Overview and Methods of Algebraic Map Theory

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July 23, 2007


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Chapter 1

Basic Map Theory

1.1 Thin Chamber Systems and Morphisms

**Definition 1.1.1 (Chamber System).** A chamber system $\mathcal{C} = (C, \sim_i \mid i \in I)$ consists of a set $C$ (of chambers), together with a family of equivalence relations $\sim_i$, $i \in I$. The cardinality of $I$ is called the rank of the chamber system. Arguably the most often quoted example of a rank $n$ chamber system is the following. Let $V$ be an $(n + 1)$-dimensional vector space over the field $\mathbb{F}$. Recall that a flag in $V$ is a collection $\mathcal{F} = \{W_1, W_2, \ldots, W_l\}$ of subspaces, with $W_1 \subseteq W_2 \subseteq \cdots \subseteq W_l$. A maximal flag in $V$ is a flag $\mathcal{F} = \{V_1, V_2, \ldots, V_n\}$, where $\dim V_j = j$, $j = 1, 2, \ldots, n$. We then can define a chamber system over the set $I = \{1, 2, \ldots, n\}$ to consist of the maximal flags $\mathcal{F}$ in $V$ and where $\mathcal{F} \sim_j \mathcal{F}'$ when the constituent subspaces of $\mathcal{F}$ and $\mathcal{F}'$ are the same, except possibly in dimension $j$.

Notice that in the above example, if $\mathbb{F}$ is the finite field of order $q$, and if $\mathcal{F}$ is a fixed maximal flag, then $\mathcal{F} \sim_j \mathcal{F}'$ for precisely $q + 1$ maximal flags $\mathcal{F}'$ in $V$. In general, a chamber system $\mathcal{C} = (C, \sim_i \mid i \in I)$ is called quasithin if all of the $\sim_i$-equivalence classes have cardinality at most 2. If each $\sim_i$-equivalence classes has cardinality exactly 2, then $\mathcal{C}$ is called thin. Thus, the “flag complex” given above is not thin—instead, it is called thick. When the chamber system is thin, then an alternative, but equivalent definition can be given as follows:

**Definition 1.1.2 (Thin Chamber System).** A thin chamber system
\( \mathcal{C} = (\{C, a_i \mid i \in I\} \) consists of a set \( C \) of chambers and a family of involutory permutations \( s_i, i \in I \) on \( C \). As above, the cardinality of the index set \( I \) is the rank of \( \mathcal{C} \). The prototypical thin chamber system of rank \( n \) is modeled on the above example, as follows. Instead of an \((n + 1)\)-dimensional vector space \( V \), start with a set \( S \) of cardinality \( n + 1 \). Again, define flags (and maximal flags) in \( S \) in terms of subsets of \( S \), in analogy with taking subspaces of \( V \). Thus, if \( F \) is a maximal flag in \( S \), it is readily seen that other than itself, \( F \) is \( \sim_i \) equivalent to exactly one other maximal flag \( F' \) in \( S \), for each \( i \in I \). Thus, the corresponding involution satisfies \( a_i(F) = F' \).

**Definition 1.1.3 (Morphisms of Chamber Systems over \( I \)).** Let \( \mathcal{C} = (\{C, \sim_i \mid i \in I\}, \mathcal{D} = (\{D, \sim_i \mid i \in I\}) \) be chamber systems over the set \( I \), and let \( \phi : C \to D \) be a mapping. We say that \( \phi \) is a morphism from \( \mathcal{C} \) to \( \mathcal{D} \), and write \( \phi : \mathcal{C} \to \mathcal{D} \), if \( c \sim_i \phi c' \) in \( \mathcal{C} \) implies that \( c\phi \sim_i \phi c' \).

**Definition 1.1.4 (Monodromy Group of Thin Chamber System).** Let \( \mathcal{C} = (\{C, a_i \mid i \in I\}) \) be a thin chamber system. We set \( G = \text{Mon}(\mathcal{C}) = \langle a_i \mid i \in I \rangle \), the group generated by the involutions \( a_i, i \in I \), and call it the monodromy group of \( \mathcal{C} \).

**Exercise.** If \( \mathcal{C} = (\{C, a_i \mid i = 1, 2, \ldots, n\}) \) is the thin chamber system defined above in terms of subsets of \( \{1, 2, \ldots, n + 1\} \), show that \( \text{Mon}(\mathcal{C}) \cong S_{n+1} \), the symmetric group on \( n + 1 \) symbols.

**Remark.** (Morphisms and Automorphisms of Thin Chamber Systems over \( I \)) Let \( \mathcal{C} = (\{C, a_i \mid i \in I\}, \mathcal{D} = (\{D, b_i \mid i \in I\}) \) be thin chamber systems over \( I \), where \( a_i, i \in I \) are involutions on \( C \), and \( b_i, i \in I \) are involutions on \( D \). If we proceed in strict analogy with chamber systems in general, then a morphism \( \phi : \mathcal{C} \to \mathcal{D} \) of chamber systems should be a mapping \( \phi : C \to D \) satisfying \( (a_i c) \phi \in \{c\phi, b_i(c\phi)\} \) for all \( i \in I \), and all \( c \in C \). However, we shall adhere to a stronger requirement for a mapping \( \phi : C \to D \) to define a morphism, namely, that it intertwine the actions of the monodromy groups: \( (a_i c)\phi = b_i(c\phi) \).
for all \( i \in I, c \in C \).\(^1\) If \( \phi : C \to D \) is a bijective morphism, then we call \( \phi : C \to D \) an isomorphism. Note that any isomorphism is automatically a 1-covering. An isomorphism \( \phi : C \to C \) is called an automorphism of \( C \); the set of all such is clearly a group under composition and is denoted \( \text{Aut}(C) \).

**Definition 1.1.5 (Connectivity).** We say that the chamber system \( C = (C, \sim_i \mid i \in I) \) is connected if and only if the transitive closure of the equivalence relations \( \sim_i, i \in I \) represents \( C \) as a single equivalence class. Equivalently, if \( c, c' \) are chambers, then there is a "path" from \( c \) to \( c' \) of the form

\[
c = c_0 \sim_{i_1} c_1 \sim_{i_2} c_2 \sim \cdots \sim_{i_k} c_k = c'.
\]

In case \( C \) is a thin chamber system, this can be stated more succinctly simply by stating that the monodromy group acts transitively on the set \( C \) of chambers.

**Lemma 1.1.1.** Let \( C = (C, a_i \mid i \in I), D = (D, b_i \mid i \in I) \) be chamber systems over \( I \), and let \( \phi : C \to D \) be a 1-morphism. If \( D \) is connected, then \( \phi : C \to D \) is surjective.

**Proof.** Assume that \( d, d' \in D \) with \( d' = b_i d \) for some \( i \in I \) and that \( d = c\phi \) for some \( c \in C \). If \( c' = a_i c \) then \( c'\phi = (a_i c)\phi = b_i (c\phi) = b_i d = d' \). The result follows.

Likewise, we have the following.

**Lemma 1.1.2.** Let \( \phi : C = (C, a_i \mid i \in I) \to D = (D, b_i \mid i \in I) \) be a surjective 1-morphism of chamber systems over \( I \), then the mapping defined by \( \phi_*(a_i) = b_i, i \in I \) determines a well-defined homomorphism of monodromy groups.

\(^1\)In the more general context of incidence geometries, such a mapping would be called a 1-morphism.
PROOF. Suppose that $i_1, i_2, \ldots, i_k \in I$ and that $a_{i_1}a_{i_2} \cdots a_{i_k} = 1 \in \text{Mon}(C)$. It suffices to prove that $b_{i_1}b_{i_2} \cdots b_{i_k} = 1 \in \text{Mon}(D)$. If $d \in D$, then $d = c\phi$ for some $c \in C$ and so
\[
b_{i_1}b_{i_2} \cdots b_{i_k}d = b_{i_1}b_{i_2} \cdots b_{i_k}(c\phi) = (a_{i_1}a_{i_2} \cdots a_{i_k}c)\phi = c\phi = d.\]
Since $d \in D$ was arbitrary, it follows that $b_{i_1}b_{i_2} \cdots b_{i_k} = 1$.

EXAMPLE. (The Connected Thin Chamber System $C(G/H) = (G/H, a_i, i \in I)$) Let $G$ be a group generated by a set of involutions $a_i \in I$, and let $H$ be a subgroup of $G$. We may form the connected thin chamber system $C(G/H) = (G/H, a_i, i \in I)$, where the involutions $a_i, i \in I$ act on $G/H$ via left multiplication.

The following is easy:

**Lemma 1.1.3.** Let $C = (C, a_i, i \in I)$ be a connected thin chamber system. If $c \in C$ is a fixed chamber, and if $H$ is the stabilizer in $G = \text{Mon}(C)$ of $c$, then the mapping
\[
\phi : C(G/H) \to C, \quad gH \mapsto g(c)
\]
is an isomorphism of thin chamber systems over $I$.

Henceforth, we shall, for convenience, stick to thin chamber systems, even though the definitions and many of the results are valid more generally.

**Lemma 1.1.4.** Let $C$ be a connected thin chamber system, with monodromy group $G$.

(i) $\text{Aut}(C)$ acts semiregularly on $C$. 
(ii) If \( c \in C \) is a fixed chamber and if \( H \) is the stabilizer of \( c \) in \( G \), then
\[
\text{Aut} \ (C) \cong N_G(H)/H.
\]

**Proof.** Let \( c \in C \) and let \( \alpha \in \text{Aut}(C) \) with \( c\alpha = c \). We shall show that \( \alpha \) fixes every chamber \( c' \in C \), i.e., that \( \alpha = 1 \). If \( c' \in C \), then by connectivity there exists an element \( g \in G \) with \( gc = c' \). But then \( c'\alpha = (gc)\alpha = g(c\alpha) = gc = c' \), which proves part (i). For part (ii), we may, by Lemma 1.1.3, assume that \( C = C(G/H) \). We define
\[
\psi : A = \text{Aut} \ (C(G/H)) \to N_G(H)/H
\]
as follows. If \( \alpha \in A \), then we set \( \alpha\psi = gH \), where \( (H)\alpha = gH \). We show first that \( g \in N_G(H) \). If \( h \in H \), then \( hgH = h(H\alpha) = (hH)\alpha = (H)\alpha = gH \), proving that \( g \in N_G(H) \).

Next, we prove:

\( \psi \) is a homomorphism. If \( H\alpha_1 = g_1H \), \( H\alpha_2 = g_2H \) then \( H(\alpha_1\alpha_2) = (H\alpha_1)\alpha_2 = (g_1H)\alpha_2 = g_1(H\alpha_2) = g_1(g_2H) = (g_1g_2)H \). Therefore, \( (\alpha_1\alpha_2)\psi = g_1g_2H = (g_1H)(g_2H) = (\alpha_1\psi)(\alpha_2\psi) \).

\( \psi \) is injective. If \( \alpha\psi = 1 \), then \( H\alpha = 1 \cdot H = H \) and so \( \alpha = 1 \) since \( A \) acts semiregularly on \( C(G/H) \).

\( \psi \) is surjective. If \( g \in N_G(H) \), then the mapping defined by \( (g'H)\tilde{g} = g'gH \) is a well-defined automorphism of \( C(G/H) \). Clearly \( \tilde{g} \mapsto gH \), and we’re done.

**Corollary 1.1.4.1.** The connected thin chamber system \( C \) has a transitive automorphism group if and only if the monodromy group acts regularly on the chambers of \( C \), in which case the automorphism group does, as well.

**Definition 1.1.6** (Regular Thin Chamber System). The thin chamber system \( C = (C, a_i \mid i \in I) \) is said to be regular if and only if the monodromy group acts regularly on the set \( C \) of chambers of \( C \). In this case the automorphism group also acts regularly on the chambers \( C \).
As a result of Corollary 1.1.4 we see that the chamber system $\mathcal{C}(G/H) = (G/H, a_i, \ i \in I)$ is regular if and only if $H \leq G$.

**Definition 1.1.7 (Regular Cover of a Thin Chamber System).** Let 
$\mathcal{C} = (C, a_i, \ i \in I)$ be a connected thin chamber system over the set $I$, and let $G = \text{Mon}(\mathcal{C})$. Define the chamber system $\mathcal{C}_G = (G, a_i, \ i \in I)$ where the monodromy involutions act on $G$ by left multiplication. It is clear that by right multiplication $G$ acts as a regular group of automorphisms of $\mathcal{C}_G$ and so $\mathcal{C}_G$ is regular. Note that for any choice of $c \in C$ there is a surjective morphism $\mathcal{C}_G \to \mathcal{C}$ given by $g \mapsto gc$. We call $\mathcal{C}_G$ the regular cover of the thin chamber system $\mathcal{C}$.

More generally, if $G_0 = \langle s_i | i \in I \rangle$ is any group such that the mapping $G_0 \to G = \text{Mon}(\mathcal{C})$ given by $s_i \mapsto a_i, i \in I$ is a homomorphism, then via this homomorphism $G_0$ acts on $C$ and so there is an obvious commutative diagram:

$$
\begin{array}{ccc}
\mathcal{C}_{G_0} & \longrightarrow & \mathcal{C}_G \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{C}
\end{array}
$$

**Lemma 1.1.5.** Let $G_0 \to G$ be as above, inducing $\mathcal{C}_{G_0} \to \mathcal{C}$. Assume that $\mathcal{C}$ is connected, that $c \in C$ is a fixed chamber, and that $H$ is the stabilizer in $G$ of $c$. Then 

$$\text{Aut}(\mathcal{C}) \cong N_{G_0}(H_0)/H_0,$$

where $H_0$ is the inverse image in $G_0$ of the subgroup $H \leq G$ under the map $G_0 \to G$.

**Definition 1.1.8 (Orbits).** Let $\mathcal{C} = (C, a_i | i \in I)$ be a connected thin chamber system over the set $I$, and let $K$ be a subgroup of $\text{Aut}(\mathcal{C})$. 

Then we may form the orbit system $C/K = (C/K, a_i \mid i \in I)$ where $C/K$ consists of the $K$-orbits in $C$, and the involutions $a_i$, $i \in I$ act on $C/K$ by left multiplication. This makes sense, because the actions of $G = \text{Mon}(C)$ and $\text{Aut}(C)$ commute, and so each element $g \in G$ permutes the $K$-orbits in $C$.

If the connected thin chamber system $C = (C, a_i \mid i \in I)$ has chamber stabilizer $H$, then a subgroup of $\text{Aut}(C)$ can be identified with a subgroup $H \leq K \leq G = \text{Mon}(C)$, where $K$ normalizes $H$. In turn, automorphisms of $C/K$ correspond to elements of $G$ that normalize $K$. As such elements need not also normalize $H$, we see that an automorphism $\alpha \in \text{Aut}(C/K)$ need not “lift” to an automorphism of $C$. If, on the other hand, we knew that, for instance, $H$ is characteristic in $K$, then all automorphisms of $C/K$ would certainly lift to automorphisms of $C$. We’ll have more to say about the lifting problem later. A more typical situation is as follows; we shall display the result as a lemma:

**Lemma 1.1.6.** Assume that $H \leq K \leq \text{Aut}(C)$ normalizing a chamber stabilizer $H$ as above. Assume also that $H = K \cap N$, where $N \leq G$. Then that every automorphism of $C/K$ will lift to one of $C$.

### 1.2 Hypermaps and Maps

**Definition 1.2.1 (Hypermaps and Maps).** A thin rank 3 chamber system $C = (C, a_1, a_2, a_3)$ is called a hypermap or sometimes an algebraic hypermap. If, in addition $a_1 a_3 = a_3 a_1$, we call $C$ a map, or sometimes an algebraic map.\(^2\) In the context of maps, we often refer to the underlying chambers as blades. The category of hypermaps and morphisms is denoted $\text{HMap}$, and the category of maps and morphisms is denoted $\text{Map}$.

\(^2\) The terms algebraic hypermap and algebraic map are often to emphasize distinctions from the notions of topological hypermap and topological map. However, in these notes we won’t consider topological hypermaps or topological maps, so for simplicity we’ll simply speak of hypermaps and maps. For a lucid treatment of algebraic and topological hypermaps, see D. Corn and D. Singerman, Regular hypermaps, *Europ. J. Combin.* 9 (1988), no. 4, 337–351.
If \( G = \langle a, b, c \rangle \) is a group generated by involutions such that \( ac = ca \), and if \( H \leq G \) is a subgroup, we define the connected map \( \mathcal{M}(G/H, a, b, c) \) to have set of blades \( G/H \) and monodromy involutions \( a, b, c \) (acting on \( G/H \)).

**Definition 1.2.2 (Ramified and Unramified Coverings).** Let \( \mathcal{C} = (C, a_1, a_2, a_3), \mathcal{C}' = (C', a_1', a_2', a_3') \) be hypermaps and let \( \phi : \mathcal{C} \to \mathcal{C}' \) be a morphism. Therefore, for all \( i, j \in \{1, 2, 3\} \) and any chamber \( c \in C \) we have that \( \phi \) restricts to a mapping \( \phi : \langle a_i, a_j \rangle c \to \langle a_i', a_j' \rangle (c\phi) \). If for each pair \( i, j \in \{1, 2, 3\} \) and any chamber \( c \in C \) the restricted map \( \phi : \langle a_i, a_j \rangle c \to \langle a_i', a_j' \rangle (c\phi) \) is bijective, we call \( \phi \) an **unramified covering**; otherwise we call \( \phi \) a **ramified covering**.

**Definition 1.2.3 (Nondegenerate Hypermaps).** We say that the hypermap \( \mathcal{C} = (C, a_1, a_2, a_3) \) is **nondegenerate** (or without boundary components) if each of the involutions \( a_1, a_2, a_3 \) acts fixed-point freely on \( C \).

**Example.** As an easy example of a degenerate map, consider a regular nondegenerate map \( \mathcal{M} = (B, a, b, c) \) and form the orbit map \( \mathcal{M}/\langle a \rangle \) (we have identified \( B \) with the monodromy group \( G = \langle a, b, c \rangle \)). Then the monodromy permutation in \( \mathcal{M}/\langle a \rangle \) given by left multiplication by \( a \) will fix the coset \( 1 \cdot \langle a \rangle \).

**Definition 1.2.4 (Uniform Hypermaps).** Let \( \mathcal{C} = (C, a_1, a_2, a_3) \) be a hypermap. If for each choice of indices \( i, j \in \{1, 2, 3\} \), all \( \langle a_i, a_j \rangle \)-orbits in \( C \) have the same cardinality, we say that \( \mathcal{C} \) is uniform.

**Lemma 1.2.1.** Let \( \mathcal{C} = (C, a_1, a_2, a_3) \) be a nondegenerate hypermap. Then \( \mathcal{C} \) is uniform if and only if for all \( i, j \in \{1, 2, 3\} \), \( \langle a_i, a_j \rangle \) acts semiregularly on \( C \).

---

3If \( a_1, a_2, a_3 \) do not all act without fixed points, then the hypermap \( \mathcal{C} \) is sometimes said to have **boundary**; blades in the fixed point sets of one of the involutions \( a_1, a_2, a_3 \) are said to be in the boundary of \( \mathcal{C} \). In the same spirit, if \( \mathcal{C} \) is nondegenerate, it is sometimes said to be **without boundary**.
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Proof. Clearly if each subgroup \( \langle a_i, a_j \rangle \) acts semiregularly on \( C \), then \( C \) is uniform. Conversely assume that \( C \) is uniform, and fix \( i, j \in \{1, 2, 3\} \), and set \( H = \langle a_i, a_j \rangle \). If \( v \subseteq C \) is an \( H \)-orbit, then since \( C \) is nondegenerate, we must have that \( v = v_0 \cup v'_0 \), where \( v_0, v'_0 \) are the \( N = \langle a_i, a_j \rangle \)-orbits in \( v \). Furthermore, left multiplication by \( a_i \) gives a bijection of \( v_0 \) onto \( v'_0 \). Therefore, it suffices to prove that \( N \) acts semiregularly on \( C \). If \( c \in C \) and \( N_c \) is the stabilizer in \( N \) of \( c \), then \( [N : N_c] \) is independent of \( c \) and \( N_c \) is normal in \( N \). But then \( N_c = N_{c'} \) for all \( c \in C \) and so \( N_c \) is in the kernel of the action of the monodromy group \( G = \langle a_1, a_2, a_3 \rangle \) on \( C \), which is trivial (by definition). The result follows.

**Question.** I don’t know if “nondegenerate” can be dropped from the hypotheses. This would be desirable, for then combinatorial regularity would imply nondegeneracy.

As a result of the above lemma, if \( \phi : C \rightarrow C' \) is a covering of uniform hypermaps, then \( \phi \) is an unramified covering if and only if the homomorphism \( \phi_* : \text{Mon}(C) \rightarrow \text{Mon}(C') \) of monodromy groups restricts to isomorphisms \( \phi_* |_{\langle a_i, a_j \rangle} \langle a_i, a_j \rangle \xrightarrow{\cong} \langle a'_i, a'_j \rangle \), for all \( i, j \in \{1, 2, 3\} \).

**Henceforth, we shall confine our considerations to maps.**

**Definition 1.2.5 (Vertices, Edges and Faces).** If \( M = (B, a, b, c) \) is a map, we have **varieties** in \( M \) as follows:

- **vertices:** These are the \( \langle b, c \rangle \)-orbits in \( B \);
- **edges:** These are the \( \langle a, c \rangle \)-orbits in \( B \);
- **faces:** These are the \( \langle a, b \rangle \)-orbits in \( B \).

**Definition 1.2.6 (Euler Characteristic).** If \( M = (B, a, b, c) \) is a finite map, i.e., if \( B \) is finite, we define the **Euler Characteristic** of \( M \) by setting \( \chi(M) = \# \text{vertices} - \# \text{edges} + \# \text{faces} \).

\[\text{The same definitions apply to hypermaps and give the hypervertices, hyperedges and hyperfaces.}\]
We shall say more about the Euler Characteristic in Subsection 4.1.

**Definition 1.2.7 (Vertex and Face Valency).** If \( \mathcal{M} = (B, a, b, c) \) is a map, and if \( \mathbf{v} \subseteq B \) is a variety, we call the cardinality \( |\mathbf{v}| \) the valency of this variety. The **vertex valency** of \( \mathcal{M} \) is the least common multiple of the valencies of the vertices in \( \mathcal{M} \). Likewise the **face valency** of \( \mathcal{M} \) is the least common multiple of the valencies of the faces in \( \mathcal{M} \).

**Definition 1.2.8 (Underlying Graph of a Map).** Recall that a graph is a quadruple \( \Gamma = (D, V, I, L) \) where \( D \) and \( V \) are sets (darts and vertices), \( I : D \rightarrow V \) is a surjective mapping, and \( L : D \rightarrow D \) is an involutory bijection. Morphisms of graphs are defined in the obvious way, and the valency of a vertex \( v \in V \) is the cardinality of the set \( vI^{-1} \). We say that \( \Gamma \) is connected if any pair \( d, d' \) of darts can be connected by a sequence \( d = d_0, d_1, d_2, \ldots, d_r = d' \), where for any index \( i = 0, 1, 2, \ldots, r - 1 \) we have \( Id_i = Id_{i+1} \), or \( Ld_i = d_{i+1} \).

If \( \mathcal{M} = (B, a, b, c) \) is a map, define the underlying graph \( \Gamma = (D, V, I, L) \) where \( D \) is the set of \( \langle c \rangle \)-orbits in \( B \), \( V \) is the set of vertices (of the map), \( I : D \rightarrow V \) is the natural map (the \( \langle c \rangle \)-orbit containing \( \beta \in B \) is mapped to the \( \langle b, c \rangle \)-orbit containing \( \beta \in B \)), and \( L : D \rightarrow D \) is the map given by left multiplication by \( a \). Since \( a, c \) commute, this makes sense.

Note that any automorphism of \( \mathcal{M} \) will clearly induce an automorphism of the underlying graph \( \Gamma \). Also we have:

**Lemma 1.2.2.** Let \( \mathcal{M} = (B, a, b, c) \) be a map with underlying graph \( \Gamma = (D, V, I, L) \). Then \( \mathcal{M} \) is connected if and only if \( \Gamma \) is connected.

**Definition 1.2.9 (Ramification).** Let \( \mathcal{M}_0 = (B_0, a_0, b_0, c_0) \xrightarrow{\phi} \mathcal{M} = (B, a, b, c) \) be a morphism of maps. Therefore, \( \phi \) must map vertices of \( \mathcal{M}_0 \) to vertices of \( \mathcal{M} \); similarly for edges and faces. If for the variety \( \mathbf{v}_0 \subseteq B_0 \) the mapping the mapping \( \phi : \mathbf{v}_0 \rightarrow \mathbf{v} = \mathbf{v}_0\phi \) is a bijective
mapping of sets, we often refer to \( \phi \) as unramified at \( v_0 \). If for all vertices \( v_0 \subseteq B_0 \), \( \phi \) is unramified at \( v_0 \), we say that \( \phi \) is unramified over vertices. Clearly, in this case the vertex valencies of \( M_0 \) and \( M \) are the same. Similar comments apply to the faces. As already observed above, if \( M_0, M \) are both uniform, \( \phi \) is unramified over vertices precisely when

\[
\phi_s : \langle b_0, a_0 \rangle \rightarrow \langle b, c \rangle.
\]

Likewise, one can speak of unramification over faces. Except in degenerate situations, a morphism of maps will be unramified over edges.

If \( \phi : M_0 \rightarrow M \) is unramified over all varieties of \( M \), then as in Definition 1.2.2 we call the morphism \( \phi \) an unramified morphism. If, in addition, \( \phi \) is surjective on the underlying blades, we call \( \phi \) an unramified covering. Finally, we say that the morphism \( \phi : M_0 \rightarrow M \) is totally ramified if it is impossible to factor this morphism nontrivially as \( \phi : M_0 \rightarrow M_{un} \rightarrow M \), where \( M_{un} \rightarrow M \) is an unramified morphism. A totally ramified morphism which is surjective on blades is called a totally ramified covering.
Chapter 2

Fundamental Groups

Definition 2.0.10 (Extended $k, l$-Triangle Group). If $k, l$ are positive integers (possibly $\infty$), we define the extended $(k, l)$-Triangle Group via the presentation:

$$\Delta(k, l) = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^k = (s_1s_3)^2 = (s_2s_3)^l = 1 \rangle.$$ 

Therefore, if $\mathcal{M} = (B, a, b, c)$ is a map, if $o(ab) = m, o(bc) = n$, and if $m|k, n|l$, then we have a surjective morphism $\theta : \Delta(k, l) \rightarrow G = \text{Mon} (\mathcal{M})$, given by $s_1 \mapsto a, s_2 \mapsto b, s_3 \mapsto c$. Correspondingly, there is the morphism of maps $\mathcal{M}_\Delta \rightarrow \mathcal{M}$, where $\mathcal{M}_\Delta$ has the meaning of the last chapter.

Inside $\Delta(k, l)$ is a normal subgroup $\Delta^+(k, l) = \langle s_2s_3, s_1s_3 \rangle$ of index 2. As noted above, we have $(s_2s_3)^l = (s_1s_3)^2 = 1$ and so $\Delta^+(k, l)$ is a homomorphic image of the $(k, l)$-triangle group

$$\Gamma = \Gamma(k, l) = \langle x, y \mid x^k = y^l = (xy)^2 = 1 \rangle.$$ 

Lemma 2.0.3. The natural mapping $\phi : \Gamma \rightarrow \Delta^+$ given by $\phi(x) = s_1s_2, \phi(y) = s_2s_3$ is an isomorphism.

Proof. We form the semidirect product $\Gamma \rtimes \langle t \rangle$, where $t$ is an involution acting on $\Gamma$ by $x \mapsto x^{-1}, y \mapsto xy^{-1}x^{-1}$; one checks easily that this action defines an automorphism of $\Gamma$. Next define a mapping $\psi : \Gamma \rtimes \langle t \rangle \rightarrow \Delta$ by $\psi(g, t^r) = \phi(g)s_2^r$. Note that $\phi(tx) = \phi(x^{-1}) = s_2s_1 = s_1\phi(x)s_1$, and
that \( \phi(ty) = \phi(y^{-1}x^{-1}) = s_1s_2(s_1s_2 \cdot s_2s_3)^{-1} = s_1s_2s_3s_1 = s_1\phi(y)s_1 \).

Therefore it follows that for all \( g \in \Gamma \), \( \phi(tg) = s_1\phi(g)s_1 \), from which it follows immediately that for all \( g \in \Gamma \), \( \phi(t^rg) = s_1'\phi(g)s_1' \). Therefore, we have

\[
\psi((g_1, t^{r_1}) \cdot (g_2, t^{r_2})) = \psi(g_1(t^{r_1}g_2), t^{r_1+r_2}) \\
= \phi(g_1(t^{r_1}g_2))s_1^{r_1+r_2} \\
= \phi(g_1)\phi(t^{r_1}g_2)s_1^{r_1+r_2} \\
= \phi(g_1)s_1^{r_1}\phi(g_2)s_1^{r_1+s_1^{r_1+r_2}} \\
= (\phi(g_1)s_1^{r_1})(\phi(g_2)s_1^{r_2}) \\
= \psi(g_1, t^{r_1})\psi(g_2, t^{r_2}).
\]

Therefore, \( \psi : \Gamma \rtimes \langle t \rangle \to \Delta \) is a homomorphism, obviously surjective.

Next, note that the mapping \( \psi' : \Delta \to \Gamma \rtimes \langle t \rangle \) with \( \psi'(s_1) = (1, t), \psi'(s_2) = (x^{-1}, t), \psi'(s_3) = (xy, t) \) defines a homomorphism (check the relations), and

\[
\psi' \circ \psi = 1_{\Gamma \rtimes \langle t \rangle} \quad \text{and} \quad \psi \circ \psi' = 1_\Delta.
\]

Therefore, it follows that \( \psi : \Gamma \rtimes \langle t \rangle \to \Delta \) is an isomorphism. Since \( \phi = \psi|_\Gamma : \Gamma \to \Delta^+ \), it follows that \( \phi \) is also an isomorphism.

**Corollary 2.0.3.1.** The relations \((s_1s_2)^k = (s_2s_3)^l = (s_1s_3)^2 = 1\) is a set of defining relations for \( \Delta^+(k, l) \leq \Delta(k, l) \).

### 2.1 Orbifold Fundamental Groups

**Definition 2.1.1 (Orbifold Fundamental Group).** Assume that \( \mathcal{M} = (B, a, b, c) \) is a connected map with monodromy group \( G \). Therefore, if we set \( \Delta = \Delta(\infty, \infty) \) then there is a surjective homomorphism \( \theta : \Delta \to G \) given by \( s_1 \mapsto a, s_2 \mapsto b, \) and \( s_3 \mapsto c \). By connectivity, this gives a transitive permutation representation of \( \Delta \) on the set \( B \) of blades in \( \mathcal{M} \). Conversely, if \( B \) is a set acted on transitively by \( \Delta \), and if \( a, b, c \) are the permutations of \( B \) induced by \( s_1, s_2, s_3 \), then
\(\mathcal{M} = (B, a, b, c)\) is clearly a connected map. Thus, it is clear the the full subcategory \(_{\text{Map}}\mathcal{M}\) of \(\text{Map}\) consisting of connected maps is isomorphic with the category \(\text{Perm}(\Delta)\) of transitive permutation representations of \(\Delta^1\)

If \(\beta \in B\) is a fixed blade, then we define \(\pi_1^\infty(\mathcal{M}, \beta)\) to be the stabilizer in \(\Delta\) of the blade \(\beta\). Equivalently, \(\pi_1^\infty(\mathcal{M}, \beta) = \theta^{-1}(H)\), where \(H\) is the stabilizer in \(G\) of the blade \(\beta\). The group \(\pi_1^\infty(\mathcal{M}, \beta)\) is called the orbifold fundamental group of \(\mathcal{M}\), based at \(\beta\).

From the above, we see that the given connected map \(\mathcal{M}\) can be exhibited as an orbit map of the form \(\mathcal{M}_\Delta/\pi_1^\infty(\mathcal{M}, \beta)\). Furthermore, we see from Lemma 1.1.4 that \(\mathcal{M}\) is regular precisely when \(\pi_1^\infty(\mathcal{M}, \beta) \subseteq \Delta\). More generally, continuing to assume that \(\mathcal{M}\) is connected, we see that if \(\beta, \beta'\) are blades of \(\mathcal{M}\), then \(\pi_1^\infty(\mathcal{M}, \beta)\) and \(\pi_1^\infty(\mathcal{M}, \beta')\) are conjugate subgroups of \(\Delta\).

