphi(E)}{\mu_\varphi(S)} \leq C_0 \left( \frac{|E|}{|S|} \right)^{\theta_0}.
\]

Indeed, by Lemma 5.2(b) from [3] for every \( x \in \mathbb{R}^n \), \( t > 0 \) and \( \delta \in (0,1) \) we have

\[
|S_\varphi(x,t) \setminus S_\varphi(x,\delta t)| \leq n(1-\delta)|S_\varphi(x,t)|,
\]

so that, given \( R > 0 \) and \( \epsilon \in (0,1) \), by taking \( t := R(1+\epsilon) \) and \( \delta := (1+\epsilon)^{-1} \) we get

\[
|S_\varphi(x,(1+\epsilon)R) \setminus S_\varphi(x,R)| = |S_\varphi(x,t) \setminus S_\varphi(x,\delta t)| \leq n(1-\delta)|S_\varphi(x,t)|
\]

\[
= \frac{n\epsilon}{1+\epsilon} |S_\varphi(x,(1+\epsilon)R)| \leq n\epsilon |S_\varphi(x,(1+\epsilon)R)|.
\]

Hence, by using (5.58) with \( S := S_\varphi(x,(1+\epsilon)R) \) and \( E := S_\varphi(x,(1+\epsilon)R) \setminus S_\varphi(x,R) \) we obtain (1.8) with \( \zeta(\epsilon) := C_0 n^{\theta_0}\epsilon^{\theta_0} \) for every \( \epsilon \in (0,1) \).

The condition (5.58) is equivalent to the \( \mu_{\infty} \)-condition for \( \mu_\varphi \) mentioned in Section 1.3 of the Introduction, see [3, Section 5] for further characterizations. In Sections 11 and 12 we will see that the condition (5.58) is satisfied by the power functions \(|x|^p\), with \( p > 1 \), as well as their tensor sums.

5.4. **The Aleksandrov–Bakelman–Pucci maximum principle.** We will use the following version of the ABP maximum principle. See [15, Section 9.1 and Exercise 9.3] and [4, Chapter 6, pp.79-87].

**Lemma 10.** Suppose that \( \varphi \) satisfies H1 and H4. Fix an open, convex, bounded set \( S \subset \mathbb{R}^n \) and consider the operator

\[
L(h)(x) := \text{trace}(A(x)D^2h(x)) + \langle D^2\varphi(x)^{-\frac{1}{2}}b(x), \nabla h(x) \rangle + c(x)h(x),
\]

where \( A(x) \) is a symmetric, nonnegative-definite \( n \times n \) matrix and \( c(x) \leq 0 \) for a.e. \( x \in S \). Given \( E \subset S \) define \( N(b,E) \) as

\[
N(b,E)^n := \exp \left[ \frac{2^{n-2}}{n^n \omega_n} \left( 1 + \int_E \frac{|D^2\varphi(x)^{-\frac{1}{2}}b(x)|^n \, dx}{\det A(x)} \right) \right] - 1.
\]

Then, there exists a dimensional constant \( C_{abp} > 0 \) such that for every \( h \in C(S) \cap W^{2,n}_{\text{loc}}(S) \) satisfying \( L(h)(x) \geq g(x) \) a.e. \( x \in S \) the following hold true:

(i) If \( \sup_{\partial S} h \leq 0 \) then

\[
\frac{\max_{S} h}{\sqrt{|S|}} \leq C_{abp} N(b,S)^{\frac{n}{2}} \left\| \frac{g}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(S)}.
\]
Remark \(x,y\) such that for every 

\[ |\nabla(5.63)| \]

\[ |\nabla(5.64)| \]

The role of the convexity of \( S \) in Lemma 10 is to allow for \(|S|^{\frac{1}{n}}\), instead of the diameter of \( S \) (as in [15, Theorem 9.1] and [4, Theorem 1.9]), on the right-hand side of (5.61). Indeed, the convexity of \( S \) and F. John’s lemma imply the existence of an affine transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) such that \( B(0,n^{-1/2}) \subset T(S) \subset B(0,1) \) and the change of variable \( y := Tx \) yields (5.61).

This technique also proves a version of Lemma 10 on sets of the form \( S^0 := S \setminus S_0 \) for any \( S_0 \subset S \). Namely, if \( h \in C(S^0) \cap W^{2,n}_{loc}(S^0) \) with \( h \leq 0 \) on \( \partial S^0 \) satisfying \( L(h)(x) \geq g(x) \), for a.e. \( x \in S^0 \), with \( c(x) \leq 0 \) a.e. \( x \in S^0 \), we have

\[ \sup_{S^0} h \leq C_{abp}N(b,S)|S|^{\frac{1}{n}} \left\| \frac{g}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(S)}. \]

Notice the factor \(|S|^{\frac{1}{n}}\) (as opposed to \(|S^0|^{\frac{1}{n}}\)) on the right-hand side of (5.62). The estimate (5.62) will be used in the proof of Theorem 20 with \( S = S_\varphi(x_0,R) \) and \( S^0 \) an annulus of the form \( S_\varphi(x_0,R) \setminus S_\varphi(x_0,r) \) for some \( x_0 \in \mathbb{R}^n \) and \( 0 < r < R \).

Remark 12. Notice the misprint on [15, p.224] (repeated on [4, p.87]) where, instead of the right-hand side of (5.59), it reads

\[
\exp \left[ \frac{2^{n-2}}{n^2 \omega_n} \int_{\Gamma_h(S)} \left( 1 + \frac{|D^2 \varphi(x)^{-\frac{1}{2}} b(x)|^n}{\det A(x)} \right) dx \right] - 1,
\]

which would make for a spurious \(|\Gamma_h(S)|\).

5.5. A Morrey-type estimate. The next lemma is a consequence of the existence of weak Poincaré inequalities in quasi-metric spaces, see for instance [1, Proposition 5.48], we include a proof of the Monge-Ampère version in the Appendix in Section 13.

Lemma 13. Suppose that \( \varphi \) satisfies H1 and H2. Given \( q > 2n \), there exists a constant \( C_{q,K} > 0 \), depending only on \( q \) and \( K \), where \( K \geq 1 \) is the quasi-triangle constant in (1.7), such that for every \( x,y \in \Omega \) with \( S_\varphi(x,2K\delta_\varphi(x,y)) \subset \subset \Omega \) and \( h \in C^1(S_\varphi(x,2K\delta_\varphi(x,y))) \) we have

\[ |h(x) - h(y)| \leq C_{q,K} \delta_\varphi(x,y)^{\frac{1}{2}} \left( \int_{S_\varphi(x,2K\delta_\varphi(x,y))} |\nabla \varphi h(z)|^q dz \right)^{\frac{1}{q}}. \]

Remark 14. Lemma 13 will be used as follows: For \( j = 1, \ldots, n \), notice that the function \( \varphi_j \) satisfies

\[ |\nabla \varphi_j|^2 = (D^2 \varphi)^{-1} \nabla \varphi_j, \nabla \varphi_j = \varphi_{ij} \leq \Delta \varphi, \]

hence, by applying (5.63) with \( h = \varphi_j \) we obtain

\[ |\nabla \varphi(x) - \nabla \varphi(y)| \leq n^{1/2} C_{q,K} \delta_\varphi(x,y)^{\frac{1}{2}} \left( \int_{S_\varphi(x,2K\delta_\varphi(x,y))} |\Delta \varphi(z)|^{q/2} dz \right)^{\frac{1}{q}}. \]
In particular, given a section $S_\varphi(x_0, 2Kr) \subset \Omega$, the inequality (5.64) with $q := 2p$, where $p > n$ is as in the hypothesis $H5$, implies that

$$\lvert \nabla \varphi(x_0) - \nabla \varphi(x) \rvert \leq K(p)r^{\frac{1}{2}} \left( \frac{1}{\int_{S_\varphi(x_0, 2Kr)} \lVert D^2 \varphi(z) \rVert^p dz} \right)^{\frac{1}{p}},$$

for every $x \in S_\varphi(x_0, 2Kr)$, where $K(p) := n^{3/2}C_{2p,K}$. We will repeatedly use the inequality (5.65) to deal with presence of the unbounded drift $b$ in the first-order term of $L_\phi^\varphi$.

Notice that if we set $\rho_\varphi(x, y) := \delta_\varphi(x, y)^{\frac{1}{2}}$ for every $x, y \in \Omega$, then the inequality (5.65) gives that $\nabla \varphi$ is locally Lipschitz in $\Omega$ with respect to the quasi-distance $\rho_\varphi$.

5.6. **A local Vitali covering lemma under** $\mu_\varphi \in (DC)_\varphi$. Monge-Ampère versions of Vitali’s covering lemma under the hypothesis that $\mu_\varphi \sim 1$ (in the sense of (1.21)) have appeared, for instance, in [7, Lemma 1] and [20, Lemma 2.15]. For the sake of completeness, in the Appendix we include a proof for the following local version of Vitali’s covering lemma under the assumption $\mu_\varphi \in (DC)_\varphi$ only and with an explicit value of the Vitali constant $K_V > 1$ in terms of the quasi-triangle constant $K$ from (1.7).

**Lemma 15.** (Local Vitali covering lemma for Monge-Ampère sections.) Suppose that $\varphi$ satisfies $H1$ and $H2$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. Fix $K_V > 2K^2 + K$, where $K$ is the quasi-triangle constant from (1.7). Given a section $S_0 := S_\varphi(x_0, R_0)$ with $S_\varphi(x_0, 2R_0) \subset \subset \Omega$, a subset $E \subset S_0$, and a covering $\{S_\varphi(x, r)\}_{x \in I}$ of $E$ such that $S_\varphi(x, 2r) \subset \subset S_0$ for every $x \in I$, there exists a finite/countable sub-collection $\{S_\varphi(x_j, r_{x_j})\}_{j \in J}$ of pairwise disjoint sections such that

$$\bigcup_{x \in I} S_\varphi(x, r_x) \subset \bigcup_{j \in J} S_\varphi(x_j, K_Vr_{x_j}).$$

6. **On the construction of Monge-Ampère geodesics and the segment properties for $\delta_\phi$**

The results proved in this Section 6 will be used in Section 10, but they contribute to the study of geometric properties of sections of general convex functions. As such they are of independent interest and will require no doubling conditions on the Monge-Ampère measure.

Let $\Omega_0 \subset \mathbb{R}^n$ be an open convex set. For a strictly convex function $\phi \in C^1(\Omega_0)$ set

$$S_\phi(x, r) := \{x \in \Omega_0 : \delta_\phi(x, y) < r\}$$

where

$$\delta_\phi(x, y) := \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle \quad \forall x, y \in \Omega_0.$$

**Theorem 16.** Let $\Omega_0 \subset \mathbb{R}^n$ be an open convex set and $\phi \in C^1(\Omega_0)$ be a strictly convex function. Then, the Monge-Ampère quasi-distance $\delta_\phi$ possesses the following two properties:

(i) **The segment property:** Given $x, z \in \Omega_0$ with $S_\phi(x, \delta_\phi(x, z)) \subset \subset \Omega_0$ and $0 < r < \delta_\phi(x, z)$, there exists $y \in S_\phi(x, \delta_\phi(x, z))$ such that $\delta_\phi(x, y) = r$ and

$$\delta_\phi(x, y) + \delta_\phi(y, z) = \delta_\phi(x, z). \tag{6.66}$$

(ii) **The segment-prolongation property:** Given $x, z \in \Omega_0$ and $R > \delta_\phi(x, z)$ with $S_\phi(x, R) \subset \subset \Omega_0$, there exists $y \in \partial S_\phi(x, R)$ (that is, $\delta_\phi(x, y) = R$) such that

$$\delta_\phi(x, y) = \delta_\phi(x, z) + \delta_\phi(z, y). \tag{6.67}$$
Proof. In order to prove (6.66), given \( x, z \in \Omega_0 \) with \( S_{\phi}(x, \delta_{\phi}(x, z)) \subset \subset \Omega_0 \) and \( 0 < r < \delta_{\phi}(x, z) \) let \( P \) be any hyperplane passing through \( z \) and tangent to \( \partial S_{\phi}(x, r) \) and let \( y \in P \cap \partial S_{\phi}(x, r) \), see Figure 1.

Since \( y \in \partial S_{\phi}(x, r) \) if and only if \( y \) belongs to the level set \( \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle = r \), it follows that the vector \( \nabla \phi(y) - \nabla \phi(x) \) is orthogonal to \( P \) and then to \( z - y \). That is, \( \langle \nabla \phi(x) - \nabla \phi(z), z - y \rangle = 0 \), so that

\[
\delta_{\phi}(x, y) + \delta_{\phi}(y, z) = \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle + \phi(z) - \phi(y) - \langle \nabla \phi(y), z - y \rangle \\
= \phi(z) - \phi(x) - \langle \nabla \phi(x), z - x \rangle + \langle \nabla \phi(x), z - x \rangle - \langle \nabla \phi(x), y - x \rangle - \langle \nabla \phi(y), z - y \rangle \\
= \delta_{\phi}(x, z) + \langle \nabla \phi(x) - \nabla \phi(z), z - y \rangle = \delta_{\phi}(x, z) + 0 = \delta_{\phi}(x, z).
\]

Notice that the choice of \( y \) as in (6.66) is by no means unique since each hyperplane \( P \) will produce one such \( y \). Similarly, in order to prove (6.67), given \( x, z \in \Omega_0 \) and \( R > \delta_{\phi}(x, z) \) with \( S_{\phi}(x, R) \subset \subset \Omega_0 \), let \( P \) be the hyperplane tangent to \( S_{\phi}(x, \delta_{\phi}(x, z)) \) at \( z \) (notice that \( z \in \partial S_{\phi}(x, \delta_{\phi}(x, z)) \)), that is, \( z \) belongs to the level set \( \phi(z) - \phi(x) - \langle \nabla \phi(x), z - x \rangle = r_{xz} \), where \( r_{xz} := \delta_{\phi}(x, z) \), see Figure 2.

Now, pick any \( y \in P \cap \partial S_{\phi}(x, R) \) to obtain \( \delta_{\phi}(x, y) = R \) as well as the fact that the vector \( \nabla \phi(z) - \nabla \phi(x) \) is orthogonal to \( P \) and then to \( z - y \). That is, \( \langle \nabla \phi(x) - \nabla \phi(z), y - z \rangle = 0 \),
so that
\[
\delta_\phi(x, z) + \delta_\phi(z, y) = \phi(z) - \phi(x) - \langle \nabla \phi(x), z - x \rangle + \phi(y) - \phi(z) - \langle \nabla \phi(z), y - z \rangle
\]
\[
= \phi(y) - \phi(x) - \langle \nabla \phi(x), y - x \rangle + \langle \nabla \phi(x), y - x \rangle - \langle \nabla \phi(x), z - x \rangle - \langle \nabla \phi(z), y - z \rangle
\]
\[
= \delta_\phi(x, y) + \langle \nabla \phi(x) - \nabla \phi(z), y - z \rangle = \delta_\phi(x, y) + 0 = \delta_\phi(x, y),
\]
and the proof of Theorem 16 is complete. □

**Theorem 17.** Let \( \Omega_0 \subset \mathbb{R}^n \) be an open convex set and \( \phi \in C^1(\Omega_0) \) be a strictly convex function. Given any section \( S := S_\phi(x_0, R) \subset \Omega_0 \) and \( y_0, y_1 \in S \) there exists a continuous curve \( \gamma : [0, 1] \rightarrow S \) such that \( \gamma(0) = y_0, \gamma(1) = y_1 \) and

\[
\delta_\phi(\gamma(t), \gamma(t')) = (t' - t)\delta_\phi(y_0, y_1) \quad \forall \, 0 \leq t \leq t' \leq 1.
\]

**Proof.** Let us start by noticing that given any \( x, z \in S_\phi(x_0, R) \) it is always possible to find a midpoint \( y \) from \( x \) to \( z \) (in the sense that \( \delta_\phi(x, y) = \delta_\phi(x, z)/2 \)) such that \( y \in S_\phi(x_0, R) \). Otherwise, the whole section \( S_\phi(x, \delta_\phi(x, z)/2) \) would lie outside \( S_\phi(x_0, R) \) contradicting the fact that \( x \in S_\phi(x_0, R) \), see Figure 3.

![Figure 3](image-url)

Figure 3. Both \( y \) and \( y' \) are midpoints from \( x \) to \( z \), but \( y \in S_\phi(x_0, R) \).

