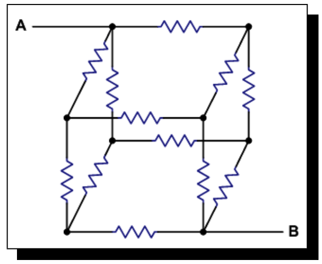
# The Resistor Distance

David Surowski[1]




**Abstract**

In **A simple method for computing resistance distance**, the authors R.B. Bapat, Ivan Gutman, and Wenjun Xiao obtained a very simple formula for the effective resistance between any two nodes in a resistor network with all resistances being $1\,\Omega$. Their treatment is "almost" ab initio save for the representation of the effective resistance in terms of an extremization of the reciprocal of a quadratic form in terms of the graph. (A reference to Bollobás' book, Modern Graph Theorey, Springer-Verlag, New York, 1998 is given.) However, all that is really needed are the linear equations which result from the partial derivatives of this quadratic form; in turn these follow instantly from Kirchhoff's first law. The actual mathematical derivation given below is formally identical (modulo some notational inversions) to that of Bapat, *et al.*

Let $\Gamma$ be a simple undirected graph, with vertices $v_0, v_1, \ldots, v_n$. For any vertex $v_j$, the set $\Gamma(v_j)$ simply denotes those vertices adjacent to $v_j$. The **Laplacian matrix** $L(\Gamma)$ of $\Gamma$ is the symmetric matrix whose diagonal elements are the vertex degrees $d_0, d_1, \ldots, d_n$, and whose off-diagonal elements are $-a_{ij}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \in \Gamma(v_j) \\ 0 & \text{otherwise.} \end{cases}$$

We now view $\Gamma$ as a resistor network, with each edge being replaced by a $1\,\Omega$ resistor. We would like to compute the effective resistance between nodes $v_0$ and $v_1$. To do this we apply a voltage of $+1\,V$ at $v_1$ and ground the node $v_0$ (at $+0\,V$). By Ohm's law the effective resistance is $R$, where $1 = R \cdot I$, and where $I$ is the total current leaving node $v_1$ (or arriving at node $v_0$). The above-specified applied voltage will result in voltages $V_2, V_3, \ldots, V_n$ at the remaining nodes. As a result, the effective current can be seen to be given by

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$$I = d_1 - \sum_{v_k \in \Gamma(v_1)} V_k = \sum_{v_l \in \Gamma(v_0)} V_l.$$

By Kirchhoff's first law we have, for any node $v_k,\ 2 \le k \le n$, that

$$d_k V_k - \sum_{l=0}^{n} a_{kl} V_l = 0.$$

This results in the matrix equation

$$L(\Gamma) \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} -I \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Written in partitioned form, this becomes

$$\begin{bmatrix} D & B \\ B^t & L_2 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} -I \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $D = \begin{bmatrix} d_0 & -a_{01} \\ -a_{10} & d_1 \end{bmatrix}$ and $L_2$ is the $(n-1) \times (n-1)$ submatrix of $L(\Gamma)$ obtained by deleting the first two rows and columns. From the above we extract the matrix equation

$$B^t \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} + L_2 \begin{bmatrix} V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As the matrix $L_2$ is invertible, and recalling that $V_0 = 0,\ V_1 = 1$, we can solve for the voltages via

$$\begin{bmatrix} V_2 \\ V_3 \\ \vdots \\ V_n \end{bmatrix} = -L_2^{-1} B^t \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} = L_2^{-1} \begin{bmatrix} a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{bmatrix}$$

If we write $L_2^{-1} = [t_{ij}]$ then we have the equations

$$V_k = \sum_{v_l \in \Gamma(v_1)} t_{kl}.$$

Next, by the familiar method of cofactors, together with symmetry, we have that

$$t_{kl} = (-1)^{k+l} \frac{\det L_2(l,k)}{\det L_2},$$

where $L_2(k,l)$ is the submatrix of $L_2$ obtained by deleting row $k$ and column $l$.

From all of this it follows that

$$\begin{aligned} I &= d_1 - \sum_{v_k \in \Gamma(v_1)} V_k = d_1 - \sum_{v_k \in \Gamma(v_1)} \sum_{v_l \in \Gamma(v_1)} t_{kl} \\ &= d_1 - \sum_{v_k \in \Gamma(v_1)} \sum_{v_l \in \Gamma(v_1)} (-1)^{k+l} \frac{\det L_2(l,k)}{\det L_2}. \end{aligned}$$

Next, let $L_1$ be the $n \times n$ submatrix of $L(\Gamma)$ obtained by deleting the first row and column. By expanding about the first row of $L_1$ we get

$$\det L_1 = d_1 \det L_2 - \sum_{v_k \in \Gamma(v_1)} (-1)^{1+k} \det L_1(1,k)$$

where the submatrix $L_1(1,k)$ is obtained from $L_1$ by deleting the first row and the $k$-th column. Next, we compute each $\det L_1(1,k)$ by expanding about the first column: this results in

$$\det L_1(1,k) = \sum_{v_l \in \Gamma(v_1)} (-1)^{l+1} \det L_2(l,k).$$

Therefore

$$\begin{aligned}
\det L_1 &= d_1 \det L_2 - \sum_{v_k \in \Gamma(v_1)} (-1)^{1+k} \det L_1(1,k) \\
&= d_1 \det L_2 - \sum_{v_k \in \Gamma(v_1)} \sum_{v_l \in \Gamma(v_1)} (-1)^{1+k} (-1)^{l+1} \det L_2(l,k) \\
&= d_1 \det L_2 - \sum_{v_k \in \Gamma(v_1)} \sum_{v_l \in \Gamma(v_1)} (-1)^{k+l} \det L_2(l,k) \\
&= I \cdot \det L_2.
\end{aligned}$$

That is to say, $I = \dfrac{\det L_1}{\det L_2}$ which implies that the effective resistance is given by

$$\boxed{R = \frac{\det L_2}{\det L_1}.}$$