As for the functoriality, assume that \(\phi : \mathcal{M} \rightarrow \mathcal{M}'\) is a morphism of maps, that \(\beta\) is a blade of \(\mathcal{M}\), \(\beta'\) is a blade of \(\mathcal{M}'\), and that \(\beta\phi = \beta'\). Thus, \(\Delta\) acts on the blades of both \(\mathcal{M}\) and \(\mathcal{M}'\) with the stabilizer of \(\beta\) being \(\pi_1^\infty(\mathcal{M}, \beta)\), and the stabilizer of the blade \(\beta'\) being \(\pi_1^\infty(\mathcal{M}', \beta')\). Note that if \(\gamma \in \pi_1^\infty(\mathcal{M}, \beta)\), \(\gamma\beta' = \gamma(\beta\phi) = (\gamma\beta)\phi = \beta\phi = \beta'\), i.e., \(\gamma\) also stabilizes \(\beta'\). That is to say, \(\gamma \in \pi_1^\infty(\mathcal{M}', \beta')\), as well. In other words, we have \(\pi_1^\infty(\mathcal{M}, \beta) \subseteq \pi_1^\infty(\mathcal{M}', \beta')\) and so we simply define

\[\phi_* : \pi_1^\infty(\mathcal{M}, \beta) \hookrightarrow \pi_1^\infty(\mathcal{M}', \beta').\]

Note that this says in particular that if \(\phi\) is an automorphism of \(\mathcal{M}\) with \(\beta\phi = \beta'\), then \(\pi_1^\infty(\mathcal{M}, \beta) = \pi_1^\infty(\mathcal{M}, \beta')\), and conversely. Therefore, if \(_{\text{Map}}\) is the category of pairs \((\mathcal{M}, \beta)\) where \(\mathcal{M}\) is a map and \(\beta\) is a blade of \(\mathcal{M}\), then we obtain a functor from \(_{\text{Map}}\) \(\rightarrow\) \text{Group}, the category of groups. The fact that all induced morphisms of the orbifold fundamental groups are injections is particular to this theory, and is reminiscent of the theory of covering space projections, where it is also true that induced mappings of the (topological) fundamental groups are injections.\(^2\)

\(^1\)This is stronger than simply saying that the categories \(_{\text{Map}}\mathcal{M}\) and \(\text{Perm}(\Delta)\) are equivalent.

This point of view, together with the preceeding discussion, makes the following result, familiar from the algebraic topology of path-connected covering spaces, almost a tautology:\footnote{op. cit., Theorem 2.5.2, page 79.}

**Theorem 2.1.1. (Lifting Theorem)** Let $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ be in $\mathcal{CMap}$, and let $\phi' : \mathcal{M}' \to \mathcal{M}$, $\phi'' : \mathcal{M}'' \to \mathcal{M}$ be morphisms. Let $\beta'$ be a blade in $\mathcal{M}'$, $\beta''$ be a blade in $\mathcal{M}''$, and assume that $\beta' \phi' = \beta'' \phi''$. Then there exists a morphism $\phi : \mathcal{M}' \to \mathcal{M}''$ such that $\phi' = \phi \circ \phi''$ if and only if

$$\phi'_* \pi_1^\infty(\mathcal{M}', \beta') \leq \phi''_* \pi_1^\infty(\mathcal{M}'', \beta'').$$

**Definition 2.1.2 (The Category $\mathcal{Map}(\mathcal{M})$).** If $\mathcal{M}$ is a connected map, we define $\mathcal{Map}(\mathcal{M})$ to be the category of all pairs $(\mathcal{M}', p')$, where $\mathcal{M}'$ is a map (not necessarily connected), and where $p' : \mathcal{M}' \to \mathcal{M}$ is a morphism. Morphisms $(\mathcal{M}', p') \to (\mathcal{M}'', p'')$ in $\mathcal{Map}(\mathcal{M})$ consist of morphisms $\phi : \mathcal{M}' \to \mathcal{M}''$ making the triangle below commute:

Note that if $(\mathcal{M}', p')$ is in $\mathcal{Map}(\mathcal{M})$, and if $\beta$ is a blade in $\mathcal{M}$, then $\pi_1^\infty(\mathcal{M}, \beta)$ acts (on the left) on each of the the connected components
of \( \mathcal{M}' \) (since \( \Delta \) does). Furthermore, if \( \mathcal{M}'_0 \subseteq \mathcal{M}' \) is a connected component of \( \mathcal{M}' \), then \( \pi_1^\infty (\mathcal{M}, \beta) \) acts transitively on \( \mathcal{M}'_0 \cap \beta \mathcal{p}'^{-1} \), the fibre in \( \mathcal{M}'_0 \) over the blade \( \beta \) in \( \mathcal{M} \). Furthermore, a typical stabilizer of this action is (by definition) of the form \( \pi_1^\infty (\mathcal{M}', \beta'_0) \) (:= \( \pi_1^\infty (\mathcal{M}'_0, \beta'_0) \)) where \( \beta'_0 \) is a blade in \( \mathcal{M}'_0 \cap \beta \mathcal{p}'^{-1} \).

The following result has a very classical origin:\footnote{For the topological setting, see Spanier, Theorem 2.5.2, page 79.}

**Theorem 2.1.2.** Let \( \mathcal{M} \) be a connected map, let \( (\mathcal{M}', \mathcal{p}'), (\mathcal{M}'', \mathcal{p}'') \) be connected coverings of \( \mathcal{M} \) and fix a blade \( \beta \in \mathcal{B} \). Then \( \mathcal{M}' \cong_\mathcal{M} \mathcal{M}'' \) if and only if the fundamental groups \( \pi_1^\infty (\mathcal{M}', \beta') \), \( \pi_1^\infty (\mathcal{M}'', \beta'') \), \( \beta' \in \beta \mathcal{p}'^{-1} \), \( \beta'' \in \beta \mathcal{p}''^{-1} \) are conjugate subgroups in \( \pi_1^\infty (\mathcal{M}, \beta) \).

**Proof.** Let \( H' = \pi_1^\infty (\mathcal{M}', \beta') \), \( H'' = \pi_1^\infty (\mathcal{M}'', \beta'') \) and note that \( \mathcal{M}' \) and \( \mathcal{M}'' \) can be identified with transitive permutation representations of \( \Delta \) with stabilizers \( H', H'' \), respectively. Therefore, to say that \( \mathcal{M}' \cong \mathcal{M}'' \) is equivalent with saying that \( H', H'' \) are conjugate subgroups in \( \Delta \). However, to say that \( \mathcal{M}' \cong_\mathcal{M} \mathcal{M}'' \) is to say that \( H', H'' \) are conjugate in \( \Delta \) by an element that fixes \( \beta \), i.e., by an element of \( \pi_1^\infty (\mathcal{M}, \beta) \).

Conversely, if \( H', H'' \) are conjugate in \( \Delta \) by an element of \( \gamma \in \pi_1^\infty (\mathcal{M}, \beta) \), say \( H'' = \gamma H' \gamma^{-1} \), then the mapping \( \mathcal{M}' \to \mathcal{M}'' \) defined by \( \sigma H' \mapsto \sigma \gamma^{-1} H'' \), is easily checked to define an isomorphism over \( \mathcal{M} \), i.e., \( \mathcal{M}' \cong_\mathcal{M} \mathcal{M}'' \).

The following result fits very naturally into this framework.

**Theorem 2.1.3.** Let \( \mathcal{M} \) be a uniform connected map, and let \( p : \mathcal{M}' \to \mathcal{M} \) be a covering by the connected map \( \mathcal{M}' \). Then there exists a unique map (up to \( \mathcal{M} \)-isomorphism) \( \mathcal{M}_u \) and a factorization \( p : \mathcal{M}' \to \mathcal{M}_u \to \mathcal{M} \) such that \( \mathcal{M}' \to \mathcal{M}_u \) is totally ramified and \( \mathcal{M}_u \to \mathcal{M} \) is unramified.

**Proof.** Let \( \mathcal{M} \) have vertex valency \( l \) and face valency \( k \). Since \( \mathcal{M} \) is uniform, the action of \( \Delta = \Delta (\infty, \infty) \) on the blades of \( \mathcal{M} \) factors
through the action of $\Delta(k,l)$; set $K = \ker(\Delta \to \Delta(k,l))$. By connectivity, we may identify the blades of $\mathcal{M}'$ and $\mathcal{M}$ with $\Delta/\pi_1^\infty(\mathcal{M}', \beta')$ and $\Delta/\pi_1^\infty(\mathcal{M}, \beta)$, respectively, for suitable blades $\beta'$, $\beta$ with $\beta'p = \beta$. Therefore, factorizations of the form $\mathcal{M}' \to \mathcal{M}_1 \to \mathcal{M}$ are in correspondence with subgroups of $\pi_1^\infty(\mathcal{M}, \beta)$ containing $\pi_1^\infty(\mathcal{M}', \beta')$. Furthermore, we see that $\mathcal{M}_1 \to \mathcal{M}$ is unramified if and only if $K \leq \pi_1^\infty(\mathcal{M}_1, \beta_1)$, where $\beta_1$ is a blade of $\mathcal{M}_1$ projecting to the blade $\beta$ of $\mathcal{M}$. As a result, if $\mathcal{M}_{un}$ corresponds to the subgroup $K \cdot \pi_1^\infty(\mathcal{M}', \beta')$, then $\mathcal{M}_{un} \to \mathcal{M}$ is unramified, $\mathcal{M}' \to \mathcal{M}_{un}$ is totally ramified and $\mathcal{M}_{un}$ is uniquely determined up to $\mathcal{M}$-isomorphism.

Automorphisms in $\text{Map}(\mathcal{M})$ are called covering (or deck) transformations. We denote the group of covering transformations of $(\mathcal{M}', p')$ by $A(\mathcal{M}'/\mathcal{M})$. Clearly, if $\alpha \in A(\mathcal{M}'/\mathcal{M})$, then $\alpha$ will permute the blades in each fibre $\beta p^{-1}$, where $\beta$ is a blade of $\mathcal{M}$. Since automorphisms of connected maps act without fixed blades, we see that if $\mathcal{M}'$ is connected, $A(\mathcal{M}'/\mathcal{M})$ acts semi-regularly on each fibre $\beta p^{-1}$, $\beta \in B$.

**Definition 2.1.3 (Regular Morphisms).** The morphism $p' : \mathcal{M}' \to \mathcal{M}$ is called regular if and only if $A(\mathcal{M}'/\mathcal{M})$ acts transitively (and hence regularly) on each fibre $\beta p'^{-1}$, $\beta \in B$. We shall sometimes refer to the regular morphism as a regular covering. (Note that this is to be distinguished from the concept of regular cover as defined on page 6.)

If the connected map $(\mathcal{M}', p')$ is in $\text{Map}(\mathcal{M})$, then we see that $A(\mathcal{M}'/\mathcal{M})$ consists of those automorphisms of $\mathcal{M}'$ that stabilize each fibre over $\mathcal{M}$. Since the automorphism group of $\mathcal{M}'$ can be identified with the normalizer in $\Delta$ of $\pi_1^\infty(\mathcal{M}', \beta')$ for any blade $\beta'$ in $\mathcal{M}'$, we conclude that if $\beta = \beta'p'$, then a covering transformation determines, modulo $\pi_1^\infty(\mathcal{M}', \beta')$, an element of the normalizer in $\pi_1^\infty(\mathcal{M}, \beta)$ of $\pi_1^\infty(\mathcal{M}', \beta')$. Conversely, it is easy to see that any such element of $\pi_1^\infty(\mathcal{M}, \beta) \leq \Delta$ will determine an automorphism of $\mathcal{M}'$ that stabilizes each fiber of $p'$ over $\mathcal{M}$. In other words, we obtain the following classical isomorphism:

\[ * \text{op. cit., Theorem } 2.6.2, \text{ page } 85. * \]
\[ N_{\pi_1^\infty(\mathcal{M}, \beta)}(\pi_1^\infty(\mathcal{M}', \beta')) \cong \Lambda(\mathcal{M}'/\mathcal{M}). \]

The following is now immediate:

**Theorem 2.1.4.** If \((\mathcal{M}', p')\) is in \(\text{Map}(\mathcal{M})\), with \(\mathcal{M}'\) connected, then \(p' : \mathcal{M}' \to \mathcal{M}\) is a regular morphism if and only if
\[
\pi_1^\infty(\mathcal{M}', \beta') \leq \pi_1^\infty(\mathcal{M}, \beta),
\]
in which case
\[
\Lambda(\mathcal{M}'/\mathcal{M}) \cong \pi_1^\infty(\mathcal{M}, \beta)/\pi_1^\infty(\mathcal{M}', \beta').
\]

In case \(p' : \mathcal{M}' \to \mathcal{M}\) with \(\mathcal{M}'\) disconnected, then for each connected component \(\mathcal{M}'_0 \subseteq \mathcal{M}'\) and each choice of blade \(\beta'_0 \in \mathcal{M}' \cap \beta p'^{-1}\), we have a surjective homomorphism \(N_{\pi_1^\infty(\mathcal{M}, \beta)}(\pi_1^\infty(\mathcal{M}'_0, \beta_0)) \to \Lambda(\mathcal{M}'_0/\mathcal{M}) \leq \Lambda(\mathcal{M}'/\mathcal{M})\) with kernel \(\pi_1^\infty(\mathcal{M}'_0, \beta_0)\).

**Definition 2.1.4. (Characteristic Homomorphism Corresponding to a Morphism)** Retaining the notation as above, if \(\mathcal{M}' \to \mathcal{M}\) is a regular morphism, then so is \(p'|_{\mathcal{M}'_0} : \mathcal{M}'_0 \to \mathcal{M}\) for any connected component \(\mathcal{M}'_0\) of \(\mathcal{M}'\). Therefore, the above theorem guarantees that there exists a surjective homomorphism
\[
\chi_{\mathcal{M}'_0/\mathcal{M}} : \pi_1^\infty(\mathcal{M}, \beta) \to \Lambda(\mathcal{M}'_0/\mathcal{M})
\]
whose kernel is \(\pi_1^\infty(\mathcal{M}'_0, \beta_0)\) (for any \(\beta'_0 \in \mathcal{M}'_0 \cap \beta p'^{-1}\)). This is called the *characteristic homomorphism* of the regular morphism. We shall have more to say about the characteristic homomorphism in Section 5.3.

**Definition 2.1.5. (Restricted Orbifold Fundamental Groups)** If \(\mathcal{M} = (B, a, b, c)\) is a map with vertex valency \(l\) and face valency \(k\), then it is immediate that \((ab)^k = (bc)^l = 1\), and so, in the obvious way, the monodromy group \(G = \langle a, b, c \rangle\) is a homomorphic image of the group
\[ \Delta(m, n) \], for any pair \((m, n)\) where \(m\) is a multiple of \(k\) and \(n\) is a multiple of \(l\). By restricting our attention to connected maps whose vertex and face valencies are, respectively, divisors of \(n\) and \(m\), then we are reduced to considering transitive permutation representations of \(\Delta(m, n)\). Correspondingly, relative to a fixed blade \(\beta\) in \(\mathcal{M}\), a we have the (restricted) orbifold fundamental group \(\pi_1^{\text{orb}}(\mathcal{M}, \beta) = \text{Stab}_{\Delta(m, n)}(\beta)\). It is clear that all of the appropriately-modified results presented above are true in the category \(\text{Map}^{(m,n)}\) of maps having vertex valency dividing \(n\) and face valency dividing \(m\).

### 2.2 The (Combinatorial) Fundamental Group

As above, we let \(\mathcal{M}\) be a connected map with monodromy group \(G = \langle a, b, c \rangle\). In this subsection we present “fundamental group” construction that is somewhat more closely tied to the combinatorial structure of the given map as well as being closer in spirit to the classical construction of the fundamental group in the topological (or simplicial) setting.

If \(\beta, \beta'\) are blades in \(\mathcal{M}\), then a gallery \(\sigma\) from \(\beta\) to \(\beta'\) is a sequence of blades

\[ \sigma = (\beta' = \beta_r, \beta_{r-1}, \ldots, \beta_1, \beta_0 = \beta) \]

in \(\mathcal{M}\), where for each \(j = 1, 2, \ldots, r\), \(\beta_j = a_j \beta_{j-1}\), for some \(a_j \in \{a, b, c\} \cup \{1\} \subseteq G\). (Note that the galleries are developed from right to left. This departure from custom is intentional and stems from the fact that the monodromy group \(G\) acts on the blades of \(\mathcal{M}\) on the left.) If \(\sigma\) is a gallery from \(\beta\) to \(\beta'\), and if \(\sigma'\) is a gallery from \(\beta'\) to \(\beta''\), we may, in the obvious way, form the juxtaposed gallery \(\sigma' \sigma\) from \(\beta\) to \(\beta''\). Two galleries \(\sigma, \sigma'\) from \(\beta\) to \(\beta'\) are said to be contiguous, and written \(\sigma \sim \sigma'\)

\[ \sigma = \tau \epsilon \tau, \sigma' = \tau' \epsilon' \tau, \]

where either \(\epsilon = (\beta'', \beta''')\), \(\epsilon' = (\beta''')\) for some blade \(\beta'''\), or if for some variety \(x \in V \cup E \cup F\), and for blades \(\beta_1, \beta_2 \in x\), \(\epsilon\) and \(\epsilon'\) are galleries from \(\beta_1\) to \(\beta_2\) totally contained in \(x\). Finally, if \(\sigma, \sigma'\) are galleries from
\( \beta \) to \( \beta' \), we say that \( \epsilon \) and \( \epsilon' \) are homotopic and write \( \sigma \simeq \sigma' \) if there is a sequence
\[
\sigma = \sigma_0 \sim \sigma_1 \sim \cdots \sim \sigma_q = \sigma'.
\]
Thus, we see that homotopy of galleries from \( \beta \) to \( \beta' \) is an equivalence relation.

Note next that if \( \sigma \simeq \sigma' \) are galleries from \( \beta \) to \( \beta' \), and if \( \tau \simeq \tau' \) are galleries from \( \beta' \) to \( \beta'' \), then \( \tau \sigma \simeq \tau' \sigma' \). Finally, if \( \sigma = (\beta' = \beta_r, \ldots, \beta_1, \beta_0 = \beta) \), and if \( \sigma^{-1} = (\beta = \beta_0, \beta_1, \ldots, \beta_r = \beta') \), then \( \sigma^{-1} \sigma \simeq (\beta) \).

**Definition 2.2.1. (Groupoids)** A groupoid\(^6\) is a small category in which every morphism is an equivalence.

The prototype here is that of a group: if \( X \) is a group, then the corresponding groupoid has the elements of \( X \) as the objects. If \( x_1, x_2 \in X \) then there is a unique morphism \( x_1 \to x_2 \) and is given by \( \mu_{x_2x_1^{-1}} = \) left multiplication by \( x_2x_1^{-1} \). Thus, we see that if \( X, X' \) are groups regarded as groupoids, then a homomorphism determines a functor \( X \to X' \) (and conversely!). Indeed, if \( F : X \to X' \) is a functor from the groupoid \( X \) to the groupoid \( X' \), then define \( f : X \to X' \) by setting \( \mu_{f(x)} = F(\mu_x), x \in X \). Conversely, given a homomorphism \( f : X \to X' \), we define \( F : X \to X' \) by setting \( F(x) = f(x), F(\mu_x) = \mu_{f(x)}, x \in X \).

A highly pertinent groupoid is the fundamental groupoid \( \mathcal{P}(\mathcal{M}) \) of the map \( \mathcal{M} \), as follows. The objects are the blades and the morphisms from \( \beta \) to \( \beta' \) are the homotopy classes of galleries from \( \beta \) to \( \beta' \). From the above discussion, we see that \( \mathcal{P}(\mathcal{M}) \) is indeed a groupoid.

A gallery \( \sigma \) which begins and ends at the same blade \( \beta_0 \) is said to be a closed gallery based at \( \beta_0 \). Thus, we see that the homotopy classes \( \pi_1(\mathcal{M}, \beta_0) \) of galleries based at the fixed blade \( \beta_0 \) is not only a subcategory of \( \mathcal{P}(\mathcal{M}) \), it is actually a group, called the (combinatorial) fundamental group of \( \mathcal{M} \) based at \( \beta_0 \) and denoted \( \pi_1(\mathcal{M}, \beta_0) \).

Assume now that \( \mathcal{M} \) has vertex valency \( l \) and face valency \( k \), and that \( m, n \) are (possibly infinite) multiples of \( k, l \), respectively. As

\(^6\)Relevant facts about groupoids and their relationship with homotopy and the fundamental group are summarized in op. cit., Chapter 1, Sections 7, 8.
above, set
\[ \Delta(m, n) = \langle s_1, s_2, s_3 \mid s_1^3 = s_2^3 = s_3^3 = (s_1s_2)^m = (s_1s_3)^n = 1 \rangle. \]
We have the restricted orbifold fundamental group \( \pi_1^{(m,n)}(\mathcal{M}, \beta_0) = \text{Stab}_\Delta(\beta_0) \) of the blade \( \beta_0 \). Regarding \( \Delta(m, n) \) as a groupoid as above, we determine a well-defined functor \( F : \Delta(m, n) \to \mathcal{P}(\mathcal{M}) \), as follows. If \( \gamma \in \Delta(m, n) \), we may express \( \gamma \) as a product of fundamental involutions:
\[ \gamma = s_{i_r}s_{i_{r-1}} \cdots s_{i_2}s_{i_1}, \]
where \( s_{i_j} \in \{s_1, s_2, s_3\}, j = 1, 2, \ldots, r \). We now set
\[ F(\gamma) = [(s_i, s_{i_{r-1}} \cdots s_{i_2}s_{i_1} \beta_0, s_{i_{r-1}} \cdots s_{i_2}s_{i_1} \beta_0, \cdots, s_{i_2}s_{i_1} \beta_0, s_{i_1} \beta_0, \beta_0)], \]
where for any gallery \( \sigma \), \( [\sigma] \) is the homotopy class of the gallery containing \( \sigma \). Checking that this functor is well-defined is tantamount to checking that the various relations are satisfied. For example, for any blade \( \beta \in B \) we have that
\[ [(s_i s_{i} \beta, s_i \beta, \beta)] = [(\beta, \beta)] = [(\beta)]. \]
In the same spirit, if \( i \neq j \) and if
\[ m_{ij} = \begin{cases} 
  m & \text{if } \{i, j\} = \{1, 2\} \\
  2 & \text{if } \{i, j\} = \{1, 3\} \\
  n & \text{if } \{i, j\} = \{2, 3\},
\end{cases} \]
then
\[ F((s_i s_j)^{m_{ij}}) = [\beta = (s_i s_j)^{m_{ij}} \beta, s_j(s_i s_j)^{m_{ij}-1} \beta, \cdots s_j \beta, \beta)] = [(\beta, \beta, \cdots, \beta)] = [(\beta)]. \]
This proves that \( F : \Delta \to \mathcal{P}(\mathcal{M}) \) is a well-defined functor. In particular, recalling that \( \pi_1^{(m,n)}(\mathcal{M}, \beta_0) = \text{Stab}_\Delta(\beta_0) \), we see that \( F \) restricts to a functor of the subcategories \( \pi_1^{(m,n)}(\mathcal{M}, \beta_0) \to \pi_1(\mathcal{M}, \beta_0) \). Such a functor, as already observed above, determines a homomorphism of the underlying groups. We shall denote this homomorphism by
\[ F^{(m,n)} : \pi_1^{(m,n)}(\mathcal{M}, \beta_0) \to \pi_1(\mathcal{M}, \beta_0); \]
If
\[ \gamma = s_{i_r} s_{i_{r-1}} \cdots s_{i_2} s_{i_1}, \]
where \( s_{i_j} \in \{s_1, s_2, s_3\}, \ j = 1, 2, \ldots, r \), then
\[
F^{(m,n)}(\gamma) = \left[ (\beta_0 = s_{i_r} s_{i_{r-1}} \cdots s_{i_2} s_{i_1} \beta_0, s_{i_{r-1}} \cdots s_{i_2} s_{i_1} \beta_0, \ldots, s_{i_2} s_{i_1} \beta_0, s_{i_1} \beta_0) \right] \\
\in \pi_1(\mathcal{M}, \beta_0).
\]

2.2.1 Further Functorial Properties

In the subsection we investigate some of the functorial properties of the (combinatorial) fundamental group. As many, if not most, of the results are proved exactly as in the topological category, our proofs will be a bit sketchy.

First of all, if \((\mathcal{M}, \beta_0), (\mathcal{M}', \beta'_0)\) are pairs consisting of maps and base blades, then any morphism \( \phi : \mathcal{M} \to \mathcal{M}' \) satisfying \( \beta_0 \phi = \beta'_0 \) gives rise to a homomorphism \( \phi_* : \pi_1(\mathcal{M}, \beta_0) \to \pi_1(\mathcal{M}', \beta'_0) \) in the obvious way, namely by applying \( \phi \) to the blades of a given gallery and proving that \( \phi \) carries homotopic galleries to homotopic galleries.

**Theorem 2.2.1. (Unique Gallery Lifting Theorem)**\(^7\) Let \( p : \mathcal{M}' \to \mathcal{M} \) be an unramified morphism, and let \( \sigma \) be a gallery in \( \mathcal{M} \) with initial blade \( \beta_0 \). If \( \beta'_0 \in \beta_0^{-1} \), then there exists a unique gallery \( \sigma' \) in \( \mathcal{M}' \) with initial blade \( \beta'_0 \) in \( \mathcal{M}' \).

**Proof.** Let \( \mathcal{M} = (B, a, b, c), \mathcal{M}' = (B', a', b', c') \). We argue by induction on the length of the gallery \( \sigma = (\beta_r, \ldots, \beta_1, \beta_0) \). Thus, if \( \beta_1 = a_i \beta_0 \), where \( a_i \in \{a, b, c\} \), then we set \( \beta'_1 = a'_i \beta'_0 \). Note that if also \( \beta_1 = a_j \beta_0 \), then since \( p : \mathcal{M}' \to \mathcal{M} \) is a covering, and since \( a'_i \beta'_0, a'_j \beta'_0 \) are in the same variety, then \( a'_i \beta'_0 \neq a'_j \beta'_0 \) implies that \( a_i \beta_0 = a'_i \beta'_0 p \neq a'_j \beta'_0 p = a_j \beta_0 \). Thus, \( \beta'_1 \) is uniquely determined and the result follows by induction.

\(^7\)Cf. op. cit., Theorem 2.2.2, page 67.
The gallery $\sigma'$ constructed above is called a lift of the gallery $\sigma$ to $\mathcal{M}'$.

**Theorem 2.2.2. (Homotopy Lifting Theorem)**\(^8\) Let $p : \mathcal{M}' \to \mathcal{M}$ be a covering. If $\sigma \simeq \tau$ are two galleries in $\mathcal{M}$ with initial blade $\beta_0$, with lifts $\sigma', \tau'$ both having initial blade $\beta'_0$, then $\sigma' \simeq \tau'$.

**Proof.** It clearly suffices to prove that if $\sigma_1 \sim \sigma_2$, then $\sigma'_1 \sim \sigma'_2$. Thus, assume that $\sigma_1 = v e_1 \tau$, $\sigma_2 = v e_2 \tau$, where $e$, $e'$ both have the same initial blade and the same terminal blade and lie in the variety $v$. Applying (2.2.1) we lift $\sigma_1$, $\sigma_2$ uniquely to galleries $\sigma'_1 = v' e'_1 \tau' \sigma'_2 = v' e'_2 \tau'$, where, since $\mathcal{M}' \to \mathcal{M}$ is a covering, $e'_1$, $e'_2$ are galleries having the same initial and terminal blades and lie in a common variety of $\mathcal{M}'$. Therefore, it follows that $\sigma'_1 \sim \sigma'_2$ and the result follows.

**Corollary 2.2.2.1.** If $\phi : \mathcal{M}' \to \mathcal{M}$ is a covering and if $\beta'_0$, $\beta_0$ are blades of $\mathcal{M}'$, $\mathcal{M}$ with $\beta'_0 \phi = \beta_0$, then $\phi_* : \pi_1(\mathcal{M}', \beta'_0) \to \pi_1(\mathcal{M}, \beta_0)$ is an injection.

**Proof.** Indeed, if $\sigma'$ is a closed gallery in $\mathcal{M}'$ based at $\beta'_0$, then $\sigma = \sigma' \phi \simeq (\beta_0)$ implies by (2.2.2) that $\sigma' \simeq (\beta'_0)$.

We have the following analog of the Lifting Theorem (2.1.1):

**Theorem 2.2.3.** Let $\mathcal{M}, \mathcal{M}', \mathcal{M}''$ be in $\mathsf{Map}$, and let $\phi' : \mathcal{M}' \to \mathcal{M}$ be a morphism, and let $\phi'' : \mathcal{M}'' \to \mathcal{M}$ be a covering. Let $\beta'_0$ be a blade in $\mathcal{M}'$, $\beta_0'$ be a blade in $\mathcal{M}''$, and assume that $\beta'_0 \phi' = \beta_0' \phi''$. Then there exists a morphism $\phi : \mathcal{M}' \to \mathcal{M}''$ such that $\phi' = \phi \circ \phi''$ if and only if

$$\phi'_* \pi_1(\mathcal{M}', \beta'_0) \leq \phi''_* \pi_1(\mathcal{M}'', \beta'_0).$$

**Proof.** Assume that $\beta'$ is a blade in $\mathcal{M}'$, and (by connectivity) that $\sigma'$ is a gallery in $\mathcal{M}'$ from $\beta'_0$ to $\beta'$. Thus $\sigma' \phi'$ is a gallery in $\mathcal{M}$, which by (2.2.1) lifts uniquely to a gallery in $\mathcal{M}''$ with initial blade

\(^8\)Cf. op. cit., *Theorem 2.2.3, page 67.*
\(\beta''\). If \(\beta''\) is the terminal blade, we set \(\beta' \phi = \beta''\). Using (2.2.2), we see that \(\phi : \mathcal{M}' \rightarrow \mathcal{M}''\) is well-defined. Clearly, we have that \(\phi \phi'' = \phi'\). Finally, we need to show that \(\phi : \mathcal{M}' \rightarrow \mathcal{M}''\) is a morphism, i.e., if \(\mathcal{M}' = (B', a', b', c')\), \(\mathcal{M}'' = (B'', a'', b'', c'')\), then for all \(\beta' \in B'\), we have \(\phi(a'\beta') = a''(\beta' \phi)\) (with similar results for \(b', c'\)). Thus, if \(\sigma'\) is a gallery in \(\mathcal{M}'\) from \(\beta_0'\) to \(\beta'\), then \((a'\beta', \sigma')\) is a gallery in \(\mathcal{M}'\) from \(\beta_0'\) to \(a'\beta'\). Since \(\phi' : \mathcal{M}' \rightarrow \mathcal{M}\) is a morphism, we have that \(((a'\beta', \sigma')\phi' = (a(\beta' \phi'), \sigma' \phi')\) is a gallery in \(\mathcal{M}\). Also, since \(\phi'' : \mathcal{M}'' \rightarrow \mathcal{M}\) is a covering, this gallery lifts uniquely to a gallery of the form \((a''\beta'', \sigma'')\) in \(\mathcal{M}''\), where \(\sigma'' \phi'' = \sigma\), and where the terminal blade of \(\sigma''\) is \(\beta' \phi\). Therefore, it follows that \(a''\beta'' = (a'\beta')\phi\); similarly \(b''\beta'' = b''\beta'\phi\), and \(c''\beta'' = \phi' (\beta' \phi)\), proving that \(\phi : \mathcal{M}' \rightarrow \mathcal{M}''\) is a morphism, as required.

Next, If \(p : \mathcal{M}' \rightarrow \mathcal{M}\) is an covering of \(\mathcal{M}\), with \(\mathcal{M}, \mathcal{M}'\) both connected, fix blades \(\beta_0, \beta'_0\) of \(\mathcal{M}, \mathcal{M}'\), respectively, such that \(\beta'_0 \in \beta_0 p^{-1}\). We shall define a transitive action of \(\pi_1(\mathcal{M}, \beta_0)\) on the fibre \(\beta_0 p^{-1}\) over \(\beta_0\) along the following standard lines. First of all \(\beta' \in \beta_0 p^{-1}\), and if \(\gamma \in \pi_1(\mathcal{M}, \beta_0)\) is represented by the closed gallery \(\sigma\) based at \(\beta_0\), then apply (2.2.1) to obtain a unique gallery \(\sigma'\) in \(\mathcal{M}'\) whose initial blade is \(\beta'\). We define \(\gamma \cdot \beta'\) to be the terminal blade of \(\sigma'\). By (2.2.2) we see that this action is well-defined. If \(\gamma_1, \gamma_2 \in \pi_1(\mathcal{M}, \beta_0)\) are represented by closed galleries \(\sigma_1, \sigma_2\), and if \(\beta' \in \beta_0 p^{-1}\) then these galleries may be lifted to galleries \(\sigma'_1, \sigma'_2\) in \(\mathcal{M}'\) having initial blades \(\beta'\) and \(\beta'_2\), where \(\beta'_2\) is the terminal blade of \(\sigma'_2\). Since the juxtaposition \(\sigma'_1 \sigma'_2\) is a lift of \(\gamma_1 \gamma_2\), we conclude that \((\gamma_1 \gamma_2)(\beta') = \gamma_1(\gamma_2(\beta'))\), i.e., that we have an action of \(\pi_1(\mathcal{M}, \beta_0)\) on \(\beta_0 p^{-1}\).

Finally if \(\beta'_1, \beta'_2 \in \beta_0^{-1}\), then by connectivity there is a gallery \(\sigma'\) joining \(\beta'_1\) to \(\beta'_2\). Thus \(\sigma = \sigma' p\) is a closed gallery in \(\mathcal{M}\) based at \(\beta_0\) and hence represents an element \(\gamma = [\sigma] \in \pi_1(\mathcal{M}, \beta_0)\). By construction, \(\gamma(\beta'_1) = \beta'_2\), proving that the action is transitive.