Now, given \( y_0, y_1 \in S \) set \( r := \delta_\phi(y_0, y_1) \). Let \( y_{1/2} \in S \) be a mid-point between \( y_0 \) and \( y_1 \), let \( y_{1/4} \in S \) be mid-point between \( y_0 \) and \( y_{1/2} \), let \( y_{3/4} \in S \) a mid-point between \( y_{1/2} \) and \( y_1 \), and so on. This defines a function \( \gamma : \text{Dom}(\gamma) \subset [0, 1] \rightarrow S \), where \( \text{Dom}(\gamma) \) consists of the numbers of the form \( k2^{-j} \) for \( j \in \mathbb{N}_0 \) and \( k \in \{0, 1, \ldots, 2^j\} \), and \( \gamma(k2^{-j}) = y_{k2^{-j}} \). For a fixed \( j \in \mathbb{N}_0 \) and \( k \leq k' \) we have

\[
\delta_\phi(\gamma(k2^{-j}), \gamma(k'2^{-j})) = \delta_\phi(y_{k2^{-j}}, y_{k'2^{-j}}) = (k' - k)2^{-j}r.
\]

For \( j, j' \in \mathbb{N}_0 \) and \( k \in \{0, 1, \ldots, 2^j\} \) and \( k' \in \{0, 1, \ldots, 2^{j'}\} \) with \( k2^{-j} < k'2^{-j'} \) we have

\[
k2^{-j} = k2'^{-j'}2^{-j'} =: k''2^{-j'} < k'2^{-j'},
\]

so that

\[
\delta_\phi(\gamma(k2^{-j}), \gamma(k'2^{-j'})) = \delta_\phi(\gamma(k'2^{-j'}), \gamma(k'2^{-j'})) = (k' - k'')2^{-j'}r = (k'2^{-j'} - k2^{-j})r.
\]

That is, for every \( t, t' \in \text{Dom}(\gamma) \) with \( t < t' \) we have \( \delta_\phi(\gamma(t), \gamma(t')) = (t' - t)r \).

The fact that \( \phi \) is strictly convex makes the topology generated by the sections equivalent to the Euclidean topology, which makes \( S_\phi(x_0, R) \) a complete topological subspace of \( \Omega_0 \). Therefore, the mapping \( \gamma \) has a continuous extension to all of \([0, 1]\) (that we also denote as \( \gamma \)) such that the \( \delta_\phi \)-geodesic condition (6.68) holds. □
**Remark 18.** In other words, Theorem 17 says that the Monge–Ampère sections are geodesically convex.

**Corollary 19.** (Chain condition). For every section \( S := S_\phi(x_0, R) \subset \Omega \) and every \( x, y \in S \) and \( N \in \mathbb{N} \) there exists a finite sequence \( \{y_j\}_{j=0}^N \subset S \) such that \( y_0 = x, y_N = y \) and
\[
\delta_\phi(y_{j-1}, y_j) = \frac{1}{N} \delta_\phi(x, y) \quad \forall j = 1, \ldots, N.
\]

**Proof.** Let \( \gamma : [0,1] \to S \) be a geodesic joining \( x \) to \( y \) as in Theorem 17, then for each \( j \in \{0, \ldots, N\} \) set \( y_j := \gamma(j/N) \in S \). \( \square \)

7. The “passage to the double section” for supersolutions

The hypothesis \( \mathbf{H6} \) will play a central role in the proof of the following “doubling property” for \( \inf u \).

**Theorem 20.** Suppose that \( \varphi \) satisfies \( \mathbf{H1}, \mathbf{H2}, \mathbf{H4}, \mathbf{H5}, \mathbf{H6} \) and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). Let \( S_r := S_\varphi(x_0, r) \) be a section with \( S_{2r} := S_\varphi(x_0, 2r) \subset \subset \Omega \) and let \( u \in C(\overline{S_{2r}}) \cap W^{2,n}(S_{2r}) \) satisfy \( u \geq 0 \) in \( S_{2r} \) and \( L_0^A(u) \leq f \) a.e. in \( S_{2r} \). Then, there exist structural constants \( \varepsilon_1, \gamma \in (0,1) \), such that the inequalities
\[
|S_{2r}| \frac{1}{2} \|f\|_{L^n(S_{2r}, d\mu_{\varphi})} \leq \varepsilon_1, \tag{7.69}
\]
\[
r^{-\frac{1}{2}}|S_{2r}| \frac{1}{2} \|D^2 \varphi\| \frac{1}{2} \|D^2 \varphi\|_{L^n(S_{2r}, d\mu_{\varphi})} \leq \varepsilon_1, \tag{7.70}
\]
\[
|S_{2r}| \frac{1}{2} \|c\|_{L^n(S_{2r}, d\mu_{\varphi})} \leq \varepsilon_1, \tag{7.71}
\]
and \( \inf_{S_{\varphi}(x_0,r)} u \geq 1 \) imply \( \inf_{S_{\varphi}(x_0,3r/2)} u \geq \gamma \).

**Proof.** For \( t > r \) with \( S_\varphi(x_0, t) \subset \subset \Omega \), let \( \text{Ann}_\varphi(x_0, t, r) \) denote the annulus \( \text{Ann}_\varphi(x_0, t, r) := S_\varphi(x_0, t) \setminus S_\varphi(x_0, r) \). For \( s > 1 \) to be determined later, set
\[
v(x) := \left( \frac{\delta_\varphi(x_0, x)}{r} \right)^{-s} - 2^{-s}
\]
so that
\[
0 < v(x) < 1 \quad \forall x \in \text{Ann}_\varphi(x_0, 2r, r), \tag{7.72}
\]
\[
(3/2)^{-s} - 2^{-s} \leq v(x), \quad \forall x \in \text{Ann}_\varphi(x_0, 3r/2, r), \tag{7.73}
\]
\[
\nabla v(x) = -s \left( \frac{\delta_\varphi(x_0, x)}{r} \right)^{-s-1} (\nabla \varphi(x) - \nabla \varphi(x_0)) \quad \forall x \in \text{Ann}_\varphi(x_0, 2r, r) \tag{7.74}
\]
and, for a.e. \( x \in \text{Ann}_\varphi(x_0, 2r, r) \),
\[
D^2 v(x) = s(s - 1) \left( \frac{\delta_\varphi(x_0, x)}{r} \right)^{-s-2} (\nabla \varphi(x) - \nabla \varphi(x_0)) \otimes (\nabla \varphi(x) - \nabla \varphi(x_0))
\]
\[
- \frac{s}{r} \left( \frac{\delta_\varphi(x_0, x)}{r} \right)^{-s-1} D^2 \varphi(x).
\]
Notice that we have \( v \in C(\text{Ann}_\varphi(x_0, 2r, r)) \cap W^{2,n}(\text{Ann}_\varphi(x_0, 2r, r)) \) due to the fact that \( \|D^2 \varphi\| \in L^p(\Omega) \) with \( p > n \) by hypothesis \( \mathbf{H5} \).
Now, suppose that \( u \in C(\overline{S_{2r}}) \cap W^{2,n}_\text{loc}(S_{2r}) \) satisfies \( L^\varphi_A(u) \leq f \) and \( u \geq 0 \) in \( S_{2r} \) with inf \( u \geq 1 \). Then if \( x \in \partial S_{2r} \) we have
\[
v(x) - u(x) \leq v(x) = 2^{-s} - 2^{-s} = 0
\]
and if \( x \in \partial S_\varphi(x_0, r) \), then
\[
v(x) - u(x) \leq 1 - 2^{-s} - u(x) \leq 2^{-s} \leq 0.
\]
Therefore, \( v - u \leq 0 \) on \( \partial(\text{Ann}_\varphi(x_0, 2r, r)) \). On the other hand, for a.e. \( x \in \text{Ann}_\varphi(x_0, 2r, r) \)
\[
L^\varphi_A(v)(x) = \frac{s(s-1)}{r^2} {\varphi}(x_0, x) \right)^{-s-2} \langle A(x)(\nabla \varphi(x) - \nabla \varphi(x_0)), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle
\]
\[
- \frac{s}{r} \left( \frac{\varphi(x_0, x)}{r} \right)^{-s-1} \text{trace}(A(x)D^2 \varphi(x)) + \langle b(x), \nabla^\varphi v(x) \rangle + c(x)v(x).
\]
From (1.2) we get
\[
\langle A(x)(\nabla \varphi(x) - \nabla \varphi(x_0)), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle \geq \lambda \langle D^2 \varphi(x)^{-1}(\nabla \varphi(x) - \nabla \varphi(x_0)), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle = \lambda D\varphi(x_0, x)
\]
as well as \( \text{trace}(A(x)D^2 \varphi(x)) \leq n\Lambda \) for a.e. \( x \in S_{2r} \). Hence,
\[
\frac{s(s-1)}{r^2} \left( \frac{\varphi(x_0, x)}{r} \right)^{-s-2} \langle A(x)(\nabla \varphi(x) - \nabla \varphi(x_0)), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle
\]
\[
- \frac{s}{r} \left( \frac{\varphi(x_0, x)}{r} \right)^{-s-1} \text{trace}(A(x)D^2 \varphi(x))
\]
\[
\geq \frac{s}{r^2} \left( \frac{\varphi(x_0, x)}{r} \right)^{-s-2} ((s-1)\lambda D\varphi(x_0, x) - n\Lambda \delta_\varphi(x_0, x)).
\]
At this point fix \( s = s(n, \sigma, \Lambda/\lambda) \) so that
\[
(7.75) \quad s \geq \frac{n\Lambda}{\sigma \lambda} + 1,
\]
where \( \sigma > 0 \) is as in (1.10) from hypothesis H6. Thus, (1.10) and (7.75) now yield
\[
\frac{D\varphi(x_0, x)}{\delta_\varphi(x_0, x)} \geq \sigma \geq \frac{n\Lambda}{\lambda(s-1)}
\]
and then
\[
(s-1)\lambda D\varphi(x_0, x) - n\Lambda \delta_\varphi(x_0, x) \geq 0 \quad \text{for a.e. } x \in \text{Ann}_\varphi(x_0, 2r, r).
\]
Consequently, by writing \( c(x) \) as \( c(x) = c^+(x) - c^-(x) \), for a.e. \( x \in \text{Ann}_\varphi(x_0, 2r, r) \) we get
\[
L^\varphi_A(v)(x) \geq \langle b(x), \nabla^\varphi v(x) \rangle + c(x)v(x)
\]
\[
= -\frac{s}{r} \left( \frac{\varphi(x_0, x)}{r} \right)^{-s-1} \langle D^2 \varphi(x)^{-1/2}b(x), (\nabla \varphi(x) - \nabla \varphi(x_0)) \rangle + c(x)v(x)
\]
\[
\geq -\frac{s}{r} |D^2 \varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - c^-(x),
\]
where for the last inequality we used that for $x \in \text{Ann}_\varphi(x_0, 2r, r)$ we have $1 \leq \frac{\delta_r(x_0, x)}{r} < 2$ and $0 < v(x) < 1$, see (7.72). Next, since $L^n_A(u) \leq f$ a.e. in $S_{2r}$, it follows that

\[
L^n_A(v - u)(x) = \text{trace}(A(x)D^2(v - u)(x)) + \langle b(x), \nabla \varphi(v - u)(x) \rangle + c(x)(v - u)(x)
\]

\[
\geq -\frac{s}{r} |D^2\varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - c^{-}(x) - f(x)
\]

a.e. in $\text{Ann}_\varphi(x_0, 2r, r)$. Therefore, by writing $c(x) = c^{+}(x) - c^{-}(x)$ again and using that $u \geq 0$ in $S_{2r}$, a.e. in $\text{Ann}_\varphi(x_0, 2r, r)$ we obtain

\[
\text{trace}(A(x)D^2(v - u)(x)) + \langle b(x), \nabla \varphi(v - u)(x) \rangle - c^{-}(x)(v - u)(x) \geq g_1(x)
\]

where

\[
g_1(x) := -\frac{s}{r} |D^2\varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - |c(x)| - f(x).
\]

Now, by the ABP maximum principle (5.62) applied to (7.76) and $v - u$ on $\text{Ann}_\varphi(x_0, 2r, r)$ we get

\[
\sup_{\text{Ann}_\varphi(x_0, 2r, r)} (v - u) \leq C_{apb}N(b, S_{2r})|S_{2r}|^{\frac{1}{n}} \left\| \frac{g_1}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(S_{2r})},
\]

where, from the definition of $N(b, S_{2r})$ in (5.59) and the first inequality in (1.2) (which yields $\det A \geq \lambda^n(\det D^2\varphi)^{-1}$),

\[
N(b, S_{2r}) = \exp\left[ \frac{2n^2 - 2}{n^2\omega_n} \left( 1 + \frac{\int_{S_{2r}} |D^2\varphi(x)^{-1/2}b(x)|^n dx}{\det A(x)} \right) \right] - 1
\]

\[
\leq \exp\left[ \frac{2n^2 - 2}{n^2\omega_n} \left( 1 + \lambda^{-n} \|D^2\varphi^{-1/2}b\|_{L^n(\Omega, d\mu_\varphi)}^n \right) \right] - 1 =: C_1(n, \lambda, b)^n,
\]

with $C_1(n, \lambda, b) > 0$, depending only on $n$, $\lambda$ and $\|D^2\varphi^{-1/2}b\|_{L^n(\Omega, d\mu_\varphi)}$, is a structural constant. Again by the first inequality in (1.2),

\[
\left\| \frac{g_1}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(S_{2r})} \leq \frac{s}{\lambda r} \|D^2\varphi^{-1/2}b\|_{L^n(S_{2r}, d\mu_\varphi)} \|\nabla \varphi(\cdot) - \nabla \varphi(x_0)\|_{L^\infty(S_{2r})}
\]

\[
+ \frac{1}{\lambda} \|c\|_{L^n(S_{2r}, d\mu_\varphi)} + \frac{1}{\lambda} \|f\|_{L^n(S_{2r}, d\mu_\varphi)},
\]

so that, from (7.69) and (7.71),

\[
|S_{2r}|^{\frac{1}{n}} \left\| \frac{g_1}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(S_{2r})} \leq \frac{s|S_{2r}|^{\frac{1}{n}}}{\lambda r} \|D^2\varphi^{-1/2}b\|_{L^n(S_{2r}, d\mu_\varphi)} \|\nabla \varphi(\cdot) - \nabla \varphi(x_0)\|_{L^\infty(S_{2r})} + \frac{2\varepsilon_1}{\lambda}.
\]
On the other hand, the Morrey-type estimate (5.65) and the hypothesis (7.70) give
\[
\frac{|S_{2r}|^{\frac{1}{p}}}{r} \|D^2 \varphi^{-1/2} b\|_{L^n(S_{2r}, d\mu_\varphi)} \|\nabla \varphi(\cdot) - \nabla \varphi(x_0)\|_{L^\infty(S_{2r})} \\
\leq K(p)r^{\frac{1}{p}} |S_{2r}|^{\frac{1}{p}} \|D^2 \varphi^{-1/2} b\|_{L^n(S_{2r}, d\mu_\varphi)} \left(\int_{S_{2r}} \|D^2 \varphi(z)\|^p \, dz\right)^{\frac{1}{p}} \\
= K(p)r^{-\frac{1}{2}} |S_{2r}|^{\frac{1}{2p}} \|D^2 \varphi^{-1/2} b\|_{L^n(S_{2r}, d\mu_\varphi)} \|D^2 \varphi\|_{L^p(S_{2r}, dx)}^{\frac{1}{2}} \\
\leq K(p) \varepsilon_1.
\] (7.78)
Consequently,
\[
\sup_{Ann_\varphi(x_0, 2r, r)} (v - u) \leq C_{abp} C_1(n, \lambda, b) |S_{2r}|^{\frac{1}{p}} \frac{g_1}{(\det A)^{\frac{1}{p}}} \|
\leq C_{abp} C_1(n, \lambda, b) \lambda^{-1}(2 + sK(p)) \varepsilon_1.
\]
Thus, using (7.72), for every \(x \in Ann_\varphi(x_0, 3r/2, r) \subset Ann_\varphi(x_0, 2r, r)\) we have
\[
(3/2)^{-s} - 2^{-s} \leq v(x) \leq u(x) + C_{abp} C_1(n, \lambda, b) \lambda^{-1}(2 + sK(p)) \varepsilon_1.
\] (7.79)
Finally, by defining the structural constant
\[
\gamma := (3/2)^{-s} - 2^{-s} - C_{abp} C_1(n, \lambda, b) \lambda^{-1}(2 + sK(p)) \varepsilon_1
\]
and choosing \(\varepsilon_1 \in (0, 1)\), also structural, such that \(\gamma > 0\) we obtain \(\inf_{S_\varphi(x_0, 3r/2)} u \geq \gamma\). For instance, one could choose \(\gamma := 2^{s-13} - 2^{-2^{s+1}}\), depending only on \(s\) as in (7.75), and then \(\varepsilon_1 := \gamma [C_{abp} C_1(n, \lambda, b)(2 + sK(p))]^{-1}\).
\(\square\)

**Remark 21.** When \(b = 0\) the hypothesis \(H_5\) can be weakened to \(\|D^2 \varphi\| \in L^p_{loc}(\Omega)\) in the proof of Theorem 20. Indeed, the hypothesis that \(\|D^2 \varphi\| \in L^p_{loc}(\Omega)\), for some \(p > n\), was only used to apply the Morrey-type estimate (5.65). If \(b = 0\), then there is no need to control \(\|\nabla \varphi(\cdot) - \nabla \varphi(x_0)\|_{L^\infty(S_{2r})}\).