Continuing to assume that \(p : \mathcal{M}' \rightarrow \mathcal{M}\) is a covering of \(\mathcal{M}\), then we have the group \(A(\mathcal{M}'/\mathcal{M})\) of fibre-preserving automorphisms of \(\mathcal{M}'\) just as before. Furthermore, we have the following analog of Theorem 2.1.4:
Theorem 2.2.4. The covering \( p : \mathcal{M}' \to \mathcal{M} \) with \( \mathcal{M}' \) and \( \mathcal{M} \) both connected and \( \beta_0'p = \beta_0 \) is regular if and only if \( p_* \pi_1(\mathcal{M}', \beta_0') \leq \pi_1(\mathcal{M}, \beta_0) \), in which case

\[
A(\mathcal{M}'/\mathcal{M}) \cong \pi_1(\mathcal{M}, \beta_0)/\pi_1(\mathcal{M}', \beta_0').
\]

Proof. Note first of all that, by definition of the action of \( \pi_1(\mathcal{M}, \beta_0) \) on \( \beta_0p^{-1} \), the stabilizer in \( \pi_1(\mathcal{M}, \beta_0) \) of \( \beta_0' \) is exactly \( p_* \pi_1(\mathcal{M}, \beta_0) \). Next, if \( \beta_1 \) is a blade of \( \mathcal{M} \), and if \( \beta_1' \in \beta_1p^{-1} \), let \( \sigma_{01} \) be a gallery in \( \mathcal{M} \) from \( \beta_1 \) to \( \beta_0 \). If \( \gamma \in \pi_1(\mathcal{M}, \beta_0) \) is represented by the closed gallery \( \sigma \), then the mapping \( \gamma \mapsto [\sigma_{01}^{-1} \sigma \sigma_{01}] \in \pi_1(\mathcal{M}, \beta_1) \) is easily checked to be an isomorphism \( \pi_1(\mathcal{M}, \beta_0) \to \pi_1(\mathcal{M}, \beta_1) \). Furthermore, it is clear that the subgroup \( p_* \pi_1(\mathcal{M}, \beta_0') \) maps to \( p_* \pi_1(\mathcal{M}', \beta_0') \), where \( \beta_1' \in \beta_1p^{-1} \). This implies that if \( p_* \pi_1(\mathcal{M}', \beta_0') \leq \pi_1(\mathcal{M}, \beta_0) \) for some blade \( \beta \) in \( \mathcal{M} \), then \( p_* \pi_1(\mathcal{M}', \beta_0') \leq \pi_1(\mathcal{M}, \beta_0) \) for all blades \( \beta_0 \) in \( \mathcal{M} \). Therefore, we see that if \( p_* \pi_1(\mathcal{M}', \beta_0') \) is in the kernel of the action of \( \pi_1(\mathcal{M}, \beta_0) \) on \( \beta_0p^{-1} \), for some blade \( \beta \) in \( \mathcal{M} \), then \( p_* \pi_1(\mathcal{M}', \beta_0') \) is in the kernel of the action of \( \pi_1(\mathcal{M}, \beta_0) \) on \( \beta_0p^{-1} \) for all blades \( \beta \) in \( \mathcal{M} \).

Assuming first of all that \( A(\mathcal{M}'/\mathcal{M}) \) is a regular covering, we infer that in particular \( A(\mathcal{M}'/\mathcal{M}) \) acts transitively (hence regularly) on the fibre \( \beta_0p^{-1} \). Thus, for all pairs \( \beta_1', \beta_2' \in \beta_0p^{-1} \), we have an automorphism \( \eta \in A(\mathcal{M}'/\mathcal{M}) \) with \( \beta_1' \eta = \beta_2' \). Therefore, it follows immediately that

\[
p_* \pi_1(\mathcal{M}', \beta_1') = p_* \pi_1(\mathcal{M}', \beta_2').
\]

However, since \( \pi_1(\mathcal{M}, \beta_0) \) acts transitively on \( \beta_0p^{-1} \), all subgroups of the form \( p_* \pi_1(\mathcal{M}', \beta_0'), \beta_0' \in \beta_0p^{-1} \) are conjugate in \( p_* \pi_1(\mathcal{M}', \beta_0') \), from which it follows that \( p_* \pi_1(\mathcal{M}', \beta_0') \leq \pi_1(\mathcal{M}, \beta_0) \).

Conversely, assume that \( p_* \pi_1(\mathcal{M}', \beta_0') \leq \pi_1(\mathcal{M}, \beta_0) \). Therefore, \( \pi_1(\mathcal{M}, \beta_0) \) acts transitively on \( \beta_0p^{-1} \) with kernel \( p_* \pi_1(\mathcal{M}', \beta_0') \). We now define an action of \( \pi_1(\mathcal{M}, \beta_0) \) on all of \( \mathcal{M}' \). If \( \beta' \) is a blade of \( \mathcal{M}' \), we may, by connectivity of \( \mathcal{M}' \), construct a gallery \( \sigma' \) from \( \beta_0' \) to \( \beta' \), and set \( \sigma = \sigma'p \), a gallery in \( \mathcal{M} \) from \( \beta_0 \) to \( \beta = \beta'p \). If \( \gamma \in \pi_1(\mathcal{M}, \beta_0) \), set \( \beta_1' = \gamma \beta_0' \) and let \( \sigma_1' \) be the unique lift of \( \sigma \) to \( \mathcal{M}' \) having initial blade \( \beta_1' \). We define \( \gamma(\beta') \) to be the terminal blade of the gallery \( \sigma_1' \).

We show first that \( \gamma(\beta') \) is independent of the choice of gallery \( \sigma' \) in \( \mathcal{M}' \) connecting \( \beta_0' \) to \( \beta' \). If \( \sigma'' \) is another such gallery from \( \beta_0' \) to \( \beta' \), then the juxtaposition \( \sigma'' \sigma'^{-1} \) is a lift of the closed gallery
$(\sigma''p)(\sigma'p)^{-1}$, representing $[(\sigma''p)(\sigma'p)^{-1}] \in \pi_1(M', \beta')$. By what has already been observed above, $p_\ast \pi_1(M', \beta')$ is in the kernel of the action of $\pi_1(M, \beta)$ on $\beta p^{-1}$ and so $\sigma'' \sigma'^{-1}$ must be a closed gallery (based at $\gamma(\beta')$). Therefore, $\gamma(\beta')$ is well-defined. Showing that this defines an action of $\pi_1(M, \beta_0)$ on $\beta' p^{-1}$ is carried out exactly as above. Finally to show that for any $\gamma \in \pi_1(M, \beta_0)$ and for any blade $\beta'$ in $M'$ we have $\gamma(d'\beta') = d' \gamma(\beta')$, where $d \in \{a, b, c\}$ is entirely routine.

**Corollary 2.2.4.1.** Let $M$ be uniform with vertex valency $l$ and face valency $k$. Fix a blade $\beta_0$ in $M$ and let $\pi_1^{(k,l)}(M, \beta_0)$ be the orbifold fundamental group based at $\beta_0$. Then $F^{(k,l)} : \pi_1^{(k,l)}(M, \beta_0) \cong \pi_1(M, \beta_0)$.

**Proof.** We have the covering $M_\Delta \to M$, where $\Delta = \Delta(k, l)$. Note that for any blade $\beta_\Delta$ of $M_\Delta$, we have $\pi_1^{(k,l)}(M_\Delta, \beta_\Delta) = \pi_1(M_\Delta, \beta_\Delta) = 1$. Therefore, by (2.1.4) and (2.2.4), it follows immediately that $M_\Delta \to M$ is a regular covering acted on regularly by both $\pi_1^{(k,l)}(M_\Delta, \beta_\Delta)$ and $\pi_1(M_\Delta, \beta_\Delta)$. As it is routine to check that the action of $\pi_1^{(k,l)}(M_\Delta, \beta_\Delta)$ on $M_\Delta$ factors through that of $\pi_1(M_\Delta, \beta_\Delta)$ via the homomorphism $F^{(k,l)}$, the result follows.

Therefore, in case $M$ is uniform with vertex valency $l$ and face valency $k$, we may identify $\pi_1^{(k,l)}(M, \beta_0)$ with $\pi_1(M, \beta_0)$ and write $\pi_1(M, \beta_0)$ for both.

The following result shows that if $M$ is a regular map and if $p : M' \to M$ is a ramified covering by the connected map $M'$, then this morphism factors uniquely into a totally ramified covering over an unramified covering.

**Theorem 2.2.5.** Let $M$ be a uniform connected map, and let $p : M' \to M$ be a ramified covering by the connected map $M'$. Then there exists a unique map (up to $M$-isomorphism) $M_{\text{un}}$ and a factorization $p : M' \to M_{\text{un}} \to M$ such that $M' \to M_{\text{un}}$ is totally ramified and $M_{\text{un}} \to M$ is unramified.

**Proof.** Let $M$ have vertex valency $l$ and face valency $k$. Since $M$
is uniform, the action of $\Delta = \Delta(\infty, \infty)$ on the blades of $\mathcal{M}$ factors through the action of $\Delta(k, l)$; set $K = \ker(\Delta \to \Delta(k, l))$. By connectivity, we may identify the blades of $\mathcal{M}'$ and $\mathcal{M}$ with $\Delta/\pi_1^\infty(\mathcal{M}', \beta')$ and $\Delta/\pi_1^\infty(\mathcal{M}, \beta)$, respectively, for suitable blades $\beta', \beta$ with $\beta' p = \beta$. Therefore, factorizations of the form $\mathcal{M}' \to \mathcal{M}_1 \to \mathcal{M}$ are in correspondence with subgroups of $\pi_1^\infty(\mathcal{M}, \beta)$ containing $\pi_1^\infty(\mathcal{M}', \beta')$. Furthermore, we see that $\mathcal{M}_1 \to \mathcal{M}$ is unramified if and only if $K \leq \pi_1^\infty(\mathcal{M}_1, \beta_1)$, where $\beta_1$ is a blade of $\mathcal{M}_1$ projecting to the blade $\beta$ of $\mathcal{M}$. As a result, if $\mathcal{M}_{un}$ corresponds to the subgroup $K \cdot \pi_1^\infty(\mathcal{M}', \beta')$, then $\mathcal{M}_{un} \to \mathcal{M}$ is unramified, $\mathcal{M}' \to \mathcal{M}_{un}$ is totally ramified and $\mathcal{M}_{un}$ is uniquely determined up to $\mathcal{M}$-isomorphism.
Chapter 3

Orientation of Maps

3.1 Orientable and Oriented Maps

**Definition 3.1.1 (Orientable Maps).** Let $\mathcal{M} = (B, a, b, c)$ be a connected map with orbifold fundamental group $\pi_1^\infty(\mathcal{M}, \beta) \leq \Delta = \Delta(\infty, \infty)$. We say that $\mathcal{M}$ is orientable if $\pi_1^\infty(\mathcal{M}, \beta) \subseteq \Delta^+$. This definition is independent of the choice of blade $\beta$ in $\mathcal{M}$. Furthermore, the same definition is valid when applied to the restricted orbifold fundamental group $\pi_1^{(m,n)}(\mathcal{M}, \beta) \leq \Delta = \Delta(m, n)$, where $m$ and $n$ are multiples of the face and vertex valencies, respectively, of $\mathcal{M}$. Equivalently, a more elementary criterion for orientability is that the subgroup $G^+ = \langle ab, bc \rangle$ of the monodromy group $G = \langle a, b, c \rangle$ acts in exactly two orbits on $B$. Each of the two $G^+$-orbits $B^+$ and $B^-$ are called orientations of the orientable map $\mathcal{M}$.

**Exercise.** Consider the so-called Möbius-Kantor map $\mathcal{M}$ depicted to the right, where the opposite edges of the octagon have been identified so as to obtain an orientable surface. Show that there is an orbifold covering $\mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is the map of the cube.
The category of orientable maps and morphisms is denoted $\text{OMap}$. An automorphism of an orientable map $\mathcal{M} = (B, a, b, c)$ will either stabilize the two orientations $B^+, B^-$, in which case it is called orientation preserving, or will interchange the two orientations, in which case it is called orientation reversing.

**Definition 3.1.2 (Oriented Maps).** An oriented map is an ordered triple $\mathcal{N} = (D, P, L)$ where $D$ is a set (whose elements are often called darts) and where $P, L$ are permutations on $B$ such that $L$ is an involution. If $L$ acts fixed-point freely on $D$, we say that the oriented map $\mathcal{M}$ is nondegenerate. The group $G = \langle P, L \rangle$ is called the (oriented) monodromy group of the oriented map $\mathcal{M}$. We say that $\mathcal{M}$ is connected if the monodromy group acts transitively on $D$, and we say that $\mathcal{M}$ is regular if the monodromy group acts regularly on $D$. Morphisms and automorphisms are defined in the obvious way. As expected, if the oriented map $\mathcal{M}$ is connected, and if $d \in D$ is a fixed dart with dart stabilizer $H \leq G$, then $\text{Aut}(\mathcal{M}) \cong \text{N}_G(H)/H$; therefore, we see that $\mathcal{M}$ is regular if and only if $\text{Aut}(\mathcal{M})$ acts transitively (and hence regularly) on $D$

**Definition 3.1.3 (Vertices, Edges and Faces).** As with maps, an oriented map $\mathcal{M} = (D, P, L)$ has varieties, as follows:

- **vertices:** these are the $\langle P \rangle$-orbits in $D$;
- **edges:** these are the $\langle L \rangle$-orbits in $D$;
- **faces:** these are the $\langle PL \rangle$-orbits in $D$.

**Definition 3.1.4 (Euler Characteristic of an Oriented Map).** If $\mathcal{M} = (D, P, L)$ is a finite oriented map, i.e., $D$ is finite, we define the Euler Characteristic of $\mathcal{M}$ by setting $\chi(\mathcal{M}) = \#\text{vertices} - \#\text{edges} + \#\text{faces}$. Again, more will be said about Euler characteristic in Subsection 4.1.
**Definition 3.1.5 (Vertex and Face Valency).** If \( \mathcal{M} = (D, P, L) \) is an oriented map, and if \( \mathbf{v} \subseteq D \) is a variety, we call the cardinality \(|\mathbf{v}|\) the valency of this variety. The vertex valency of \( \mathcal{M} \) is the least common multiple of the valencies of the vertices in \( \mathcal{M} \). Likewise the face valency of \( \mathcal{M} \) is the least common multiple of the valencies of the faces in \( \mathcal{M} \).

The category of oriented maps and morphisms is denoted \( \text{OrMap} \). As in \( \text{Map} \) it makes sense to speak of ramification over varieties of a morphism in \( \text{OrMap} \).

There is a functor \( \text{OrMap} \to \text{Map} \) given as follows. Let \( \mathcal{N} = (D, P, L) \) be an oriented map, and let \( G^+ = \langle P, L \rangle \) be its monodromy group. Assume that \( P^m = 1 = (PL)^n \). Therefore, if \( k, l \) are positive integers satisfying \( n|k, m|l \), then by Corollary 2.0.3.1 there is a surjective homomorphism

\[
\theta^+ : \Delta^+ = \Delta^+(k, l) = \langle s_2 s_3, s_1 s_3 \rangle \to G^+, \ (s_2 s_3) \mapsto P, (s_1 s_3) \mapsto L.
\]

If \( d \in D \) is a fixed dart, with stabilizer \( H^+ \leq G^+ \), then we set \( H_\Delta^+ = \theta^+ \Gamma^+(H^+) \leq \Delta^+ \), and define the map \( \mathcal{M}(\Delta/H_\Delta^+) = (\Delta/H_\Delta^+, s_1, s_2, s_3) \), thereby obtaining a functor from \( \text{OrMap} \to \text{Map} \). In fact, however, the map \( \mathcal{M}(\Delta/H_\Delta^+) \) so obtained is orientable and so we actually get a functor \( \text{OrMap} \to \text{OMap} \).

There is an equivalent, and more direct description of this functor\(^1\) as follows. Given the oriented map \( \mathcal{M} \) define the map \( \mathcal{M}^\sharp = (B^\sharp, a^\sharp, b^\sharp, c^\sharp) \), where \( B^\sharp = D \times \{\pm 1\} \), and

\[
\begin{align*}
a^\sharp(d, \zeta) &= (Ld, -\zeta) \\
b^\sharp(d, \zeta) &= (P^\zeta d, -\zeta) \\
c^\sharp(d, \zeta) &= (d, -\zeta).
\end{align*}
\]

Then \( \mathcal{M}^\sharp \) is readily checked to be orientable.

The following is virtually obvious, but worth emphasizing:

Lemma 3.1.1. If $\mathcal{M}$ is an oriented map, with corresponding orientable map $\mathcal{M}^\sharp$ as above, then

$$\chi(\mathcal{M}) = \chi(\mathcal{M}^\sharp).$$

Definition 3.1.6. (Oriented Monodromy Group of a Map) If $\mathcal{M} = (B, a, b, c)$ is a map, we set $G^+ = \langle bc, ac \rangle$ and call $G^+$ the oriented monodromy group of $\mathcal{M}$.

Note that in the above, the sets $D \times \{1\}$ and $D \times \{-1\}$ are invariant under the oriented monodromy group $G^\sharp = \langle a^\sharp b^\sharp, b^\sharp c^\sharp \rangle$ of $\mathcal{M}^\sharp$. If $\mathcal{M}$ is a connected oriented map, then $D \times \{1\}$ and $D \times \{-1\}$ are the two orientations of the orientable map $\mathcal{M}^\sharp$.

Remark. If $\mathcal{M}^\sharp$ is regular, it is clear that $\mathcal{M}$ is also regular. However, the converse fails. Here, take $\Delta = \Delta(3, 3) \cong S_4$, and so $\Delta^+ \cong A_4$, the alternating group of degree 4. Let $Z_2 \times Z_2 \cong H = H^+ \leq A_4$ and note that $H \leq \Delta^+$, but that $H \not\cong \Delta$. Therefore, $\mathcal{M} = \mathcal{M}(\Delta^+/H)$ is regular, but $\mathcal{M}^\sharp \cong \mathcal{M}(\Delta/H)$ is not regular.

If $\mathcal{M} = (B, a, b, c)$ is orientable, then the oriented monodromy group $G^+$ has index 2 in $G$ and decomposes the blade set $B$ into two $G^+$-orbits: $B = B^+ \cup B^-$. Thus an orientable map $\mathcal{M}$ gives rise to two oriented maps: $\mathcal{M}^+ = (B^+, bc, ac)$ and $\mathcal{M}^- = (B^-, bc, ac)$.

On the other hand, if $\mathcal{M} = (B, a, b, c)$ is nonorientable, then $G^+$ acts in one orbit on $B$ (even when $[G : G^+] = 2$, which is possible for a nonregular nonorientable map). Thus, one has the well-defined oriented map $\mathcal{M}^+ = (B, bc, ac)$ whose monodromy group is $G^+$. Therefore, if we denote by $\text{NMMap}$ the subcategory of $\text{Map}$ consisting of of nonorientable maps and morphisms, then there results a well-defined functor $\text{NMMap} \to \text{OrMap}$, and consequently a functor $\text{NMMap} \to \text{OMap}$. This latter functor constructs the “orientable double cover” of the nonorientable map; we’ll look at this in more detail later.

Remark. If the nonorientable map $\mathcal{M}$ is regular, it is clear that the oriented map $\mathcal{M}^+$ is also regular. The converse fails. As a example,
consider the group $G = \langle a, b, c \rangle \cong \Delta(3, 4)$, which is known to be a finite group of order 48.\footnote{This is the Coxeter group of type $B_3$.} Now let $\omega \in G - G^+$ be any non-central involution and set $H = \langle \omega \rangle$. Then the map $\mathcal{M} = (G/H, a, b, c)$ is nonorientable and nonregular (as $H \not\cong G$). However, $G = G^+H$ implies that $G^+$ acts regularly on the set $G/H$. Therefore, the oriented map $\mathcal{M}^+ = (G/H, bc, ac)$ is regular.

### 3.2 Reflexibility

**Definition 3.2.1 (Reflexible Orientable Maps)** Let $\mathcal{M} = (B, a, b, c)$ be an orientable map with monodromy group $G = \langle a, b, c \rangle$. Set $G^+ = \langle bc, ac \rangle$ and let $B = B^+ \cup B^-$ be the decomposition of $B$ into two $G^+$-orbits. We say that $\mathcal{M}$ is reflexible if and only if there exists an automorphism $\alpha \in \text{Aut}(\mathcal{M})$ with $B^+\alpha = B^-$ (in which case $B^-\alpha = B^+$). Note that if the orientable map $\mathcal{M}$ is regular, it is certainly reflexible.

**Definition 3.2.2. (Orientably Regular Maps)** Let $\mathcal{M} = (B, a, b, c)$ be an orientable map with monodromy group $G = \langle a, b, c \rangle$. Set $G^+ = \langle bc, ac \rangle$ and let $B = B^+ \cup B^-$ be the decomposition of $B$ into two $G^+$-orbits. We say that $\mathcal{M}$ is orientably regular if $G^+$ acts regularly on each of $B^+$ and $B^-$. In this case, if $\text{Aut}(\mathcal{M})^+$ is the stabilizer in $\text{Aut}(\mathcal{M})$ of $B^+$, then $\text{Aut}(\mathcal{M})^+$ acts transitively (and hence regularly) on $B^+$ and $B^-$. Note that if $\mathcal{M}$ is reflexible and orientably regular, it is regular.

**Definition 3.2.3 (Reflexible Oriented Maps)** If $\mathcal{N} = (D, P, L)$ is an oriented map, we say that $\mathcal{N}$ is reflexible if and only if there exists a bijection $\psi : D \to D$ such that $(Pd)\psi = P^{-1}(d\psi), (Ld)\psi = L(d\psi)$, that is if and only if there is an isomorphism of oriented maps $(D, P, L) \isom (D, P^{-1}, L)$. Note that if $\mathcal{M}$ is orientably regular, then $\mathcal{N}$ is reflexible precisely when there is an automorphism of $G^+ = \langle P, L \rangle$ that inverts $P$ and fixes $L$. 
Lemma 3.2.1. Let $\mathcal{M} = (B, a, b, c)$ be an orientable map with corresponding oriented maps $\mathcal{M}^+$ and $\mathcal{M}^-$. The following are equivalent:

1. $\mathcal{M}$ is reflexible.
2. $\mathcal{M}^+ \cong \mathcal{M}^-$. 
3. $\mathcal{M}^+$ is reflexible.
4. $\mathcal{M}^-$ is reflexible.

Proof. (1) $\Rightarrow$ (2) follows by definition. For (2) $\Rightarrow$ (3), let $\phi : \mathcal{M}^+ \xrightarrow{\cong} \mathcal{M}^-$ and define $\psi : B^+ \to B^+$ by setting $\beta \psi = c(\beta \phi), \beta \in B^+$. If $P = bc$ and $L = ac$, then $(P \beta) \psi = c((P \beta) \phi) = cP(\beta \phi) = P^{-1}c(\beta \phi) = P^{-1}(\beta \psi)$. Similarly, one shows that $(L \beta) \psi = L(\beta \psi)$, and so $\mathcal{M}^+$ is reflexible. In exactly the same way, one proves that (2) $\Rightarrow$ (4). For (3) $\Rightarrow$ (1), assume that there exists a bijection $\psi : B^+ \to B^+$ such that $(P \beta) \psi = P^{-1}(\beta \psi)$ and $(L \beta) \psi = L(\beta \psi)$, where $P = bc$ and $L = ac$. Define $\phi : B \to B$ by setting

$$\beta \phi = \begin{cases} c(\beta \psi) & \text{if } \beta \in B^+ \\ (c\beta) \psi & \text{if } \beta \in B^- \end{cases}$$

Then $\beta \in B^+$ implies that $(a\beta) \phi = (ca\beta) \psi = ca(\beta \psi) = a(\beta \phi)$. Also $(b\beta) \phi = (cb\beta) \psi = bc(\beta \psi) = bcc(\beta \phi) = b(\beta \phi)$. Finally, $(c\beta) \phi = (\beta) \psi = c(\beta \phi)$. Next, assume that $\beta \in B^-$. First, note that $a(a\beta) \phi = ac((a\beta) \psi) = (c\beta) \psi = c((c\beta) \phi)$. Therefore, $(a\beta) \phi = a \cdot a((a\beta) \phi) = ac((c\beta) \phi) = a(\beta \phi)$. Next, $(b\beta) \phi = c((b\beta) \psi) = b((cb\beta) \psi) = b((c\beta) \psi) = b(\beta \phi)$. Finally, $(c(c\beta) \phi) = a((a\beta) \phi) = (\beta) \phi$, and so $(c\beta) \phi = c(\beta \phi)$. Therefore, we see that $\phi \in \text{Aut}(\mathcal{M})$. By definition, $\phi : B^+ \to B^-$, and so $\mathcal{M}$ is reflexible.

Therefore, if we have any of conditions (1), (2) or (3), we have condition (2), which we have already seen implies that $\mathcal{M}^-$ is reflexible, i.e., that condition (4) obtains. Finally, to see that (4) $\Rightarrow$ (1), let $\psi : B^- \to B^-$ be given and define $\phi : B \to B$ by

$$\beta \phi = \begin{cases} c(\beta \psi) & \text{if } \beta \in B^- \\ (c\beta) \psi & \text{if } \beta \in B^+ \end{cases}$$
Then, exactly as above, one proves that \( \phi \in \text{Aut}(\mathcal{M}) \) and interchanges \( B^+ \) and \( B^- \). The proof follows.

### 3.3 Chirality Groups of Orientably Regular Maps

Let \( \mathcal{M} = (B, a, b, c) \) be an orientably regular group with vertex valency \( l \), face valency \( k \), and monodromy group \( G = \langle a, b, c \rangle \). Thus we have a surjective homomorphism \( \Delta(k, l) \to G \); if \( \beta \in B \) is a fixed blade in \( \mathcal{M} \), then the orbifold fundamental group \( H = \pi_1^{(k,l)}(\mathcal{M}, \beta) \), is a normal subgroup of \( \Delta(k, l)^+ \). Furthermore, in this case we have that \( \mathcal{M} \) is reflexible if and only if \( H \leq \Delta(k, l) \). Writing \( \Delta(k, l) = \langle s_1, s_2, s_3 \rangle \), we see that when \( \mathcal{M} \) is orientably regular, \( H^{s_1} = H^{s_2} = H^{s_3} \); in this case we denote this conjugate subgroup by \( H^s \). For the remainder of this subsection, we shall assume that the maps in question are orientably regular.

**Definition 3.3.1.** The *chirality group* of the orientably regular map \( \mathcal{M} \) is the quotient group \( X(\mathcal{M}) = H/(H \cap H^s) \). The *chirality index* of \( \mathcal{M} \) is the order of the chirality group: \( \kappa(\mathcal{M}) = |X(\mathcal{M})| \). Whenever the chirality index \( \kappa(\mathcal{M}) > 1 \), we say that the orientably regular map \( \mathcal{M} \) is *chiral*.\(^3\)

There are other description of the chirality group; for convenience we set \( H^\Delta = HH^s \), \( H_\Delta = H \cap H^s \). Therefore the chirality group of \( \mathcal{M} \) is \( H/H_\Delta \). By the Nöther Isomorphism Theorem, one has

**Lemma 3.3.1.** Relative to the above notation, \( X(\mathcal{M}) = H/H_\Delta \cong H^\Delta/H \).

Note that the chirality group of \( \mathcal{M} \) can be identified with a subgroup of the oriented monodromy group \( G^+ \); indeed, \( G^+ \cong \Delta^+/H \geq H^\Delta/H \cong X(\mathcal{M}) \). Furthermore, since \( H \leq \Delta^+ \), it follows that \( H^\Delta \leq \Delta \), and so

---

\(^3\)The definition of the chirality group and chirality index of an orientably regular map and its basic properties are due to G.A. Jones, R. Nedela, and M. Škoviera.
certainly $H^\Delta \leq \Delta^+$. Therefore, it follows that $X(M)$ is isomorphic to a normal subgroup of $G^+$. Finally, note that the chirality index $\kappa(M)$ divides $|G^+| = |B^+|$.

As an instructive example of interesting chirality groups, we start with the following family of orientably regular maps.\(^4\) Let $F = F_q$ be the finite field of $q = p^e$ elements, where $p$ is a prime, and let $t \in F^\times$ be a fixed generator of $F^\times$. Define the oriented map $M_t(F) = (D, P, L)$ by setting

\[
D = \{(a, b) \in F \times F | a \neq b\},
\]

\[
P(a, b) = (a, a + (b - a)t),
\]

\[
L(a, b) = (b, a).
\]

It is clear that the oriented map $M_t(F)$ is connected. To see that it’s orientably regular, consider the mappings $\gamma_a, \mu_b : D \to D$, $a, b \in F$, $b \neq 0$, given by

\[
\gamma_a(a', b') = (a' + a, b' + a),
\]

\[
\mu_b(a', b') = (ab', b'b).
\]

It is routine to check that all $\gamma_a, \mu_b \in \text{Aut}(M_t(F))$, from which one easily concludes that $|\text{Aut}(M_t(F))| \geq q(q-1) = |D|$, proving regularity.

However, it turns out that very few of the above maps are reflexible.\(^5\) To keep the development self-contained, we shall prove the following result from James and Jones’ paper:

**Proposition 3.3.2.** $M_t(F) \cong M_{t'}(F)$ if and only if $t$ and $t'$ are conjugate under some power of the Frobenius map $F : F \to F$, $s \mapsto s^p$. In other words, $M_t \cong M_{t'}$ if and only if $t' = t^{p^j}$ for some $j$, $0 \leq j < e$.


\(^5\)op. cit.
PROOF. Write $\mathcal{M}_t = \mathcal{M}_t(\mathbb{F})$, $\mathcal{M}_{t'} = \mathcal{M}_{t'}(\mathbb{F})$, and assume that there exists an isomorphism $\tau : \mathcal{M}_t \xrightarrow{\cong} \mathcal{M}_{t'}$. Note that the vertex sets of both maps have the form $\{(a, b) \mid b \in F\}$. Therefore, the vertices can be identified with the elements of $\mathbb{F}$, via $(a, b) \mapsto a$. Furthermore, as $\tau$ also induces a bijection of the underlying vertex sets, we may regard $\tau$ as a bijection $\tau : \mathbb{F} \rightarrow \mathbb{F}$. If $P, P'$ are the rotations on $\mathcal{M}_t$, $\mathcal{M}_{t'}$ corresponding to the generators $t, t' \in \mathbb{F}$, respectively, we have a commutative diagram

\[
\begin{array}{ccc}
{B} & \xrightarrow{\tau} & {B} \\
\downarrow & & \downarrow \\
{P} & \xrightarrow{} & {P'} \\
\uparrow & & \uparrow \\
{B} & \xrightarrow{\tau} & {B} 
\end{array}
\]

As a result, we have for all $a \neq b$ in $\mathbb{F}$ that

\[(a\tau, (a + (b - a)t)\tau) = (a\tau, a\tau + (b\tau - a\tau)t'),\]

and so $\tau : \mathbb{F} \rightarrow \mathbb{F}$ satisfies

\[(a + (b - a)t)\tau = a\tau + (b\tau - a\tau)t' \quad (3.1)\]

for all $a \neq b$ in $\mathbb{F}$.

Next, since $\text{Aut}(\mathcal{M}_{t'})$ acts on the vertices (identified with $\mathbb{F}$) of $\mathcal{M}_t$, with stabilizers of order $q$, we infer that, in fact, $\text{Aut}(\mathcal{M}_{t'})$ acts doubly transitively on $\mathbb{F}$. Therefore, we may compose $\tau$ with an automorphism $\phi \in \text{Aut}(\mathcal{M}_{t'})$ so that

\[0(\tau \phi) = 0, \quad 1(\tau \phi) = 1.\]

As $\tau \phi : \mathcal{M}_t \xrightarrow{\cong} \mathcal{M}_{t'}$, we may replace $\tau$ by $\tau \phi$; i.e., we may assume from the start that

\[0\tau = 0, \quad 1\tau = 1.\]

Therefore, from Equation (3.1) above with $a = 0$, $b = 1$, we obtain $t\tau = t'$. 
CLAIM. \( \tau : \mathbb{F}^\times \rightarrow \mathbb{F}^\times \) is a group automorphism.

PROOF OF CLAIM. We have \( 1\tau = 1, t\tau = t' \). We shall argue by induction on \( m \) that \( t^m\tau = (t')^m \). Indeed, we have

\[
t^m\tau = (t^{m-1}t)\tau \\
= (0 + (t^{m-1} - 0)t)\tau \\
= 0\tau + (t^{m-1}\tau - 0\tau)t' \\
= (t')^{m-1}t' \\
= t'^m.
\]

Since \( t, t' \) both generate \( \mathbb{F}^\times \), the claim follows.

CLAIM. \( \tau : \mathbb{F} \rightarrow \mathbb{F} \) is a field automorphism.

PROOF OF CLAIM. We show that for all \( a, b \in \mathbb{F} \) \( (a + b)\tau = a\tau + b\tau \). Since \( \tau \) intertwines the rotations \( P, P' \), we have, for all integers \( s \), that

\[
(a + (b - a)t^s)\tau = a\tau + (b\tau - a\tau)t^s;
\]

since \( t\tau = t' \), this implies

\[
(a + (b - a)t^s)\tau = a\tau + (b\tau - a\tau)(t\tau)^s. \tag{3.2}
\]

In Equation 3.2, set \( a = 1, b = 0 \) and obtain

\[
(1 - t^s)\tau = 1 - (t\tau)^s \\
= 1 - t^s\tau
\]

where we have used the fact that \( \tau : \mathbb{F}^\times \rightarrow \mathbb{F}^\times \) is a group automorphism. Therefore, it follows that for all \( a \in \mathbb{F} \),

\[
(1 - a)\tau = 1 - a\tau.
\]

Note that \( (-1)\tau = -1 \), as \( -1 \) is the unique involution in \( \mathbb{F}^\times \) (where \( q \) is odd). (If \( q \) is even, there is no issue here.) Therefore, we obtain

\[
(1 + a)\tau = 1 - (-a)\tau \\
= 1 - (-1)a\tau \\
= 1 + a\tau.
\]
Finally, if \( b \neq 0 \), then

\[
(a + b)\tau = ((1 + ab^{-1})b)\tau \\
= (1 + ab^{-1})\tau(b) \\
= (1 + (ab^{-1})\tau)(b) \\
= b\tau + a\tau = a\tau + b\tau,
\]

proving that \( \tau : \mathbb{F} \to \mathbb{F} \) is a field automorphism. Therefore, we know that \( \tau = F^j \), for some integer \( j \), \( 0 \leq j < e \), where \( F : \mathbb{F} \to \mathbb{F} \) is the Frobenius automorphism of \( \mathbb{F} \), given by \( x \mapsto x^p \). This proves that if \( \mathcal{M}_t \cong \mathcal{M}_r \), then \( t' = t\tau = t^{j} \) for some \( j \), \( 0 \leq j < e \).

Conversely, if \( t' = t^{j} \), define \( \mathcal{M}_t \to \mathcal{M}_r \) by \( (a, b) \mapsto (F^j(a), F^j(b)) \) and show that this mapping realizes \( \mathcal{M}_t \cong \mathcal{M}_r \).

Note that \( p^{-1}(a, b) = (a, a + (b - a)t^{-1}) \); therefore, we deduce that \( \mathcal{M}_t \) is reflexible if and only if \( t^{j} = t^{-1} \) for some \( j \), \( 0 \leq j < e \). This is equivalent to the condition \( p^{e} - 1|p^{j} + 1 \). However, it’s easy to see that this can happen only for \( p^{e} = 2, 3, 4 \), with \( j = 0, 0, 1 \), respectively. In no other cases can \( \mathcal{M}_t \) be reflexible.

Therefore, it follows immediately that whenever \( p^{e} > 4 \), the the orientably regular maps \( \mathcal{M}_t(\mathbb{F}) \) are chiral. We proceed to determine the chirality groups of the chiral maps \( \mathcal{M}_t \). Let the oriented monodromy group be \( G^+ = \langle P, L \rangle \), with surjective homomorphism \( \Delta^+ \to G^+ \), with kernel \( H \). From the work above, it follows easily that \( G^+ \) has the structure of a semidirect product \( G^+ = TN \), where \( T \cong \mathbb{F}^x \) and \( N \cong \mathbb{F}^+ \) (here, \( \mathbb{F}^+ \) is the additive group of \( \mathbb{F} \)). Let \( K \) be the inverse image of \( N \) under the homomorphism \( \Delta^+ \to G^+ \). Thus, we have \( \Delta^+ \geq K \geq H \), and \( K \triangleleft \Delta^+ \). Since \( \mathcal{M} \) is chiral, \( H \) is not normal in \( \Delta \), and the chirality group is given by \( X(\mathcal{M}) = H/(H \cap H^*) \cong HH^*/H = H^\Delta/H \). On the other hand, \( \Delta^+/K \cong G^+/N \cong T \), which is cyclic and hence, a fortiori is abelian. Next, if \( \Delta = \langle s_1, s_2, s_3 \rangle \), then we have \( \Delta^+ = \langle s_2s_3, s_1s_3 \rangle \cong G^+ \) via \( s_2s_3 \mapsto P, s_1s_3 \mapsto L \). Therefore, we see that conjugation by \( s_3 \) will invert both \( s_2s_3 \) and \( s_1s_3 \). Since \( \Delta^+/K \) is abelian, we see that conjugation by \( s_3 \) will induce the automorphism given by inversion of each element; in particular, it follows that \( s_3 \) must
normalize the subgroup $K \leq \Delta^+$. This proves already that $K \leq \Delta$, from which it follows immediately that $K \geq HH^*$, and so $X(M)$ is isomorphic to a subgroup of $N$. On the other hand, it is routine to show that $N$ is a minimal normal subgroup of $G^+$; since we have already observed above that $X(M)$ can be identified with a normal subgroup of $G^+$, it follows that we must have equality, i.e., $X(M_t) \cong N \cong \mathbb{F}^+$.

We summarize as follows:

**Theorem 3.3.3.** For the orientably regular maps $M = M_t(\mathbb{F})$, where $\mathbb{F}$ is the finite field of $q = p^e$ elements, we have

(i) if $q \leq 4$, then $M$ is reflexible;

(ii) if $q > 4$, then $M$ is chiral, with chirality group $X(M) \cong \mathbb{F}^+$, which is elementary abelian of order $q = p^e$.

**Definition 3.3.2.** The orientably regular map $M$ is called *totally chiral* if $H^\Delta = \Delta^+$; this is equivalent to saying that $X(M) \cong G^+$, where $G^+$ is the oriented monodromy group of $M$.

The following is routine.

**Proposition 3.3.4.** If the orientably regular map $M$ is totally chiral, then its monodromy group $G^+$ is perfect.

**Proof.** Let $\Delta^+ \to G^+$ be the canonical surjection, with kernel $H$ and let $K = [G^+, G^+]$ be the commutator subgroup of $G^+$. If $\widetilde{K} \leq \Delta^+$ maps to $K$, then $\widetilde{K}$ is characteristic in $\Delta^+$, hence is normal in $\Delta$. It follows immediately that $H^\Delta \leq \widetilde{K}$, contrary to the fact that $M$ is totally chiral.

The following example gives a family of totally chiral maps.

**Theorem 3.3.5.** For $n \geq 4$ there is a totally chiral map with monodromy group isomorphic to $A_{2n}$. 
Proof. Let $P = (1, 2, \ldots, 2n-1)$ and let $L = (12n)(23)$. It is clear that the group $\langle P, L \rangle$ is doubly transitive on the set $\{1, 2, \ldots, 2n\}$, hence is certainly primitive. Next, a direct computation shows that $P^{-1}LPL = (12n3^42)$, and so $\langle P, L \rangle$ contains a 5-cycle. Now a theorem of Jordan\(^6\) states that a primitive group of degree $m$ and containing a $p$-cycle for some prime $p \leq m-3$ must contain $A_m$. In the present context, this guarantees that $\langle P, L \rangle = A_{2n}$. Since all automorphisms of $A_{2n}$ are induced by conjugation by elements of $S_{2n}$, and since clearly no element of $S_{2n}$ can invert $P$ and at the same time centralize $L$, it follows that the orientably regular map $\mathcal{M} = (A_{2n}, P, L)$ is not reflexible. However, the chirality group of $\mathcal{M}$ is isomorphic to a normal subgroup of $A_{2n}$, and hence must be trivial (as $A_{2n}$ is simple). Therefore, the only possibility for $\mathcal{M}$ is to be totally chiral.

3.4 The Double Cover Functor NMap $\rightarrow$ OMap

Let $\mathcal{M} = (B, a, b, c)$ be a non-orientable map, with monodromy group $G = \langle a, b, c \rangle$, and subgroup $G^+ = \langle bc, ac \rangle$. If $k, l$ are positive integers satisfying $(ab)^k = 1 = (bc)^l$ then for some blade $\beta \in B$, we may form the orbifold fundamental group $H = \pi_1^{(k, l)}(\mathcal{M}, \beta) \leq \Delta = \Delta(k, l)$. If we set $H^+ = H \cap \Delta^+$, then we may identify $\mathcal{M}$ with $(\Delta/H, s_1, s_2, s_3)$ and form the map $\tilde{\mathcal{M}} = (\Delta/H^+, s_1, s_2, s_3)$; furthermore, the natural mapping $\Delta/H^+ \rightarrow \Delta/H$ gives rise to the mapping $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$.

As $\mathcal{M}$ is non-orientable, $H \nless \Delta^+$ and so we see that this map is two-to-one. The covering $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$ so obtained is called the orientable double cover of the non-orientable map $\mathcal{M}$.

A much more direct, but equivalent construction can be obtained as follows. Given the non-orientable map $\mathcal{M} = (B, a, b, c)$, we define, as on page 32, the corresponding connected oriented map $\mathcal{M}^+ = (B^+, bc, ac)$ ($B^+ = B$), and correspondingly the connected map $\tilde{\mathcal{M}} = \mathcal{M}^{+\sharp} = (B^\sharp, a^\sharp, b^\sharp, c^\sharp)$ as on page 31. Therefore, $\mathcal{M}^{+\sharp} = (B^\sharp, a^\sharp, b^\sharp, c^\sharp)$, where $B^\sharp = B \times \{\pm 1\}$, and

$$a^2(\beta, \zeta) = (ac\beta, -\zeta)$$
$$b^2(\beta, \zeta) = ((bc)^c\beta, -\zeta)$$
$$c^2(\beta, \zeta) = (\beta, -\zeta).$$

The mapping $\widetilde{M} \to M$ given by $(\beta, \zeta) \mapsto c^{(1+\zeta)/2}\beta$ realizes $\widetilde{M} \to M$ as a double cover of $M$. Furthermore, the map $\widetilde{M}$ is orientable and has orientations $B \times \{1\}$ and $B \times \{-1\}$. Note that $(\cdot) : \text{NMap} \to \text{OMap}$ is clearly functorial.

**Remark.** Combining the above two remarks, we see that if $M$ is regular, so is $\widetilde{M} \to M$, but that the converse fails.

**Definition 3.4.1 (Antipodal Maps).** The orientable map $M$ is called antipodal if and only if $M$ is the orientable double cover of a non-orientable map.

**Lemma 3.4.1.** Let $M$ be an orientable map. If $M$ is antipodal, it is reflexible.

**Proof.** Let $M$ be the orientable double cover of a non-orientable map $\mathcal{N} = (B, a, b, c)$. Thus we may write $M = \widehat{\mathcal{N}} = (\widehat{B}, \widehat{a}, \widehat{b}, \widehat{c})$, where $\widehat{B} = B \times \{\pm 1\}$, and $\widehat{a}, \widehat{b}, \widehat{c}$ are given by

$$\widehat{a}(\beta, \zeta) = (ac\beta, -\zeta)$$
$$\widehat{b}(\beta, \zeta) = ((bc)^c\beta, -\zeta)$$
$$\widehat{c}(\beta, \zeta) = (\beta, -\zeta).$$

The orientations of $M = \widehat{\mathcal{N}}$ are given by $\widehat{B}^+ = B \times \{1\}$, and $\widehat{B}^- = B \times \{-1\}$, and the map $(\beta, \zeta) \mapsto (c\beta, -\zeta)$ is an automorphism of $M$ that maps $\widehat{B}^+$ to $\widehat{B}^-$. 
Remark. The converse of the above is patently false. To see this, one need only consider the regular orientable map $\mathcal{M} = (B, a, b, c)$ of the torus, consisting of 8 blades, depicted as below:

If $\mathcal{M} = \tilde{\mathcal{N}}$ for some nonorientable map $\mathcal{N}$, then the automorphism group $\text{Aut}(\mathcal{M})$ ($\cong D_8$) must have a central orientation-reversing automorphism. A moment’s thought reveals that this is impossible.

The following result is really a corollary to the construction of the orientable double cover of a non-orientable map.

Corollary 3.4.1.1. Let $\mathcal{M}$ be an antipodal map. Then there exists $\tau \in G = \langle a, b, c \rangle$, such that $\tau$ inverts $bc$ and centralizes $a$ and $c$.

Proof. Indeed, write $\mathcal{M} = \tilde{\mathcal{N}} = (\tilde{B}, \tilde{a}, \tilde{b}, \tilde{c})$, as above and note that setting $\tau = \tilde{c}$ does the job.

The following is well-known.\footnote{See Nedela and Škoviera, op. cit., Theorem 7.2. See also P. Bergau and D. Garbe, Non-orientable and orientable regular maps, Proceedings of Groups – Korea 1988 (Pusan, August 1988), Lecture Notes in Mathematics 1398 (Springer, Berlin, 1989), pp. 29-42, Lemma 1.}

Theorem 3.4.2. Let $\mathcal{M} = (B, a, b, c)$ be a regular orientable map, with monodromy group $G = \langle a, b, c \rangle$. Then $\mathcal{M}$ is antipodal if and only if $G = G^+ \times \langle \tau \rangle$ for some central involution $\tau \in G$.

Proof. If $\mathcal{M} = \tilde{\mathcal{N}} = (\tilde{B}, \tilde{a}, \tilde{b}, \tilde{c})$ as above, then the mapping $\tau : (\beta, \zeta) \mapsto (c\beta, -\zeta)$ is central in $G = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle$, and $G = G^+ \times \langle \tau \rangle$, as
required. Conversely, let \( \tau \in G \) be given, satisfying the requirements. Set \( N = (\overline{B}, \overline{a}, \overline{b}, \overline{c}) \), where \( \overline{B} = B/\langle \tau \rangle \), and where \( \overline{a}, \overline{b}, \overline{c} : \overline{B} \to \overline{B} \) are determined by \( a, b, c : B \to B \). Thus, \( N \) is regular, as \( \tau \in G \) is central, and is non-orientable, as \( \tau \not\in G^+ \). Since \( M \) is orientable, we may write \( B = B^+ \cup B^- \), where by regularity and orientability, \( B^+ \) and \( B^- \) are the \( G^+ \)-orbits in \( B \), and, by assumption \( \tau : B^+ \to B^- \) is a bijection commuting with the \( G^+ \)-actions on each. If we form the orientable double cover \( \tilde{N} = (\overline{B}, \tilde{a}, \tilde{b}, \tilde{c}) \) as usual, then the mapping \( B \to \tilde{B} = \overline{B} \times \{ \pm 1 \} \) given by

\[
\tau^i \beta \mapsto (\tilde{c}^{(1-i)/2} \tilde{\beta}, (-1)^i), \quad \beta \in B^+,
\]

can be checked to provide an isomorphism \( M \to \tilde{N} \).

**Proposition 3.4.3.** Let \( N \) be a non-orientable map, with automorphism group \( A \). Let \( M = \tilde{N} \) be the orientable double cover of \( N \), with automorphism group \( \tilde{A} \), and group of orientation-preserving automorphisms \( \tilde{A}^+ \). Then there is an injection \( A \to \tilde{A}^+ \).

We write \( N = (B, a, b, c) \), and identify \( M \) with the map \( (B \times \{ \pm 1 \}, a^i, b^i, c^i) \) as above. If \( \phi \in A \), it is routine to verify that the map \( (\beta, \zeta) \mapsto (\beta \phi, \zeta) \) is in \( \tilde{A}^+ \).

**Exercise.** Let \( M \) be a non-orientable map and let \( \tilde{M} \to M \) be the orientable double cover of \( M \). If \( g \) is the genus of \( M \) and \( \tilde{g} \) is the genus of \( \tilde{M} \), show that \( \tilde{g} = g - 1 \).
Chapter 4

Euler Characteristic and Genus

4.1 Euler Characteristic and Genus

The results of this subsection are due to R. Cori and A. Machi.\footnote{Maps, hypermaps and their automorphisms: a survey. I, Exposition. Math. vol. 10 (1992), no. 5, 403-427. See especially pp. 419-422.}

**Definition 4.1.1.** Let $B$ be a set and assume that $\sim_1$, $\sim_2$ are equivalence relations on $B$ with corresponding partitions $X_1, X_2$. We say that the triple $(B, X_1, X_2)$ is connected if for any pair of elements $b, b' \in B$ there is a sequence

$$b = b_0 \sim_{i_1} b_1 \sim_{i_2} b_2 \cdots \sim_{i_k} b_k = b',$$

where $i_j \in \{1, 2\}$ for all $j$.

The following is virtually obvious.

**Lemma 4.1.1.** Assume that $\sim_1$, $\sim_2$ are equivalence relations on $B$ with corresponding partitions $X_1, X_2$, and assume that $(B, X_1, X_2)$ is connected. If $X_1$ consists of more than one equivalence class, then there exist elements $b, b' \in B$ such that

(i) $b \not\sim_1 b'$,

(ii) $b \sim_2 b'$.
Lemma 4.1.2. Let $(B, X_1, X_2)$ be connected and assume that $B$ is finite with $|B| = n$. Then

$$|X_1| + |X_2| \leq n + 1.$$  

Proof. We shall argue by induction of $|X_1|$. Note that if $|X_1| = 1$, then since $|X_2| \leq n$, the result follows in this case. Next, if $|X_1| > 1$, we apply Lemma 4.1.1 and find elements $b, b' \in B$ such that $b \not\sim_1 b'$ and $b \sim_2 b'$. Form new partitions $X'_1, X'_2$ of $B$ as follows:

(a) If $[b], [b']$ are the equivalence classes of $X_1$ containing $b, b'$, respectively, then declare that $[b] \cup [b']$ is an equivalence class of $X'_1$. The remaining equivalence classes of $X_1$ continue to be equivalence classes in $X'_1$. (Therefore $X'_1$ has one fewer equivalence class than does $X_1$.)

(b) Let $x_2$ be the equivalence class of $X_2$ containing $b, b'$, and arbitrarily decompose $x_2$ into two disjoint subsets $x_2 = y_2 \cup y'_2$ such that $b \in y_2$ and $b' \in y'_2$. Declare that $y_2, y'_2$ are equivalence classes of $X'_2$. The remaining equivalence classes of $X'_2$ are those of $X_2$. (Therefore $X'_2$ has one more equivalence class than does $X_2$.)

Claim. $(B, X'_1, X'_2)$ is connected.

Proof of Claim. Let $\sim'_1, \sim'_2$ be the equivalence relations induced by $X'_1$ and $X'_2$. Note that if $c, c' \in B$, then $c \sim_1 c'$ clearly implies that $c \sim'_1 c'$. Also, $c \sim_2 c'$, implies that $c \sim_1 c'$ unless $[c]_2' \cup [c']_2' = [b]_2 \cup [b']_2'$, where $[x]_2'$ denotes the equivalence class in $X'_2$ containing $x$. Therefore, it suffices to prove that we can join such a pair $c, c'$, with a sequence of $\sim'_1, \sim'_2$. But this is true if we can join $b, b'$ by such a sequence; note that this latter statement is obvious as $b \sim'_1 b'$, proving the claim.

Therefore, we can apply induction to conclude that $|X'_1| + |X'_2| \leq n + 1$. Since $|X'_1| = |X_1| - 1$, and $|X'_2| = |X_2| + 1$, we’re finished.

Next, if $\sigma$ is a permutation on the finite set $B$, we denote by $z(\sigma)$ the number of cycles of $\sigma$ on $B$. Equivalently, $z(\sigma)$ is the number of $\langle \sigma \rangle$-orbits in $B$. 
The following is routine:

**Lemma 4.1.3.** Let $\sigma$ be a permutation on the finite set $B$, and let $\tau = (i \, j)$ be a transposition on $B$. Then

$$z(\tau \sigma) = \begin{cases} z(\sigma) + 1 & \text{if } i, j \text{ are in the same } \sigma\text{-cycle} \\ z(\sigma) - 1 & \text{if } i, j \text{ are not in the same } \sigma\text{-cycle.} \end{cases}$$

The next result involves the following setup. We are given a finite set $B$ acted on by permutations $\sigma$ and $\alpha$. The cycles of these permutations induce partitions $X_\sigma, X_\alpha$ of $B$, and it’s easy to see that $(B, X_\sigma, X_\alpha)$ is connected precisely when $\langle \sigma, \alpha \rangle$ acts transitively on $B$.

**Proposition 4.1.4.** Under the assumption that $\langle \sigma, \alpha \rangle$ acts transitively on the finite set $B$, where $|B| = n$, we have

$$z(\sigma) + z(\alpha) + z(\sigma \alpha) \leq n + 2.$$  

**Proof.** We argue by induction on $z(\sigma)$. If $z(\sigma) = 1$, then since $\langle \sigma, \alpha \rangle = \langle \alpha, \sigma \alpha \rangle$, we have, by Lemma 4.1.2 that

$$z(\alpha) + z(\sigma \alpha) \leq n + 1.$$  

Therefore, the result follows in this case. Next, let $z(\sigma) > 1$ and note that $\langle \sigma, \sigma \alpha \rangle = \langle \sigma, \alpha \rangle$; by Lemma 4.1.1 there are elements $i, j \in B$ in the same $\sigma \alpha$-cycle but in different $\sigma$-cycles. Therefore, if $\tau = (i \, j)$, then Lemma 4.1.3 tells us that

$$z(\tau \sigma) = z(\sigma) - 1, \quad z(\tau \sigma \alpha) = z(\sigma \alpha) + 1.$$  

Note that since the cycles of $\sigma$ are contained in the cycles of $\tau \sigma$ (direct verification), and since $\langle \sigma, \alpha \rangle$ is transitive, it follows that $\langle \tau \sigma, \alpha \rangle$ is also transitive. It follows by induction that

$$z(\tau \sigma) + z(\alpha) + z(\tau \sigma \alpha) \leq n + 2.$$  

Since we’ve already seen that $z(\tau \sigma) = z(\sigma) - 1$ and that $z(\tau \sigma \alpha) = z(\sigma \alpha) + 1$, the result follows.
Now assume that $\mathcal{M} = (D, P, L)$ is a connected oriented map. Thus the numbers of vertices, edges, and faces of $\mathcal{M}$ are given, respectively, by the numbers $z(P), z(L),$ and $z(PL).$ Note also that $|D| = 2z(L)$ and that the Euler characteristic of $\mathcal{M}$ is given by

$$\chi(\mathcal{M}) = z(P) - z(L) + z(PL).$$

As an immediate corollary to Proposition 4.1.4, we have

**Corollary 4.1.4.1.** For the connected oriented map $\mathcal{M} = (D, P, L),$ we have

$$\chi(\mathcal{M}) \leq 2.$$

Finally, note that if $\sigma$ is a permutation acting on the finite set $B$ or order $n,$ then the parity $\text{sgn}(\sigma)$ of $\sigma$ is given by

$$\text{sgn}(\sigma) = (-1)^{n - z(\sigma)}.$$

**Theorem 4.1.5.** Let $\mathcal{M} = (D, P, L)$ be the oriented map and let $\chi(\mathcal{M})$ be the Euler characteristic. Then there exists a non-negative integer $g$ such that

$$\chi(\mathcal{M}) = 2 - 2g.$$ 

**Proof.** By Corollary 4.1.4.1 it suffices to prove that the Euler characteristic is even. Note that of the permutations $P, L, PL$ either one of these is an even permutation or all three are even permutations. Therefore, if $|D| = n,$ then the quantity

$$(n - z(P)) + (n - z(L)) + (n - z(L))$$

is an even integer. On the other hand, since $n = 2z(L),$ we see that the above quantity is equal to $2n - \chi(\mathcal{M}).$ The result follows.

**Definition 4.1.2 (Genus of an Oriented Map).** The number $g$ in Theorem 4.1.5 is called the genus of $\mathcal{M}.$
**Definition 4.1.3.** (Genus of a Non-orientable Map) In case the map $\mathcal{M}$ is non-orientable, define the genus $g = g(\mathcal{M})$ through the equation $\chi(\mathcal{M}) = 2 - g$.

**Example 4.1.** The diagram to the right depicts the standard embedding of the complete graph $K_5$ into $\mathbb{R}P^2$, the real projective plane. One quickly computes that the resulting nonorientable map has genus 1.

### 4.2 The Hurwitz 84($g-1$) Theorem and the Riemann-Hurwitz Formula

Let $\mathcal{M} = (B, a, b, c)$ be a connected regular map, and let $\text{Aut}(\mathcal{M})$ be its automorphism group. We have already seen that $|\text{Aut}(\mathcal{M})| \leq |B|$ with equality if and only if $\mathcal{M}$ is regular. One has the following result, due originally to Hurwitz in the context of compact Riemann surfaces:²

**Theorem 4.2.1.** If $\mathcal{M}$ is an oriented map, then $|\text{Aut}(\mathcal{M})| \leq 84(g - 1)$.

**Proof.** By the work of Jones and Singerman³ one has that $\text{Aut}(\mathcal{M})$ is isomorphic to a group of conformal automorphisms of a compact Riemann surface, also of genus $g$. Thus, the result follows by Hurwitz’s original theorem.

If one is content to assume that $\mathcal{M} = (D, P, L)$ is uniform, say with vertex valency $l$ and face valency $k$, and if $\mathcal{M}$ has $V$ vertices, $E$ edges

---


and $F$ faces, then

$$2 - 2g = V - E + F = |D|\left(\frac{1}{l} - \frac{1}{2} + \frac{1}{k}\right),$$

and so

$$|\text{Aut}(\mathcal{M})| \leq |D| = \frac{2(g - 1)}{\frac{1}{2} - \frac{1}{l} - \frac{1}{k}}.$$

One then shows that the minimum positive value of

$$\frac{1}{2} - \frac{1}{l} - \frac{1}{k}$$

occurs for $\{k, l\} = \{3, 7\}$, in which case

$$\frac{1}{2} - \frac{1}{l} - \frac{1}{k} = \frac{1}{42},$$

and the result follows.

**Theorem 4.2.2.** Let $\mathcal{M}$ be a connected map. Then one has the upper bounds on the automorphism group of $\mathcal{M}$:

1. If $\mathcal{M}$ is orientable, then $|\text{Aut}(\mathcal{M})| \leq 168(g - 1)$;

2. If $\mathcal{M}$ is non-orientable, then $|\text{Aut}(\mathcal{M})| \leq 84(g - 2)$.

**Proof.** Part (1) is obvious as the group of orientation-preserving automorphisms of $\mathcal{M}$ has index at most two in $\text{Aut}(\mathcal{M})$. If $\mathcal{M}$ is non-orientable, let $\widetilde{\mathcal{M}}$ be the orientable double cover of $\mathcal{M}$. From Proposition 3.4.3 we have that $|\text{Aut}(\mathcal{M})| \leq |\text{Aut}(\widetilde{\mathcal{M}})^+| \leq 84(\tilde{g} - 1)$, where $\tilde{g}$ is the genus of $\widetilde{\mathcal{M}}$. Since $\tilde{g} = g - 1$ (apply the exercise on page 44), we're done.

Maps for which the above upper bounds obtain are called extremal maps and have attracted much attention.

We turn now to the Riemann-Hurwitz formula. This relates the genus of the map $\mathcal{M}$ to the genus of the orbit map $\mathcal{M}/K$, where $K$ is a subgroup of $\text{Aut}(\mathcal{M})$; see page 6.
We begin with a general result. Let $X$ be a set and let $A$, $B$ be groups acting on $X$, with $A$ acting on the left, $B$ acting on the right and with the actions commuting:

$$a(xb) = (ax)b,$$

for all $a \in A$, $x \in X$, $b \in B$.

**Lemma 4.2.3.** With the hypotheses as above, then

(i) $A$ acts on the $B$-orbits in $X$;

(ii) $B$ acts on the $A$-orbits in $X$;

(iii) The number of orbits of the action of $A$ on $X/B$ is the same as the number of orbits of the action of $B$ on $A\backslash X$.

**Proof.** Parts (i), (ii) are routine. For part (iii), we let $B^{\text{opp}}$ be the group having underlying set $B$, but "opposite multiplication," i.e., $b \cdot b' := b' b$, $b, b' \in B$. Thus, $B^{\text{opp}}$ acts on $X$ on the left via $b \cdot x = xb$, $b \in B$, $x \in X$. Clearly the orbits of $B^{\text{opp}}$ on $X$ coincide with the orbits of $B$ on $X$. Now define an action of $A \times B^{\text{opp}}$ on $X$ via $(a, b) \cdot x = axb$, $a \in A$, $x \in X$, $b \in B$. One easily checks that

the number of $A$ - orbits on $X/B = \text{the number of } A \times B^{\text{opp}} - \text{orbits on } X = \text{the number of } B - \text{orbits on } A\backslash X$.

Next, if $G$ is a group acting on the finite set $X$, we denote by $\text{Fix}(g)$ the set of elements of $X$ fixed by the element $g \in G$. We have the well-known *Burnside formula*:

**Theorem 4.2.4.** Let $G$ be a group acting on the finite set $X$. Then the number $N$ of orbits of $G$ in $X$ is given by

$$N = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$
Proof. First of all, note that it suffices to assume \( G \) acts transitively on \( X \), so that \( N = 1 \). Therefore, all stabilizers of elements of \( X \) have the same order, viz., \(|G|/|X|\). Next, let \( \Omega = \{(g, x) \in G \times X \mid gx = x\} \). Then

\[
|G| = \frac{|G|}{|X|} |X| = \sum_{x \in X} |\text{Stab}_G(x)| = |\Omega| = \sum_{g \in G} |\text{Fix}(g)|,
\]

and so the result follows.

Now let \( \mathcal{M} = (B, a, b, c) \) be a map and let \( K \leq \text{Aut}(\mathcal{M}) \). We denote the quotient map by \( \mathcal{M}/K = (B/K, \hat{a}, \hat{b}, \hat{c}) \). Let \( V, E, F \) be, respectively the vertices, edges and faces in \( \mathcal{M} \); similarly, let \( \hat{V}, \hat{E}, \hat{F} \) be the vertices, edges and faces in \( \mathcal{M}/K \). By Lemma 4.2.3, the number of orbits of \( K \) on \( V \) is the number of orbits of \( \langle b, c \rangle \) on \( B/K \), i.e., is the number \( |\hat{V}| \) of vertices in \( \mathcal{M}/K \). On the other hand, by Burnside’s formula, the number of orbits of \( K \) on \( V \) is given by \( \frac{1}{|K|} \sum_{k \in K} |\text{Fix}_V(k)| \), where \( \text{Fix}_V(k) \) is the set of fixed vertices of \( k \) in \( V \). Therefore, we have the following equality:

\[
|\hat{V}| = \sum_{k \in K} |\text{Fix}_V(k)|.
\]

Since \( |\text{Fix}_V(1)| = |V| \), we write this as

\[
|\hat{V}| = \frac{|V|}{|K|} + \frac{1}{|K|} \sum_{1 \neq k \in K} |\text{Fix}_V(k)|.
\]

In an entirely similar fashion, we conclude that

\[
|\hat{E}| = \frac{|E|}{|K|} + \frac{1}{|K|} \sum_{1 \neq k \in K} |\text{Fix}_E(k)|,
\]

and that

\[
|\hat{F}| = \frac{|F|}{|K|} + \frac{1}{|K|} \sum_{1 \neq k \in K} |\text{Fix}_F(k)|.
\]
This can be expressed as

$$|\hat{V}| + |\hat{E}| + |\hat{F}| = \frac{|V| + |E| + |F|}{|K|} + \frac{1}{|K|} \sum_{1 \neq k \in K} |\text{Fix}(k)|,$$

where in the above sum, $\text{Fix}(k)$ is the set of fixed vertices, edges and faces of $k$ in $\mathcal{M}$.

Next, if $g = g(\mathcal{M})$, $\hat{g} = g(\mathcal{M}/K)$, then we have seen that

$$|V| + |E| + |F| = \chi(\mathcal{M}) + |B|/2;$$

similarly

$$|\hat{V}| + |\hat{E}| + |\hat{F}| = \chi(\mathcal{M}/K) + |B/K|/2$$

$$= \chi(\mathcal{M}/K) + |B|/2|K|.$$

It follows, therefore, that

$$|K|\chi(\mathcal{M}/K) = \chi(\mathcal{M}) = \sum_{1 \neq k \in K} |\text{Fix}(k)|.$$ 

Therefore, we have proved the Riemann-Hurwitz formula:

**Theorem 4.2.5.** Let $\mathcal{M}$ be a map, let $K \leq \text{Aut}(\mathcal{M})$, and let $\mathcal{M}/K$ be the quotient map. Then the genera $g$, $\hat{g}$ of $\mathcal{M}$, $\mathcal{M}/K$, respectively, are related by the formula

$$2g - 2 = |K|(2\hat{g} - 2) + \sum_{1 \neq k \in K} |\text{Fix}(k)|,$$

where $\text{Fix}(k)$ is the set of vertices, edges and faces in $\mathcal{M}$ left fixed by the element $k \in K$. 

Chapter 5

Derived Maps

5.1 Voltages and Derived Maps in Map

**Definition 5.1.1 (Cochains and Voltages).** Let $\mathcal{M} = (B, a, b, c)$ be a map, and let $Z$ be a group. A function $z : B \rightarrow Z$ is called a $Z$-valued cochain; if, in addition $z(a\beta) = z(\beta)^{-1}$ and $z(c\beta) = z(\beta)$ for any $\beta \in B$, we call $z$ a $Z$-valued voltage on $\mathcal{M}$. We denote the set of voltages on $\mathcal{M}$ by $C(\mathcal{M}; Z)$ and frequently write $z_\beta$ in place of $z(\beta)$.