Notice, however, that even in the case \(b = 0\) the hypothesis \(\|D^2 \varphi\| \in L^p_{loc}(\Omega)\) is required to guarantee that the test function \(v - u\) belongs to \(W^{2, p}_{loc}(S_{2r})\), as required by the ABP maximum principle in Lemma 10.

The next corollary quantifies iterations of Theorem 20 to pass from a section \(S_\varphi(y_0, \rho)\) to \(S_\varphi(y_0, N_0 \rho)\) for an arbitrary \(N_0 > 1\).

**Corollary 22.** Suppose that \(\varphi\) satisfies \(H_1, H_2, H_4, H_5, H_6\) and fix \(A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)\). For every \(N_0 > 1\) there exists a constant \(\gamma(N_0) \in (0, 1)\), depending only on the structural constant \(\gamma\) from Theorem 20 as well as on \(N_0\), such that for every section \(S_{2N_0 \rho} := S_\varphi(y_0, 2N_0 \rho) \subset \subset \Omega\) and every \(u \in C(S_{2N_0 \rho}) \cap W^{2, n}_{loc}(S_{2N_0 \rho})\) satisfying \(u \geq 0\) in \(S_{2N_0 \rho}\) and \(L^\varphi_A(u) \leq f\) a.e. in \(S_{2N_0 \rho}\) the assumptions
\[
\|S_{2N_0 \rho}\|^{\frac{1}{2}} \|f\|_{L^n(S_{2N_0 \rho}, d\mu_\varphi)} \leq \varepsilon_1 \gamma^{\frac{ln N_0}{ln 3/2}},
\] (7.81)
\[
\rho^{-\frac{1}{2}} |S_{2N_0 \rho}|^{\frac{1}{2}} (D^2 \varphi)^{-1/2} b\|_{L^n(S_{2N_0 \rho}, d\mu_\varphi)} \|D^2 \varphi\|_{L^p(S_{2N_0 \rho}, dx)} \leq \varepsilon_1,
\] (7.82)
\[
|S_{2N_0 \rho}|^{\frac{1}{2}} \|c\|_{L^n(S_{2N_0 \rho}, d\mu_\varphi)} \leq \varepsilon_1,
\] (7.83)
and \(\inf_{S_\varphi(y_0, \rho)} u \geq 1\) imply \(\inf_{S_\varphi(y_0, N_0 \rho)} u \geq \gamma(N_0)\). Here \(\varepsilon_1 \in (0, 1)\) is the structural constant from Theorem 20.
Proof. The proof of Corollary 22 is based on an iteration of Theorem 20. Notice that the hypotheses (7.81), (7.82), and (7.83) imply

\[
\begin{align*}
|S_{2t}|^{\frac{1}{n}} \|f\|_{L^n(S_{2t}, d\mu_{\varphi})} &\leq \varepsilon_1 \ln^N_{\infty} \rho_N, \\
\ell^{-\frac{1}{p}} |S_{2t}|^{\frac{1}{n} - \frac{1}{2p}} \|(D^2 \varphi)^{-1/2} b\|_{L^n(S_{2t}, d\mu_{\varphi})} \|D^2 \varphi\|_{L^p(S_{2t}, dx)} &\leq \varepsilon_1, \\
|S_{2t}|^{\frac{1}{n}} \|c\|_{L^n(S_{2t}, d\mu_{\varphi})} &\leq \varepsilon_1,
\end{align*}
\tag{8.84}
\]

for every \( t \in [\rho, \rho_0] \) (recall that \( \frac{1}{n} - \frac{1}{2p} > 0 \) because \( p > n \) from hypothesis \textbf{H5}). Thus, given \( N_0 > 1 \), let \( \ell_0 \in \mathbb{N} \) such that \( (3/2)^{\ell_0} < N_0 \leq (3/2)^{\ell_0} \). Now, Theorem 20 applied to \( u, f, S_{\varphi}(y_0, \rho) \) (that is, \( t = \rho \) in (8.84)) and \( \varepsilon_1 \) gives \( \inf_{S_{\varphi}(y_0, \rho/2)} u \geq \gamma \), then Theorem 20 applied to \( u/\gamma, f/\gamma, \varepsilon_1 \gamma \), and \( S_{\varphi}(y_0, 3\rho/2) \) (that is, \( t = 3\rho/2 \) in (8.84)) gives \( \inf_{S_{\varphi}(y_0, (3/2)^2 \rho)} u \geq \gamma^2 \), etc. At the \((\ell_0 - 1)\)-th iteration we apply Theorem 20 to \( u/\gamma^{\ell_0 - 1}, f/\gamma^{\ell_0 - 1}, \varepsilon_1 \gamma^{\ell_0 - 1} \), and \( S_{\varphi}(y_0, (3/2)^{\ell_0 - 1} \rho) \) (that is, \( t = (3/2)^{\ell_0 - 1} \rho \) in (8.84) and notice that \( 2(3/2)^{\ell_0 - 1} < 2N_0 \) so that \( S_{\varphi}(y_0, 2(3/2)^{\ell_0 - 1} \rho) \subset \subset \Omega \)) to obtain \( \inf_{S_{\varphi}(y_0, (3/2)^{\ell_0} \rho)} u \geq \gamma^{\ell_0} \). Next, since \( N_0 \leq (3/2)^{\ell_0} \), it follows that \( \inf_{S_{\varphi}(y_0, N_0 \rho)} u \geq \gamma^{\ell_0} \geq 1^{\ln N_0 \rho} =: \gamma(N_0) \). Finally, notice that the definition of \( \ell_0 \) implies \( \gamma^{\ln N_0 \rho} \leq \gamma^{\ell_0} \). \qed

8. A CRITICAL-DENSITY ESTIMATE FOR SUPERSOLUTIONS

In this section we prove a critical-density estimate for nonnegative supersolutions (Theorem 25). The proof differs from the ones for similar previous results (e.g. \cite[Theorem 1]{3}, \cite[Lemma 3.1]{20}, and \cite[Theorem 2]{26}) in the following ways: the hypothesis \( \varphi \in C^2(\Omega) \) with \( D^2 \varphi > 0 \) is not used, the Caffarelli-Gutiérrez normalization technique from \cite[Section 1]{3} is not used, the first-order term of \( L_{\varphi} A \) is handled by means of the Morrey-type estimate in Lemma 13 and the hypothesis \textbf{H5}.

As another difference with previous proofs, by considering infima over the boundaries of sections (see (8.88) below) we first prove Theorem 23, which is a weaker version of the critical-density estimate better suited to the ABP maximum principle in Lemma 10. Then, by means of Corollary 22 and the hypothesis \textbf{H3}, the condition on the infima over boundaries of sections is removed.

**Theorem 23.** Suppose that \( \varphi \) satisfies \textbf{H1}, \textbf{H2}, \textbf{H4}, \textbf{H5} and fix \( A \in \mathcal{E}(\lambda, \varphi, \Omega) \). Let \( S_r := S_{\varphi}(x_0, r) \) be a section with \( S_{2r} := S_{\varphi}(x_0, 2r) \subset \subset \Omega \) and let \( u \in C(S_{2r}) \cap W^{2,n}(S_{2r}) \) satisfy \( u \geq 0 \) in \( S_{2r} \) and \( L_{\varphi}^2(u) \leq f \) a.e. in \( S_{2r} \). Then, there are structural constants \( \varepsilon_0, \varepsilon_2 \in (0, 1) \) such that the inequalities

\[
\begin{align*}
|S_{2r}|^{\frac{1}{n}} \|f\|_{L^n(S_{2r}, d\mu_{\varphi})} &\leq \varepsilon_2, \\
r^{-\frac{1}{p}} |S_{2r}|^{\frac{1}{n} - \frac{1}{2p}} \|(D^2 \varphi)^{-1/2} b\|_{L^n(S_{2r}, d\mu_{\varphi})} \|D^2 \varphi\|_{L^p(S_{2r}, dx)} &\leq \varepsilon_2, \\
|S_{2r}|^{\frac{1}{n}} \|c\|_{L^n(S_{2r}, d\mu_{\varphi})} &\leq \varepsilon_2,
\end{align*}
\tag{8.85-8.87}
\]

imply \( \mu_{\varphi}(\{x \in S_{2r} : u(x) < 5\}) \geq \varepsilon_0 \mu_{\varphi}(S_{2r}) \).
Proof. Introduce the test function

\begin{equation}
(8.89) 
 h(x) := -u(x) - 4 \left( \frac{\delta_x(x_0, x)}{2r} - 1 \right) + \inf_{\partial S_{2r}} u \quad \forall x \in \overline{S_{2r}}.
\end{equation}

Then, \( h \in C(\overline{S_{2r}}) \cap W^{2,n}(S_{2r}) \) with \( D^2 h(x) = -D^2 u(x) - \frac{2}{r} D^2 \varphi(x) \) a.e. in \( S_{2r} \) and, since \( \delta_x(x_0, x) = 2r \) for every \( x \in \partial S_{2r} \), we have \( \sup_{\partial S_{2r}} h = 0 \). By the hypothesis \( \inf u < 1 \), there exists \( x_1 \in S_r \) (i.e., \( \delta_x(x_0, x_1) < r \)) such that \( u(x_1) < 1 \) and then

\begin{equation}
(8.90) 
 h(x_1) = -u(x_1) - 4 \left( \frac{\delta_x(x_0, x_1)}{2r} - 1 \right) > -1 - 4 \left( \frac{1}{2} - 1 \right) = 1.
\end{equation}

Also,

\[
L_A^\varphi(h)(x) = -L_A^\varphi(u)(x) - \frac{2}{r} \operatorname{trace}(A(x)D^2 \varphi(x)) 
- \frac{2}{r}(b(x), D^2 \varphi(x)^{-1/2}(\nabla \varphi(x) - \nabla \varphi(x_0))) - 4c(x) \left( \frac{\delta_x(x_0, x)}{2r} - 1 \right).
\]

Now, by the second inequality in (1.2) we have \( \operatorname{trace}(A(x)D^2 \varphi(x)) \leq n \Lambda \) a.e. \( x \in S_{2r} \) and, since \( \delta_x(x_0, x) < 2r \) for every \( x \in S_{2r} \) and \( c(x) = c^+(x) - c^-(x) \),

\[ 0 \geq -4c(x) \left( \frac{\delta_x(x_0, x)}{2r} - 1 \right) \geq -4c^-(x) \quad \text{a.e. } x \in S_{2r}.
\]

Therefore, by using that \( L_A^\varphi(u) \leq f \) a.e. in \( S_{2r} \) we get that, for a.e. \( x \in S_{2r} \),

\[
L_A^\varphi(h)(x) = \operatorname{trace}(A(x)D^2 h(x)) + \langle b(x), \nabla^\varphi h(x) \rangle + c(x)h(x)
\geq -f(x) - \frac{2n \Lambda}{r} - \frac{2}{r} |D^2 \varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - 4c^-(x) - c^+(x)h(x),
\]

which implies

\[
\operatorname{trace}(A(x)D^2 h(x)) + \langle b(x), \nabla^\varphi h(x) \rangle - c^-(x)h(x)
\geq -f(x) - \frac{2n \Lambda}{r} - \frac{2}{r} |D^2 \varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - 4c^-(x) - c^+(x)h(x),
\]

and, since \( u \geq 0 \) in \( S_{2r} \),

\[
c^+(x)h(x) = c^+(x) \left( -u(x) - 4 \left( \frac{\delta_x(x_0, x)}{2r} - 1 \right) + \inf_{\partial S_{2r}} u \right) \leq c^+(x)(4 + \inf_{\partial S_{2r}} u) < 5c^+(x),
\]

where for the last inequality we used (8.88). Hence,

\[
\operatorname{trace}(A(x)D^2 h(x)) + \langle b(x), \nabla^\varphi h(x) \rangle - c^-(x)h(x)
\geq -f(x) - \frac{2n \Lambda}{r} - \frac{2}{r} |D^2 \varphi(x)^{-1/2}b(x)||\nabla \varphi(x) - \nabla \varphi(x_0)| - 5|c(x)| =: g,
\]

and by the ABP estimate (5.61) applied to \( h \) in \( S_{2r} \) it follows that

\begin{equation}
(8.91) 
 1 \leq C_{abp}N(b, \Gamma^+_h(S_{2r}))(\overline{S_{2r}})^\frac{1}{n} \left\| \frac{g}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(\Gamma^+_h(S_{2r}))},
\end{equation}
where $\Gamma_+(S_{2r})$ is the contact set from Lemma 10. In particular, recalling the definitions of $h$ and $\Gamma_+(S_{2r})$, we have the inclusions

$$
\Gamma_+(S_{2r}) \subset \{ x \in S_{2r} : h(x) \geq 0 \} \setminus \{ x \in S_{2r} : u(x) < 4 - 2\delta_{\varphi}(x_0, x) / r + \inf_{\partial S_{2r}} u \}
$$

where for the last inclusion we used the hypothesis $\inf_{\partial S_{2r}} u < 1$. Also, notice that the definition of $C_1(n, \lambda, b)$ in (7.77) gives $N(b, \Gamma_+(S_{2r})) \leq N(b, S_{2r}) \leq C_1(n, \lambda, b)$.

Using again that the first inequality in (1.2) implies $\det A \geq \lambda n (\det D^2 \varphi)^{-1}$ we can write

$$
|S_{2r}|^{1/n} \left\| \frac{g}{(\det A)^{\frac{1}{n}}} \right\|_{L^n(\Gamma_+(S_{2r}))} \leq \frac{1}{\lambda} |S_{2r}|^{\frac{1}{n}} \| f \|_{L^n(S_{2r}, d\mu_{\varphi})} + \frac{2n\Lambda}{\lambda r} |S_{2r}|^{\frac{1}{n}} \mu_{\varphi}(\Gamma_+(S_{2r}))^{\frac{1}{n}}
$$

$$
+ \frac{2}{\lambda r} |S_{2r}|^{\frac{1}{n}} \| D^2 \varphi^{-1/2} b \|_{L^n(S_{2r}, d\mu_{\varphi})} \| \nabla \varphi(\cdot) - \nabla \varphi(x_0) \|_{L^\infty(S_{2r})}
$$

$$
+ \frac{5}{\lambda} |S_{2r}|^{\frac{1}{n}} \| c \|_{L^n(S_{2r}, d\mu_{\varphi})}.
$$

Now, from (5.54) and (5.57) we get

$$
\frac{|S_{2r}|^{\frac{1}{n}}}{r} = \left( \frac{\mu_{\varphi}(S_{2r}) |S_{2r}|^{\frac{1}{n}}}{r \mu_{\varphi}(S_{2r})^{\frac{1}{n}}} \right) \leq \frac{2K_2}{\mu_{\varphi}(S_{2r})^{\frac{1}{n}}},
$$

which, along with the inclusion (8.92), gives

$$
\frac{|S_{2r}|^{\frac{1}{n}}}{r} \mu_{\varphi}(\Gamma_+(S_{2r}))^{\frac{1}{n}} \leq 2K_2 \left( \frac{\mu_{\varphi}(\Gamma_+(S_{2r}))}{\mu_{\varphi}(S_{2r})} \right)^{\frac{1}{n}} \leq 2K_2 \left( \frac{\mu_{\varphi}(\{ x \in S_{2r} : u(x) < 5 \})}{\mu_{\varphi}(S_{2r})} \right)^{\frac{1}{n}}.
$$

On the other hand, from the Morrey-type estimate (5.65) and in the same way we obtained (7.78), but now using the hypothesis (8.86), we get

$$
\frac{1}{r} |S_{2r}|^{\frac{1}{n}} \| D^2 \varphi^{-1/2} b \|_{L^n(S_{2r}, d\mu_{\varphi})} \| \nabla \varphi(\cdot) - \nabla \varphi(x_0) \|_{L^\infty(S_{2r})} \leq K(p) \varepsilon_2.
$$

Thus, coming back to (8.91) and by using the hypotheses (8.85) and (8.87), we deduce

$$
1 \leq \frac{C_{abp}C_1(n, \lambda, b)}{\lambda} \left[ (2K(p) + 6) \varepsilon_2 + 4nK_2 \Lambda \left( \frac{\mu_{\varphi}(\{ x \in S_{2r} : u(x) < 5 \})}{\mu_{\varphi}(S_{2r})} \right)^{\frac{1}{n}} \right],
$$

so that by taking $\varepsilon_2 > 0$ in (8.85)–(8.87) as, for instance, the structural constant defined by the equality $C_{abp}C_1(n, \lambda, b)(2K(p) + 6) \varepsilon_2 := \lambda/2$ we obtain

$$
\frac{4nK_2C_{abp}C_1(n, \lambda, b)}{\lambda} \left( \frac{\mu_{\varphi}(\{ x \in S_{2r} : u(x) < 5 \})}{\mu_{\varphi}(S_{2r})} \right)^{\frac{1}{n}} \geq \frac{1}{2}
$$

and the theorem follows with $\varepsilon_0 := \lambda n (8nK_2C_{abp}C_1(n, \lambda, b)\Lambda)^{-n}$, a structural constant.