**Definition 5.1.2 (Principal Derived Maps).** Let $\mathcal{M} = (B, a, b, c)$ be a map, $Z$ be a group, and let $z \in C(\mathcal{M}; Z)$. We define the map $\mathcal{M}_z = (B \times Z, a_z, b_z, c_z)$ by setting

$$a_z(\beta, \zeta) = (a\beta, \zeta z\beta), \quad b_z = b \times 1_Z \quad \text{and} \quad c_z = c \times 1_Z.$$  

Since $z$ is a voltage, it follows that $a_z$ and $a_z c_z$ are both involutions. We call $\mathcal{M}_z$ the principal derived map corresponding to the voltage $z \in C(\mathcal{M}; Z)$. Note that projection onto the first coordinate, $(\beta, \zeta) \mapsto \beta$, defines a morphism $\pi : \mathcal{M}_z \rightarrow \mathcal{M}$. Furthermore, for each $\zeta_0 \in Z$ the mapping on $\mathcal{M}_z$ defined by $(\beta, \zeta)l_{\zeta_0} = (\beta, \zeta_0^{-1}\zeta)$ is an automorphism in $A(\mathcal{M}_z/\mathcal{M})$, and the mapping $\zeta_0 \mapsto l_{\zeta_0}$ defines an injective homomorphism $Z \rightarrow A(\mathcal{M}_z/\mathcal{M})$. Therefore, we see that $A(\mathcal{M}_z/\mathcal{M})$ acts transitively on each fibre $\beta \pi^{-1}$, $\beta \in B$, and so $\mathcal{M}_z \rightarrow \mathcal{M}$ is a regular covering. Furthermore, if $\mathcal{M}_z$ is connected, it follows that $A(\mathcal{M}_z/\mathcal{M}) \cong Z$ via $\zeta \mapsto l_\zeta$, $\zeta \in Z$. 

55
Note that the morphism $\pi : \mathcal{M}_z \to \mathcal{M}$ is a regular covering that is unramified over vertices. The converse is also true:

**Theorem 5.1.1.** If $p_0 : \mathcal{M}_0 = (B_0, a_0, b_0, c_0) \to \mathcal{M} = (B, a, b, c)$ is a regular connected covering that is unramified over vertices, then $\mathcal{M}_0 \cong \mathcal{M}_z$ for some voltage $z \in C(\mathcal{M}; Z)$.

**Proof.** First of all, we set $Z = \Lambda(\mathcal{M}_0/\mathcal{M})$, and so $Z$ acts regularly on each fibre $B_0^{-1} \subseteq B_0$. We shall construct a voltage $z \in C(\mathcal{M}; Z)$ and an isomorphism $\phi : \mathcal{M}_0 \cong \mathcal{M}_z$, making the diagram below commute:

$$
\begin{array}{ccc}
\mathcal{M}_0 & \xrightarrow{\phi} & \mathcal{M}_z \\
\downarrow{p_0} & & \downarrow{z} \\
\mathcal{M} & & \\
\end{array}
$$

For each vertex $v$ in $\mathcal{M}$, fix a vertex $v_0$ in $\mathcal{M}_0$ with $v_0p_0 = v$. Since $p_0 : \mathcal{M}_0 \to \mathcal{M}$ is unramified over vertices, $p_0|_{v_0} : v_0 \to v$ is bijective, and so we may invert this and obtain a mapping

$$
\sigma_v : v \to v_0
$$

such that $\sigma_v p_0 = 1_v$. Since $B$ is the disjoint union of its vertices, we may define a “section” $\sigma : B \to B_0$ where $\sigma|_v = \sigma_v$, and where $v$ ranges over the vertices of $\mathcal{M}$. For any $\beta_0 \in B_0$, define the element $\zeta_0 \in Z$ by the requirement that $\beta_0p_0\sigma = \beta_0\zeta_0$. By regularity, $\zeta_0$ is uniquely defined. Furthermore, $\zeta_0$ depends only on the vertex in $\mathcal{M}_0$ determined by $\beta_0$, i.e., $b_0\beta_0p_0\sigma = b_0\beta_0\zeta_0$ and $c_0\beta_0p_0\sigma = c_0\beta_0\zeta_0$. We now define

$$
\phi : B_0 \to B \times Z \quad \text{by} \quad \beta_0\phi = (\beta_0p_0, \zeta_0).
$$

Let $\beta \in B$ and let $\beta_0 \in B_0$ with $\beta_0p_0 = \beta$. Let $\beta_0\phi = (\beta_0p_0, \zeta_0)$, $(a_0\beta_0)\phi = (a_0\beta_0p_0, \zeta_{a_0})$. Define $z_\beta = \zeta_0^{-1}\zeta_{a_0}$. We show that $z_\beta$ is well defined. If $\beta'_0 \in B_0$ with $\beta'_0p_0 = \beta$, let $\beta'_0\phi = (\beta'_0p_0, \zeta'_0)$, $(a_0\beta'_0)\phi = (a_0\beta'_0p_0, \zeta'_{a_0})$, and let $\mu_0 \in Z$ satisfy $\beta'_0 = \beta_0\mu_0$. Therefore, $\beta_0p_0\sigma = \beta_0p'_0\sigma = \beta_0\mu_0\zeta'_0$.}
\( \beta_0 \zeta_0 \), which implies that \( \beta'_0 \zeta'_0 = \beta'_0 p_0 \sigma = \beta_0 \mu_0 p_0 \sigma = \beta_0 p_0 \sigma = \beta_0 \zeta_0 = \beta_0 \mu_0 \cdot \mu_0^{-1} \zeta_0 = \beta'_0 \mu_0^{-1} \zeta_0 \) and so \( \zeta'_0 = \mu_0^{-1} \zeta_0 \) by regularity. Likewise, setting \( a_0 \beta_0 p_0 \sigma = a_0 \beta_0 \zeta_0 \alpha, a_0 \beta'_0 p_0 \sigma = a_0 \beta'_0 \zeta'_0 \alpha, \) we get \( a_0 \beta'_0 \zeta'_0 \alpha = (a_0 \beta'_0) p_0 \sigma = a_0 \beta_0 p_0 \sigma = a_0 \beta_0 \zeta_0 \alpha = a_0 \beta_0 \mu_0 \cdot \mu_0^{-1} \zeta_0 \alpha = a_0 \beta'_0 \mu_0^{-1} \zeta_0 \alpha, \) and so \( \zeta'_0 \alpha = \mu_0^{-1} \zeta_0 \alpha. \) Therefore, \( \zeta'_0^{-1} \zeta'_0 \alpha = \zeta_0^{-1} \mu_0 \mu_0^{-1} \zeta_0 \alpha = \zeta_0^{-1} \zeta_0 \alpha \) and so \( z_{\beta} \) is well defined.

Next, we show that the mapping \( \phi : B_0 \to B \times Z \) determines a morphism \( \mathcal{M}_0 \to \mathcal{M}_z. \) First of all, for \( \beta_0 \in B_0, \) we have

\[
(a_0 \beta_0) \phi = ((a_0 \beta_0) p_0, \zeta_0 \alpha) \\
= (a(\beta_0 p_0), \zeta_0 \alpha) \\
= (a(\beta_0 p_0), \zeta_0 z_{\beta}) \quad (\beta = \beta_0 p_0) \\
= a_z(\beta_0 p_0, \zeta_0) \\
= a_z(\beta_0 \phi).
\]

If \( \beta_0 \in B_0, \) then, as observed above, and so \( b_0 \beta_0 p_0 \sigma = b_0 \beta_0 \zeta_0, \) we have

\[
(b_0 \beta_0) \phi = ((b_0 \beta_0) p_0, \zeta_0) \\
= (b(\beta_0 p_0), \zeta_0) \\
= b_z(\beta_0 p_0, \zeta_0) \\
= b_z(\beta_0 \phi).
\]

Similarly \( (c_0 \beta_0) \phi = c_z(\beta_0 \phi). \) It follows, therefore, that \( \phi \) determines an isomorphism \( \phi : \mathcal{M}_0 \xrightarrow{\cong} \mathcal{M}_z. \)

**Definition 5.1.3 (Voltage Equivalence).** We say that voltages \( z, z' \in C(M; Z) \) are equivalent, and write \( z \sim z' \), if there exists a cochain \( f : B \to Z \), which is constant-valued on vertices, satisfying

\[
f(\beta) z'_\beta f(a_\beta)^{-1} = z_\beta
\]

for all \( \beta \in B \).

The following is basic but important.
Lemma 5.1.2. If \( z, z' \in C(M; Z) \) are equivalent voltages, then \( M_z \cong_{M} M_{z'} \).

Proof. Let \( f : B \to Z \) be constant-valued on vertices and satisfying
\[
f(\beta)z'_\beta f(a\beta)^{-1} = z_\beta
\]
for all \( \beta \in B \), where \( B \) is the set of blades in \( M \). Now define
\[
\phi : B \times Z \to B \times Z, \quad (\beta, \zeta) \mapsto (\beta, \zeta \cdot f(\beta)).
\]
That this defines an \( M \)-isomorphism \( M_z \to M_{z'} \) is entirely routine to check.

The converse of the above lemma is easily seen to be false. Indeed, if \( Z \) is an abelian group and if \( z \in C(M; Z) \) then it is easy to check that \( M_z \cong_{\text{Aut}(M)} M_{z^{-1}} \), but that \( z \not\cong z' \). We shall return to this phenomenon shortly.

Definition 5.1.4 (Functoriality). If \( \phi : M \to M' \) is a morphism of maps, then for any group \( Z \), \( \phi \) induces a mapping
\[
\phi^* = C(\phi, 1_Z) : C(M'; Z) \to C(M; Z),
\]
where if \( z' \in C(M'; Z) \), and if \( \beta \) is a blade in \( M \), then \( (\phi^* z')_\beta = z'_{\beta \phi} \in Z \). Similarly, if \( Z, Z' \) are groups and if \( \alpha : Z \to Z' \) is a homomorphism of groups (written exponentially: \( x \mapsto x^\alpha \)), then
\[
\alpha_* = C(1_M, \alpha) : C(M; Z) \to C(M; Z')
\]
is given by \( (z\alpha)_\beta = (z_\beta)^\alpha \in Z' \), where \( \beta \) is a blade in \( M \). Note that if \( z'_1 \sim z'_2 \in C(M'; Z) \), then \( \phi^* z'_1 \sim \phi^* z'_2 \). Similarly, if \( z_1 \sim z_2 \in C(M; Z) \), then \( z_1 \alpha_* \sim z_2 \alpha_* \). An easy calculation reveals that for all \( z \in C(M; Z) \), \( \phi^*(z \alpha_*) = (\phi^* z)\alpha_* \). Therefore, we may regard
\[
C : \text{Map} \times \text{Group} \to \text{Set}
\]
as a bifunctor, where, of course, \( \text{Group} \) is the category of groups and group homomorphisms, and \( \text{Set} \) is the category of sets and set mappings. If \( \phi : M' \to M \) is a morphism of maps, and if \( \alpha : Z \to Z' \) is a
homomorphism of groups, then we may define
\[ C(\phi, \alpha) : C(\mathcal{M}; Z) \to C(\mathcal{M}'; Z'), \]
where \( C(\phi, \alpha)(z) = \phi^* (z \alpha_s) = (\phi^* z) \alpha_s. \)

If \( \mathcal{M} \) is a map and \( Z \) is a group, set \( D(\mathcal{M}; Z) = C(\mathcal{M}; Z)/ \sim \), where \( \sim \) is the relation of voltage equivalence. If \( z \in C(\mathcal{M}; Z) \), we denote by \( [z] \in D(\mathcal{M}; Z) \) the corresponding equivalence class. Thus, we see that Lemma 5.1.2 says that if \( [z] = [z'] \in D(\mathcal{M}; Z) \), then \( \mathcal{M}_z \cong \mathcal{M}_{z'} \). Next, if \( \phi : \mathcal{M}' \to \mathcal{M} \) is a morphism of maps, and if \( \alpha : Z \to Z' \) is a homomorphism of groups, we denote (again) by \( \phi^* : D(\mathcal{M}; Z) \to D(\mathcal{M}'; Z) \) the mapping \( \phi^*[z] = [\phi^* z] \), and by \( \alpha_s : D(\mathcal{M}; Z) \to D(\mathcal{M}; Z') \) the mapping \( [z] \alpha_s = [z \alpha_s] \). By the above discussion, these mappings are well-defined. It follows that for any \( \zeta \in D(\mathcal{M}; Z) \), \( \phi^*(\zeta \alpha_s) = (\phi^* \zeta) \alpha_s \), and so setting \( D(\phi, \alpha) : D(\mathcal{M}, Z) \to D(\mathcal{M}', Z') \), \( \zeta \mapsto \phi^*(\zeta \alpha_s) = (\phi^* \zeta) \alpha_s \) shows that
\[
D : \text{Map} \times \text{Group} \to \text{Set}
\]
is also a bifunctor.

Note, finally, that if \( Z \) is an (additive) abelian group, then not only is \( C(\mathcal{M}; Z) \) an abelian group with respect to pointwise addition, so is \( D(\mathcal{M}; Z) \). In this case, we may regard \( C \) and \( D \) as functors \( \text{Map} \times \text{AbGroup} \to \text{AbGroup} \). Furthermore, we see that \( C(\mathcal{M}; Z) \) and \( D(\mathcal{M}; Z) \) are \( (\text{Aut}(\mathcal{M}), \text{Aut}(Z)) \)-bimodules.

The following generalizes Lemma 5.1.2.

**Theorem 5.1.3.** Let \( \mathcal{M} = (B, a, b, c) \) be a map, let \( Z, Z' \) be groups and let \( z \in C(\mathcal{M}; Z) \), \( z' \in C(\mathcal{M}; Z') \) be voltages. Assume that for some isomorphism \( \alpha : Z \to Z' \), we have \( [z] \alpha_s = [z'] \in D(\mathcal{M}; Z') \). Then \( \mathcal{M}_z \cong \mathcal{M}_{z'} \).

**Proof.** By hypothesis there exists a cochain \( f : B \to Z' \) which is constant-valued on vertices and satisfying
\[
f(\beta) z'_\beta f(a\beta)^{-1} = (z_\beta)^\alpha
\]
for all $\beta \in B$. Now define $\phi : M_z \to M_{z'}$ by setting $(\beta, \zeta) \phi = (\beta, \zeta^a f(\beta))$. It is routine to check that $\phi$ realizes $M_z \cong_{M} M_{z'}$.

The converse of the above is a bit more subtle, and is complicated by the fact that the principal derived map $M_z$ need not be connected. If it is, then the converse does hold:

**Proposition 5.1.4.** Let $M = (B, a, b, c)$ be a map, let $Z, Z'$ be groups and let $z \in C(M; Z)$, $z' \in C(M; Z')$ be voltages. Assume that $M_z, M_{z'}$ are connected and that $M_z \cong_{M} M_{z'}$. Then there exists an isomorphism $\alpha : Z \to Z'$ such that $[z]_\alpha = [z']$ in $D(M; Z')$.

**Proof.** Assume that $\phi : M_z \cong_{M} M_{z'}$ is an isomorphism over $M$. Note that for any $\zeta \in Z$, $\phi$ determines an $M$-automorphism of $M_{z'}$ via the composition

$$\phi^{-1}l_{\zeta} : M_{z'} \xrightarrow{\phi^{-1}} M_z \xrightarrow{l_{\zeta}} M_z \xrightarrow{\phi} M_{z'}.$$ 

By connectivity, $A(M_{z'}/M) \cong Z'$, and so there exists $\zeta^a \in Z'$ such that $\phi^{-1}l_{\zeta} = l_{\zeta^a}$; clearly the mapping $\zeta \mapsto \zeta^a$ defines an isomorphism $Z \to Z'$. Therefore, for all $\beta \in B$, and all $\zeta, \zeta_1 \in Z$, we have

$$(\beta, \zeta_1)\phi^{-1}l_{\zeta} = (\beta, \zeta_1)l_{\zeta^a} = (\beta, \zeta^a \zeta_1).$$

Write

$$(\beta, \zeta_1)\phi = (\beta, f(\beta, \zeta_1)), $$

for some function $f : B \times Z \to Z'$. Note that for all $\zeta, \zeta_1 \in Z$,

$$(\beta, f(\beta, \zeta_1)) = (\beta, \zeta \zeta_1)\phi = (\beta, \zeta_1)l_{\zeta^{-1}}\phi = (\beta, \zeta_1)\phi l_{\zeta_1^{-1}} = (\beta, \zeta^a f(\beta, \zeta_1)), $$

i.e., the function $f : B \times Z \to Z'$ satisfies $f(\beta, \zeta_1) = \zeta^a f(\beta, \zeta_1)$ for all $\beta \in B$, and all $\zeta, \zeta_1 \in Z$. In particular, if we write $f(\beta) = f(\beta, 1)$, then
\[ f(\beta, \zeta) = \zeta^\alpha f(\beta, 1) = \zeta^\alpha f(\beta). \]
Since \( \phi : M_z \to M_{z'} \) is an isomorphism, we have, for any \( \beta \in B \) and any \( \zeta \in Z \), that
\[
(a \beta, \zeta^\alpha f(\beta)z'_\beta) = a_{z'}(\beta, \zeta^\alpha f(\beta))
= a_{z'}(\beta, f(\beta, \zeta))
= a_{z'}((\beta, \zeta)\phi)
= (a_z(\beta, \zeta))\phi
= (a\beta, \zeta z_\beta)\phi
= (a\beta, f(a\beta, \zeta z_\beta))
= (a\beta, \zeta^\alpha(z_\beta)^\alpha f(a\beta)).
\]

Therefore, for all \( \beta \in B \), \( f(\beta)z'_\beta = (z_\beta)^\alpha f(a\beta) \), i.e., \( f(\beta)z'_\beta f(a\beta)^{-1} = (z_\beta)^\alpha \). Similarly, one proves that \( f \) is constant-valued on vertices, i.e., that for all \( \beta \in B \), we have \( f(b\beta) = f(\beta) = f(c\beta) \). The result follows.

To handle the disconnected case, we need a bit more notation. Let \( Z \) be a group and let \( Z_0 \) be a subgroup of \( Z \). Then there is the obvious inclusion \( C(M; Z_0) \to C(M; Z) \), which we shall denote by \( z_0 \mapsto z_0 \).

**Theorem 5.1.5. (Reduction Theorem)** Let \( M = (B, a, b, c) \) be a map, let \( Z \) be a group, and let \( z \in C(M; Z) \). Assume that \( M_{z_0} \) is a connected component of \( M_z \) and that \( Z_0 \) is the stabilizer in \( Z \) of \( M_{z_0} \). Then there exists \( z_0 \in C(M; Z_0) \) such that \( z \sim z_0 \). In particular, \( M_z \cong M_{z_0} \).

**Proof.** We may identify \( Z_0 \) with \( A(M_{z_0}/M) \) and apply Theorem 5.1.1 to obtain a voltage \( w_0 \in C(M; Z_0) \) and an isomorphism \( \phi : M_{w_0} \to M_{z_0} \). Next, if \( \zeta_0 \in Z_0 \), then there exists an element \( \zeta_0^\alpha \in Z_0 \) such that
\[
\phi^{-1}l_{\zeta_0} \phi = l_{\zeta_0^\alpha} : M_{z_0} \to M_{z_0}.
\]
Clearly the mapping \( \zeta \mapsto \zeta^\alpha \) defines an automorphism of \( Z_0 \).

Set \( (\beta, \zeta)\phi = (\beta, f(\beta, \zeta)) \in M_{z_0} \), where \( f : B \times Z_0 \to Z \) is some function. Exactly as above, one shows that for all \( \zeta, \zeta' \in Z_0 \), one has \( f(\beta, \zeta' \zeta') = \zeta^\alpha f(\beta, \zeta') \), and then that (setting \( f(\beta) = f(\beta, 1) \))
\[
f(\beta) z_\beta f(a\beta)^{-1} = f(\beta) z_\beta f(ac\beta)^{-1} = f(\beta) z_\beta f(ba\beta)^{-1} = (w_0\beta)^\alpha, \ \beta \in B.
\]
Set \( z_0 = w_0\alpha \), and the result follows.
Corollary 5.1.5.1. Let $\mathcal{M}$ be a map and let $Z, Z'$ be groups. Assume that $z \in C(\mathcal{M}; Z)$, $z' \in C(\mathcal{M}; Z')$ are voltages with $\mathcal{M}_z \cong \mathcal{M}_z'$. Then there exist subgroups $Z_0 \leq Z$, $Z_0' \leq Z'$, voltages $z_0 \in C(\mathcal{M}; Z_0)$, $z_0' \in C(\mathcal{M}; Z_0')$, and an isomorphism $\alpha : Z_0 \to Z_0'$ such that $z \sim \tilde{z}_0$, $z' \sim \tilde{z}_0'$ and $z_0\alpha \sim z_0'$.

The following, while quite restrictive, is useful in certain special cases.

Corollary 5.1.5.2. Let $\mathcal{M}$ be a map, let $Z, Z'$ be groups such that for every subgroup $Z_0$ of $Z$ and every monomorphism $Z_0 \to Z'$, there is an extension to an isomorphism $\alpha : Z \to Z'$. If $z \in C(\mathcal{M}; Z)$, $z' \in C(\mathcal{M}; Z')$ are such that $\mathcal{M}_z \cong \mathcal{M}_z'$, then there exists an isomorphism $\alpha : Z \to Z'$ such that $[z]\alpha = [z'] \in D(\mathcal{M}; Z')$.

Proof. Let $Z_0 \leq Z$, $Z_0' \leq Z'$, $z_0 \in C(\mathcal{M}; Z_0)$, $z_0' \in C(\mathcal{M}; Z_0')$ and $\alpha : Z_0 \cong Z_0'$ be as in Corollary 5.1.5.1. By assumption, $\alpha$ extends to an isomorphism (which we still denote) $\alpha : Z \to Z'$. We have $z \sim \tilde{z}_0$, which implies that $z\alpha \sim \tilde{z}_0\alpha = \tilde{z}_0\alpha \sim \tilde{z}_0' \sim z'$.

The following is also restrictive, but will be useful in Chapter 8.

Theorem 5.1.6. Let $\mathcal{M}$ be a map, let $Z$ be a cyclic group, and let $z, z' \in C(\mathcal{M}; Z)$. Then $\mathcal{M}_z \cong \mathcal{M}_z'$ if and only if there exists an automorphism $\alpha \in \text{Aut}(Z)$, such that $[z]\alpha = [z'] \in D(\mathcal{M}; Z)$.

Proof. Indeed, we need only note that any injection of a subgroup $Z_0 \leq Z$ into $Z$ clearly extends to an automorphism of $Z$.

Problem. Find voltages $z, z' \in C(\mathcal{M}; Z)$ such that $\mathcal{M}_z \cong \mathcal{M}_z'$, but such that $z\alpha \neq z'$ for any automorphism $\alpha \in \text{Aut}(Z)$.

A likely candidate might be found as follows. Let $Z \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, and let $Z_0 \leq Z$ where $Z_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (this uniquely places $Z_0$ inside $Z$). Find $\mathcal{M}$ and $z_0 \in C(\mathcal{M}; Z_0)$ such that $\mathcal{M}_{z_0}$ is connected. Let $\alpha : Z_0 \to Z_0$
be an automorphism such that $\alpha$ doesn’t extend to an automorphism of $Z$. Then setting $z = \tilde{z}_0$, $z' = \tilde{z}_0\alpha$, would yield $\mathcal{M}_z \cong_{\mathcal{M}} \mathcal{M}_{z'}$, but (probably) $z\gamma \not\cong z'$ for any automorphism $\gamma$ of $Z$.

We close this section with a question of possible interest.

**Question.** Let $\mathcal{M}$ be a non-orientable map, let $Z$ be a cyclic group of order two, and let $z \in C(\mathcal{M}; Z)$. Under what conditions can we say that the principal derived map $\mathcal{M}_z$ is orientable? What about more general coefficient groups?\(^1\)

## 5.2 Lifting of Automorphisms

**Definition 5.2.1 (Lift of an Automorphism).** Let $p : \tilde{\mathcal{M}} \rightarrow \mathcal{M}$ be a ramified covering of maps, and let $\tilde{\phi} \in \text{Aut}(\tilde{\mathcal{M}}), \phi \in \text{Aut}(\mathcal{M})$. We say that $\tilde{\phi}$ is a lift of $\phi$ if the following square commutes:

$$
\begin{array}{c}
\tilde{\mathcal{M}} \\
\downarrow p \\
\mathcal{M}
\end{array}
\xymatrix{ 
\tilde{\mathcal{M}} \ar[rr]^{\tilde{\phi}} \ar[d]_{p} & & \tilde{\mathcal{M}} \\
\mathcal{M} \ar[rr]_{\phi} & & \mathcal{M} \\
\end{array}

$$

In the above, we sometimes say that $\phi$ lifts to the automorphism $\tilde{\phi} \in \text{Aut}(\tilde{\mathcal{M}})$, or that $\tilde{\phi} \in \text{Aut}(\tilde{\mathcal{M}})$ covers $\phi$.

The universal criterion is as follows.

Theorem 5.2.1. Let \( \mathcal{M} \) be a map, and let \( z \in C(\mathcal{M}; Z) \). If \( \phi \in \text{Aut}(\mathcal{M}) \), then \( \phi \) lifts to an automorphism \( \phi_z \in \text{Aut}(\mathcal{M}_z) \) if and only if \( \mathcal{M}_z \cong \mathcal{M} \mathcal{M}_{\phi^z} \).

Proof. Assume that \( \psi : \mathcal{M}_z \stackrel{\cong}{\rightarrow} \mathcal{M}_{\phi^z} \) realizes \( \mathcal{M}_z \cong \mathcal{M} \mathcal{M}_{\phi^z} \). Write

\[(\beta, \zeta)\psi = (\beta, f(\beta, \zeta)),\]

for some function \( f : B \times Z \rightarrow Z \). Then as \( \psi \) is an isomorphism, we have, for all \( \beta \in B, \ \zeta \in Z \), that

\[(a\beta, f(a\beta, \zeta\beta)) = (a\beta, \zeta\beta)\psi = (a_z(\beta, \zeta))\psi = a_{\phi^z}((\beta, \zeta)\psi) = a_{\phi^z}(\beta, f(\beta, \zeta)) = (a\beta, f(\beta, \zeta)z_{\beta\phi}),\]

and so \( f(\beta, \zeta)z_{\beta\phi} = f(a\beta, \zeta\beta) \). Similarly, \( f(b\beta, \zeta) = f(\beta, \zeta) = f(c\beta, \zeta). \)

Define \( \phi_z : B \times Z \rightarrow B \times Z \) by setting \( (\beta, \zeta)\phi_z = (\beta\phi, f(\beta, \zeta)) \) and verify that \( \phi_z \) is an automorphism of \( \mathcal{M}_z \) that covers \( \phi \).

Conversely, assume that \( \phi_z : \mathcal{M}_z \stackrel{\cong}{\rightarrow} \mathcal{M}_z \) exists and covers \( \phi : \mathcal{M} \stackrel{\cong}{\rightarrow} \mathcal{M} \). Write

\[(\beta, \zeta)\phi_z = (\beta\phi, g(\beta, \zeta)),\]

for some function \( g : B \times Z \rightarrow Z \). Then as \( \phi_z \) is an isomorphism, it follows that for all \( \beta \in B, \ \zeta \in Z \), one has \( g(\beta, \zeta)z_{\beta\phi} = g(a\beta, \zeta\beta), \ g(\beta, \zeta) = g(b\beta, \zeta), \ g(\beta, \zeta) = g(c\beta, \zeta). \) Now define \( \psi : B \times Z \rightarrow B \times Z \) by setting \( (\beta, \zeta)\psi = (\beta, g(\beta, \zeta)) \); exactly as above, \( \psi \) realizes \( \mathcal{M}_z \cong \mathcal{M} \mathcal{M}_{\phi^z}. \)

Corollary 5.2.1.1. Let \( \mathcal{M} \) be a map and let \( \phi \in \text{Aut}(\mathcal{M}) \). Let \( z \in C(\mathcal{M}; Z) \), and assume that there exists an automorphism \( \alpha \) of \( Z \) such that \( \phi^*[z] = [z]\alpha_* \in D(\mathcal{M}; Z) \). Then \( \phi \) lifts to an automorphism of \( \mathcal{M}_z \).

Proof. By Theorem 5.1.3, together with Lemma 5.1.2, we above we have that \( \mathcal{M}_z \cong \mathcal{M} \mathcal{M}_{\alpha_*}, \cong \mathcal{M} \mathcal{M}_{\phi^z}. \) Now apply Theorem 5.2.1.
**Remark.** Assume that $\phi \in \text{Aut}(\mathcal{M})$, $z \in C(\mathcal{M}; Z)$, and that $\alpha \in \text{Aut}(Z)$ is such that $z \alpha_s \sim \phi^* z$. Thus there exists a cochain $f : B \to Z$, constant-valued on vertices, such that for all blades $\beta \in B$, 

$$f(\beta)z_{\beta\phi}f(\alpha\beta)^{-1} = z_\beta^\alpha.$$ 

Then an explicit lift of $\phi$ to an automorphism $\phi_z \in \text{Aut}(\mathcal{M}_z)$ is given by 

$$(\beta, \zeta)\phi_z = (\beta\phi, \zeta^\alpha f(\beta)), \quad \beta \in B, \ \zeta \in Z.$$ 

Therefore, the inverse $\phi_z^{-1}$ is given by 

$$(\beta, \zeta)\phi_z^{-1} = (\beta^{-1}, \zeta^{-1} f(\beta)^{-\alpha^{-1}}), \quad \beta \in B, \ \zeta \in Z.$$ 

Thus, a direct computation reveals that $\phi_z^{-1} l_\zeta \phi_z = l_\zeta^\alpha$.

As an immediate corollary to Corollary 5.2.1.1, we have the following.

**Corollary 5.2.1.2.** Assume that $\mathcal{M}$ is a map with automorphism group $\text{Aut}(\mathcal{M})$, and that $Z$ is a group with automorphism group $\text{Aut}(Z)$. Assume that $z \in C(\mathcal{M}; Z)$ is a voltage such that for some homomorphism $\alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z)$ we have $\phi^*[z] = [z] \alpha(\phi)$, for all $\phi \in \text{Aut}(\mathcal{M})$. Then every automorphism of $\text{Aut}(\mathcal{M})$ lifts to one of $\mathcal{M}_z$.

We can obtain a converse, either by imposing some restrictions on the coefficient group, or by imposing connectivity on the derived map.

**Corollary 5.2.1.3.** Let $\mathcal{M}$ be a map and let $\phi \in \text{Aut}(\mathcal{M})$. Let $Z$ be a group and assume that $\phi$ lifts to an automorphism of $\mathcal{M}_z$. Then there exists an automorphism $\alpha$ of $Z$ such that $z \alpha_s \sim \phi^* z$ in either of the following cases:

1. $\mathcal{M}_z$ is connected.
2. $Z$ is cyclic;
Before leaving this section, the following reformulation of the lifting problem shall prove convenient. If \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \) is a homomorphism, set
\[
D(\mathcal{M}; Z)_\alpha = \{ \zeta \in D \mid \phi^*\zeta = \zeta\alpha(\phi) \text{, for all } \phi \in \text{Aut}(\mathcal{M}) \},
\]
and call \( D(\mathcal{M}; Z)_\alpha \) the \( \alpha \)-isotypical voltage classes on \( \mathcal{M} \). Put slightly differently, we see that \( \zeta \in D(\mathcal{M}; Z)_\alpha \) if and only if \( D(\phi, \alpha(\phi)^{-1})(\zeta) = \zeta \) for all \( \phi \in \text{Aut}(\mathcal{M}) \). Note that if \( Z \) is an abelian group, then, in fact, \( D(\mathcal{M}; Z)_\alpha \) is an \( \text{Aut}(\mathcal{M}) \)-submodule of \( D(\mathcal{M}; Z) \). Furthermore, if \( \zeta \in D(\mathcal{M}; Z)_\alpha \), then every element of \( \text{Aut}(\mathcal{M}) \) lifts to one of \( \mathcal{M}_z \), where \( \zeta = [z] \). In case \( \mathcal{M} \) is regular, we see that \( \mathcal{M}_z \) is regular if and only if \( [z] \in D_\alpha \) for some \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \).

Note that if \( \mathcal{M}_z \) is connected, then the group of covering transformations is \( \{ l_\zeta \mid \zeta \in Z \} \cong Z \). Furthermore, continuing to assume that \( Z \) is abelian, one can apply the remark on page 65 to infer that for any lift \( \tilde{\phi} \) of the automorphism \( \phi \in \text{Aut}(\mathcal{M}) \), \( \tilde{\phi}^{-1}l_\zeta\tilde{\phi} = l_{\zeta \alpha(\phi)} \). Therefore, we see that \( \text{Aut}(\mathcal{M}_z) \) fits into the short exact sequence
\[
1 \to Z \to \text{Aut}(\mathcal{M}_z) \to \text{Aut}(\mathcal{M}) \to 1,
\]
where \( \text{Aut}(\mathcal{M}) \) acts on \( Z \) via the homomorphism \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \), i.e., \( \phi^{-1}\zeta\phi = \zeta^{\alpha(\phi)} \), \( \phi \in \text{Aut}(\mathcal{M}) \), \( \zeta \in Z \).

We may summarize our findings as follows:

**Theorem 5.2.2.** Let \( \zeta = [z] \in D(\mathcal{M}; Z) \), where \( Z \) is an abelian group, and assume that every automorphism of \( \mathcal{M} \) lifts to one of \( \mathcal{M}_z \). If either \( \mathcal{M}_z \) is connected or \( Z \) is cyclic, then \( \zeta \in D(\mathcal{M}; Z)_\alpha \) for some homomorphism \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \).