**Remark 24.** As in Remark 21, we point out that when $b = 0$ the hypothesis H5 can be weakened to $\| D^2 \varphi \| \in L^2_{\text{loc}}(\Omega)$ in the proof of Theorem 23 as well. Again, even in the case $b = 0$ the hypothesis $\| D^2 \varphi \| \in L^2_{\text{loc}}(\Omega)$ is required to guarantee that the test function $h$ in (8.89) belongs to $W^2_{\text{loc}}(S_{2r})$, as required by the ABP maximum principle in Lemma 10.
The next result strengthens Theorem 23 by removing the hypothesis \( \inf u < 1 \) from (8.88). The price to pay is that the critical density increases from \( (1 - \varepsilon_0) \) to \( (1 - \frac{\varepsilon_0}{2}) \) and that, instead of having \( \inf u \geq 1 \), we obtain \( \inf u \geq \gamma_0 \) for some structural constant \( \gamma_0 \in (0, 1) \). Here the hypothesis \( H3 \) makes its first appearance.

**Theorem 25.** Suppose that \( \varphi \) satisfies \( H1 \)–\( H6 \) and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). There exist geometric constants \( \kappa_0 \in (0, 1) \), \( K_3 \geq 2 \) and structural constants \( \varepsilon_3, \gamma_0 \in (0, 1) \) such that for every section \( S_r := S_\varphi(x_0, r) \) with \( S_{Kr} := S_\varphi(x_0, K_3r) \subset \subset \Omega \) and every \( u \in C(S_{Kr}) \cap W^{2,n}(S_{Kr}) \) with \( u \geq 0 \) in \( S_{Kr} \) and \( L_\varphi^\omega(u) \leq f \ a.e. \ in \ S_{Kr} \), the inequalities

\[
(8.93) \quad |S_{Kr}| \frac{1}{2} \|f\|_{L^n(S_{Kr}, d\mu_\varphi)} \leq \varepsilon_3,
\]

\[
(8.94) \quad r^{-\frac{1}{2}}|S_{Kr}|^{\frac{1}{2} - \frac{1}{n}} \|D^2 \varphi\|_{L^n(S_{Kr}, d\mu_\varphi)} \|D^2 \varphi\|_{L^p(S_{Kr}, dx)} \leq \varepsilon_3,
\]

\[
(8.95) \quad |S_{Kr}|^{\frac{1}{2}} \|c\|_{L^n(S_{Kr}, d\mu_\varphi)} \leq \varepsilon_3,
\]

and \( \mu_\varphi(\{x \in S_{2r} : u(x) \geq 5\}) \geq (1 - \frac{\varepsilon_0}{2}) \mu_\varphi(S_{2r}) \) imply \( \inf u \geq \gamma_0 \). Here \( \varepsilon_0 \in (0, 1) \) is the structural constant from Theorem 23.

**Proof.** Fix \( \varepsilon_0 \in (0, 1) \) such that

\[
\zeta(\varepsilon_0) = \frac{\varepsilon_0}{2(1 - \varepsilon_0)},
\]

where \( \zeta \) is the non-decreasing function with \( \zeta(0^+) = 0 \) from the hypothesis \( H3 \). Hence, for every \( \varepsilon \in (0, \varepsilon_0) \), the annular-decay property (1.8) implies

\[
\frac{\mu_\varphi(S_{1+\varepsilon}) \setminus S_r}{\mu_\varphi(S_{2r})} \leq \zeta(\varepsilon) \leq \frac{\varepsilon_0}{2(1 - \varepsilon_0)},
\]

which yields

\[
\frac{\mu_\varphi(S_{1+\varepsilon})}{\mu_\varphi(S_{2r})} \leq \frac{\varepsilon_0}{2(1 - \varepsilon_0)} + 1 = \frac{1 - \varepsilon_0}{1 - \varepsilon_0} \quad \forall \varepsilon \in (0, \varepsilon_0).
\]

Then, from the hypothesis \( \mu_\varphi(\{x \in S_{2r} : u(x) \geq 5\}) \geq (1 - \frac{\varepsilon_0}{2}) \mu_\varphi(S_{2r}) \), for every \( \varepsilon \in (0, \varepsilon_0) \) we get

\[
\mu_\varphi(\{x \in S_{2(1+\varepsilon)r} : u(x) \geq 5\}) \geq \mu_\varphi(\{x \in S_{2r} : u(x) \geq 5\}) \geq (1 - \frac{\varepsilon_0}{2}) \mu_\varphi(S_{2r})
\]

\[
> (1 - \varepsilon_0) \mu_\varphi(S_{1+\varepsilon}),
\]

so that \( \mu_\varphi(\{x \in S_{2(1+\varepsilon)r} : u(x) < 5\}) < \varepsilon_0 \mu_\varphi(S_{1+\varepsilon}) \) for every \( \varepsilon \in (0, \varepsilon_0) \). Choose \( K_3 \geq 2 \) and \( \varepsilon_3 \leq \varepsilon_2 \), where \( \varepsilon_2 \in (0, 1) \) is the structural constant from Theorem 23, and \( t = 2r \) in (8.93), (8.94), and (8.95). Given \( \varepsilon \in (0, \varepsilon_0) \), by applying Theorem 23 (in its contrapositive) to the section \( S_{2(1+\varepsilon)r} \) (notice that \( S_{2(1+\varepsilon)r} \subset S_{3r} \subset S_{Kr} \subset \subset \Omega \)) it follows that either

\[
\inf_{S_{1+\varepsilon}} u \geq 1 \quad or \quad \inf_{S_{2(1+\varepsilon)r}} u \geq 1.
\]

Now, if \( \inf_{S_{1+\varepsilon}} u \geq 1 \) for some \( \varepsilon \in (0, \varepsilon_0) \), then \( \inf u \geq 1 \) and the theorem gets proved with \( \gamma_0 := 1, \varepsilon_3 := \varepsilon_2, K_3 := 2 \) and, say, \( \kappa_0 := 1/2 \).

On the other hand, if \( \inf_{S_{1+\varepsilon}} u < 1 \) for every \( \varepsilon \in (0, \varepsilon_0) \), then we must have \( \inf_{S_{2(1+\varepsilon)r}} u \geq 1 \) for every \( \varepsilon \in (0, \varepsilon_0) \). In particular, it follows that \( u \geq 1 \) in the annulus \( S_{2(1+\varepsilon_0)r} \setminus S_{2r} = S_\varphi(x_0, 2(1 + \varepsilon_0)r) \setminus S_\varphi(x_0, 2r) \). Next, by Theorem 3.3.10 on [16, p.60] there exists a constant \( \kappa_0 \in (0, 1) \), depending only on \( \varepsilon_0 \) and geometric constants, such that for every \( y \in \)
\( \partial S_\varphi(x_0, 2(1 + \epsilon_0/2)r) \) (the section \( S_\varphi(x_0, 2(1 + \epsilon_0/2)r) \) being the intermediate section between \( S_\varphi(x_0, 2r) \) and \( S_\varphi(x_0, 2(1 + \epsilon_0)r) \)) we have the inclusion
\[
S_\varphi(y, \kappa_0r) \subset S_\varphi(x_0, 2(1 + \epsilon_0)r) \setminus S_\varphi(x_0, 2r).
\]
Thus, by fixing a \( y \in \partial S_\varphi(x_0, 2(1 + \epsilon_0/2)r) \), we obtain \( \inf_{S_\varphi(y, \kappa_0r)} u \geq 1 \). Also, since \( y \in \partial S_\varphi(x_0, 2(1 + \epsilon_0/2)r) \subset S_3r \), the quasi-triangle inequality (1.7) gives the inclusions
\[
(8.96) \quad S_\varphi(x_0, r) \subset S_\varphi(y, 4Kr)
\]
and
\[
(8.97) \quad S_\varphi(y, 8Kr) \subset S_\varphi(x_0, (8K + 6)Kr).
\]
The idea is to use Corollary 22 with the sections \( S_\varphi(y, \kappa_0r) \) and \( S_\varphi(y, 4Kr) \) to pass from \( \inf_{S_\varphi(y, \kappa_0r)} u \geq 1 \) to \( \inf_{S_\varphi(y, 4Kr)} u \geq \gamma_0 \) so that the latter along with the inclusion (8.96) proves \( \inf_u u \geq \gamma_0 \) and the theorem.

All that is left is the tuning of the constants to apply Corollary 22 to the sections \( S_\varphi(y, \kappa_0r) \) and \( S_\varphi(y, 4Kr) \). In the notation from Corollary 22 put \( y_0 := y, \rho := \kappa_0r \) and \( N_0 := 4K/\kappa_0 \) so that \( 4Kr = N_0\rho \).

By choosing \( K_3 := (8K + 6)K \) and using the inclusion (8.97) we obtain \( S_\varphi(y, 2N_0\rho) = S_\varphi(y, 8Kr) \subset S_\varphi(x_0, K_3r) \subset \Omega \) so that the inequalities (8.93) and (8.95) imply (7.81) and (7.83), respectively, provided that \( \varepsilon_3 \in (0, 1) \) satisfies \( \varepsilon_3 \leq \varepsilon_1 \gamma^{\frac{\ln(4K/\kappa_0)}{\ln(3/2)}} \) and, regarding (7.82), we get
\[
\rho^{-\frac{1}{2}}|S_\varphi(y, 2N_0\rho)|^{\frac{1}{2}} \sup_{x \in S_\varphi(x_0, 2r)} \frac{1}{\rho}(D^2\varphi)^{-\frac{1}{2}}b||L^n(S_\varphi(y, 2N_0\rho), d\mu_\varphi)||D^2\varphi||L^p(S_\varphi(y, 2N_0\rho), dx) \leq \varepsilon_1,
\]
where the last inequality holds true provided that \( \varepsilon_3 \leq \varepsilon_1 \kappa_0^{1/2} \). That is, the constants \( K_3 \) and \( \varepsilon_3 \) are given by \( K_3 := (8K + 6)K \) (geometric) and \( \varepsilon_3 := \min\{\varepsilon_3, \varepsilon_1 \kappa_0^{1/2} \}, \varepsilon_1 \gamma^{\frac{\ln(4K/\kappa_0)}{\ln(3/2)}} \} \) (structural). The structural constant \( \gamma_0 \) equals \( \gamma_0(4K/\kappa_0) := \gamma^{\frac{\ln(4K/\kappa_0)}{\ln(3/2)}} \) from Corollary 22.

We close this section by combining Theorem 25 and Corollary 22 as follows.

**Corollary 26.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). There exists a structural constant \( \varepsilon_4 \in (0, 1) \) such that for every \( N_0 \geq K_3 \) (where \( K_3 \geq 2 \) is the geometric constant from Theorem 25) there exists \( M(N_0) > 1 \), depending only on structural constants as well as on \( N_0 \), such that for every section \( S_{2N_0} := S_\varphi(x_0, 2N_0r) \subset \subset \Omega \) and every \( w \in C(S_{2N_0}) \cap W^{2,n}(S_{2N_0}) \) satisfying \( w \geq 0 \) in \( S_{2N_0} \) and \( L^2_A(w) \leq f \) a.e. in \( S_{2N_0} \), the assumptions
\[
|S_{2N_0}|^{\frac{1}{2}} ||f||L^n(S_{2N_0}, d\mu_\varphi) \leq \varepsilon_4,
\]
\[
r^{-\frac{1}{2}}|S_{2N_0}|^{\frac{1}{2}} \sup_{x \in S_{2N_0}} \frac{1}{\rho}(D^2\varphi)^{-\frac{1}{2}}b||L^n(S_{2N_0}, d\mu_\varphi)||D^2\varphi||L^p(S_{2N_0}, dx) \leq \varepsilon_4,
\]
\[
|S_{2N_0}|^{\frac{1}{2}} ||c||L^n(S_{2N_0}, d\mu_\varphi) \leq \varepsilon_4,
\]
and \( \inf_{S_{2N_0}} w < 1 \) imply \( \mu_\varphi(\{x \in S_\varphi(x_0, 2r) : w(x) < M(N_0)\}) \geq \frac{\varepsilon_0}{2} \mu_\varphi(S_\varphi(x_0, 2r)) \). Here \( \varepsilon_0 \in (0, 1) \) is the structural constant from Theorem 23.
Proof. Given \( N_0 \geq K_3 \), let \( \gamma(N_0) := \gamma^{|\ln N_0|/m_2} \) be the structural constant from Corollary 22. Now, the assumption \( \inf_{S_\varphi(x_0,N_0r)} w < 1 \) means \( \inf_{S_\varphi(x_0,N_0r)} \gamma(N_0)w < \gamma(N_0) \) so that by applying Corollary 22 to \( \gamma(N_0)w \) on the section \( S_\varphi(x_0,N_0r) \) (think \( y_0 := x_0 \) and \( \rho := r \) in the notation from Corollary 22) we obtain \( \inf_{S_\varphi(x_0,r)} \gamma(N_0)w < 1 \), provided that the inequalities (7.81), (7.82), and (7.83) hold true, and this can be accomplished by choosing \( \varepsilon_4 \leq \varepsilon_1 \gamma^{|\ln N_0|/m_2} \).

Next, we use that \( \inf_{S_\varphi(x_0,r)} \gamma(N_0)w < 1 \) to write \( \inf_{S_\varphi(x_0,r)} \gamma_0 \gamma(N_0)w < \gamma_0 \), where \( \gamma_0 \in (0,1) \) is the structural constant from Theorem 25. By applying Theorem 25 to \( \gamma_0 \gamma(N_0)w \) on the section \( S_\varphi(x_0,r) \) we obtain

\[
\mu_\varphi(\{x \in S_\varphi(x_0,2r) : \gamma_0 \gamma(N_0)w(x) < 5\}) \geq \frac{\varepsilon_4}{2} \mu_\varphi(S_\varphi(x_0,2r)),
\]

provided that \( \varepsilon_4 \leq \varepsilon_3 \) and that

\[
|S_{2Kr}|^{\frac{1}{n}} \|\gamma_0 \gamma(N_0)f\|_{L^n(S_{2Kr},d\mu_\varphi)} \leq \varepsilon_3
\]

which follows from \( \varepsilon_4 \leq \varepsilon_3 \) since \( \gamma_0 \gamma(N_0) \in (0,1) \). Hence, Corollary 26 is proved with \( M(N_0) := \frac{5}{\gamma_0 \gamma(N_0)} \) and \( \varepsilon_4 := \min(\varepsilon_1 \gamma^{|\ln N_0|/m_2}, \varepsilon_3) \). \( \square \)

9. Mean-Value Inequalities for Subsolutions

As a consequence of the critical-density estimate in Theorem 25, in this section we prove a mean-value property for nonnegative subsolutions. The proof of Theorem 27 below follows from a combination of the proofs of Lemmas 5 and 6 from [26] and we sketch it in Section 13.3 of the Appendix.

Let \( \varepsilon_3 \in (0,1) \) and \( K_3 > 1 \) be the structural and geometric constants from Theorem 25.

**Theorem 27.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). There exist structural constants \( 0 < \varepsilon_5 < 1 < C_6 \) such that for every section \( S_\varphi(z, r) \) with \( S_{2Kr} := S_\varphi(z, 2Kr) \subset \subset \Omega \) and every \( u \in C(S_{2K_r}) \cap W^{2,n}(S_{2Kr}) \) with \( u \geq 0 \) in \( S_{2Kr} \) and \( L_A^\varphi(u) \geq f \) a.e. in \( S_{2Kr} \), the inequalities

\[
|S_{2Kr}|^{\frac{1}{n}} \|f\|_{L^n(S_{2Kr},d\mu_\varphi)} + |S_{2Kr}|^{\frac{1}{n}} \|c\|_{L^n(S_{2Kr},d\mu_\varphi)} \leq \varepsilon_5,
\]

(9.98)

\[
\sup_{x \in S_{\varphi}(z,r)} \left( p^{-\frac{1}{2}} |S_{\varphi}(x, \rho)|^{\frac{1}{n}} \|D^2 \varphi\|_{L^n(S_{\varphi}(x, \rho),d\mu_\varphi)} \right) \leq \varepsilon_5,
\]

(9.99)

and \( f_{S_{2Kr}} u(x) d\mu_\varphi(x) \leq 1 \) imply \( \sup_{S_{\varphi}(z,r/2)} u \leq C_6 \).

**Corollary 28.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). Let \( S_\varphi(z, r) \) be a section with \( S_{2Kr} := S_\varphi(z, 2Kr) \subset \subset \Omega \) and let \( u \in C(S_{2Kr}) \cap W^{2,n}(S_{2Kr}) \) satisfy \( u \geq 0 \) in \( S_{2Kr} \) and \( L_A^\varphi(u) \geq f \) a.e. in \( S_{2Kr} \). Suppose that \( |S_{2Kr}|^{\frac{1}{n}} \|c\|_{L^n(S_{2Kr},d\mu_\varphi)} \leq \varepsilon_5 \) and that (9.99) hold true. Then,

\[
\sup_{S_{\varphi}(z,r/2)} u \leq C_6 \int_{S_{2Kr}} u(x) d\mu_\varphi(x) + 2C_6 \varepsilon_5^{-1} |S_{2Kr}|^{\frac{1}{n}} \|f\|_{L^n(S_{2Kr},d\mu_\varphi)}.
\]

(9.100)

**Proof.** Just divide the inequality \( L_A^\varphi(u) \leq f \) by \( Q(u, f) \), where

\[
Q(u, f) := \int_{S_{2Kr}} u(x) d\mu_\varphi(x) + 2 \varepsilon_5^{-1} |S_{2Kr}|^{\frac{1}{n}} \|f\|_{L^n(S_{2Kr},d\mu_\varphi)}.
\]
and apply Theorem 27 to \( u/Q(u, f) \).

\[ \square \]

**Corollary 29.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). Then, there exist geometric constants \( 0 < \kappa_1 < \kappa_2 \leq 1 \) such that for every \( 0 < q < \infty \) there is a constant \( C_7 > 0 \), depending only on structural constants as well as on \( q \), so that for every section \( S_r := S_\varphi(z, r) \) with \( S_{2K_r} := S_\varphi(z, 2K_r) \subset \subset \Omega \) and every \( u \in C(S_{2K_r}) \cap W^{2,n}(S_{2K_r}) \) that satisfies \( u \geq 0 \) in \( S_{2K_r} \) and \( L^n_A(u) \geq f \) a.e. in \( S_{2K_r} \), we have that the inequalities

\[
|S_{2K_r}| \frac{1}{n} \| f \|_{L^n(S_{2K_r}, d\mu_\varphi)} \leq \varepsilon_4, \\
R^{-\frac{1}{2}} |S_{2K_r}| \frac{1}{n-\frac{1}{2}} \| (D^2 \varphi)^{-1} \|_{L^n(S_{2K_r}, d\mu_\varphi)} \| D^2 \varphi \|^2_{L^p(S_{2K_r}, dx)} \leq \varepsilon_4,
\]

and \( \inf_{S_\varphi(z, r)} u \leq 1 \) imply

\[
|S_{2K_r}| \frac{1}{n} \| f \|_{L^n(S_{2K_r}, d\mu_\varphi)} \leq \varepsilon_4,
\]

\[
|S_{2K_r}| \frac{1}{n-\frac{1}{2}} \| (D^2 \varphi)^{-1} \|_{L^n(S_{2K_r}, d\mu_\varphi)} \| D^2 \varphi \|^2_{L^p(S_{2K_r}, dx)} \leq \varepsilon_4,
\]

for a structural constant \( \eta_0 \in (0, 1) \). In particular, there exist structural constants \( \beta_2 > 0 \) and \( M_2 > 0 \) such that

\[
\int_{S_\varphi(z, R)} u^{\beta_2}(x) d\mu_\varphi(x) \leq M_2^{\beta_2}.
\]

10. **ON THE POWER-LIKE DECAY PROPERTY AND THE WEAK-HARNACK INEQUALITY**

The next theorem establishes a power-like decay property for the distribution function of nonnegative supersolutions. Its proof is based on Lemma 4 and Theorem 7 from [18] and the segment properties from Section 6 will play a key role.

Let us choose \( K_V := 2K(1+1) > 2K^2 + K \) as the Vitali constant in Lemma 15 and introduce the geometric constants \( K_0, K_4, \) and \( K_5 \) as

\[
K_0 := 2K(1 + KK_V), \quad K_4 := \frac{1 + KK_V}{2(K+1)} + \frac{1}{2K_V}, \quad K_5 := K(K_4 K_V + 1),
\]

where \( K_3 := (8K + 6)K \) is the geometric constant from Theorem 25. In particular, it follows that \( K_3 \leq K_0 \leq K_5 \) and \( K_4 > K_2 \). Next, put \( M_0 := M(K_0) > 1 \), where \( M(K_0) \) is as in Corollary 26 for \( N_0 := K_0 \) as in (10.101).

Let \( \varepsilon_4 \in (0, 1) \) be the structural constant from Corollary 26.

**Theorem 30.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). For every section \( S_R := S_\varphi(z, R) \) with \( S_{2K_3 R} := S_\varphi(z, 2K_3 R) \subset \subset \Omega \) and every \( u \in C(S_{2K_3 R}) \cap W^{2,n}(S_{2K_3 R}) \) satisfying \( u \geq 0 \) in \( S_{2K_3 R} \) and \( L^n_A(u) \leq f \) a.e. in \( S_{2K_3 R} \), the assumptions

\[
|S_{2K_3 R}| \frac{1}{n} \| f \|_{L^n(S_{2K_3 R}, d\mu_\varphi)} \leq \varepsilon_4, \\
R^{-\frac{1}{2}} |S_{2K_3 R}| \frac{1}{n-\frac{1}{2}} \| (D^2 \varphi)^{-1} \|_{L^n(S_{2K_3 R}, d\mu_\varphi)} \| D^2 \varphi \|^2_{L^p(S_{2K_3 R}, dx)} \leq \varepsilon_4,
\]

and \( \inf_{S_\varphi(z, R)} u \leq 1 \) imply

\[
\mu_\varphi(\{ x \in S_\varphi(z, R) : u(x) > M_0^{k+1} \}) \leq (1 - \eta_0)^k \mu_\varphi(S_\varphi(z, R)), \quad \forall k \in \mathbb{N}_0,
\]

for a structural constant \( \eta_0 \in (0, 1) \). In particular, there exist structural constants \( \beta_2 > 0 \) and \( M_2 > 0 \) such that

\[
\int_{S_\varphi(z, R)} u^{\beta_2}(x) d\mu_\varphi(x) \leq M_2^{\beta_2}.
\]
Proof. Set $S := S_{\varphi}(z, R)$ and for $k \in \mathbb{N}_0$ define
\[ D_k := \{ y \in S : u(y) \leq M_0^k \}. \]
Notice the hypothesis $\inf_{S_{\varphi}(z, R)} u \leq 1$ makes $D_k \neq \emptyset$ for every $k \in \mathbb{N}_0$. Fix $k \in \mathbb{N}_0$ and pick $x \in S \setminus D_k$. It follows that $\delta_{\varphi}(x, D_k) < 2KR$. Indeed, given $x' \in D_k$ we have
\[\delta_{\varphi}(x, D_k) \leq \delta_{\varphi}(x, x') \leq K(\delta_{\varphi}(z, x) + \delta_{\varphi}(z, x')) < 2KR. \]
For $x \in S \setminus D_k$ define
\[ r_x := \frac{K_V}{2K} \delta_{\varphi}(x, D_k) \]
so that, in particular, from the fact that $K \geq 1$, we have
\[ S_{\varphi}(x, r_x/K_V) \cap D_k = \emptyset. \]
Now, either: $r_x \leq 2KK_V\delta_{\varphi}(x, z)$ or $r_x > 2KK_V\delta_{\varphi}(x, z)$. In case of $r_x \leq 2KK_V\delta_{\varphi}(x, z)$, by the segment property in Theorem 16 there exists $y \in S$ such that
\[\frac{r_x}{2KK_V} + \delta_{\varphi}(y, z) = \delta_{\varphi}(x, y) + \delta_{\varphi}(y, z) = \delta_{\varphi}(x, z) < KR, \]
where the last inequality is due to fact that $x \in S = S_{\varphi}(z, R)$. On the other hand, if $r_x > 2KK_V\delta_{\varphi}(x, z)$, by the segment-prolongation property in Theorem 16 there exists $y \in \partial S_{\varphi}(x, r_x/(2KK_V))$ such that
\[\delta_{\varphi}(x, z) + \delta_{\varphi}(z, y) = \delta_{\varphi}(x, y) = \frac{r_x}{2KK_V}. \]
Next, we claim that in either case we have the inclusions
\[ S_{\varphi}\left(y, \frac{r_x}{2KK_V}\right) \subset S_{\varphi}\left(x, \frac{r_x}{K_V}\right) \subset S_{\varphi}(z, 2KR), \]
so that by intersecting (10.108) with $D_{k+1}$ we get
\[ S_{\varphi}\left(y, \frac{r_x}{2KK_V}\right) \cap D_{k+1} \subset S_{\varphi}\left(x, \frac{r_x}{K_V}\right) \cap D_{k+1} \subset S. \]
In order to prove the first inclusion in (10.108), take $x' \in S_{\varphi}(y, \frac{r_x}{2KK_V})$, and by using either (10.106) or (10.107) (since they both yield $\delta_{\varphi}(x, y) = r_x/(2KK_V)$) we get
\[\delta_{\varphi}(x, x') \leq K(\delta_{\varphi}(y, x') + \delta_{\varphi}(x, y)) < K\left(\frac{r_x}{2KK_V} + \frac{r_x}{2KK_V}\right) = \frac{r_x}{K_V}. \]
To see the second inclusion in (10.108), take $x'' \in S_{\varphi}(x, r_x/K_V)$ and do
\[\delta_{\varphi}(z, x'') \leq K(\delta_{\varphi}(z, x) + \delta_{\varphi}(x, x'')) \leq K(R + r_x/K_V) = K\left(R + \frac{\delta_{\varphi}(x, D_k)}{2K}\right) < 2KR, \]
where we used the definition of $r_x$ in (10.104) and (10.103). In addition, along the lines of the proof of the inclusions (10.108), by putting $t := (1 + KK_V)r_x$, we also have
\[ S_{\varphi}(y, t/K_V) \subset S_{\varphi}(x, K_4r_x) \subset S_{\varphi}(z, K_5R) \subset \Omega, \]
so that all the sections involved remain strictly inside \( \Omega \). The next claim is that, in either case (10.106) or (10.107), we have

\[
\delta_p(y, D_k) \leq \frac{t}{4K} := \frac{(1 + KK_V)r_x}{4K}.
\]

In order to prove (10.111), recall that \( K_V = 2K(K + 1) \) and write

\[
\delta_p(y, D_k) \leq K(\delta_p(x, y) + \delta_p(x, D_k))
\]

\[
= K \left( \frac{r_x}{2K} + \frac{2Kr_x}{K} \right) = \frac{(1 + 4K^2)r_x}{2K} \leq \frac{(1 + KK_V)r_x}{4K} = \frac{t}{4K}.
\]

In particular, (10.111) yields \( y' \in S_p(y, t/(2K_V)) \) such that \( u(y') \leq M_0^k \) (indeed, notice that for \( y'' \in \partial S_p(y, t/(2K_V)) \) we have \( \delta_p(y, y'') = t/(2K_V) > t/(4K_V) \geq \delta_p(y, D_k) \) by (10.111), so that for \( y' \in S_p(y, t/(2K_V)) \) but near \( \partial S_p(y, t/(2K_V)) \), we still have \( \delta_p(y, y') > \delta_p(y, D_k) \), which implies \( u(y') \leq M_0^k \). Therefore, by setting \( w := u/M_0^k \) we get

\[
\inf_{S_p(y, t/(2K_V))} w \leq 1.
\]

Hence, since \( K_5 \geq K_0 \geq K_3 \), we can apply Corollary 26 to \( w \) on the section \( S_p(y, K_0r) \) with \( r := t/(2K_VK_0) \) (think \( x_0 := y \) and \( N_0 := K_0 \) in Corollary 26) and noticing that \( K_0 \geq 2K(1 + KK_V) \) (by its definition in (10.101)) we have

\[
2r = \frac{t}{K_VK_0} = \frac{(1 + KVK)r_x}{2KK_V} \leq \frac{r_x}{2KK_V},
\]

so that

\[
\mu_p \left( S_p \left( y, \frac{r_x}{2KK_V} \right) \cap D_{k+1} \right) \geq \mu_p \left( S_p(y, 2r) \cap D_{k+1} \right)
\]

\[
= \mu_p (S_p(y, 2r) \cap \{ \tilde{x} \in S : w(\tilde{x}) \leq M_0 \}) = \mu_p (\{ \tilde{x} \in S_p(y, 2r) : w(\tilde{x}) \leq M_0 \})
\]

\[
\geq \frac{\varepsilon_0}{2} \mu_p (S_p(y, 2r)) = \frac{\varepsilon_0}{2} \mu_p \left( S_p \left( y, \frac{t}{2KK_V} \right) \right) \geq \frac{\varepsilon_0}{2KK_V} \mu_p (S_p(y, t/K_V)),
\]

where the second equality is due to the inclusion (10.109) (which implies \( S_p(y, 2r) \cap D_{k+1} \subset S \)), the second inequality is due to Corollary 26 and the last one is due to the doubling property (5.56). Thus, by introducing the structural constant

\[
\eta_0 := \frac{\varepsilon_0}{2KK_V} \]

we obtain

\[
(10.112) \quad \mu_p \left( S_p \left( y, \frac{r_x}{2KK_V} \right) \cap D_{k+1} \right) \geq \eta_0 \mu_p (S_p(y, t/K_V)).
\]

On the other hand, we have the inclusion

\[
(10.113) \quad S_p(x, r_x) \subset S_p(y, t/K_V),
\]

since, given \( x' \in S_p(x, r_x) \),

\[
\delta_p(y, x') \leq K(\delta_p(x, x') + \delta_p(x, y)) \leq K \left( r_x + \frac{r_x}{2KK_V} \right) = \frac{r_x}{K_V} (KK_V + \frac{1}{2})
\]

\[
\leq \frac{r_x}{K_V} (1 + KK_V) = \frac{t}{K_V}.
\]
Therefore, from the first inclusion in (10.109), the inequality (10.112), and the inclusion (10.113) we get

\[(10.114) \quad \mu_\varphi(S_\varphi(x, r_x/K_V) \cap D_{k+1}) \geq \eta_0 \mu_\varphi(S_{\varphi}(x, r_x)).\]

We now look at the collection of sections \(\{S_\varphi(x, r_x/K_V)\}_{x \in S \setminus D_k}\) and resort to Vitali’s covering lemma to obtain a pairwise disjoint subcollection \(\{S_\varphi(x_j, r_{x_j}/K_V)\}_{j \in \mathbb{N}}\) such that \(\{S_{\varphi}(x_j, r_{x_j})\}_{j \in \mathbb{N}}\) covers \(S \setminus D_k\). Next, from (10.105), it follows that

\[
\bigcup_{j \in \mathbb{N}} S_\varphi(x_j, r_{x_j}/K_V) \cap S_\varphi(z, R) \subset S \setminus D_k,
\]

and, after intersecting with \(D_{k+1}\),

\[
\bigcup_{j \in \mathbb{N}} S_\varphi(x_j, r_{x_j}/K_V) \cap D_{k+1} \subset D_{k+1} \setminus D_k,
\]

and then

\[
\mu_\varphi(S \setminus D_k) \leq \sum_{j \in \mathbb{N}} \mu_\varphi(S_\varphi(x_j, r_{x_j})) \leq \frac{1}{\eta_0} \sum_{j \in \mathbb{N}} \mu_\varphi(S_\varphi(x_j, r_{x_j}/K_V) \cap D_{k+1}) = \frac{1}{\eta_0} \mu_\varphi(D_{k+1} \setminus D_k).
\]

That is, for every \(k \in \mathbb{N}_0\), we have obtained \(\eta_0 \mu_\varphi(S \setminus D_k) \leq \mu_\varphi(D_{k+1} \setminus D_k)\), which means

\[
\mu_\varphi(S \setminus D_{k+1}) \leq (1 - \eta_0) \mu_\varphi(S \setminus D_k), \quad \forall k \in \mathbb{N}_0,
\]

from whose iterations (10.102) follows. \(\square\)

**Corollary 31.** Suppose that \(\varphi\) satisfies H1–H6 and fix \(A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega)\). For every section \(S_R := S_\varphi(z, R)\) with \(S_{2K_5R} := S_\varphi(z, 2K_5R) \subset \subset \Omega\) and every \(u \in C(S_{2K_5R}) \cap W^{2,n}(S_{2K_5R})\) satisfying \(u \geq 0\) in \(S_{2K_5R}\) and \(L^\varphi_A(u) \leq f\ a.e.\ in \ S_{2K_5R}\) the assumptions

\[
R^{-\frac{1}{2}}|S_{2K_5R}|^{\frac{1}{2n}} - \frac{1}{2} \|(D^2\varphi)^{-1/2}b\|_{L^n(S_{2K_5R}, d\mu_\varphi)} \|D^2\varphi\|_{L^p(S_{2K_5R}, dz)}^{\frac{1}{2}} \leq \varepsilon_4,
\]

and \(|S_{2K_5R}|^{\frac{1}{2n}} \|c\|_{L^n(S_{2K_5R}, d\mu_\varphi)} \leq \varepsilon_4
\]

imply the inequality

\[
\left(\int_{S_\varphi(z, R)} u^{\beta_2}(x) d\mu_\varphi(x)\right)^{\frac{1}{\beta_2}} \leq M_2 \inf_{S_\varphi(z, R)} u + M_2 \varepsilon_4^{-1} |S_{2K_5R}|^{\frac{1}{n}} \|f\|_{L^n(S_{2K_5R}, d\mu_\varphi)},
\]

where \(\beta_2 \in (0, 1)\) and \(M_2 > 0\) are the structural constants from Theorem 30.