We shall use the following form of the above result in Chapter 8:

**Corollary 5.2.2.1.** Let \( \mathcal{M} \) be a regular map, let \( Z \) be an abelian group and let \( \zeta = [z] \in D(\mathcal{M}; Z) \). If \( \mathcal{M}_z \) is connected, then \( \mathcal{M}_z \) is regular if and only if there exists a homomorphism \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \) such that \( \zeta \in D(\mathcal{M}; Z)_\alpha \), (i.e., such that \( \phi^*\zeta = \zeta\alpha(\phi) \), for all \( \phi \in \text{Aut}(\mathcal{M}) \)).
If $\mathcal{M}$ is orientably regular (see page 33), we see that $\mathcal{M}_z$ is orientably regular if and only if $[z] \in D(\mathcal{M}; Z)_{\alpha^+}$ for some homomorphism $\alpha^+ : \text{Aut}(\mathcal{M})^+ \to \text{Aut}(Z)$, where, of course,

$$D(\mathcal{M}; Z)_{\alpha^+} = \{ \zeta \in D \mid \phi^\ast \zeta = \zeta \alpha(\phi)^\ast, \text{ for all } \phi \in \text{Aut}(\mathcal{M})^+ \}.$$

### 5.3 The Characteristic Homomorphism and Principal Derived Maps

Let $\mathcal{M}$ be a map, $Z$ be a group, and let $z \in C(\mathcal{M}; Z)$. Let $l$ be the vertex valency of $\mathcal{M}_z$ (= vertex valency of $\mathcal{M}$) and $k$ be the face valency of $\mathcal{M}_z$. As $p : \mathcal{M}_z \to \mathcal{M}$ is a regular ramified covering, we see that $k$ and $l$ can be taken as the face and vertex valencies of any connected component of $\mathcal{M}_z$. If $\beta_0$ is a fixed blade in $\mathcal{M}$ we let $\beta_0' = (\beta_0, 1) \in \beta_0 p^{-1}$, and let $\mathcal{M}_{z0}$ be the connected component in $\mathcal{M}_z$ containing $\beta_0'$. We recall the action of $Z$ on $\mathcal{M}_z$ given on page 55 and let $Z_0$ be the stabilizer in $Z$ of $\mathcal{M}_{z0}$. Set $\pi = \pi^{(k,l)}_1(\mathcal{M}, \beta_0)$, $\pi' = \pi^{(k,l)}_1(\mathcal{M}_z, \beta_0') = \pi^{(k,l)}_1(\mathcal{M}_{z0}, \beta_0')$ and set $\Delta = \Delta(k, l)$. As we saw in Section 2.1, $\pi' \leq \pi$ giving rise to the characteristic homomorphism

$$\chi_{\mathcal{M}_{z0}/\mathcal{M}} : \pi = \pi(\mathcal{M}, \beta_0) \longrightarrow \Lambda(\mathcal{M}_{z0}/\mathcal{M}) = Z_0 \to Z,$$

whose kernel is $\pi' = \pi(\mathcal{M}_z, \beta_0')$. Next, if $\mathcal{M}_{z1}$ is another connected component of $\mathcal{M}_z$, then by regularity, there exists an element $\zeta \in Z$ such that $\mathcal{M}_{z1} = \mathcal{M}_{z0} l_\zeta$ and so $\Lambda(\mathcal{M}_{z1}/\mathcal{M}) = l_\zeta^{-1} \Lambda(\mathcal{M}_{z0}) l_\zeta$. Therefore, we see that the homomorphisms

$$\chi_{\mathcal{M}_{z0}/\mathcal{M}}, \chi_{\mathcal{M}_{z1}/\mathcal{M}} : \pi \longrightarrow Z$$

differ only by an inner automorphism of $Z$. We may therefore denote $\chi_z = \chi_{\mathcal{M}_{z0}/\mathcal{M}}$, with the understanding that $\chi_z : \pi \to Z$ is well defined up to an inner automorphism of $Z$.

We wish to make this homomorphism more explicit. Note that as subgroups of $\Delta$, $\pi$ and $\pi'$ both act on the left on $\mathcal{M}_{z0}$ (which we shall identify with $\Delta/\pi'$) as monodromy transformations. With this identification, $\pi'$ is the stabilizer of the blade $(\beta_0, 1) = \beta_0' = 1 \cdot \pi'$, and $\pi$ transitively permutes the blades in $\beta_0 p^{-1} \cap \mathcal{M}_{z0}$.
Lemma 5.3.1. If $\gamma \in \pi$, then $\gamma \beta'_0 = \beta'_0 \chi_z(\gamma)$.

Proof. This boils down to recalling how $\text{Aut}(\mathcal{M}_{s_0}) = \text{Aut}(\Delta/\pi')$ is identified with the normalizer in $\Delta$ of $\pi'$, modulo $\pi'$. Indeed, if $\gamma \in N(\pi')$, then $\gamma$ acts on the right on $\Delta/\pi'$ via

$$x \cdot \pi' \mapsto x\gamma \cdot \pi', \quad x \in \Delta.$$ 

Since $\beta'_0$ has been identified with the coset $1 \cdot \pi'$, we see that $\beta'_0 \chi_z(\gamma) = \gamma \pi' = \gamma(1 \cdot \pi') = \gamma \beta'_0$.

Using the above, we can determine how to explicitly calculate the value of $\chi_z(\gamma)$, $\gamma \in \pi$, as an element of the coefficient group $Z$. That is to say, if $\chi_z(\gamma) \mapsto \zeta \in Z$, where $\beta'_0 \chi_z(\gamma) = \beta'_0 l_\zeta = (\beta_0, \zeta^{-1})$, our task is to explicitly calculate $\zeta$ in terms of $\gamma$. To do this, we write $\gamma$ as a product of the generators $s_1, s_2, s_3 \in \Delta$, and recall that for all blades $\beta$ of $\mathcal{M}$,

$$s_i(\beta, \zeta) = \begin{cases} (a\beta, \zeta z_\beta) & \text{if } i = 1 \\ (b\beta, \zeta) & \text{if } i = 2 \\ (c\beta, \zeta) & \text{if } i = 3. \end{cases}$$

The above then gives explicitly the element $\zeta \in Z$, such that $\gamma \beta'_0 = (\beta_0, \zeta^{-1})$. For instance, if $\gamma$ is the element $(s_2 s_1)^m \in \pi$, where $m$ is the face valency of $\mathcal{M}$, then

$$\gamma \beta'_0 = \gamma(\beta_0, 1) = (s_2 s_1)^m s_2(a\beta_0, z_{\beta_0}) = (s_2 s_1)^m(ba\beta_0, z_{\beta_0}) = \vdots$$

$$= (\beta_0, z_{\beta_0} z_{ba} z_{ba^2} z_{ba^3} \cdots z_{(ba)^{m-1}}).$$

Therefore, we infer that when $\gamma = (s_2 s_1)^m$

$$\chi_z(\gamma) = (z_{\beta_0} z_{ba\beta_0} z_{(ba)^2} \beta_0 \cdots z_{(ba)^{m-1}})^{-1}.$$ 

In general, the value of $\chi_z(\gamma) \in Z$ can be calculated as follows. First of all, let $W = \langle s_2, s_3 \rangle \leq \Delta$, and note that any element $\gamma \in \Delta$ can
be expressed in the form \( \gamma = wu_r u_{r-1} \cdots u_1 \), where \( w, s_i u_i \in W, i = 1, 2, \ldots, r \). We then have, for any blade \( \beta \), that

\[
\gamma(\beta, \zeta) = (\gamma \beta, \zeta z_{au_1 \beta} z_{au_2 u_1 \beta} \cdots z_{au_r u_{r-1} \cdots u_1 \beta}).
\]

In particular, if \( \gamma \in \pi \), then

\[
\chi_z(\gamma) = (z_{au_1 \beta} z_{au_2 u_1 \beta} \cdots z_{au_r u_{r-1} \cdots u_1 \beta})^{-1}
= z_{u_r u_{r-1} \cdots u_1 \beta} \cdots z_{u_2 u_1 \beta} z_{u_1 \beta},
\]

where we recall that for any blade \( \beta \), \( z_{a \beta} = z_{\beta}^{-1} \).

** Remark. (Effect of Voltage Equivalence) ** Now assume that we have equivalent voltages \( z, z' \in C(M; Z) \). We would like to compare the values of the characteristic homomorphisms \( \chi_z(\gamma), \chi_{z'}(\gamma) \in Z \). Thus there exists a mapping \( f : B \to Z \), constant-valued on vertices, such that for any \( \beta \in B = B(M) \) we have

\[
f(\beta) z'_\beta f(a_\beta)^{-1} = z_\beta.
\]

Note that if \( \langle s_2, s_3 \rangle = W \leq \Delta \), then for any blade \( \beta \) and for any element \( w \in W \), we have \( f(\beta) = f(w \beta) \). Therefore, if the voltage \( z' \sim z \) is as above, then the above shows that if \( \gamma \in \pi \), then

\[
\chi_{z'}(\gamma) = (z'_{au_1 \beta} z'_{au_2 u_1 \beta} \cdots z'_{au_r u_{r-1} \cdots u_1 \beta})^{-1}
= (f(a_1 \beta_0) z_{a_1 \beta_0} f(u_1 \beta_0)^{-1} f(a_2 u_1 \beta_0) z_{a_2 u_1 \beta_0} f(u_2 u_1 \beta_0)^{-1})
\cdots (f(a_r u_{r-1} \cdots u_1 \beta_0) z_{a_r u_{r-1} \cdots u_1 \beta_0} f(u_r u_{r-1} \cdots u_1 \beta_0)^{-1})^{-1}.
\]

However, since \( s_i u_i \in W \) for all \( i = 1, 2, \ldots, r \), we have

\[
\begin{align*}
f(\beta_0) &= f(a_1 \beta_0), \\
f(u_1 \beta_0) &= f(a_2 u_1 \beta_0), \\
f(u_2 u_1 \beta_0) &= f(a_3 u_2 u_1 \beta_0), \\
&\vdots \\
f(u_{r-1} \cdots u_2 u_1 \beta_0) &= f(a_r u_{r-1} \cdots u_1 \beta_0), \\
f(u_r u_{r-1} \cdots u_1 \beta_0) &= f(w_r u_{r-1} \cdots u_1 \beta_0)
= f(\gamma \beta_0) \\
&= f(\beta_0).
\end{align*}
\]
Therefore, we conclude that
\[
\chi_{z'}(\gamma) = (f(\beta_0)(z_{au_1\beta_0}z_{au_2\beta_0}\cdots z_{au_{r-1}\beta_0})f(\beta_0)^{-1})^{-1} = f(\beta_0)\chi_z(\gamma)f(\beta_0)^{-1}.
\]

In conclusion, we see that if \(z, z' \in C(M; Z)\) are equivalent, then the corresponding characteristic homomorphisms \(\chi_z, \chi_{z'} : \pi_1^{(n,l)}(M, \beta_0) \to Z\) differ only by an inner automorphism of \(Z\). In particular, if \(Z\) is an abelian group of coefficients, then \(\chi_z = \chi_{z'}\) whenever \(z \sim z'\).

We close this section with the following result of practical interest. It’s well-known, but is a natural observation at this juncture; it follows immediately from our observations in Section 2.1.

**Proposition 5.3.2.** Let \(M = (B, a, b, c)\) be a connected map. Fix a blade \(\beta_0 \in B\) and a voltage \(z \in C(M; Z)\). Then for any multiple \(l\) of the vertex valency of \(M\) and any multiple \(n\) of the face valency of \(M_z\), \(M_z\) is connected if and only if the characteristic homomorphism \(\chi_z : \pi_1^{(n,l)}(M, \beta_0) \to Z\) is surjective. More generally, the number of connected components of \(M_z\) is the index of the image of \(\pi_1^{(n,l)}(M, \beta_0)\) in \(Z\).

As a result, we have the following:

**Corollary 5.3.2.1.** Let \(M = (B, a, b, c)\) be a connected map, let \(z \in C(M; Z)\) be a voltage, and assume that the principal derived map \(M_z\) is connected. Then for any blade \(\beta_0\) of \(M\) we have that \(\pi_1^{(n,l)}(M_z, (\beta_0, 1)) = \ker(\chi_z : \pi_1^{(n,l)}(M, \beta_0) \to Z)\), where \(l\) is a multiple of the vertex valency of \(M\), and \(n\) is a multiple of the face valency of \(M_z\).

**Proof.** Of course, one only needs to observe that \(\pi_1^{(n,l)}(M_z, (\beta_0, 1))\) is contained in \(\ker(\chi_z)\).
Finally, we have another approach to the lifting question, as it relates to the fundamental group. In particular, we have the following result:\footnote{P. Gvozdjak and J. Širáň, Regular maps from voltage assignments, in: \textit{Graph Structure Theory}, N. Robertson and P. Seymour (eds), \textit{Contemporary Mathematics} (AMS Series), vol. 147, 1993, 441-454.}

\textbf{Theorem 5.3.3.} Let $\mathcal{M}$ be a map, let $Z$ be a group, let $z \in C(\mathcal{M}; Z)$, and assume that the principal derived map $\mathcal{M}_z$ is connected. Let $\phi \in \text{Aut}(\mathcal{M})$, and set $z' = z\phi^z \in C(\mathcal{M}; Z)$. Fix a blade $\beta$ in $\mathcal{M}$, set $K_\beta = \ker \chi_z : \pi_1^\infty(\mathcal{M}, \beta) \to Z$, $K'_\beta = \ker \chi_{z'} : \pi_1^\infty(\mathcal{M}, \beta) \to Z$. If $K_\beta = K'_\beta$, then $\phi$ lifts to an automorphism of $\mathcal{M}_z$.

\textbf{Proof.} Indeed, from the above corollary, we have that $K_\beta = \pi_1^{(n,l)}(\mathcal{M}_z, (\beta, 1))$ and $K'_\beta = \pi_1^{(n,l)}(\mathcal{M}_z', (\beta, 1))$ and so (for suitable $(n, l)$), $\pi_1^{(n,l)}(\mathcal{M}_z, (\beta, 1)) = \pi_1^{(n,l)}(\mathcal{M}_z', (\beta, 1))$. Using Theorem 2.1.1, we conclude that there exists an isomorphism (over $\mathcal{M}$) $\mathcal{M}_z \to \mathcal{M}_z' = \mathcal{M}_{z\phi^z}$. Now apply Theorem 5.2.1 to complete the proof.

\section{5.4 Some Categorical Considerations}

Let $\mathcal{M} = (B, a, b, c)$ be a map, let $Z$ be a group, and let $z \in C(\mathcal{M}; Z)$. From the set of $Z$-valued voltages, we may, as on page 59, form the quotient set $D(\mathcal{M}; Z) = C(\mathcal{M}; Z)/\sim$, where, as usual, “$\sim$” is voltage equivalence. The assignment $\mathcal{M} \mapsto D(\mathcal{M}; Z)$ is clearly functorial in that a map morphism $\phi : \mathcal{M} \to \mathcal{M}'$ induces a “pull-back” of voltages, $z' \mapsto z = z'\phi$, where $(z'\phi)_d = z'_d\phi \in Z$. It is easily checked that this factors through equivalence, giving the mapping $\phi^* : D(\mathcal{M}'; Z) \to D(\mathcal{M}; Z)$.

Next, if $G_1, G_2$ are groups, we denote by $[G_1, G_2] = \text{Hom}(G_1, G_2)/\sim$ where $f \sim f' : G_1 \to G_2$ if $f, f'$ differ by an inner automorphism of $G_2$. In other words, $f \sim f'$ if and only if there exists $g \in G_2$ such that for all $x \in G_1$, $f(x) = g^{-1}f'(x)g$. Therefore, we obtain a category $\textbf{IGroup}$ whose objects are groups and whose morphisms are homomorphisms modulo inner automorphisms. Next, as we saw in the preceding section, if $[z]$ is the equivalence class in $D(\mathcal{M}; Z)$ determined by the voltage $z \in$
\( C(\mathcal{M}; Z) \) then \([z]\) determines an element of \([\pi_1^\infty(\mathcal{M}, \beta_0), Z]\). Therefore, we have a mapping

\[
\chi_\mathcal{M} : D(\mathcal{M}; Z) \to [\pi_1^\infty(\mathcal{M}, \beta_0), Z], \quad \chi_\mathcal{M}([z]) = \chi_z
\]

which is functorial in both \( \mathcal{M} \) and \( Z \).

Put somewhat differently, we can interpret \( \chi \) as a natural transformation from the functor \((\mathcal{M}, \beta_0) \mapsto D(\mathcal{M}; Z)\) to the functor \((\mathcal{M}, \beta_0) \mapsto [\pi_1^\infty(\mathcal{M}, \beta_0), Z]\), such that if \( \phi : \mathcal{M} \to \mathcal{M}' \) is a morphism of maps, \( \beta_0 \phi = \beta'_0 \); then the diagram below commutes:

\[
\begin{array}{ccc}
D(\mathcal{M'}; Z) & \xrightarrow{\chi_{\mathcal{M'}}} & [\pi_1^\infty(\mathcal{M'}, \beta'_0), Z] \\
\phi^* \downarrow & & \downarrow \text{Hom}(\phi^*, 1_Z) \\
D(\mathcal{M}; Z) & \xrightarrow{\chi_{\mathcal{M}}} & [\pi_1^\infty(\mathcal{M}, \beta_0), Z]
\end{array}
\]

When we restrict our attention to a fixed abelian coefficient group \( Z \), then the above can be read as a natural transformation between functors from the category \( \text{Map}_* \) to the category \( \text{Ab} \) of abelian groups.

Next we show that if \( \mathcal{M} \) is a connected map, with fixed blade \( \beta_0 \) and coefficient group \( Z \), then the mapping \( \chi_\mathcal{M} : D(\mathcal{M}; Z) \to [\pi_1^\infty(\mathcal{M}, \beta_0), Z] \) is injective.\(^3\) Thus, let \([z], [z'] \in D(\mathcal{M}; Z)\), and assume that \( \chi_\mathcal{M}([z]) = \chi_\mathcal{M}([z']) \) in \([\pi_1^\infty(\mathcal{M}, \beta_0), Z]\). Thus, there exists an element \( g \in Z \) such that \( \chi_\mathcal{M}([z]) = g^{-1}\chi_\mathcal{M}([z'])g \). First of all, if \( \tau = (w, u_r, u_{r-1}, \ldots, u_2, u_1) \) is a sequence in \( \Delta \) such that \( w, s_1 u_i \in W = \langle s_2, s_3 \rangle \leq \Delta, i = 1, 2, \ldots, r \), we set

\[
\chi_z(\tau) = z_{u_r u_{r-1} \ldots u_1 \beta_0} \cdots z_{u_2 u_1 \beta_0} z_{u_1 \beta_0} \in Z.
\]

For such a sequence \( \tau \) we set \( |\tau| = r = w u_r u_{r-1} \ldots u_2 u_1 \). Next if \( \beta \in B \), then by connectivity there exists \( \tau \in \Delta \) such that \( \tau(\beta) = \beta \). We may factor \( \tau \) as \( \tau = w u_r u_{r-1} \ldots u_2 u_1, w, s_1 u_i \in W, i = 1, 2, \ldots, r \); we set \( \tau = (w, u_r, u_{r-1}, \ldots, u_2, u_1) \) and then set

\[
f(\beta) = \chi_z(\tau) g^{-1} \chi_{z'}(\tau)^{-1} \in Z.
\]

\(^3\)This proof is adapted from That of Theorem 4.3 of my earlier paper, Covers of simplicial complexes and applications to geometry, \textit{Geom. Dedicata}, 16 (1984), 35–62.
We start by showing that $f(\beta)$ is well-defined. Thus, assume that
\[ \tau = (w, u_r, u_{r-1}, \ldots, u_2, u_1), \quad \tau' = (w', u'_s, u'_{s-1}, \ldots, u'_2, u'_1), \]
w, w', s_1 u_i, s_1 u'_j \in W, i = 1, 2, \ldots, r, j = 1, 2, \ldots, s, \quad \tau = |\tau|, \quad \tau' = |\tau'| \]
where $\tau(\beta_0) = \tau'(\beta_0)$. We must show that
\[ \chi_{z}(\tau')g^{-1}\chi_{z'}(\tau')^{-1} = \chi_{z}(\tau)g^{-1}\chi_{z'}(\tau')^{-1}, \]
i.e.,
\[ \chi_{z}(\tau)^{-1}\chi_{z}(\tau') = g^{-1}\chi_{z'}(\tau)^{-1}\chi_{z'}(\tau')g. \]
Since $\gamma = \tau^{-1}\tau' \in \pi_{\infty}(\mathcal{M}, \beta_0)$, we have by assumption $\chi_{z}(\gamma) = g^{-1}\chi_{z'}(\gamma)g$.

Next we have
\[ \tau^{-1} \tau' = u_1^{-1} \cdots u_r^{-1} w^{-1} w' u'_s \cdots u'_1 \]
\[ = (u_1^{-1} s_1)(s_1 u_2^{-1} s_1) \cdots (s_1 u_r^{-1} s_1)(s_1 w^{-1} w') u'_s \cdots u'_1 \]
from which it follows that
\[ z_{s_1 u_2^{-1} \cdots u_r^{-1} w^{-1} w' u'_s \cdots u'_1 \beta_0} z_{s_1 u_3^{-1} \cdots u_r^{-1} w^{-1} w' u'_s \cdots u'_1 \beta_0} \cdots \]
\[ = g^{-1} z'_{s_1 u_2^{-1} \cdots u_r^{-1} w^{-1} w' u'_s \cdots u'_1 \beta_0} z'_{s_1 u_3^{-1} \cdots u_r^{-1} w^{-1} w' u'_s \cdots u'_1 \beta_0} \cdots \]
\[ = g^{-1} z'_{s_1 w^{-1} w' u'_s \cdots u'_1 \beta_0} \chi_{z'}(\tau')g. \]

Using the fact that $z_{s_1 \beta} = z_{\beta}^{-1}$, $z'_{s_1 \beta} = z'_{\beta}^{-1}$ for all $\beta \in B$, together with the fact that $u_i^{-1} \cdots u_{r+1}^{-1} w^{-1} w' u'_s \cdots u'_1 \beta_0 = u_i \cdots u_1 \beta_0$, we infer from the above that
\[ z_{u_1 \beta_0} z_{u_2 \beta_0} \cdots z_{u_r \cdots u_1 \beta_0} \chi_{z}(\tau') = g^{-1} z'_{u_1 \beta_0} z'_{u_2 \beta_0} \cdots z'_{u_r \cdots u_1 \beta_0} \chi_{z'}(\tau')g, \]
i.e.,
\[ \chi_{z}(\tau)^{-1}\chi_{z}(\tau') = g^{-1}\chi_{z'}(\tau)^{-1}\chi_{z'}(\tau')g, \]
and so $f(\beta) \in Z$ is well-defined.

We continue with the proof that $\chi_M : D(M; Z) \rightarrow [\pi_{\infty}^1(M, \beta_0), Z]$ is injective. Thus, let $\beta \in B$, let $\tau(\beta_0) = \beta$, and write $\tau = w u_r \cdots u_2 u_1$ as above. Again, set $\tau = (w, u_r, \ldots, u_2, u_1)$, $(s_1, \tau) = (s_1, w, u_r, \ldots, u_2, u_1)$

Note first that
\[ \chi_{z'}(s_1, \tau) = z'_{s_1 w u_r \cdots u_2 u_1 \beta_0} z'_{u_r \cdots u_2 u_1 \beta_0} \cdots z'_{u_1 \beta_0} \]
\[ = z'_{s_1 w u_r \cdots u_2 u_1 \beta_0} \chi_{z'}(\tau) \]
(with a similar result with the voltage \( z \) replacing \( z' \)). From this, it follows that

\[
f(\beta) z' \beta f(s_1 \beta)^{-1} = \chi_z(\tau) g^{-1} \chi_{z'}(\tau)^{-1} z' \beta \chi_{z'}(s_1, \tau) g \chi_z(s_1, \tau)^{-1}
\]

\[
= \chi_z(\tau) g^{-1} \chi_{z'}(\tau)^{-1} z' \beta^s z_1 \beta^s \chi_{z'}(\tau) g \chi_z(s_1, \tau)^{-1}
\]

\[
= \chi_z(\tau) \chi_z(s_1, \tau)^{-1}
\]

\[
= z_{s_1 \beta}^{-1}
\]

\[
= z_\beta,
\]

proving that \([z'] = [z]\), i.e., that \(\chi_M : D(M; Z) \rightarrow [\pi_1^\infty(M, \beta_0), Z]\) is injective.

In general, we cannot expect \(\chi_M : D(M; Z) \rightarrow [\pi_1^\infty(M, \beta_0), Z]\) to be surjective as coverings \(M_z\) of \(M\) based on \(D(M; Z)\) are unramified over vertices, whereas those of the form \(\Delta / \pi_1^\infty(M, \beta_0)\) (\(\Delta = \Delta(\infty, \infty)\)) can indeed ramify over vertices. If we consider coverings of the form \(\Delta(\infty, l) / \pi_1^{(\infty, l)}(M, \beta_0)\), then such coverings are unramified over vertices precisely when \(M\) is uniform with respect to vertices, i.e., all vertices of \(M\) have vertex valency \(l\). In this case, we note that, the above recipe gives \(\chi_M : D(M; Z) \rightarrow [\pi_1^{(\infty, l)}(M, \beta_0), Z]\), and the same proof shows that this mapping is injective. We now show that it is surjective.\(^4\)

Thus, let \(\theta \in \text{Hom}(\pi_1^{(\infty, l)}(M, \beta_0), Z)\); we shall construct a voltage \(z \in D(M; Z)\) such that \(\chi_z\) and \(\theta\) determine the same element of \([\pi_1^{(\infty, l)}(M, \beta_0), Z]\). For the fixed blade \(\beta_0\), set \(v_0 = [\beta_0] V\), the vertex in \(M\) determined by \(\beta_0\). Next, for each vertex \(v \in V\), let \(\phi_v = (v_0, v_1, \ldots, v_r = v)\) be a fixed path in the underlying graph of \(M\) from \(v_0\) to \(v\). For each vertex \(v \in V\), we fix a blade \(\beta_v \in V\) with \(\beta_{v_0} = \beta_0\) and fix an element \(\gamma_v \in G = \text{Mon}(M)\) satisfying

(i) \(\gamma_v(\beta_0) = \beta_v\);

(ii) \(\gamma_v = w u_r \cdots u_1\), where \(w, s_1 u_j \in W = \langle s_2, s_3 \rangle, j = 1, 2, \ldots, r\).

Therefore, we see that \(u_j \cdots u_2 u_1(\beta_0) \in v_j, j = 1, 2, \ldots, r\). Finally, for any blade \(\beta \in B\), let \(v = [\beta] V\) and let \(w_\beta \in W\) satisfy \(\beta = w_\beta \beta_v\). It is important to note here that \(w_\beta \in W\) is unique since \(W\) maps

\(^4\text{Again, this proof is similar in spirit to that of Theorem 4.3 in op. cit.}\)
isomorphically to the dihedral subgroup \( \langle b, c \rangle \) of the monodromy group \( G \) and acts regularly on the blades of each vertex by uniformity. For any blade \( \beta \in B \) we now set
\[
\gamma_\beta = w_\beta \gamma_\nu \in G, \quad z_\beta = \theta(\gamma_\beta^{-1}s_1\gamma_{s_1\beta}).
\]
It follows that if \( w \in W \), then \( \gamma_{w\beta} = w\gamma_\beta \) for any \( \beta \in B \). Finally, if \( \gamma \in \pi_{1}^{(\infty,1)}(M, \beta_0) \), then write \( \gamma = w u_r \cdots u_2 u_1, w, s_1 u_j \in W, j = 1, 2, \ldots, r \). We have
\[
\chi_z(\gamma) = z_{u_r \cdots u_2 u_1 \beta_0} \cdots z_{u_2 u_1 \beta_0} z_{u_1 \beta_0} \\
= \theta(\gamma_{u_r \cdots u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_r \cdots u_2 u_1 \beta_0}) \cdots \theta(\gamma_{u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_2 u_1 \beta_0}) \theta(\gamma_{u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_1 \beta_0}) \\
= \theta(\gamma_{u_r \cdots u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_r \cdots u_2 u_1 \beta_0} \cdots \gamma_{u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_2 u_1 \beta_0} \gamma_{u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_1 \beta_0}).
\]
Next, using the fact observed above that \( \gamma_{s_1 u_r \cdots u_2 u_1 \beta_0} = s_1 u_j \gamma_{u_j-1 \cdots u_2 u_1 \beta_0}, \gamma_{\beta_0} = w \gamma_{u_r \cdots u_2 u_1 \beta_0}, \) we see that
\[
\chi_z(\gamma) = \theta(\gamma_{u_r \cdots u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_r \cdots u_2 u_1 \beta_0} \cdots \gamma_{u_2 u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_2 u_1 \beta_0} \gamma_{u_1 \beta_0}^{-1}s_1\gamma_{s_1 u_1 \beta_0}) \\
= \theta(\gamma_{u_r }^{-1}ws_1(s_1 u_r)s_1(s_1 u_{r-1}) \cdots s_1(s_1 u_1) \gamma_{\beta_0}) \\
= \theta(\gamma_{\beta_0}^{-1}w u_r \cdots u_2 u_1 \gamma_{\beta_0}) \\
= \theta(\gamma_{\beta_0})-1 \theta(\gamma) \theta(\gamma_{\beta_0}),
\]
proving that \( \chi_z \) and \( \theta \) determine the same element of \([\pi_{1}^{(\infty,1)}(M, \beta_0), Z]\).

We summarize all of the above in the following theorem.

**Theorem 5.4.1.** For a fixed coefficient group \( Z \), there are functors from the category \( c\text{Map}_* \) of pairs \( (M, \beta_0) \) where \( M \) is connected to the category \( \text{IGroup} \) given by \( (M, \beta_0) \mapsto D(M; Z) \) and \( (M, \beta_0) \mapsto [\pi_{1}^{\infty}(M, \beta_0), Z] \). Furthermore,

(i) \( \chi : D(-; Z) \rightarrow [\pi_{1}^{\infty}(-), Z] \) is an injective natural transformation between these functors.

(ii) On the subcategory \( \text{Map}_*[l] \) of pairs \( (M, \beta_0) \) where \( M \) is connected and has uniform vertex valency \( l \), the functor \( \chi : D(-; Z) \rightarrow [\pi_{1}^{(\infty,l)}(-), Z] \) is a natural equivalence. Therefore, on this subcategory, \( D(-; Z) \) is a representable functor.
5.4.1 Cocycles, Cohomology, and the Fundamental Group

**Definition 5.4.1 (Cocycles and Cohomology).** Let \( \mathcal{M} \) be a map, let \( Z \) be a group, and let \( z \in C(\mathcal{M}; Z) \). We say that \( z \) is a \( Z \)-valued cocycle if the net voltage around any face is 1. That is, if \( \beta \) is a blade of \( \mathcal{M} \) and if the valency of the face containing \( \beta \) is \( k \), then

\[
z_\beta z_{\beta a} z_{\beta a} \cdots z_{(\beta a)^{k-1} \beta} = 1 \in Z.
\]

Equivalently, \( z \) is a cocycle if the covering \( \mathcal{M}_z \to \mathcal{M} \) is an unramified covering of maps. The set of \( Z \)-valued cocycles is denoted \( Z(\mathcal{M}; Z) \). Note that if \( z, z' \) are equivalent voltages, and if \( z \) is a cocycle, so is \( z' \), and so it makes sense to restrict equivalence of voltages \( \sim \) to \( Z(\mathcal{M}; Z) \) and form the quotient set \( H(\mathcal{M}; Z) = Z(\mathcal{M}; Z)/\sim \), called the \( Z \)-valued cohomology set of \( \mathcal{M} \). As usual, if \( Z \) is an abelian group, then relative to pointwise operations, the above constructions are abelian groups; in particular, we can speak of the cohomology group of \( \mathcal{M} \) with coefficients in \( Z \).\(^5\)

Next, assume that \( \mathcal{M} \) is a uniform map with vertex valency \( l \) and face valency \( k \). Therefore \( \mathcal{M} \) has fundamental group \( \pi_1(\mathcal{M}, \beta_0) = \pi_1^{(k,l)}(\mathcal{M}, \beta_0) \). Note that this is unambiguous by **Corollary 2.2.4.1**. Thus, if \( \Delta = \Delta(k, l) \), then \( \mathcal{M} \cong \Delta/\pi_1(\mathcal{M}, \beta_0) \), and \( \mathcal{M} \) is regular if and only if \( \pi_1(\mathcal{M}, \beta_0) \leq \Delta \).

We note first that if \( \text{UMap}_s \) is the category of uniform based maps, then the fundamental group construction \( \pi_1 \) is functorial. Indeed, let \( \phi : \mathcal{M} \to \mathcal{M}' \) be a morphism \( \mathcal{M} = (B, a, b, c) \to \mathcal{M}' = (B', a', b', c') \). Then as in **Lemma 1.1.2**, we have the morphism of monodromy groups

\[
\phi_* : G = \langle a, b, c \rangle \to G' = \langle a', b', c' \rangle \text{ where } \phi_*(a) = a', \phi_*(b) = b', \phi_*(c) = c'.
\]

If \( \mathcal{M}, \mathcal{M}' \) have vertex valencies \( l, l' \) and face valencies \( k, k' \), respectively, then \( l'|l \) and \( k'|k \). Thus, if we set \( \Delta = \Delta(k, l) \) and \( \Delta' = \Delta(k', l') \), then \( \phi_* \) lifts to a homomorphism \( \Delta \to \Delta' \), where if \( \Delta = \langle s_1, s_2, s_3 \rangle \) and \( \Delta' = \langle s'_1, s'_2, s'_3 \rangle \) then \( \phi_*(s_i) = s'_i, i = 1, 2, 3 \), giving rise to a commutative diagram

\(^5\)We shall have much more to say about cohomology in **Chapter 7**.
If $\beta_0 \in B$ is fixed, and if $\beta'_0 = \beta_0 \phi$, then $\gamma \in \pi_1(\mathcal{M}, \beta_0) (= \text{Stab}_\Delta(\beta_0))$ implies that $\phi_*(\gamma)(\beta'_0) = (\gamma(\beta_0))\phi = \beta_0\phi = \beta'_0$, and so $\phi_* : \pi_1(\mathcal{M}, \beta_0) \to \pi_1(\mathcal{M}', \beta'_0)$.