**Proof.** Divide the inequality \(L^\varphi_A(u) \leq f\) by \(Q(f, u)\) with

\[
Q(f, u) := \inf_{S_\varphi(z, R)} u + \varepsilon_4^{-1} |S_{2K_5R}|^{\frac{1}{n}} \|f\|_{L^n(S_{2K_5R}, d\mu_\varphi)}
\]

and apply Theorem 30 to \(u/Q(f, u)\). \(\square\)
10.1. Proof of Theorem 1. Given a section $S_r := S_\varphi(z,r)$ with $S_{2Kr} = S_\varphi(z,2Kr) \subset \subset \Omega$ such that $|S_{2Kr}|^{\frac{1}{n}}\|e\|_{L^n(S_{2Kr},d\mu_\varphi)} \leq \varepsilon_H$ and
\begin{equation}
(10.115) \sup_{x \in S_\varphi(z,r)} \left( \rho^{-\frac{1}{2}} |S_\varphi(x,\rho)|^{\frac{1}{n}} - \frac{1}{n} \frac{1}{2\pi} \|D^2 \varphi\|_{L^n(S_\varphi(x,\rho),d\mu_\varphi)} \|D \varphi\|_{L^p(S_\varphi(x,\rho),dx)}^{\frac{1}{2}} \right) \leq \varepsilon_H,
\end{equation}
Corollary 29 applied with $q := \beta_2$, where $\beta_2 \in (0,1)$ is the structural constant from Corollary 31, gives
\begin{equation}
(10.116) \sup_{S_\varphi(z,\kappa_1 r)} u \leq C_7 \left( \int_{S_\varphi(z,\kappa_2 r)} u(x)^{\beta_2} d\mu_\varphi(x) \right)^{\frac{1}{\beta_2}} + C_7 |S_r|^{\frac{1}{n}} \|f\|_{L^n(S_{2Kr},d\mu_\varphi)},
\end{equation}
provided that $\varepsilon_H \leq \varepsilon_5/2$. On the other hand, the condition (10.115) used with $x = z$ and $\rho = r$ gives
\begin{equation}
r^{-\frac{1}{2}} |S_\varphi(z,r)|^{\frac{1}{n}} - \frac{1}{n} \frac{1}{2\pi} \|D^2 \varphi\|_{L^n(S_\varphi(z,r),d\mu_\varphi)} \|D \varphi\|_{L^p(S_\varphi(z,r),dx)}^{\frac{1}{2}} \leq \varepsilon_H
\end{equation}
and we can use Corollary 31 with $R := r/(2K_5)$ to obtain
\begin{equation}
(10.117) \left( \int_{S_\varphi(z,r/(2K_5))} u^{\beta_2}(x) d\mu_\varphi(x) \right)^{\frac{1}{\beta_2}} \leq M_2 \inf_{S_\varphi(z,r/(2K_5))} u + M_2 \varepsilon_4^{-1} |S_r|^{\frac{1}{n}} \|f\|_{L^n(S_r,d\mu_\varphi)},
\end{equation}
provided that $\varepsilon_H \leq \varepsilon_4/(2K_5)^{1/2}$. Since we can always make $\kappa_1$ smaller in (10.116), as well as $\kappa_2$ close to $\kappa_1$, we can assume that $\kappa_2 \leq 1/(2K_5)$, so that the inequalities (10.116) and (10.117) imply the Harnack inequality (1.17) with $\tau := \kappa_1$, $\varepsilon_H := \min\{\varepsilon_4/(2K_5)^{1/2},\varepsilon_5/2\}$, and $M_H := C_7(1 + M_2 \varepsilon_4^{-1})$. \qed

11. Examples of $\varphi$ satisfying $\textbf{H1}–\textbf{H6}$: The functions $|x|^p$ with $p \geq 2$

For each $p > 1$ set
\begin{equation}
(11.118) \varphi_p(x) := \frac{1}{p}|x|^p \quad \forall x \in \mathbb{R}^n.
\end{equation}
In this section prove that $\varphi_p$ satisfies $\textbf{H1}, \textbf{H2}, \textbf{H3},$ and $\textbf{H4}$ whenever $p > 1$; $\textbf{H5}$ whenever $p > 2-n/q$ for some $q > n$; and $\textbf{H6}$ whenever $p \geq 2$. Therefore, $\varphi_p$ satisfies all the hypotheses $\textbf{H1}–\textbf{H6}$ when $p \geq 2$.

When $1 < p < \infty$, the strictly convex functions $\varphi_p$ have been considered in [13] as prototypical examples of quasi-uniformly convex functions, see Example 2.4 in [13]. That is, the sections of $\varphi_p$ are round in the sense that there exist constants $0 < c_p \leq 1 \leq C_P < \infty$, depending only on $p$ and $n$, such that they relate to Euclidean balls through the following inclusions
\begin{equation}
(11.119) \quad B(x_0,c_p R^{\frac{1}{n}}) \subset S_{\varphi_p}(x_0,R) \subset B(x_0,C_p R^{\frac{1}{n}}) \quad \forall x_0 \in \mathbb{R}^n, R > 0.
\end{equation}
This means that $\nabla \varphi_p : \mathbb{R}^n \to \mathbb{R}^n$ is a quasi-conformal mapping and, as a consequence (see [13, Corollary 3.8]), the Monge-Ampère measure $\mu_{\varphi_p}$ satisfies the $(\mu_\infty)$-condition of Caffarelli-Gutiérrez. Namely, there exist constants $C_0, \theta_0 > 0$, depending only on $p$ and $n$, such that for every section $S_{\varphi_p} \subset \subset \Omega$ and every Borel subset $E \subset S_{\varphi_p}$ we have
\begin{equation}
(11.120) \quad \frac{\mu_{\varphi_p}(E)}{\mu_{\varphi_p}(S_{\varphi_p})} \leq C_0 \left( \frac{|E|}{|S_{\varphi_p}|} \right)^{\theta_0}.
\end{equation}
For every hypothesis, we have that \( p > 1 \) we have \( \mu_{\varphi_p} \in (DC)_{\varphi_p} \), with doubling constants depending only on \( p \) and \( n \), and the hypothesis H2 is met. Also, from (11.120) and the discussion in Section 5.3 it follows that \( \varphi_p \) satisfies H3 as well.

Next, regarding the local \( L^q \)-integrability of \( D^2 \varphi_p \), since \( \|D^2 \varphi_p(x)\| \sim \Delta \varphi_p(x) \sim |x|^{p-2} \), we have that \( D^2 \varphi_p \in L^q(B(0,1)) \) if and only if \( q(p-2) > -n \), that is, \( p > 2 - n/q \). In particular, if \( p \geq 2 \), then \( D^2 \varphi_p \in L^q(B(0,1)) \) for every \( q > 0 \). On the other hand, since the required \( L^q \)-integrability asks for some \( q > n \), such \( q \) can be found (close enough to \( n \)) whenever \( p > 1 \).

In order to prove condition (1.10) when \( p \geq 2 \) we will use the next observation.

**Lemma 32.** Given \( v, w \in \mathbb{R}^n \setminus \{0\} \) let \( N_{v,w} : (0, \infty) \to [0, \infty) \) be the function

\[
N_{v,w}(q) := \left| \frac{v}{|v|} - \left( \frac{|w|}{|v|} \right)^q \frac{w}{|w|} \right|.
\]

If \( |v| < |w| \) then \( N_{v,w}(q) > 0 \) for every \( q \in (0, \infty) \). In particular, since \( N_{v,w} \equiv 0 \) when \( |v| = |w| \), it follows that \( N_{v,w}(q) \geq 0 \) whenever \( |v| \leq |w| \).

**Proof.** Notice that, given \( q \in (0, \infty) \), we have \( N_{v,w}(q) = 0 \) if and only if \( v = w \), so that the hypothesis \( |v| < |w| \) implies \( N_{v,w}(q) > 0 \). Now, by considering \( N_{v,w}(q)^2 \) we get

\[
\frac{d(N_{v,w}(q)^2)}{dq} = \sum_{j=1}^n \frac{d}{dq} \left[ v_j \left( \frac{|w|}{|v|} \right)^q \frac{w_j}{|w|} \right] \left( \frac{|w|}{|v|} \right)^2 \\
= 2 \sum_{j=1}^n \left[ v_j \left( \frac{|w|}{|v|} \right)^q \frac{w_j}{|w|} \right] \left[ - \frac{w_j}{|w|} \left( \frac{|w|}{|v|} \right)^q \log \left( \frac{|w|}{|v|} \right) \right] \\
= 2 \left( \frac{|w|}{|v|} \right)^q \log \left( \frac{|w|}{|v|} \right) - \langle v, w \rangle \left( \frac{|w|}{|v|} \right)^q + \left( \frac{|w|}{|v|} \right)^q.
\]

Next, from the hypothesis \( |v| < |w| \) and the Cauchy-Schwarz inequality we obtain

\[
\left( \frac{|w|}{|v|} \right)^q > 1 \geq \frac{\langle v, w \rangle}{|v||w|},
\]

\( \log(|w|/|v|) > 0 \) and \( N_{v,w}(q) > 0 \) for every \( q \in (0, \infty) \), which make \( N_{v,w}(q) > 0 \) as claimed. \( \square \)

**Theorem 33.** For every \( p \geq 2 \) there exists \( \sigma(p, n) > 0 \), depending only on \( p \) and dimension \( n \), such that

\[
D_{\varphi_p}(x_0, x) \geq \sigma(p, n) \delta_{\varphi_p}(x_0, x) \quad \forall x_0 \in \mathbb{R}^n, x \in \mathbb{R}^n \setminus \{0\}.
\]

**Proof.** Since \( \mu_{\varphi_p} \in (DC)_{\varphi_p} \) with constants depending only on \( p \) and \( n \), by Theorems 1 and 4 from [9] there exists a constant \( \kappa_{p,n} \in (0, 1) \), also depending only on \( p \) and \( n \), such that

\[
\kappa_{p,n} \delta_{\varphi_p}(x, y) \leq \langle \nabla \varphi_p(x) - \nabla \varphi_p(y), x - y \rangle \quad \forall x, y \in \mathbb{R}^n.
\]

On the other hand,

\[
D_{\varphi_p}(x_0, x) := \langle D^2 \varphi_p(x)^{-1}(\nabla \varphi_p(x_0) - \nabla \varphi_p(x)), \nabla \varphi_p(x_0) - \nabla \varphi_p(x) \rangle.
\]

Now, we have \( \nabla \varphi_p(0) = 0 \) and, given \( x \in \mathbb{R}^n \setminus \{0\} \), \( \nabla \varphi_p(x) = |x|^{p-1} \frac{x}{|x|} \) and

\[
D^2 \varphi_p(x) = (p-1)|x|^{p-2} \left( \frac{x}{|x|} \otimes \frac{x}{|x|} \right) + |x|^{p-2} \left( I - \frac{x}{|x|} \otimes \frac{x}{|x|} \right).
\]
Thus, for $x \in \mathbb{R}^n \setminus \{0\}$, the eigenvalues of $D^2 \varphi_p(x)$ are $(p - 1)|x|^{p-2}$ with multiplicity 1 and eigenvector $x$ and $|x|^{p-2}$ with multiplicity $n - 1$ and eigenspace $x^\perp$.

Let us first consider the case $x_0 = 0$. In this case we have

$$D_{\varphi_p}(0, x) = \langle D^2 \varphi_p(x)^{-1} \nabla \varphi_p(x), \nabla \varphi_p(x) \rangle = \frac{1}{(p - 1)|x|^{p-2}} |\nabla \varphi_p(x)|^2 = \frac{|x|^p}{p - 1}$$

and

$$\langle \nabla \varphi_p(x) - \nabla \varphi_p(0), x - 0 \rangle = \langle \nabla \varphi_p(x), x \rangle = |x|^p,$$

so that by (11.123) we get

$$\text{(11.124)}$$

$$D_{\varphi_p}(0, x) \geq \frac{\kappa_{p,n}}{p - 1} \delta_{\varphi_p}(0, x) \ \forall x \in \mathbb{R}^n \setminus \{0\}.$$  

Next, consider the case $x, x_0 \in \mathbb{R}^n \setminus \{0\}$ with $|x| < |x_0|$. We have

$$\langle \nabla \varphi_p(x) - \nabla \varphi_p(x_0), x - x_0 \rangle \leq |x|^{p-1} \frac{x}{|x|} - |x_0|^{p-1} \frac{x_0}{|x_0|} |x - x_0|$$

$$\text{(11.125)}$$

$$= |x|^p \left| \frac{x}{|x|} - \left( \frac{x_0}{|x_0|} \right)^{p-1} \frac{x_0}{|x_0|} \right| |x - x_0| = |x|^p \left| \frac{x}{|x|} - \frac{x_0}{|x_0|} \right| N_{x,x_0}(p - 1),$$

where $N_{x,x_0}$ is as in (11.121). By using that the smallest eigenvalue of $D^2 \varphi_p(x)^{-1}$ is $\frac{1}{(p - 1)|x|^{p-2}}$ we can write

$$D_{\varphi_p}(x_0, x) = \langle D^2 \varphi_p(x)^{-1} (\nabla \varphi_p(x_0) - \nabla \varphi_p(x)), \nabla \varphi_p(x_0) - \nabla \varphi_p(x) \rangle$$

$$\geq \frac{1}{(p - 1)|x|^{p-2}} |\nabla \varphi_p(x_0) - \nabla \varphi_p(x)|^2 = \frac{|x|^{p-1} \frac{x}{|x|} - |x_0|^{p-1} \frac{x_0}{|x_0|} |x_0|}{(p - 1)|x|^{p-2}},$$

and by expressing $p \geq 2$ as $p - 1 \geq 1$ and using Lemma 32 to get $N_{x,x_0}(p - 1) > N_{x,x_0}(1)$, we obtain

$$\left| \frac{|x|^{p-1} \frac{x}{|x|} - |x_0|^{p-1} \frac{x_0}{|x_0|}}{(p - 1)|x|^{p-2}} \right|^2 = \frac{|x|^p}{(p - 1)} N_{x,x_0}(p - 1)^2$$

$$> \frac{|x|^p}{(p - 1)} N_{x,x_0}(1) N_{x,x_0}(p - 1) = \frac{|x|^p}{(p - 1)} \left| \frac{x}{|x|} - \frac{x_0}{|x_0|} \right| N_{x,x_0}(p - 1),$$

which combined with (11.125), (11.126), and (11.123), gives

$$\text{(11.127)}$$

$$D_{\varphi_p}(x_0, x) \geq \frac{\kappa_{p,n}}{p - 1} \delta_{\varphi_p}(x_0, x) \ \forall x, x_0 \in \mathbb{R}^n \setminus \{0\}, \text{with } |x| < |x_0|.$$  

Next, we split the case $0 < |x_0| < |x|$ into two subcases: namely, $N_{x,x_0}(p - 1) \geq 1/2$ and $N_{x,x_0}(p - 1) < 1/2$. If $N_{x,x_0}(p - 1) \geq 1/2$, as in (11.126) we write

$$D_{\varphi_p}(x_0, x) \geq \frac{|x|^{p-1} \frac{x}{|x|} - |x_0|^{p-1} \frac{x_0}{|x_0|}}{(p - 1)|x|^{p-2}} N_{x,x_0}(p - 1)^2 \geq \frac{|x|^p}{4(p - 1)},$$

and, as in (11.125) and using that $|x_0| \leq |x|$, the}
Thus, in the subcase $0 < |x_0| \leq |x|$ and $N_{x,x_0}(p-1) \geq 1/2$, we obtain
\begin{equation}
(11.128) \quad D_\varphi(x_0, x) \geq \frac{1}{16(p-1)}(\nabla \varphi_p(x) - \nabla \varphi_p(x_0), x - x_0).
\end{equation}

Now, for the subcase $0 < |x_0| \leq |x|$ and $N_{x,x_0}(p-1) < 1/2$ we first notice that $N_{x,x_0}(p-1) < 1/2$ implies that
\begin{equation}
(11.129) \quad \left(\frac{|x_0|}{|x|}\right)^{p-1} > \frac{1}{2}
\end{equation}
because
\[
0 \leq 1 - \left(\frac{|x_0|}{|x|}\right)^{p-1} \leq \left|\frac{x}{|x|} - \left(\frac{|x_0|}{|x|}\right)^{p-1} \frac{x_0}{|x_0|}\right| = N_{x,x_0}(p-1) < \frac{1}{2}.
\]