With respect to cohomology, one has the following version of Theorem 5.4.1 (attention restricted to $\text{UMap}$):

**Theorem 5.4.2. (Duality Theorem)** For any coefficient group $Z$, the natural transformation

$$\chi : \text{H}(-; Z) \to [\pi_1(-), Z]$$

is a natural equivalence of functors.

**Proof.** Both the injectivity and surjectivity of $\chi_\mathcal{M} : \text{H}(\mathcal{M}; Z) \to [\pi_1(\mathcal{M}, \beta_0), Z]$ are proved exactly as in the proof of Theorem 5.4.1.

**Corollary 5.4.2.1.** The cohomology functor $\mathcal{M} \mapsto \text{H}(\mathcal{M}; Z)$ is a representable functor.

### 5.4.2 Non-Uniform Maps, the Combinatorial Fundamental Group and the Characteristic Homomorphism

For non-uniform maps the following line needs to be worked out. What is needed first is an explicit computation of the characteristic map $\chi_z : \pi_1(\mathcal{M}, \beta_0) \to Z$ given an element $[z] \in \text{H}(\mathcal{M}; Z)$. This will involve unique gallery lifting, and the like as initiated in the new chapter on "Fundamental Groups." In this case, recall the construction of
the fundamental group $\pi_1(M, \beta_0)$ of Subsection 2.2. (To be continued ...)

Chapter 6

Exponents of Graphs and Maps

6.1 Exponents of Graphs and Maps

Definition 6.1.1 (Exponents). If $\Gamma = (D, V, I, L)$ is a graph of valency $k$ (=least common multiple of the local valencies), we set $\text{Ex}(\Gamma) = U_k$, the group of units in the multiplicative group of integers modulo $k$. If $e \in \text{Ex}(\Gamma)$, and if $M = (B, a, b, c)$ is a map whose underlying graph is $\Gamma$, we define the $e$-dual of $M$ to be the map $M[e] = (B, a', b', c')$, where $a'\beta = a\beta$, $b'\beta = (bc)^e c\beta$, $c'\beta = c\beta$, $\beta \in B$. As it is easy to check that $a', b', c'$ are involutions, $M[e]$ is indeed a map.\footnote{The maps $M[e]$, where $e$ is an exponent of the underlying graph of $M$ were first defined in Steve Wilson’s paper, Operators over regular maps, Pacific Journal of Mathematics, 81, no. 2 (1979), 559-568. See in particular, Theorem 2, page 563, and denoted $H_e(M)$. Unfortunately his treatment, while intuitive, is difficult to follow.} Furthermore, if $e, f \in \text{Ex}(\Gamma)$, then it is clear that $M[e][f] = M[e f]$. Note that since $e$ is relatively prime to the order of $bc$, then the monodromy group $\langle a, (bc)^e c, c \rangle$ of $M[e]$ agrees with $\langle a, b, c \rangle$, the monodromy group of $M$. As a result, $M$ and $M[e]$ have the same automorphism groups. In particular, $M$ is regular if and only if $M[e]$ is regular. If $M \cong M[e]$, we say that $e$ is an exponent of $M$. The exponents of $M$ form a subgroup $\text{Ex}(M) \leq \text{Ex}(\Gamma)$. Any isomorphism $\phi : M \to M[e]$ that realizes $M \cong M[e]$ is called an exomorphism of $M$. The set of all exomorphisms of $M$ forms a group, denoted $\text{Exo}(M)$, and contains $\text{Aut}(M)$ as a normal subgroup. Note that if $e, f \in \text{Ex}(M)$ and if $\phi_e, \phi_f$ are the corresponding exomorphisms, then $\beta \in B$ implies that $(b\beta)\phi_e = (bc)^e c(\beta\phi_e)$ and $(b\beta)\phi_f = (bc)^e c(\beta\phi_f)$, from which it follows
that

\[(b\beta)\phi_e\phi_f = ((bc)^ec(\beta\phi_e))\phi_f = (bc)^e\phi_f(\beta\phi_e)\phi_f,\]

and so it follows that \(\phi_e\phi_f = \phi_{ef}\). From this it follows immediately that

\[
\text{Exo}(\mathcal{M})/\text{Aut}(\mathcal{M}) \cong \text{Ex}(\mathcal{M}).
\]

**Lemma 6.1.1.** For any exponent \(e \in \text{Ex}(\Gamma)\), \(\mathcal{M}\) is orientable if and only if \(\mathcal{M}[e]\) is orientable.

**Proof.** Let \(G = \langle a, b, c \rangle\) be the monodromy group of \(\mathcal{M}\), and let \(G[e] = \langle a, (bc)^ec, c \rangle\) be the monodromy group of \(\mathcal{M}[e]\). Then \(G^+ = \langle ac, bc \rangle\) and \(G[e]^+ = \langle ac, (bc)^e \rangle = G^+\), since \(e\) is relatively prime to the order of \(bc\). Therefore, \(G^+\) acts transitively on \(B\) if and only if \(G[e]^+\) does, which says that \(\mathcal{M}\) is non-orientable if and only if \(\mathcal{M}[e]\) is.

**Proposition 6.1.2.** Let \(\mathcal{M} = (B, a, b, c)\) be a map with monodromy group \(G = \langle a, b, c \rangle\). Then \(e\) is an exponent of \(\mathcal{M}\) if and only if there exists an automorphism \(\tau\) of \(G\) such that \(\tau(bc) = (bc)^e\) and \(\tau(a) = a, \tau(c) = c\), and such that \(\tau\) permutes the blade stabilizers in \(\mathcal{M}\).

**Proof.** Let \(\phi: \mathcal{M} \xrightarrow{\cong} \mathcal{M}[e]\) be an exomorphism realizing \(\mathcal{M} \cong \mathcal{M}[e]\), and let \(\phi_b: G \rightarrow G\) be the corresponding map (as in **Lemma 1.1.2**). Then setting \(\tau = \phi_b\), we see that \(\tau(a) = a, \tau(c) = c\) and \(\tau(b) = (bc)^ec = (bc)^e\tau(c)\) from which we conclude that \(\tau(bc) = (bc)^e\).

Conversely, assume that the automorphism \(\tau\) of \(H\) exists as claimed. Fix a blade \(\beta_0 \in B\) and let \(H\) be the stabilizer in \(G\) of \(\beta_0\). By assumption, \(\tau(H) = gHg^{-1}\) for some \(g \in G\). If \(\beta \in B\), let \(g' \in G\) satisfy \(g'\beta_0 = \beta\). Now define \(\beta\phi = \tau(g')g\beta_0\). It is routine to check that \(\phi\) is a well-defined exomorphism of \(\mathcal{M}\) with the correct properties.

The above proposition motivates the following.

**Definition 6.1.2.** Let \(\mathcal{M} = (B, a, b, c)\) be a map. The exponent \(e \in \text{Ex}(\mathcal{M})\) is called an **inner exponent** if there exists an inner automorphism \(\tau\) of \(G\) such that \(\tau(bc) = (bc)^e\) and \(\tau(a) = a, \tau(c) = c\).
By Corollary 3.4.1.1, the following is immediate:

**Corollary 6.1.2.1.** If $\mathcal{M}$ is antipodal, then $-1$ is an inner exponent of $\mathcal{M}$.

**Remark.** The converse of the above is false – use the same example given on page 43.

**Exercise.** If $\mathcal{M}$ is the map of the icosahedron, compute all relevant parameters (vertex and face valencies, as well as the genus) of the map $\mathcal{M}[2]$. This map is sometimes called the *great dodecahedron*.

**Definition 6.1.3 (Involutory Exponents).** We say that the exponent $e \in \text{Ex}(\Gamma)$ is *involutory* if $e^2 = 1$. In this case it is easy to check that if $\mathcal{M}$ is a map with underlying graph $\Gamma$, then $\mathcal{M}[e][e] = \mathcal{M}[e^2] = \mathcal{M}$. The group of involutory exponents of $\Gamma$ is denoted $\text{Ex}_2(\Gamma)$ and the group of involutory exponents of $\mathcal{M}$ is denoted $\text{Ex}_2(\mathcal{M})$.

**Definition 6.1.4 (Derived Exponent Maps).** Let $\mathcal{M} = (B, a, b, c)$ be a map, let $Z$ be a group and let $z \in C(\mathcal{M}; Z)$. Assume that $\Gamma$ is the underlying graph of $\mathcal{M}$, and that $\epsilon : Z \to \text{Ex}(\Gamma)$ is a homomorphism. Then we may form the new map $\mathcal{M}^{(e)}_z = (B \times Z, a^{(e)}_z, b^{(e)}_z, c^{(e)}_z)$, where

\[
\begin{align*}
a^{(e)}_z(\beta, \zeta) &= (a\beta, \zeta z\beta) \\
b^{(e)}_z(\beta, \zeta) &= ((bc)^e(\zeta)c\beta, \zeta) \\
c^{(e)}_z(\beta, \zeta) &= (c\beta, \zeta).
\end{align*}
\]

If $Z = \{\pm 1\}$ and $\epsilon(-1) = e \in \text{Ex}(\Gamma)$, we sometimes write $\mathcal{M}^{(e)}_z$ in place of $\mathcal{M}^{(e)}_z$.

**Remark.** Note first that if $Z$ is a group and if $z \in C(\mathcal{M}; Z)$, then the maps of the form $\mathcal{M}^{(1)}_z$ are precisely the principal derived maps over $\mathcal{M}$. However, notice that in contrast with the principal derived
maps, projection onto the first coordinate in \( \mathcal{M}_x^{(e)} \to \mathcal{M} \) need not be a morphism of maps. Furthermore, the isomorphism class of \( \mathcal{M}_x^{(e)} \) is not invariant under equivalence of voltages. In other words, it is entirely possible for \( z \sim z' \) in \( C(\mathcal{M}; Z) \) and yet \( \mathcal{M}_x^{(e)} \not\cong \mathcal{M}_{z'}^{(e)} \). This needs to be explored further.

**Remark.** Let \( \mathcal{M} = (B, a, b, c) \) be a map, with monodromy group \( G = \langle a, b, c \rangle \). Let \( Z \) be cyclic of order 2, and let \(-1 \in C(\mathcal{M}; Z)\) denote the voltage having constant value \(-1\) on each blade of \( \mathcal{M} \). If \( \epsilon : Z \to \text{Ex}(\Gamma) \) is a homomorphism, and if \( p(\cdot) \) denotes Petrie dual\(^2\) note that \( p(\mathcal{M}_x^{(e)}) = (p(\mathcal{M}))_{\epsilon}^{-1} \) as \( a_{\epsilon}^{(-1)} c_{\epsilon}^{(-1)} = (ac)^{(-1)} : B \times Z \to B \times Z \).

**Exercise.** As above, let \( \mathcal{M} = (B, a, b, c) \) be a map, with monodromy group \( G = \langle a, b, c \rangle \). Let \( Z \) be cyclic of order 2, and let \(-1 \in C(\mathcal{M}; Z)\) denote the voltage having constant value \(-1\) on each blade of \( \mathcal{M} \). Let \( \epsilon : Z \to \text{Ex}(\Gamma) \) be a homomorphism. Show that \( p(\mathcal{M}_x^{(e)}) \cong \mathcal{M}_{\epsilon}^{(-e)} \).

**Solution.** Let \( \sigma : Z \to \langle c \rangle \) be \(-1 \mapsto c\), and define \( \phi : B \times Z \to B \times Z \) by setting \( (\beta, \zeta) \phi = (\sigma(\zeta), \zeta) \). Show that \( \phi \) realizes \( p(\mathcal{M}_x^{(e)}) \cong \mathcal{M}_{\epsilon}^{(-e)} \). Conclude that also \( (p(\mathcal{M}))_{\epsilon}^{-1} \cong \mathcal{M}_{\epsilon}^{(-e)} \), and that \( \mathcal{M}_{\epsilon}^{(e)}, \mathcal{M}_{\epsilon}^{(-e)} \) have the same automorphisms groups.

In what follows, we retain the above notation: \( \mathcal{M} = (B, a, b, c) \) is a map with monodromy group \( G = \langle a, b, c \rangle, Z = \{ \pm 1 \} \) is a cyclic group of order 2, and \(-1 \in C(\mathcal{M}; Z)\) denotes the voltage having constant value \(-1\) on each blade of \( \mathcal{M} \). We wish to study the symmetry properties of \( \mathcal{M}_{\epsilon}^{(e)} \), relative to a fixed homomorphism \( \epsilon : Z \to \text{Ex}(\Gamma) \).

Let \( \phi \in \text{Aut}(\mathcal{M}) \) and define \( \tilde{\phi} : B \times Z \to B \times Z \) by setting \( (\beta, \zeta) \phi = (\beta \phi, \zeta) \). Since \( \phi \) commutes with \( a, b, c \), it follows easily that \( \tilde{\phi} \) commutes with \( a_{\epsilon}^{(e)}, b_{\epsilon}^{(e)}, \) and \( c_{\epsilon}^{(e)} = c \times 1_Z \). Therefore, \( \tilde{\phi} \in \text{Aut}(\mathcal{M}_{\epsilon}^{(e)}) \) and the mapping \( \phi \mapsto \tilde{\phi}\) gives an injection \( \text{Aut}(\mathcal{M}) \hookrightarrow \text{Aut}(\mathcal{M}_{\epsilon}^{(e)}) \).

Next, let \( e = \epsilon(-1) \in \text{Ex}(\Gamma) \), and assume that, in fact, \( e \) is an

\(^2\)If \( \mathcal{M} = (B, a, b, c) \), the **Petrie dual** is the map \( p(\mathcal{M}) = (B, ac, b, c) \). We shall have much more to say about the Petrie dual in Chapter 9.
exponent of $\mathcal{M}$. Therefore, there exists an isomorphism

$$\alpha : \mathcal{M} \rightarrow \mathcal{M}[e].$$

In particular, $\alpha : B \rightarrow B$ commutes with $a$ and $c$, and

$$(b\beta)\alpha = (bc)^c c(\beta \alpha), \quad (6.1)$$

for all $\beta \in B$. Therefore, it follows that, for all $\beta \in B$, $(bc\beta)\alpha = (bc)^c (\beta \alpha)$, from which we conclude that $((bc)^c \beta)\alpha = ((bc)^c)^c (\beta \alpha) = (bc)^{c^2} (\beta \alpha) = (bc)(\beta \alpha)$, since $e$ is involutory. In particular, since $\alpha$ commutes with $c$ we can express this as

$$((bc)^c c\beta)\alpha = b(\beta \alpha), \quad (6.2)$$

for all $\beta \in B$. We can summarize (6.1), (6.2) by writing

$$((bc)^c c\beta)\alpha = (bc)^{c(-\zeta)} c(\beta \alpha). \quad (6.3)$$

Define $\tilde{\alpha} : B \times Z \rightarrow B \times Z$ by setting $(\beta, \zeta)\tilde{\alpha} = (\beta \alpha, -\zeta)$. We check that, in fact, $\tilde{\alpha} \in \text{Aut}(\mathcal{M}_{-1}^{e})$. First of all

$$(a_{-1}^{(e)}(\beta, \zeta))\tilde{\alpha} = (a\beta, -\zeta)\tilde{\alpha}$$

$$= (a\beta \alpha, \zeta)$$

$$= a_{-1}^{(e)}(\beta \alpha, -\zeta)$$

$$= a_{-1}^{(e)}((\beta, \zeta)\tilde{\alpha}).$$

Next,

$$(b_{-1}^{(e)}(\beta, \zeta))\tilde{\alpha} = ((bc)^{c(-\zeta)} c\beta, \zeta)\tilde{\alpha}$$

$$= (((bc)^{c(-\zeta)} c\beta)\alpha, -\zeta)$$

$$= ((bc)^{c(-\zeta)} c(\beta \alpha), -\zeta) \quad \text{(by (6.3))}$$

$$= b_{-1}^{(e)}(\beta \alpha, -\zeta)$$

$$= b_{-1}^{(e)}((\beta, \zeta)\tilde{\alpha}).$$

Obviously $\tilde{\alpha}$ commutes with $c_{-1}^{(e)} = c \times 1_Z$, and so $\tilde{\alpha} \in \text{Aut}(\mathcal{M}_{-1}^{e})$, as claimed.

Finally, note that if $\mathcal{M}$ is regular, then obviously so is $\mathcal{M}_{-1}^{e}$. We summarize the above results as follows.
Theorem 6.1.3. Let $\mathcal{M} = (B, a, b, c)$ be a map, and let $Z = \{ \pm 1 \}$ be a cyclic group of order 2. Let $-1 \in C(\mathcal{M}; Z)$ denote the voltage having constant value $-1$ on each blade of $\mathcal{M}$ and fix a homomorphism $\epsilon : -1 \mapsto \epsilon \in \text{Ex}(\Gamma)$.

1. If $\sigma \in \text{Aut}(\mathcal{M})$ then the mapping $\tilde{\sigma} : B \times Z \to B \times Z$ defined by $(\beta, \zeta)\tilde{\sigma} = (\beta \sigma, \zeta)$ defines an automorphism of $\mathcal{M}^{(e)}_{-1}$.

2. If $\epsilon \in \text{Ex}_2(\mathcal{M})$, and if $\alpha : \mathcal{M} \to \mathcal{M}[\epsilon]$, then the mapping $\tilde{\alpha} : B \times Z \to B \times Z$ defined by $(\beta, \zeta)\tilde{\alpha} = (\beta \alpha, -\zeta)$ defines an automorphism of $\mathcal{M}^{(e)}_{-1}$.

3. If $\mathcal{M}$ is regular and $\epsilon \in \text{Ex}_2(\mathcal{M})$, then so is $\mathcal{M}^{(e)}_{-1}$.

Remark. Part (3) of the above theorem was proved by Nedela and Škoviera\(^3\) when $\mathcal{M}$ is an oriented map.

As for the monodromy group of $\mathcal{M}^{(e)}_{-1}$, we can say the following.

Theorem 6.1.4. Let $\mathcal{M} = (B, a, b, c)$ be a regular map with monodromy group $G = \langle a, b, c \rangle$, and let $-1 \in C(\mathcal{M}; Z)$ be the voltage on $\mathcal{M}$ as above. If $\epsilon : -1 \mapsto \epsilon \in \text{Ex}_2(\mathcal{M})$, and if $G^{(e)}$ denotes the monodromy group of $\mathcal{M}^{(e)}_{-1}$, then $G^{(e)}$ is isomorphic with the semidirect product $G \rtimes \langle \tau \rangle$, where $\tau$ is the automorphism fixing $a$ and $c$ and satisfying $\tau(bc) = (bc)^e$.

Proof. Note that the automorphism $\tau$ satisfies $\tau^i((bc)^{\epsilon(\zeta)}) = (bc)^{\epsilon(-1)^i\zeta)}$, for all $\zeta \in Z$ and for all indices $i$. Next, by regularity of $\mathcal{M}$, we may identify the set $B$ of blades of $\mathcal{M}$ with its monodromy group $G$. With this identification, each element $g \tau^i \in G \rtimes \langle \tau \rangle$ determines a mapping $G \times Z \to G \times Z$ by the rule

$$(g', \zeta)g \tau^i = (\tau^i(g'g), (-1)^i\zeta).$$

\(^3\)Regular embeddings of canonical double coverings of graphs, J. Comb. Theory, Series B, 67 (1996), 249-277; see Theorem 5.1, page 258.
Since $\tau$ fixes $a$ and $c$, it is routine to check that this mapping commutes with the monodromy involutions $a_{-1}^{(e)}$ and $c_{-1}^{(e)}$. Next, we have that

$$
(b_{-1}^{(e)}(g', \zeta))g\tau^i = ((bc)^{(e)}_{g'}c, \zeta)g\tau^i
= (\tau^i((bc)^{(e)}_{c}g', (1)) \zeta)
= ((bc)^{(e)}_{c}(-1)^i\zeta) c_{\tau}^{i}(g' g, (-1)^i \zeta)
= b_{-1}^{(e)}(\tau^i(g' g, (-1)^i \zeta))
= b_{-1}^{(e)}((g', \zeta)g\tau^i).
$$

Therefore, we see that the element $g\tau^i$ does, indeed, determine an automorphism of $M_{-1}^{(e)}$. That the induced mapping $G \times \langle \tau \rangle \rightarrow \text{Aut}(M_{-1}^{(e)})$ is an injective homomorphism is routine. The result follows.

\section*{6.2 Canonical Double Cover of a Graph}

If $M$ is a map with underlying graph $\Gamma$, the underlying graph of the derived exponent maps of the form $M_{-1}^{(e)}$, where $e : \{\pm 1\} \rightarrow \text{Ex}(\Gamma)$ is a homomorphism, is what is called the “canonical double cover” of the graph $\Gamma$, whose definition is formalized below.

\textbf{Definition 6.2.1 (Canonical Double Cover).} Let $\Gamma = (D, V, I, L)$ be a graph. We define the canonical double cover $\tilde{\Gamma} = (\tilde{D}, \tilde{V}, \tilde{I}, \tilde{L})$, where $\tilde{D} = D \times \{\pm 1\}$, $\tilde{V} = V \times \{\pm 1\}$, and where $\tilde{I}(d, \zeta) = (I(d), \zeta)$, $\tilde{L} = (L(d), -\zeta)$. We have that $\tilde{\Gamma} \rightarrow \Gamma$ given by projection onto the first coordinate is a graph morphism.

It is clear that the canonical double cover $\tilde{\Gamma}$ of $\Gamma$ is connected precisely when $\Gamma$ is not bipartite.

Note that if $\sigma \in \text{Aut}(\Gamma)$, then $\sigma$ determines an automorphism $\tilde{\sigma} \in \text{Aut}(\tilde{\Gamma})$ via

$$
(d, \zeta)\tilde{\sigma} = (d\sigma, \zeta), \quad (v, \zeta)\tilde{\sigma} = (v\sigma, \zeta),
$$

where $d \in D, v \in V$. Likewise, there is also the “deck automorphism” $\tau \in \text{Aut}(\tilde{\Gamma})$, given by

$$
(d, \zeta)\tau = (d, -\zeta).
$$

This allows us to identify $\text{Aut}(\Gamma) \times \langle \tau \rangle$ with a subgroup of $\text{Aut}(\tilde{\Gamma})$. 
DEFINITION 6.2.2 (Stability). If the above inclusion $\text{Aut}(\Gamma) \times \langle \tau \rangle \hookrightarrow \text{Aut}(\tilde{\Gamma})$ is surjective, we say that the graph $\Gamma$ is stable.

While the interest in the stability of graphs seems to be mostly concentrated on finite arc-transitive graphs, we would like to provide here an example of an infinite arc-transitive unstable graph which particularly clearly illustrates the essence of instability.

To this end, let $(V, B)$ be a 3-dimensional real inner product space with positive-definite symmetric bilinear form $B$. We let $(\mathcal{P}, \mathcal{L})$ be the points and lines of the real projective plane $\mathbb{R}P^2$, and form the graph $\Gamma = \Gamma(V, E)$ to have vertex set $V = \mathcal{P}$, and incidence given by $p \sim q$ if and only if $p \perp q$ (relative to $B$). Note that the real orthogonal group $O(3)$ (relative to $B$) clearly acts arc-transitively on $\Gamma$.

Next let $\Gamma'$ be the bipartite graph consisting of the points and lines of $\mathbb{R}P^2$ with incidence given by containment. Thus $\Gamma'$ admits all of $\text{PGL}_3(\mathbb{R})$ as a group of automorphisms. (By the Fundamental Theorem of Projective Geometry, this is the complete group of automorphisms of $\Gamma'$ preserving the partite classes $\mathcal{P}$ and $\mathcal{L}$.) Note that $\Gamma'$ also admits the “polarity” $t : \Gamma' \rightarrow \Gamma'$ given by $t(x) = x^\perp$, where $x \in \mathcal{P} \cup \mathcal{L}$, and so $t$ interchanges the partite classes $\mathcal{P}$ and $\mathcal{L}$. Note, finally, that the centralizer of $t$ in $\text{PGL}_3(\mathbb{R})$ is precisely $\text{PCO}(3) \cong \text{PSO}(3)$, where $\text{CO}(3)$ is the subgroup of $\text{GL}_3(\mathbb{R})$ consisting of similitudes of $V$ with respect to $B$, i.e., elements $g \in \text{GL}_3(\mathbb{R})$ such that

$$B(g(v), g(v')) = \xi_g B(v, v'), \quad \xi_g \in \mathbb{R}^\times,$$

and where $\text{PSO}(3)$ is the projective special orthogonal group.

Finally, let $\tilde{\Gamma}$ be the canonical double covering of $\Gamma$ and note that $\tilde{\Gamma} \cong \Gamma'$ via the isomorphism

$$(x, \zeta) \mapsto \begin{cases} x & \text{if } \zeta = 1, \\ x^\perp & \text{if } \zeta = -1. \end{cases}$$

Under this isomorphism, the “covering transformation” $(x, \zeta) \mapsto (x, -\zeta)$ of $\tilde{\Gamma}$ corresponds to the polarity $t$ of $\Gamma'$. As $t$ is not centralized by $\text{PGL}_3(\mathbb{R})$, we conclude that $\Gamma$ is unstable.

---

I have recently found\textsuperscript{5} three infinite families of finite arc-transitive unstable graphs. Previously, the only known non-trivial examples were the graphs of the dodecahecr and of the icosahedron.

**Proposition 6.2.1.** Assume that $\Gamma$ is connected and non-bipartite (so $\tilde{\Gamma}$ is connected). Then $\Gamma$ is stable if and only if the automorphism $\tau \in \mathbb{Z}(\text{Aut}(\tilde{\Gamma}))$.

**Proof.** Note first that stability of $\Gamma$ clearly implies that $\tau \in \mathbb{Z}(\text{Aut}(\tilde{\Gamma}))$. Conversely, assume that $\tau \in \mathbb{Z}(\text{Aut}(\tilde{\Gamma}))$. We already have the injection $\text{Aut}(\Gamma) \hookrightarrow \text{Aut}(\tilde{\Gamma})$ given by $\sigma \mapsto ((d, \zeta) \mapsto (d\sigma, \zeta))$, $\sigma \in \text{Aut}(\Gamma)$. Also, if $\pi : \text{Aut}(\tilde{\Gamma}) \to \text{Aut}(\Gamma)$ is the first coordinate projection, then we define a homomorphism $\rho : \text{Aut}(\tilde{\Gamma}) \to \text{Aut}(\Gamma)$ by

$$\psi \mapsto (d \mapsto (d, \zeta)\psi\pi), \quad \psi \in \text{Aut}(\tilde{\Gamma}).$$

Note that the definition of $\rho$ is independent of our particular choice of $\zeta \in \{\pm 1\}:

$$(d, \zeta)\psi\pi = (d, \zeta)\psi\tau\pi = (d, \zeta)\tau\psi\pi = (d, -\zeta)\psi\pi$$

This is clearly a homomorphism, and there results a short exact sequence

$$1 \longrightarrow K \longrightarrow \text{Aut}(\tilde{\Gamma}) \xrightarrow{\rho} \text{Aut}(\Gamma) \longrightarrow 1,$$

where $K = \{\psi \in \text{Aut}(\tilde{\Gamma}) | (d, \zeta)\psi = (d, \zeta_d), \; d \in D\}$.

Assume that for some dart $\tilde{d} \in \tilde{D}$ we have $\tilde{d}\psi = \tilde{d}$. We shall show, in fact, that $\psi$ fixes all darts in $\tilde{\Gamma}$. If $\tilde{d} \in \tilde{D}$, the connectivity of $\tilde{\Gamma}$ implies that there exists a sequence

$$\tilde{d}_0 = \tilde{d}, \tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_r = \tilde{d},$$

where $\tilde{\mathcal{L}}(\tilde{d}_i) = \tilde{\imath}(\tilde{d}_{i+1})$, $i = 0, 1, 2, \ldots, r - 1$. By induction, it suffices to assume that $\tilde{\mathcal{L}}(\tilde{d}) = \tilde{\imath}(\tilde{d})$ and to show that $\tilde{d}\psi = \tilde{d}$. In this case,

$$\tilde{\imath}(\tilde{d}\psi) = \tilde{\mathcal{L}}(\tilde{d})\psi = \tilde{\mathcal{L}}(\tilde{d}\psi) = \tilde{\mathcal{L}}(\tilde{d}) = \tilde{\imath}(\tilde{d}),$$

and so \( \psi \) fixes \( \tilde{d'} \). Write \( \tilde{d'} = (d', \zeta) \); by assumption \( (d', \zeta)\psi = (d', \pm \zeta) \). If \( (d', \zeta)\psi = (d', -\zeta) \), then setting \( \nu' = I(d') \) gives

\[
\begin{align*}
(v', \zeta) &= \tilde{I}(d', \zeta) \\
&= \tilde{I}(\tilde{d'}) \\
&= (\tilde{I}(\tilde{d'})\psi \\
&= \tilde{I}(\tilde{d'}\psi) \\
&= \tilde{I}(d', -\zeta) \\
&= (v', -\zeta),
\end{align*}
\]

an obvious contradiction. Thus the only possibilities for \( K \) are \( 1_\Gamma \) and \( \tau \), which proves stability.

### 6.3 Regular Embeddings of the Canonical Double Cover of a Graph

One of the major industries in map theory is, for a given graph \( \Gamma \), find all regular maps \( \mathcal{M} \) whose underlying graph is \( \Gamma \), in which case we call \( \mathcal{M} \) a “regular imbedding” of \( \Gamma \). Yet the only “interesting” graphs for which a complete solution is available are the complete graphs.\(^6\) The present section deals with the following problem. If the regular embeddings of the graph \( \Gamma \) are known, are those of the canonical double cover \( \tilde{\Gamma} \) also known? Roman Nedela and Martin Škoviera\(^7\) have shown that if \( \Gamma \) is stable, then the regular embeddings of \( \tilde{\Gamma} \) are of the form \( \mathcal{M}^{(e)}_{-1} \) for some involutory exponent \( e \) (i.e. \( e = e(-1) \in \text{Ex}(\Gamma) \) is involutory). We shall summarize the relevant results in this section.

**Lemma 6.3.1.** Let \( \mathcal{M} = (B, a, b, c) \), \( \mathcal{M}' = (B, a', b', c') \) be regular maps with the same underlying graph \( \Gamma \) of valency \( k > 2 \). If \( \text{Aut}(\mathcal{M}) = \text{Aut}(\mathcal{M}') \) (as subgroups of the symmetric group on \( B \)), then there exists an exponent \( e \) such that \( b'c' = (bc)^e \).


\(^7\)Op. cit.
Proof. Set $G = \langle a, b, c \rangle$, $G' = \langle a', b', c' \rangle$, the monodromy groups of $\mathcal{M}$ and $\mathcal{M}'$, respectively. By regularity of $\mathcal{M}$, $\mathcal{M}'$ we conclude that $G, G'$ are the centralizers in the symmetric group on $B$ of $\text{Aut}(\mathcal{M}) = \text{Aut}(\mathcal{M}')$, and so $G = G'$, as well. Since $\mathcal{M}$ and $\mathcal{M}'$ have the same underlying graph, we see that the $\langle b, c \rangle$-orbits and the $\langle b', c' \rangle$-orbits are the same, and since $G$ acts regularly on $B$, it follows easily that $\langle b, c \rangle = \langle b', c' \rangle$. Since we have assumed that $k > 2$, there is a unique cyclic subgroup of order $k$ in the dihedral group $\langle b, c \rangle$, viz., $\langle bc \rangle = \langle b'c' \rangle$. It follows that $b'c' = (bc)^e$ for some exponent $e$.

Remark. When the valency of $\Gamma$ is 2, then $\Gamma$ is just an $n$-circuit for some $n$ and is the underlying graph only of the 2-sphere (with two faces) or of the real projective plane (with one face). Similarly, if $n$ is odd, then $\tilde{\Gamma}$ is the $2n$-circuit, and if $n$ is even, then $\Gamma$ is bipartite and $\tilde{\Gamma}$ is the disjoint union of two copies of $\Gamma$.

Nedela and Škoviera proved the following in the context of oriented maps.\(^8\)

Theorem 6.3.2. Let $\Gamma$ be a stable graph of valency greater than 2, and let $\tilde{\mathcal{M}}$ be a regular imbedding of the canonical double cover $\tilde{\Gamma}$ of $\Gamma$. Then $\tilde{\mathcal{M}} \cong \mathcal{M}_{-1}^{[e]}$ for some regular embedding $\mathcal{M}$ of $\Gamma$ and some involutory exponent $\epsilon : \{\pm 1\} \to \text{Ex}(\Gamma)$.