Again as in (11.126) we have
\[
D_\varphi(x_0, x) \geq \frac{|x|^{p-1} - |x_0|^{p-1} \frac{x_0}{|x_0|}}{(p-1)|x|^{p-2}},
\]
but now we use $N_{x_0,x}$ (as opposed to $N_{x,x_0}$) as well as (11.129) and Lemma 32 to write
\[
\frac{|x|^{p-1} - |x_0|^{p-1} \frac{x_0}{|x_0|}}{(p-1)|x|^{p-2}} = \frac{|x_0|^{2(p-1)}}{(p-1)|x|^{p-2}} \left(\left(\frac{|x|}{|x_0|}\right)^{p-1} \frac{x}{|x|} - \frac{x_0}{|x_0|}\right)^2.
\]

On the other hand,
\[
\langle \nabla \varphi_p(x) - \nabla \varphi_p(x_0), x - x_0 \rangle \leq \left|\frac{x}{|x|} - \left(\frac{|x_0|}{|x_0|}\right)^{p-1} \frac{x_0}{|x_0|}\right| |x - x_0|,
\]
\[
= |x_0|^p \left(\left(\frac{|x|}{|x_0|}\right)^{p-1} \frac{x}{|x|} - \frac{x_0}{|x_0|}\right) \left(\left|\frac{x}{|x_0|}\right| \frac{x}{|x|} - \frac{x_0}{|x_0|}\right) = |x_0|^p N_{x_0,x}(p-1) N_{x_0,x}(1).
\]

Thus, in the subcase $0 < |x_0| \leq |x|$ and $N_{x,x_0}(p-1) < 1/2$, we obtain
\begin{equation}
(11.130) \quad D_\varphi(x_0, x) \geq \frac{1}{2^{p-1}(p-1)} \langle \nabla \varphi_p(x) - \nabla \varphi_p(x_0), x - x_0 \rangle.
\end{equation}

Hence, (11.122) follows from (11.124), (11.127), (11.128), and (11.130) with
\begin{equation}
(11.131) \quad \sigma(p, n) := \frac{\kappa_{p,n}}{(p-1)} \min\{2^{-4}, 2^{p-2}\}.
\end{equation}

\[\square\]

Remark 34. The condition (1.10) fails for $\varphi_p$ when $1 < p < 2$. Indeed, given $x_0 \in \mathbb{R}^n \setminus \{0\}$, put $r := \delta_{\varphi_p}(x_0, 0) > 0$ so that $0 \in S_\varphi(x_0, 2r) \setminus S_\varphi(x_0, r)$. Next, notice that, when $1 < p < 2$, we have that $D_{\varphi_p}(x_0, x)$ tends to 0 as $x \to 0$. Therefore, (1.10) cannot happen for any value of $\sigma > 0$. 

12. The Preservation of $\mathbf{H1, H2, H4, H5, and H6}$ under Tensor Sums

In this section we prove that the hypotheses $\mathbf{H1, H2, H4, H5, and H6}$ are preserved, quantitatively, under tensor sums. Also, we point out that $\mathbf{H3}$ is preserved under tensor sums of (positive multiples of) the functions $\varphi_p$ in (11.118).

Fix $m \in \mathbb{N}$. For each $j = 1, \ldots, m$, consider $n_j \in \mathbb{N}$ and a convex function $\varphi^j : \mathbb{R}^{n_j} \rightarrow \mathbb{R}$. Let $\varphi$ be the tensor sum of the $\varphi^j$’s, that is, $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, with $n := \sum_{j=1}^{m} n_j$, and

$$\varphi(x) := \sum_{j=1}^{m} \varphi^j(x_j), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_j \in \mathbb{R}^{n_j}, j = 1, \ldots, m. \tag{12.132}$$

It is immediate to see that if the functions $\varphi^j$, $j = 1, \ldots, m$, satisfy $\mathbf{H1}$ and $\mathbf{H4}$, then so does $\varphi$. In order to consider $\mathbf{H2, H5, and H6}$, given an open, bounded set $\Omega \subset \mathbb{R}^n$, let $\Omega_j$ denote the projection of $\Omega$ on $\mathbb{R}^{n_j}$. Then we have the following

**Theorem 35.** If the functions $\varphi^j$, $j = 1, \ldots, m$, satisfy $\mathbf{H2}$ then so does $\varphi$, quantitatively. Moreover, $\delta_\varphi$ satisfies a quasi-triangle inequality with constant $K := \max_{j=1, \ldots, m} \{K_j\}$, where $K_j \geq 1$ is the quasi-triangle constant for $\delta_{\varphi^j}$.

If the functions $\varphi^j$, $j = 1, \ldots, m$, satisfy $\mathbf{H5}$ in $\Omega_j$ with some $p > n$, then $\varphi$ satisfies $\mathbf{H5}$ in $\Omega$ with the same $p > n$.

If the functions $\varphi^j$, $j = 1, \ldots, m$, satisfy $\mathbf{H6}$ in the following sense: For each $j = 1, \ldots, m$, there exists $\sigma_j > 0$ such that for every $x_{0,j} \in \mathbb{R}^{n_j}$ and $r > 0$ with $S_{\varphi^j}(x_{0,j}, 2K_j r) \subset \subset \Omega_j$, we have

$$D_{\varphi^j}(x_{0,j}, x_j) \geq \sigma_j \delta_{\varphi^j}(x_{0,j}, x_j) \quad \text{a.e.} \; x_j \in S_{\varphi^j}(x_{0,j}, 2mr) \setminus S_{\varphi^j}(x_{0,j}, r). \tag{12.133}$$

Then, $\varphi$ satisfies $\mathbf{H6}$ with $\sigma := \frac{1}{2m} \min_{j=1, \ldots, m} \{\sigma_j\}$ in the sense that for every $x_0 \in \mathbb{R}^n$ and $r > 0$ with $S_{\varphi}(x_0, 2Kmr) \subset \subset \Omega$ we have

$$D_{\varphi}(x_0, x) \geq \sigma \delta_{\varphi}(x_0, x) \quad \text{a.e.} \; x \in S_{\varphi}(x_0, 2r) \setminus S_{\varphi}(x_0, r). \tag{12.134}$$

**Proof.** If the functions $\{\varphi^j\}_{j=1}^{m}$ satisfy the hypothesis $\mathbf{H2}$, each with some pairs of constants $(C_{d,j}, \alpha_j)$ in (1.6), then so does $\varphi$ with some constants $(C_d, \alpha)$ depending only on the constants $\{C_{d,j}, \alpha_j, n_j\}_{j=1}^{m}$. This is so because of the characterization of the doubling property (1.6) in terms of the so-called engulfing property of the Monge-Ampère sections (see [8, Theorem 8] and [9, Theorem 4]) and the fact that the sections of $\varphi$ and the ones for $\{\varphi^j\}_{j=1}^{m}$ are related by the inclusions

$$S_{\varphi}(x, t) \subset S_{\varphi^j}(x_1, t) \times \cdots \times S_{\varphi^m}(x_m, t) \subset S_{\varphi}(x, mt), \tag{12.135}$$

for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_j \in \mathbb{R}^{n_j}, j = 1, \ldots, m$, and $t > 0$, see [10, Lemma 6]. Also, the fact that $\nabla \varphi = \sum_{j=1}^{m} \nabla \varphi^j$ and the definition of $\delta_\varphi$ in (1.4) imply that

$$\delta_{\varphi}(x_0, x) = \sum_{j=1}^{m} \delta_{\varphi^j}(x_{0,j}, x_j); \tag{12.136}$$

in particular, $\delta_\varphi$ satisfies a quasi-triangle inequality with constant $K := \max_{j=1, \ldots, m} \{K_j\}$.

Next, assume that each function $\varphi^j$, $j = 1, \ldots, m$, satisfies $\mathbf{H5}$ in $\Omega_j$ with an exponent $p > n$. Since $\Delta \varphi(x) = \sum_{j=1}^{m} \Delta_j \varphi^j(x_j)$, where $\Delta_j$ denotes the Laplacian in $\mathbb{R}^{n_j}$, by Fubini’s
theorem we get
\begin{equation}
\int_{\Omega} \Delta \varphi(x)^p \, dx \simeq_{m,p} \sum_{j=1}^{m} \int_{\Omega_j} \Delta_j \varphi^j(x_j)^p \, dx_j = \sum_{j=1}^{m} \left( \int_{\Omega_j} \Delta_j \varphi^j(x_j)^p \, dx_j \right) \mathcal{L}^{n-n_j}(\tilde{\Omega}_j) < \infty,
\end{equation}
where \( \tilde{\Omega}_j \) denotes the projection of \( \Omega \) over \( \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_j} \times \mathbb{R}^{n_j+1} \times \cdots \times \mathbb{R}^{n_m} \) and \( \mathcal{L}^{n-n_j}(\tilde{\Omega}_j) \) denotes its Lebesgue measure. Thus, \( \varphi \) satisfies \textbf{H5} in \( \Omega \) with the same \( p > n \).

Finally, regarding \textbf{H6} for \( \varphi \), suppose that each \( \varphi^j \) satisfies (12.133). Given \( x_0 \in \mathbb{R}^n \) and \( r > 0 \) with \( S_{\varphi}(x_0,2Kmr) \subset \subset \Omega \), we first notice that the inclusions (12.135) imply that \( S_{\varphi}(x_0,j,2Kj^r) \subset \subset \Omega_j \) for every \( j = 1, \ldots, m \). On the other hand, since \( D^2 \varphi = \bigoplus_{j=1}^{m} D^2 \varphi^j \), it follows that for every \( x, x_0 \in \Omega \)
\begin{align*}
D_{\varphi}(x_0, x) := & \langle D^2 \varphi(x)^{-1}(\nabla \varphi(x) - \nabla \varphi(x_0)), \nabla \varphi(x) - \nabla \varphi(x_0) \rangle \\
= & \sum_{j=1}^{m} \langle D^2 \varphi^j(x_j)^{-1}(\nabla \varphi(x_j) - \nabla \varphi(x_0,j)), \nabla \varphi(x_j) - \nabla \varphi(x_0,j) \rangle = \sum_{j=1}^{m} D_{\varphi^j}(x_0,j, x_j).
\end{align*}
Fix \( x \in S_{\varphi}(x_0,2r) \setminus S_{\varphi}(x_0, r) \subset \subset \Omega \). From the inclusion (12.135) we get that \( x_j \in S_{\varphi^j}(x_0,j, 2r) \) for every \( j = 1, \ldots, m \), and that there exists \( j_0 \) such that \( x_j \notin S_{\varphi^j}(x_0,j_0, r/m) \). That is, \( \delta_{\varphi^j}(x_0,j_0, x_j, ) > r/m \). Then, by recalling that \( \sigma := \frac{1}{2m} \min_{j=1, \ldots, m} \{ \sigma_j \} \),
\begin{align*}
D_{\varphi}(x_0, x) = & \sum_{j=1}^{m} D_{\varphi^j}(x_0,j, x_j) \geq D_{\varphi^j}(x_0,j_0, x_{j_0}) \geq \sigma_j \delta_{\varphi^j}(x_0,j_0, x_{j_0}) \\
> & \sigma_j \frac{r}{m} \sigma_j \frac{r}{2m} \delta_{\varphi}(x_0, x) \geq \sigma \delta_{\varphi}(x_0, x),
\end{align*}
which proves (12.134).

\textbf{Remark 36.} (On the preservation of the \( \mu_\infty \)-condition under tensor sums). It is not clear to the author whether the hypothesis \textbf{H3} is preserved under tensor sums. However, \textbf{H3} will be preserved, quantitatively, by tensor sums of the functions \( \varphi^j \) in (11.118). This is due to the fact that given strictly convex functions \( \varphi^j \in C^1(\mathbb{R}^n_j) \), \( j = 1, \ldots, m \), each satisfying the \( \mu_\infty \)-condition, then its tensor sum (12.132) will also satisfy the \( \mu_\infty \)-condition, quantitatively (see Lemma 6(ii) from [10]).

Now, given \( p_1, \ldots, p_m \in (1, \infty) \), each function \( \varphi_{p_j} \) satisfies (11.120), which is equivalent to the \( \mu_\infty \)-condition (see [3, Section 5]). Then, their tensor sum will also satisfy (11.120), quantitatively and, from the discussion in Section 5.3, it will satisfy \textbf{H3} as well.

\section{Appendix}

\subsection{Proof of \textbf{Lemma 13} (Morrey-type estimate).}
By [25, Theorem 1.3], there exists a geometric constant \( K_P > 0 \) such that for every section \( S := S_{\varphi}(z_0, t) \) such that for every \( h \in C^1(S) \) the following Poincaré inequality holds true
\begin{equation}
\frac{1}{|S|} \int_{S} |h(z) - h_S| \, dz \leq K_P t^{\frac{1}{2}} \left( \frac{1}{|S|} \int_{S} |\nabla \varphi h(z)|^2 \, dz \right)^{\frac{1}{2}},
\end{equation}
where \( \nabla \varphi h(z) := D^2 \varphi(z)^{-1/2} \nabla h(z) \) and
\[ h_S := \frac{1}{|S|} \int_{S} h(z) \, dz. \]
Given \( x, y \in \Omega \) and \( j \in \mathbb{N}_0 \), set \( S^x_j := S_\varphi(x, 2^{-j}\delta_\varphi(x, y)) \) and \( S^y_j := S_\varphi(y, 2^{-j}\delta_\varphi(y, x)) \). Also, define \( S_K^x := S_\varphi(x, 2K\delta_\varphi(x, y)) \) so that the \( K \)-quasi-triangle inequality gives the inclusions

\[
S_\varphi(x, \delta_\varphi(x, y)/(2K)) \subseteq S^y_0 = S_\varphi(y, \delta_\varphi(y, x)) \subseteq S_\varphi(x, 2K\delta_\varphi(x, y)) = S_K^x.
\]

In particular, (13.139) and the doubling property (5.55) imply the inequalities

\[
|S_K^x| \leq 2^n(2K)^{2n}|S_\varphi(x, \delta_\varphi(x, y)/(2K))| \leq 2^n(2K)^{2n}|S_0^y|,
\]

\[
|S_K^x| \leq 2^n(2K)^n|S_\varphi(x, \delta_\varphi(x, y))|.
\]

By the Lebesgue differentiation theorem applied to \( h \) at \( x \), the doubling property (5.54) and the Poincaré inequality (13.138) we have

\[
|h(x) - h_{S_0^x}| \leq \sum_{j \in \mathbb{N}_0} |h_{S_j^x} - h_{S_{j+1}^x}| \leq \sum_{j \in \mathbb{N}_0} \int_{S_{j+1}^x} |h(z) - h_{S_j^x}| \, dz
\]

\[
\leq 2^n \sum_{j \in \mathbb{N}_0} \int_{S_j^x} |h(z) - h_{S_j^x}| \, dz \leq 2^n K_P \delta_\varphi(x, y)^{\frac{3}{2}} \sum_{j \in \mathbb{N}_0} 2^{-j/2} \left( \frac{1}{|S_j^x|} \int_{S_j^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}.
\]

Next, given \( q > 2n \geq 2 \), by the doubling property (5.55)

\[
\sum_{j \in \mathbb{N}_0} 2^{-j/2} \left( \frac{1}{|S_j^x|} \int_{S_j^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}} \leq \sum_{j \in \mathbb{N}_0} 2^{-j/2} \left( \frac{1}{|S_j^x|} \int_{S_j^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}
\]

\[
\leq 2^{n/q} \sum_{j \in \mathbb{N}_0} 2^{-j/2} 2^{jn/q} \left( \frac{1}{|S_j^x|} \int_{S_j^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}
\]

\[
\leq 2^{n/q} \left( \sum_{j \in \mathbb{N}_0} 2^{-j(1/2-n/q)} \right) \left( \frac{1}{|S_j^x|} \int_{S_j^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}.
\]

Hence,

\[
|h(x) - h_{S_0^x}| \leq C_1 \delta_\varphi(x, y)^{\frac{1}{2}} \left( \frac{1}{|S_0^x|} \int_{S_0^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}},
\]

with \( C_1 := 2^{n+n/q} K_P \sum_{j \in \mathbb{N}_0} 2^{-j(1/2-n/q)} < \infty \), since \( q > 2n \). Switching the roles of \( x \) and \( y \) gives

\[
|h(y) - h_{S_0^y}| \leq C_1 \delta_\varphi(y, x)^{\frac{1}{2}} \left( \frac{1}{|S_0^y|} \int_{S_0^y} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}.
\]

Hence,

\[
|h_{S_0^x} - h_{S_0^y}| \leq |h_{S_0^x} - h_{S_K^x}| + |h_{S_K^x} - h_{S_0^y}| \leq 2 \int_{S_K^x} |h(z) - h_{S_K^x}| \, dz
\]

\[
\leq 2K_P K^{1/2} \delta_\varphi(x, y)^{\frac{1}{2}} \left( \frac{1}{|S_K^x|} \int_{S_K^x} |\nabla \varphi h(z)|^q \, dz \right)^{\frac{1}{q}}.
\]
Finally, (5.63) follows from (13.141), (13.142), (13.143), and (13.140) with
\[ C_{q,K} := (2^{2n}K^n)^{1/q}C_1[1 + (2K)^{n/q}] + 2K^{1/2}. \]

\[ \square \]

13.2. Proof of Lemma 15 (Local Vitali covering lemma). The following proof of Vitali’s covering lemma is modeled after Theorem 1.2 on [5, p.69]. Notice that the inclusion \( S_\varphi(x, 2r_x) \subset S_0 := S_\varphi(x_0, R_0) \) for every \( x \in I \) implies that \( \sup_{x \in I} r_x < \infty \). More precisely, given \( x \in I \) and \( y \in \partial S_\varphi(x, r_x) \), from the quasi-triangle inequality (1.7) we get
\[ r_x = \delta_\varphi(x, y) \leq K[\delta_\varphi(x_0, x) + \delta_\varphi(x_0, y)] < 2KR_0. \]

Now, given \( K \in (2K^2 + K) \) fix \( \varepsilon > 0 \) such that
\[ K_\varepsilon := \left( \frac{2 - \varepsilon}{1 - \varepsilon} \right) K + \frac{1}{1 - \varepsilon} < \frac{K}{K}. \]

and put \( \kappa := \kappa(\varepsilon) := \left( \frac{2 - \varepsilon}{1 - \varepsilon} \right) K > 1 \). Next, define \( E := E \) and, inductively, for \( k \in \mathbb{N} \) choose \( x_k \in E_k := E \setminus \bigcup_{j=1}^{k-1} S_\varphi(x_j, \kappa r_{x_j}) \) with
\[ r_{x_k} > (1 - \varepsilon) \sup_{x \in E_k} r_x. \]

Notice that for \( j \leq k \) we have \( E_k \subset E_j \) so that
\[ r_{x_k} \leq \sup_{x \in E_k} r_x \leq \sup_{x \in E_j} r_x < \frac{1}{1 - \varepsilon} r_{x_j}. \]

It now follows that for \( j < k \) we have \( S_\varphi(x_j, r_{x_j}) \cap S_\varphi(x_k, r_{x_k}) = \emptyset \). Otherwise, there would exist \( z \in S_\varphi(x_j, r_{x_j}) \cap S_\varphi(x_k, r_{x_k}) \) and the quasi-triangle inequality (1.7), the inequality (13.145), and the definition of \( \kappa \) would yield
\[ \delta_\varphi(x_j, x_k) \leq K[\delta_\varphi(x_k, z) + \delta_\varphi(x_j, z)] < K r_{x_k} + r_{x_j} \leq K \left( \frac{1}{1 - \varepsilon} \right) r_{x_j} = \kappa r_{x_j}, \]
contradicting the fact that \( x_k \notin S_\varphi(x_j, \kappa r_{x_j}) \). If the process of choosing \( x_k \)'s stops after finitely many steps, then the result follows. Let us then assume that there is an \( x_k \) for each \( k \in \mathbb{N} \) in such case we must have \( \lim_{k \to \infty} r_{x_k} = 0 \). Otherwise, there would exist \( \rho > 0 \) such that \( r_{x_j} > \rho \) for infinitely many values of \( j \). That is, there would be infinitely many pairwise-disjoint sections \( \{S_\varphi(x_j, r_{x_j})\}_{j \in \mathbb{N}} \) with
\[ S_\varphi(x_j, \rho) \subset S_\varphi(x_j, r_{x_j}) \subset S_\varphi(x_j, \kappa r_{x_j}) \subset S_0 \quad \forall j \in \mathbb{N}. \]

Let us now see that (13.147) leads to a contradiction. Indeed, given any two of these disjoint sections \( S_\varphi(x_j, \rho) \) and \( S_\varphi(x_k, \rho) \), we claim that
\[ S_\varphi(x_j, \rho) \subset S_\varphi(x_k, K(\rho + 2KR_0)) \]
so that the doubling property (5.56) yields
\[ \mu_\varphi(S_\varphi(x_j, \rho)) \leq \mu_\varphi(S_\varphi(x_k, K(\rho + 2KR_0))) \leq K_D \left( \frac{K(\rho + 2KR_0)}{\rho} \right)^n \mu_\varphi(S_\varphi(x_k, \rho)), \]
which makes the Monge-Ampère measures of the disjoint sections \( S_\varphi(x_j, \rho) \) and \( S_\varphi(x_k, \rho) \) comparable. Consequently, since (13.147) implies the inclusion \( \bigcup_{j \in \mathbb{N}} S_\varphi(x_j, \rho) \subset S_0 \), it follows that \( \mu_\varphi(S_0) = \infty \), a contradiction.
In order to prove the inclusion (13.148), given \( y \in S_\varphi(x_j, \rho) \) the quasi-triangle inequality (1.7) gives
\[
\delta_\varphi(y, x_k) \leq K[\delta_\varphi(x_j, y) + \delta_\varphi(x_j, x_k)] \leq K[\rho + \delta_\varphi(x_j, x_k)]
\]
and, on the other hand, from (1.7) and (13.147)
\[
\delta_\varphi(x_j, x_k) \leq K[\delta_\varphi(x_0, x_j) + \delta_\varphi(x_0, x_k)] < 2KR_0,
\]
and (13.148) follows. Finally, let us see that \( \bigcup_{x \in I} S_\varphi(x, r_x) < \bigcup_{j \in \mathbb{N}} S_\varphi(x_j, K_V r_{x_j}) \). Given \( z \in \bigcup_{x \in I} S_\varphi(x, r_x) \), there exists a pair \( (x_z, r_z) \) with \( z \in S_z:= S_\varphi(x_z, r_z) \). The next claim is that \( S_z \cap S_\varphi(x_j, \kappa r_{x_j}) \neq \emptyset \) for some \( j \in \mathbb{N} \). Indeed, by construction, if \( S_z \cap S_\varphi(x_j, \kappa r_{x_j}) = \emptyset \) for every \( j = 1, \ldots, k - 1 \), then
\[
r_z < \frac{1}{1 - \varepsilon} r_{x_k}.
\]
Now, the fact that \( \lim_{k \to \infty} r_{x_k} = 0 \) prevents (13.149) from happening for infinitely many \( k \)'s in \( \mathbb{N} \). Therefore, there is a first \( \ell \in \mathbb{N} \) such that \( S_z \cap S_\varphi(x_{\ell}, \kappa r_{x_{\ell}}) = \emptyset \) for every \( j = 1, \ldots, \ell - 1 \) and \( S_z \cap S_\varphi(x_\ell, \kappa r_{x_\ell}) \neq \emptyset \). In particular, \( r_{x_z} < (1 - \varepsilon)^{-1} r_{x_{\ell}} \). The final step is to prove that \( z \in S_\varphi(x_\ell, K_V r_\ell) \) which follows from taking \( y \in S_z \cap S_\varphi(x_\ell, \kappa r_{x_\ell}) \neq \emptyset \) and writing
\[
\delta_\varphi(x_\ell, z) \leq K[\delta_\varphi(x_\ell, y) + \delta_\varphi(z, y)] < K(\kappa r_{x_\ell} + r_z) < K \left( \kappa + \frac{1}{1 - \varepsilon} \right) r_{x_\ell} = KK_{x_\ell} r_{x_\ell} < K_V r_{x_\ell}.
\]
\( \square \)

13.3. Sketch of the proof of Theorem 27. The first step is to prove the following counterpart to Lemma 5 from [26] based on Theorem 25. The idea is to replace the inequality (4.43) and Theorem 2 from [26] with (5.56) and Theorem 25 in this article) and we will just sketch it. Also, the role of “\((1 - \varepsilon_0)\)” in the inequality (3.24) of [26] will be played by the “\(\varepsilon/2\)” from Theorem 25.

Let \( \gamma_0, \varepsilon_3 \in (0, 1) \) and \( K_3 > 1 \) be the structural constants from Theorem 25 and put \( M_1 := 5/\gamma_0 \) and \( \nu := 2M_1/(2M_1 - 1) > 1 \).

**Lemma 37.** Suppose that \( \varphi \) satisfies H1–H6 and fix \( A \in \mathcal{E}(\lambda, \Lambda, \varphi, \Omega) \). Let \( S_\varphi(z, r) \) be a section with \( S_{2K_r} := S_\varphi(z, 2K) \subset \subset \Omega \) and let \( u \in C(S_{2K_r}) \cap W^{2,n}(S_{2K_r}) \) satisfy \( u \geq 0 \) in \( S_{2K_r} \) and \( L^*_A(u) \geq f \ a.e. \) in \( S_{2K_r} \). Assume that the inequalities (9.98) and (9.99) and that
\[
\int_{S_\varphi(z, 2Kr)} u \, d\mu_\varphi < M_1.
\]

Suppose that there exists \( x_0 \in S_\varphi(z, r) \), \( j \in \mathbb{N} \), and \( \rho \in (0, r) \) such that \( u(x_0) \geq \nu^j \gamma_0 \) and that
\[
\mu_\varphi(S_\varphi(x_0, \rho)) \geq \left( \frac{2^{n+2}K^2_rK^n_3}{\nu^{j_0}\varepsilon_0} \right) \mu_\varphi(S_\varphi(z, r)).
\]

Then \( \sup_{S_\varphi(x_0, \rho)} u > \nu^j M_1 \).

**Proof.** By contradiction, assume that \( \sup_{S_\varphi(x_0, \rho)} u \leq \nu^j M_1 \) and define
\[
w_j(x) := \frac{\gamma_0(\nu^j M_1 - u(x))}{\nu^{j-1}(\nu - 1)M_1}
\]
(13.152)
so that \( w_j \geq 0 \) in \( S_\varphi(x_0, \rho) \). Since \( x_0 \in S_\varphi(z, r) \) the triangle-inequality (1.7) gives \( S_\varphi(x_0, \rho) \subset S_\varphi(z, 2Kr) \). Then, using that \( L^p_\nu(u) \geq f \) a.e. in \( S_\varphi(z, 2Kr) \), for a.e. in \( S_\varphi(x_0, \rho) \) we get

\[
L^p_\nu(w_j)(x) \leq \frac{\gamma_0}{\nu - 1} M_1 f(x) + 10c(x) =: f_j(x),
\]

where we have used that, by definition, \( \nu/(\nu - 1) = 2M_1 \) and \( \gamma_0 M_1 = 5 \). In addition, the assumption \( u(x_0) \geq \nu^{-1} M_1 \) gives \( \gamma_j(x_0) \leq \gamma_0 \), which implies \( \sup_{S_\varphi(x_0, \rho/(2K_3))} w_j \leq \gamma_0 \). We want to apply Theorem 25 to \( w_j \) and \( f_j \) on the section \( S_\varphi(x_0, \rho/(2K_3)) \) (think \( r = \frac{\rho}{2K_3} \) in the notation of Theorem 25). Since \( S_\varphi(x_0, \rho) \subset S_\varphi(z, 2Kr) \subset \Omega \), from the hypothesis (9.98), for \( r := \frac{\rho}{2K_3} \) we have

\[
|S_\varphi(x_0, K_3r)|^{\frac{1}{n}} \|f_j\|_{L^n(S_\varphi(x_0, K_3r), d\mu_\varphi)} = |S_\varphi(x_0, \rho/2)|^{\frac{1}{n}} \|f_j\|_{L^n(S_\varphi(x_0, \rho/2), d\mu_\varphi)}
\]

\[
\leq |S_\varphi(z, 2Kr)|^{\frac{1}{n}} \left( \frac{\gamma_0}{\nu - 1} M_1 \|f\|_{L^n(S_\varphi(z, 2Kr), d\mu_\varphi)} + 10c \|L^n(S_\varphi(z, 2Kr), d\mu_\varphi)\right)
\]

\[
\leq |S_\varphi(z, 2Kr)|^{\frac{1}{n}} \left( \frac{\gamma_0}{\nu - 1} M_1 \|f\|_{L^n(S_\varphi(z, 2Kr), d\mu_\varphi)} + 10c \|L^n(S_\varphi(z, 2Kr), d\mu_\varphi)\right) \leq 11\varepsilon_5,
\]

similarly,

\[
|S_\varphi(x_0, K_3r)|^{\frac{1}{n}} \|c\|_{L^n(S_\varphi(x_0, K_3r), d\mu_\varphi)} = |S_\varphi(x_0, \rho/2)|^{\frac{1}{n}} \|c\|_{L^n(S_\varphi(x_0, \rho/2), d\mu_\varphi)}
\]

\[
\leq |S_\varphi(z, 2Kr)|^{\frac{1}{n}} \|c\|_{L^n(S_\varphi(z, 2Kr), d\mu_\varphi)} \leq \varepsilon_5
\]

and, since \( x_0 \in S_\varphi(z, r) \) and \( \rho < r \), from (9.99)

\[
r^{-\frac{1}{2}} |S_\varphi(x_0, K_3r)|^{\frac{1}{n}} \|L^n(S_\varphi(x_0, K_3r), d\mu_\varphi)\|D^2_\varphi\|_{L^n(S_\varphi(x_0, K_3r), dx)}^\frac{1}{2}
\]

\[
= (2K_3)^{\frac{1}{2}} \rho^{-\frac{1}{2}} |S_\varphi(x_0, \rho/2)|^{\frac{1}{n}} \|L^n(S_\varphi(x_0, \rho/2), d\mu_\varphi)\|D^2_\varphi\|_{L^n(S_\varphi(x_0, \rho/2), dx)}^\frac{1}{2}
\]

\[
\leq (2K_3)^{\frac{1}{2}} \varepsilon_5.
\]

Thus, by taking \( \varepsilon_5 := (2K_3)^{-\frac{1}{2}} \varepsilon_3 \), Theorem 25 applied to \( w_j \) and \( f_j \) on the section \( S_\varphi(x_0, \rho/(2K_3)) \) (and here is where the hypothesis (9.99) comes into play) gives

\[
\mu_\varphi(\{x \in S_\varphi(x_0, \rho/K_3) : w_j(x) \geq 5\}) \leq (1 - \frac{\varepsilon_3}{2}) \mu_\varphi(S_\varphi(x_0, \rho/K_3)).
\]

Next, set

\[
A_1 := \{x \in S_\varphi(z, 2Kr) : u(x) \geq \nu M_1/2\}
\]

\[
A_2 := \{x \in S_\varphi(x_0, \rho/K_3) : w_j(x) \geq 5\} \subset S_\varphi(z, 2Kr).
\]

Then, the inclusion \( S_\varphi(x_0, \rho/K_3) \subset A_1 \cup A_2 \) holds true since given \( x_1 \in S_\varphi(x_0, \rho/K_3) \setminus A_1 \) it follows that \( w_j(x_1) \geq \frac{\gamma_0 \nu M_1}{\nu - 1} = \gamma_0 M_1 = 5 \), so that \( x_1 \in A_2 \). Consequently, Chebyshev’s inequality, the inequalities (13.150) and (13.153) give

\[
\mu_\varphi(S_\varphi(x_0, \rho/K_3)) \leq \mu_\varphi(A_1) + \mu_\varphi(A_2) \leq \frac{2}{\nu M_1} \int_{S_\varphi(z, 2Kr)} u \, d\mu_\varphi + (1 - \frac{\varepsilon_3}{2}) \mu_\varphi(S_\varphi(x_0, \rho/K_3))
\]

\[
< \frac{2\mu_\varphi(S_\varphi(z, 2Kr))}{\nu} + (1 - \frac{\varepsilon_3}{2}) \mu_\varphi(S_\varphi(x_0, \rho/K_3)).
\]
and then $\mu_{\varphi}(S_{\varphi}(x_0, \rho/K_3)) < \frac{4}{\nu' \varepsilon_0 K_3^3} \mu_{\varphi}(S_{\varphi}(z, 2Kr))$, which, along with the doubling property (5.56), implies
\[
\frac{\mu_{\varphi}(S_{\varphi}(x_0, \rho))}{K_D K_3^n} \leq \mu_{\varphi}(S_{\varphi}(x_0, \rho/K_3)) < \frac{4}{\nu' \varepsilon_0} \mu_{\varphi}(S_{\varphi}(z, 2Kr)) \leq \frac{4K_D K_3^n}{\nu' \varepsilon_0} \mu_{\varphi}(S_{\varphi}(z, r)),
\]
thus contradicting (13.151). □

The proof of Theorem 27 now follows (along the same lines as the one for [26, Lemma 6]) from Lemma 37 and the argument on [26, pp.2002-2004]. The corresponding structural constant $C_4$ (in the notation of [26, p.2002]) now can be taken as
\[
C_4 := (2K^2) \left( \frac{2^{n+2} K_D K_n^2 K_3^n}{\varepsilon_0} \right)^{1/n},
\]
and the rest of the proof follows the (purely geometric) argument on [26, pp.2002-2004]. □

Acknowledgements

The author is grateful to Nam Q. Le for comments and corrections to an early version of this article. The author is supported by the National Science Foundation under grant DMS 1361754.

References