Outline of Proof. Let $\tilde{\mathcal{M}} = (\tilde{B}, \tilde{a}, \tilde{b}, \tilde{c})$; one can identify $\tilde{B}$ with $B \times Z$, $Z = \{\pm 1\}$ in such a way that if $\pi : B \times Z \to Z$ is projection onto first coordinate, then

$$(\tilde{a}(\beta, \zeta))\pi = -\zeta, (\tilde{b}(\beta, \zeta))\pi = \zeta, (\tilde{c}(\beta, \zeta))\pi = \zeta$$

where $\beta \in B$ and $\zeta \in Z$. For $\zeta = \pm 1$, define $a_\zeta, b_\zeta, c_\zeta : B \to B$ by setting

$$\tilde{a}(\beta, \zeta) = (a_\zeta \beta, -\zeta), \tilde{b}(\beta, \zeta) = (b_\zeta \beta, \zeta), \tilde{c}(\beta, \zeta) = (c_\zeta \beta, \zeta),$$

$\beta \in B, \zeta \in Z$. One easily checks that $a_\zeta, b_\zeta, c_\zeta$ are involutions and so we have map structures $\mathcal{M}_\zeta = (B, a_\zeta, b_\zeta, c_\zeta)$ on $B$.

\(^8\)Ibid., Theorem 5.2, page 257.
Next we have $\text{Aut}(\widetilde{\mathcal{M}}) \leftrightarrow \text{Aut}(\widetilde{\Gamma}) = \text{Aut}(\Gamma) \times \langle \tau \rangle$, where we have used the fact that $\Gamma$ is stable.\footnote{That $\text{Aut}(\mathcal{M}) \to \text{Aut}(\Gamma)$ is an injection follows from the fact that the valency is greater than 2. When $\Gamma$ is a circuit bounding the two halves of a sphere, one can interchange the two faces without moving any vertex or edge, and so in this case injectivity breaks down.} Here, we identify $\text{Aut}(\Gamma)$ with those automorphisms of $\text{Aut}(\widetilde{\Gamma})$ of the form $(d, \zeta) \mapsto (d\sigma, \zeta)$, $d \in D$, $\zeta \in Z$. Therefore, $\text{Aut}(\widetilde{\mathcal{M}}) \cap \text{Aut}(\Gamma)$ has index at most 2 in $\text{Aut}(\mathcal{M})$; set $\text{Aut}_0(\mathcal{M}) = \text{Aut}(\mathcal{M}) \cap \text{Aut}(\Gamma)$. Thus, $\psi \in \text{Aut}_0(\mathcal{M})$ if and only if $\psi$ has the form $(\beta, \zeta)\psi = (\beta\theta, \zeta)$ (i.e., $\psi\pi = \pi\theta$, where $\pi$ is projection onto first coordinate), where $\theta : B \to B$ and $\psi$ commutes with $\widetilde{a}, \widetilde{b}, \widetilde{c}$.

**Claim:** The mapping $\theta : B \to B$ commutes with $a_\zeta, b_\zeta$ and $c_\zeta$, $\zeta = \pm 1$.

**Proof of Claim.** If $\beta \in B$, $\zeta = \pm 1$, we have

\[
(a_\zeta\beta)\theta = (\widetilde{a}(\beta, \zeta))\pi\theta \\
= (\widetilde{a}(\beta, \zeta))\psi\pi \\
= (\widetilde{a}(\beta, \zeta)\psi)\pi \\
= (\widetilde{a}(\beta\theta, \zeta))\pi \\
= a_\zeta(\beta\theta).
\]

Similarly, one shows that $\theta$ commutes with both $b_\zeta$ and $c_\zeta$, proving the claim.

From the above claim, we conclude that $\theta \in \text{Aut}(\mathcal{M}_1) \cap \text{Aut}(\mathcal{M}_{-1})$, from which we obtain an injective mapping

$$\text{Aut}_0(\mathcal{M}) \leftrightarrow \text{Aut}(\mathcal{M}_1) \cap \text{Aut}(\mathcal{M}_{-1}).$$

Since $\mathcal{M}$ is regular, we have that $|\text{Aut}_0(\mathcal{M})| = |B|$, and so $|\text{Aut}(\mathcal{M}_1)|$, $|\text{Aut}(\mathcal{M}_{-1})| \geq |B|$. This implies, of course, that $|\text{Aut}(\mathcal{M}_1)| = |B| = |\text{Aut}(\mathcal{M}_{-1})|$, and so both $\mathcal{M}_1$, $\mathcal{M}_{-1}$ are regular and $\text{Aut}(\mathcal{M}_1) = \text{Aut}(\mathcal{M}_{-1})$.

Now we apply Lemma 6.3.1 and infer that $b_{-1}c_{-1} = (b_{1}c_{1})^e$ for some exponent $e$.

**Claim:** $\mathcal{M}_1 \cong \mathcal{M}_{-1}$ and so $e \in \text{Exp}(\mathcal{M})$. 
**Proof.** Since \( \tilde{\mathcal{M}} \) is regular, there exists an automorphism \( \phi \in \text{Aut}(\tilde{\mathcal{M}}) - \text{Aut}_0(\tilde{\mathcal{M}}) \). Therefore, \( \phi \) is of the form

\[
(\beta, \zeta) \phi = (\beta \theta, -\zeta),
\]

for some \( \theta : B \to B \). We shall show that \( \theta \) induces an isomorphism \( \mathcal{M}_\zeta \cong \mathcal{M}_{-\zeta} \). We have

\[
(a_\zeta \beta) \theta = (\tilde{a}(\beta, \zeta) \pi) \theta \\
= (\tilde{a}(\beta, \zeta) \phi) \pi \\
= (\tilde{a}((\beta, \zeta) \phi)) \pi \\
= (\tilde{a}(\beta \theta, -\zeta)) \pi \\
= a_{-\zeta}(\beta \theta).
\]

Similarly, one proves that \( \theta \) likewise intertwines \( b_\zeta, b_{-\zeta} \) and \( c_\zeta, c_{-\zeta} \). This not only proves that \( e \) is an exponent, it’s involutory. Finally, the above work shows that \( \mathcal{M} \cong \mathcal{M}^{(e)} \), proving the theorem.

The last result of this section is an extension of Theorem 5.3 of Nedela and Škoviera’s paper\(^{10}\) to maps (as opposed to oriented maps). This result classifies the regular embeddings of the canonical double cover \( \tilde{\Gamma} \).

**Theorem 6.3.3.** Let \( \mathcal{M} \) and \( \mathcal{N} \) be regular embeddings of the stable graph \( \Gamma \) and let \( e \in \text{Ex}(\mathcal{M}) \), \( f \in \text{Ex}(\mathcal{N}) \). Then \( \mathcal{M}^{(e)} \cong \mathcal{N}^{(f)} \) if and only if \( \mathcal{M} \cong \mathcal{N} \) and \( e \equiv f \pmod{k} \), where \( k \) is the valency of \( \Gamma \).

\(^{10}\)Ibid., page 261.
Chapter 7

Homology

7.1 Integral Homology of Oriented and Unoriented Maps

We begin by reviewing the salient features of what shall be called oriented homology for oriented maps. (Actually the same can be done for hypermaps.)\(^1\) Thus, let \(\mathcal{M} = (D, P, L)\) be an oriented map, and denote by \([d]_V, [d]_E, [d]_F\) the vertex, edge and face determined by the dart \(d \in D\). Thus

\[
[d]_V = \langle P \rangle\text{-orbit of } d \text{ in } D, \\
[d]_E = \langle L \rangle\text{-orbit of } d \text{ in } D, \\
[d]_F = \langle PL \rangle\text{-orbit of } d \text{ in } D.
\]

We let \(W\) be the free abelian group on the set \(D\) of darts. Thus, \(W\) consists of functions \(w : D \to \mathbb{Z}\) of finite support (automatically satisfied if \(D\) is finite), and with pointwise addition. For each finite subset \(Y \subseteq D\), we define the characteristic function \(\chi_Y \in W\) by

\[
\chi_Y(d) = \begin{cases} 
1 & \text{if } d \in Y \\
0 & \text{if } d \notin Y.
\end{cases}
\]

If \(Y = \{d\}\), for some \(d \in D\), we often write \(\chi_d\) in place of \(\chi_{\{d\}}\). Thus, we see that \(W\) is the free abelian group with basis \(\{\chi_d \mid d \in D\}\). The monodromy group \(G = \langle P, L \rangle\) acts on \(D\) on the left and hence acts on

\(^1\)This was first defined for hypermaps, but over field coefficients by A. Machi, Homology of hypermaps, *J. London Math. Soc.*, 31 (1985), 10–16.
$W$ on the right via $wg(d) = w(gd)$, $w \in W$, $g \in G$, $d \in D$. For any subgroup $H$ of $G$, we denote the subgroup of $H$-invariants in $W$ by

$$W^H = \{ w \in W \mid wh = w \text{ for all } h \in H \}. $$

We proceed to define a chain complex $(C_*(M), \partial_*)$, functorial in $M$, as follows. We set

$$C_0(M) = W^{(P)}, \quad C_1(M) = W/W^{(L)}, \quad C_2(M) = W^{(PL)}. $$

Note that $C_i(M)$, $i = 0, 1, 2$ are free abelian groups of ranks equal to $|V|, |E|, \text{ and } |F|$, the numbers of vertices, edges and faces in $M$ (at least when these numbers are finite.)

The “boundary maps” are defined as follows. For any $w \in W$, $d \in D$, define

$$ (\partial_1w)(d) = \sum_{d' \in [d]_V} (w(d') - w(Ld')) \in \mathbb{Z}. $$

It is clear that $(\partial_1w)(Pd) = (\partial_1w)(d)$ and so it follows that $\partial_1w \in W^{(P)}$. Equivalently, this same mapping can be defined in terms of the characteristic functions via

$$ \chi_d \mapsto \chi_{[d]_V} - \chi_{[Ld]_V}, \quad d \in D. $$

Since $w \in W^{(L)}$ implies that $\partial_1w = 0$, we see that $\partial_1$ factors through $W/W^{(L)} = C_1(M)$, giving a map $\partial_1 : C_1(M) \rightarrow C_0(M)$. The map $\partial_2 : C_2(M) \rightarrow C_0(M)$ is even easier to define. Here we just compose:

$$ \partial_2 : C_2(M) = W^{(PL)} \hookrightarrow W \rightarrow W/W^{(L)} = C_1(M). $$

Finally, we show that $\partial_1\partial_2 = 0 : C_2(M) \rightarrow C_0(M)$. Indeed, if $w \in W^{(PL)}$, then from $wPL = w$, we infer that $wP = wL$. Therefore, if $d \in D$, then

$$ \begin{align*}
\partial_1\partial_2w(d) &= \sum_{d' \in [d]_V} (w(d') - w(Ld')) \\
&= \sum_{d' \in [d]_V} w(d') - \sum_{d' \in [d]_V} wP(d') = 0.
\end{align*} $$

The functoriality is as follows. Let $M = (D, P, L)$, $M' = (D', P', L')$ be oriented maps, and let $\phi : M \rightarrow M'$ be a morphism. Let $W, W'$ have
the obvious definitions and define the associated mapping $\phi_s : W \to W'$ by the recipe
\[
(w\phi_s)(d') = \sum_{d \in d' \delta^{-1}} w(d).
\]

(In the above, if the above sum is empty, i.e., if $d'$ is not in the image of $\phi$, then we agree that $(w\phi_s)(d') = 0$.) Note that the above definition is equivalent to the stipulation that $\chi_d \phi_s = \chi_d \phi$.

Let $\phi : \mathcal{M} \to \mathcal{M}'$ be a surjective morphism, where $\mathcal{M} = (D, P, L)$, $\mathcal{M}' = (D', P', L')$ are oriented maps, and recall (from Lemma 1.1.2 of page 3) that there is a well-defined homomorphism $\text{Mon}(\mathcal{M}) \to \text{Mon}(\mathcal{M}')$ determined by $P \mapsto P'$, $L \mapsto L'$. If $g \in \text{Mon}(\mathcal{M})$, and if $g \mapsto g' \in \text{Mon}(\mathcal{M}')$, then

\[
(w\phi_s)(g'd') = \sum_{d \in g'd' \delta^{-1}} w(d) = \sum_{g^{-1}d \in d' \delta^{-1}} w(d) = \sum_{d \in d' \delta^{-1}} w(gd) = \sum_{d \in d' \delta^{-1}} wg(d) = (wg)\phi_s(d'),
\]

from which it follows that $(wg)\phi_s = (w\phi_s)g'$. In turn, it follows immediately from this that if $w \in W^{(P)}_r, W^{(L)}_r$, or $W^{(PL)}_r$, respectively and if $\phi : \mathcal{M} \to \mathcal{M}'$ is a surjective morphism, then $w\phi_s \in W^{(P)}_r, W^{(L)}_r$, or $W^{(PL)}_r$, respectively. In the event that $\phi : \mathcal{M} \to \mathcal{M}'$ isn’t surjective, simply note that if $d' \notin \text{im} \phi$, then $g'd' \notin \text{im} \phi$ for all $g' \in \text{Mon}(\mathcal{M}')$ and so $((w\phi_s)g')(d') = w\phi_s(g'd') = 0 = (w\phi_s)(d')$. Therefore, it follows that, whether or not $\phi : \mathcal{M} \to \mathcal{M}'$ is surjective, it is always the case that if $w \in W^{(P)}_r, W^{(L)}_r$, or $W^{(PL)}_r$, respectively, then $w\phi_s \in W^{(P')}_r, W^{(L')}_r$, or $W^{(PL')}_r$, respectively. From this it follows immediately that $\phi$ induces mappings.
\[\phi_0 : C_0(\mathcal{M}) = W^{(P)} \to W'^{(P')} = C_0(\mathcal{M}'),\]
\[\phi_1 : C_1(\mathcal{M}) = W/W^{(L)} \to W'/W'^{(L')} = C_1(\mathcal{M}'),\]
\[\phi_2 : C_2(\mathcal{M}) = W^{(PL)} \to W'^{(P'L')} = C_2(\mathcal{M}').\]

To show that the above maps collectively define a morphism of chain complexes \(C_*(\mathcal{M}) \to C_*(\mathcal{M}')\), we must show that \(\phi_* = (\phi_i)_i\) intertwines the boundary maps, i.e., that the diagrams below commute for \(i = 1, 2:\)

\[
\begin{array}{ccc}
C_i(\mathcal{M}) & \xrightarrow{\partial_i} & C_{i-1}(\mathcal{M}) \\
\downarrow \phi_i & & \downarrow \phi_{i-1} \\
C_i(\mathcal{M}') & \xrightarrow{\partial'_i} & C_{i-1}(\mathcal{M}')
\end{array}
\]

That \(\phi_*\) intertwines \(\partial_2\) and \(\partial'_2\) is trivial. Next, if \(w \in W, d' \in D'\), we have

\[
(\partial_1 w)\phi_0(d') = \sum_{d \in d' \phi_0^{-1}} \partial_1 w(d)
= \sum_{d \in d' \phi_0^{-1}} \sum_{d_1 \in [d)_V} (w(d_1) - w(Ld_1))
= \sum_{d \in [d']_V} \sum_{d_1 \in [d)_V} (w(d) - w(Ld))
= \sum_{d' \in [d']_V} \sum_{d \in d' \phi_0^{-1}} (w(d) - wL(d))
= \sum_{d' \in [d']_V} \sum_{d'' \in d' \phi_0^{-1}} (w(d'') - wL(d'))
= \sum_{d' \in [d']_V} (w\phi_* (d'') - wL\phi_*(d))
= \sum_{d' \in [d']_V} (w\phi_*(d'') - \phi_*(L'd''))
= \partial'_1 (w\phi_1)(d').
\]
Therefore, one has the following result:

**Proposition 7.1.1.** The assignment $\mathcal{M} \mapsto C_*(\mathcal{M})$ from the category of oriented maps to the category of chain complexes of abelian groups is functorial.

In terms of the above, we can define the *homology groups* of $\mathcal{M}$ via the usual quotients:

$$H_0(\mathcal{M}) = \text{coker}(\partial_1), \quad H_1(\mathcal{M}) = \ker(\partial_1)/\text{im}(\partial_2), \quad H_2(\mathcal{M}) = \ker(\partial_2).$$

In light of the above, we have the following basic result:

**Proposition 7.1.2.** The assignment $\mathcal{M} \mapsto H_*(\mathcal{M})$ from the category of oriented maps to the category of graded abelian groups is functorial.$^2$

If $\phi : \mathcal{M} \to \mathcal{M}'$ is a morphism of oriented maps, we shall sometimes denote by $H_*(\phi) : H_*(\mathcal{M}) \to H_*(\mathcal{M}')$ the corresponding mapping induced in homology.

We define the *augmentation map* $C_0(\mathcal{M}) \to \mathbb{Z}$, as follows. If $w \in C_0(\mathcal{M}) = W^{(P)}$, then $w : B \to \mathbb{Z}$ is constant-valued on vertices; thus, for any vertex $v \in V$ we may write $w(v) = w(d) \in \mathbb{Z}$ for any $d \in v$. With this in force we define $\epsilon : C_0(\mathcal{M}) \to \mathbb{Z}$ by setting

$$\epsilon(w) = \sum_{v \in V} w(v).$$

One has the following:

**Lemma 7.1.3.** Assume that the oriented map $\mathcal{M}$ is connected. Then

1. $C_1(\mathcal{M}) \xrightarrow{\partial_1} C_0(\mathcal{M}) \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is exact.

$^2$A graded abelian group is just a collection $H_* = (H_k)_k$ of abelian groups, where $k$ ranges over some index set.
2. $H_2(\mathcal{M}) \cong H_0(\mathcal{M}) \cong \mathbb{Z}$.

**Proof.** Note that if $d \in D$, then $\varepsilon \partial_1(x_d) = \varepsilon(x_{[d]_v} - x_{[Ld]_v}) = 1 - 1 = 0$, and so $\text{im} \partial_1 \subseteq \text{ker} \varepsilon$. For the reverse inclusion, we show first that $\text{ker} \varepsilon$ is generated by elements of the form $x_v - x_{v'}$, where $v, v' \in V$. To see this, let $w \in \text{ker} \varepsilon$; we argue by induction on the number vertices in the support of $w$. Thus, assume that $v_1, v_2, \ldots, v_n$ are those vertices with $w(v_i) \neq 0, i = 1, 2, \ldots, n$. Clearly $n > 1$; thus we have $w - (w(v_n)x_{v_n} - w(v_n)x_{v_{n-1}}) \in \text{ker} \varepsilon$ and has support contained in \{v_1, v_2, \ldots, v_{n-1}\}. Thus, $w - (w(v_n)x_{v_n} - w(v_n)x_{v_{n-1}})$ is a $\mathbb{Z}$-linear combination of elements of the form $x_v - x_{v'}$ and hence so is $w$. The proof of (1) will be complete as soon as we show that for any pair of vertices $v, v' \in V$, $x_v - x_{v'} \in \text{im} \partial_1$. To this end, use connectivity to obtain a sequence of vertices $v = v_1, v_2, \ldots, v_n = v' \in V$ such that for each $i = 1, 2, \ldots, n - 1$, there exists $d_i \in v_i$ such that $Ld_i \in v_{i+1}$. Then $\partial_1(x_{d_i}) = x_{v_i} - x_{v_{i+1}}$, and so

$$\partial_1(x_{d_1} + x_{d_2} + \cdots + x_{d_{n-1}}) = x_{v_1} - x_{v_n} = x_v - x_{v'}.$$

As for (2), note first that $H_0(\mathcal{M}) \cong \mathbb{Z}$ by the exactness of the sequence in (1). Next, note that $\text{ker} \partial_2 = W^{(P/L)} \cap W^{(L)} = W^G \cong \mathbb{Z}$, by connectivity.

We expect this "homology theory" to produce the same results as does "simplicial homology" for orientable 2-manifolds. As the above lemma shows, this is true for the 0- and 2-homology. Proving that the 1-homology is free and of rank $2g$, where $g$ is the genus of $\mathcal{M}$ takes a bit of work. To prove this, I’ve appended part of a paper that I coauthored with Gareth Jones\(^3\) that deals with this issue. Again, Machi’s paper avoids this technicality since he works over fields, where life is much easier.

The relevant technical lemma (below) is a crucial ingredient in proving the above-mentioned freeness. If $V$ is a free abelian group with basis $X$, and if $Y$ is a finite subset of $X$, let $[Y]$ denote the element $\sum_{y \in Y} y$

of $V$. If $\mathcal{T}$ is a family of finite subsets of $X$, let $V(\mathcal{T}) = \langle [T] \mid T \in \mathcal{T} \rangle$ be the subgroup of $V$ generated by the elements $[T]$ ($T \in \mathcal{T}$).

We are interested in whether $V/V(\mathcal{T})$ is free. In general this need not be the case: for instance, if $\mathcal{T}$ consists of all the $k$-element subsets of $X$, for some finite $k$ such that $1 \leq k < |X|$, then $V/V(\mathcal{T})$ is easily seen to be a cyclic group of order $k$. The main result is due to Warren May (private correspondence) and gives a very useful sufficiency condition.

**Lemma 7.1.4.** Let $V$ be the free abelian group on a set $X$, and let $\mathcal{T}, \mathcal{T}_1$ and $\mathcal{T}_2$ be families of finite subsets of $X$ such that

1. $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$;

2. For each $i = 1$ or $2$, if $T, T' \in \mathcal{T}_i$ then $T = T'$ or $T \cap T' = \emptyset$.

Then $V/V(\mathcal{T})$ is free.

**Proof.** It suffices to produce a basis $\mathcal{B}$ of $V$ such that $\mathcal{B} \cap V(\mathcal{T})$ generates $V(\mathcal{T})$, since $V/V(\mathcal{T})$ is then freely generated by the images of the elements of $\mathcal{B} \setminus V(\mathcal{T})$. Let us index $X$ with the elements of a set $\mathcal{A}$, so that $X = \{ x_\alpha \mid \alpha \in \mathcal{A} \}$. We may impose a well order $<\,$ on $\mathcal{A}$, and adjoin an ordinal $\lambda$ so that $X = \{ x_\alpha \mid \alpha < \lambda \}$. For each $\mu \in \mathcal{A} \cup \{ \lambda \}$, we define $V_\mu$ to be the subgroup of $V$ generated by all $x_\gamma \in X$ with $\gamma < \mu$, and we define $V_\mu(\mathcal{T}) = V_\mu \cap V(\mathcal{T})$. In particular, $V_\lambda = V$ and $V_\lambda(\mathcal{T}) = V(\mathcal{T})$. We shall use transfinite induction to construct, for each $\mu \in \mathcal{A} \cup \{ \lambda \}$, a basis $\mathcal{B}_\mu$ of $V_\mu$ such that

(a) $\mathcal{B}_\mu \cap V_\mu(\mathcal{T})$ generates $V_\mu(\mathcal{T})$, and

(b) $\mathcal{B}_\alpha \subseteq \mathcal{B}_\beta$ whenever $\alpha < \beta$.

Taking $\mathcal{B} = \mathcal{B}_\lambda$ will then complete the proof, by condition (a).

First of all, note that if $\mu$ is a limit ordinal, and if bases $\mathcal{B}_\nu$ has been constructed as above for all $\nu < \mu$, then $\mathcal{B}_\mu = \cup_{\nu<\mu} \mathcal{B}_\nu$ clearly also satisfies (a) and (b). We may therefore assume that $\mu$ has the form $\nu + 1$ for some $\nu \in \mathcal{A}$. If $V_\mu(\mathcal{T}) = V_\nu(\mathcal{T})$ we can take $\mathcal{B}_\mu = \mathcal{B}_\nu \cup \{ x_\nu \}$. We therefore assume that $V_\mu(\mathcal{T}) \neq V_\nu(\mathcal{T})$, so there exists some $w \in V_\mu(\mathcal{T})$ of the form $w = rx_\nu + y$, where $0 \neq r \in \mathbb{Z}$ and $y \in V_\nu$. If $r = 1$ we can take $\mathcal{B}_\mu = \mathcal{B}_\nu \cup \{ w \}$, completing the proof. In the remaining case,
where \( r \neq 1 \), we will show how to replace \( w \) with an element \( w' \in V_\mu(\mathcal{T}) \) in which \( x_\nu \) does have coefficient 1, so we can take \( B_\mu = B_\nu \cup \{w'\} \).

Since \( w \in V(\mathcal{T}) \) we may write \( w = w_1 + w_2 \), where

\[
w_1 = \sum_{A \in \mathcal{T}_1} c_A[A] \quad \text{and} \quad w_2 = \sum_{B \in \mathcal{T}_2} d_B[B];
\]

here \( c_A, d_B \in \mathbb{Z} \), only finitely many non-zero. Recalling that each \( [T] = \sum_{x_\gamma \in T} x_\gamma \), let \( c \) and \( d \) be the coefficients of \( x_\nu \) in \( w_1 \) and \( w_2 \). Thus, \( c = c_A \) if \( x_\nu \in A \) for some \( A \in \mathcal{T}_1 \) (necessarily unique by (ii)), and otherwise \( c = 0 \); similarly \( d = d_B \) if \( x_\nu \in B \in \mathcal{T}_2 \), and otherwise \( d = 0 \). Since \( w_1 + w_2 = w \) we have \( c + d = r \); since \( r \neq 0 \), \( c \) and \( d \) cannot both be 0, so transposing \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) if necessary we may assume that \( c \neq 0 \).

We now define \( w' = w'_1 + w'_2 \) where

\[
w'_1 = \sum_{A \in \mathcal{T}_1} c'_A[A] \quad \text{and} \quad w'_2 = \sum_{B \in \mathcal{T}_2} d'_B[B],
\]

with

\[
c'_A = \begin{cases} c_A & \text{if } c_A \neq c, \\ 1 - d & \text{if } c_A = c, \end{cases} \quad \text{and} \quad d'_B = \begin{cases} d_B & \text{if } d_B \neq -c, \\ d - 1 & \text{if } d_B = -c. \end{cases}
\]

Since \( c \neq 0 \), any element \([A]\) or \([B]\) with non-zero coefficient \( c'_A \) or \( d'_B \) in \( w'_1 \) or \( w'_2 \) must also have non-zero coefficient \( c_A \) or \( d_B \) in \( w_1 \) or \( w_2 \); there are only finitely many such elements, so \( w' \in V(\mathcal{T}) \). Next, we show that \( w' \in V_\mu(\mathcal{T}) \) by showing that if \( \gamma \geq \mu \) then the coefficient of \( x_\gamma \) in \( w' \) is 0. If \( x_\gamma \) has non-zero coefficient in \( w'_1 \) or \( w'_2 \) then \( x_\gamma \in A \) or \( x_\gamma \in B \) for some \( A \in \mathcal{T}_1 \) or \( B \in \mathcal{T}_2 \), each necessarily unique by (ii). The corresponding element \([A]\) or \([B]\) has non-zero coefficient in \( w' \), and hence also in \( w \). But \( x_\gamma \) has coefficient 0 in \( w \) since \( w \in V_\mu(\mathcal{T}) \), so \( x_\gamma \) must lie in both \( A \) and \( B \), with \( c_A + d_B = 0 \). If \( c_A \neq c \) then \( d_B \neq -c \), so \( c'_A = c_A \) and \( d'_B = d_B \), giving \( c'_A + d'_B = c_A + d_B = 0 \); if \( c_A = c \) then \( d_B = -c \), so \( c'_A = 1 - d \) and \( d'_B = d - 1 \), again forcing \( c'_A + d'_B = 0 \). In either case, \( w' \in V_\mu(\mathcal{T}) \). Finally, \( x_\nu \) has coefficient \( 1 - d \) in \( w'_1 \), and coefficient \( d \) in \( w'_2 \) (since \( d \neq -c \)), so its coefficient in \( w' \) is 1. We can therefore take \( B_\mu = B_\nu \cup \{w'\} \), completing the proof.

From the above, one has the following:
Proposition 7.1.5. If $\mathcal{M}$ is an oriented map, all of whose faces are finite, then

(i) $C_1(\mathcal{M})/\partial_2 C_2(\mathcal{M})$ is free;

(ii) $H_1(\mathcal{M})$ is free;

(iii) If $B$ is finite and $\mathcal{M}$ is connected, then $H_1(\mathcal{M})$ has finite rank $2g$, where $g$ is the genus of $\mathcal{M}$.

Proof. (i) Note that $C_1(\mathcal{M})/\partial_2 C_2(\mathcal{M}) \cong W/(W^{(L)} + W^{(PL)})$, and $W$ is free on the set $X = \{\chi_d \mid d \in B\}$. Now let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the sets of $\langle L \rangle$- and $\langle PL \rangle$-orbits in $X$, so $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ satisfy the hypotheses of Lemma 7.1.4, and $W^{(L)} + W^{(PL)} = W(\mathcal{T})$. It follows from Lemma 7.1.4 that $W/(W^{(L)} + W^{(PL)})$ is free, and hence so is $C_1(\mathcal{M})/\partial_2 C_2(\mathcal{M})$.

(ii) Since $C_1(\mathcal{M})/\partial_2 C_2(\mathcal{M})$ is free, the subgroup $H_1(\mathcal{M}) = \ker(\partial_1)/\partial_2 C_2(\mathcal{M})$ is also free.

(iii) We have

$$\chi(\mathcal{M}) = 2 - 2g = |V| - |E| + |F| = \text{rank}(C_0(\mathcal{M})) - \text{rank}(C_1(\mathcal{M})) + \text{rank}(C_2(\mathcal{M})),$$

which by the integral version of the Hopf trace formula is equal to

$$\text{rank}(H_0(\mathcal{M})) - \text{rank}(H_1(\mathcal{M})) + \text{rank}(H_2(\mathcal{M})).$$

But $\text{rank}(H_0(\mathcal{M})) = 1 = \text{rank}(H_2(\mathcal{M}))$ (by Lemma 7.1.3), so the result follows.

I can think of at least two difficulties of the above theory:

1. It applies only to oriented maps.

2. Even when $\mathcal{M}$ admits ‘orientation-reversing” automorphisms, i.e. mappings $\phi : B \to B$ such that $\phi$ commutes with $L$ and inverts $P$, such an automorphism will still act on the homology groups.
$H_i(M)$, but won’t have the expected “orientation-reversing” effect on $H_2(M)$. To the contrary, one can show that if $\phi_2: H_2(M) \to H_2(M)$ is the induced mapping, then $\phi_2 = 1_{H_2(M)}$, as opposed to having the expected action of multiplication by $-1$.

A remedy to this is given below:

This time let $M = (B, a, b, c)$ be a map, not necessarily orientable, and let $\beta_V, \beta_E, \beta_F$ be the corresponding vertices, edges, and faces:

\[
\begin{align*}
[\beta]_V &= \langle b, c \rangle\text{-orbit of } \beta \text{ in } B, \\
[\beta]_E &= \langle a, c \rangle\text{-orbit of } \beta \text{ in } B, \\
[\beta]_F &= \langle a, b \rangle\text{-orbit of } \beta \text{ in } B.
\end{align*}
\]

Next, we define $W$ exactly as above, viz., as the abelian group of finitely-supported functions $B \to \mathbb{Z}$ with pointwise addition. Define the chain complex $C_*(M) = (C_i(M), \partial_i)_i$ of $M$ by

\[
C_0(M) = W^{(b,c)}, \quad C_1(M) = W^{(c)}/W^{(a,c)}, \quad C_2(M) = W^{(a,b)}/W^{(a,b)}.
\]

A moment’s thought reveals that these groups are free, of ranks equal to $|V|, |E|$ and $|F|$, respectively, exactly as in the oriented case.

We define the boundary maps as follows. First of all, if $\beta \in B$, let $[\beta]_{V+}$ denote the $\langle bc \rangle$-orbit of $\beta$ in $B$. If $w \in W^{(c)}$, define

\[
(\partial w)(\beta) = \sum_{\beta' \in [\beta]_{V+}} (w(\beta') - w(\alpha \beta')) \in \mathbb{Z}.
\]

Equivalently, this same mapping can be defined in terms of the characteristic functions via $\chi_{\beta} + \chi_{\alpha \beta} \mapsto \chi_{[\beta]_V} - \chi_{[\alpha \beta]_V}, \beta \in B$. Note that $\partial w \in W^{(b,c)}$, given that $w \in W^{(c)}$.

Note that if also $w \in W^{(a,c)}$ then $\partial w = 0$, and so $\partial$ factors through $W^{(c)}/W^{(a,c)}$, giving a map $\partial_1: C_1(M) \to C_0(M)$. For $w \in W$ we set
\[ \partial_2 w = w + wc + W^{(a,c)} \in W^{(c)}/W^{(a,c)}. \]

Note that if \( w \in W^{(a)} \supseteq W^{(a,b)} \), then \( w + wc \in W^{(a,c)} \) and so we obtain a mapping

\[ \partial_2 : C_2(M) = W^{(ab)}/W^{(a,b)} \to W^{(c)}/W^{(a,c)} = C_1(M). \]

**Lemma 7.1.6.** We have \( \partial_1 \partial_2 = 0 : C_2(M) \to C_0(M) \).

**Proof.** Let \( w \in W^{(ab)} \), and let \( \beta \in B \). Then using the fact that \( w \in W^{(ab)} \) implies that \( wa = wb \), we get

\[
(\partial_1 \partial_2 w)(\beta) = \sum_{\beta' \in [\beta]_V} (\partial_2 w(\beta') - \partial_2 w(a\beta'))
\]

\[
= \sum_{\beta' \in [\beta]_V} (w(\beta') + wc(\beta') - wa(\beta') - wca(\beta'))
\]

\[
= \sum_{\beta' \in [\beta]_V} (w(\beta') + wc(\beta') - wb(\beta') - wbc(\beta'))
\]

\[
= \sum_{\beta' \in [\beta]_V} w(\beta') - \sum_{\beta' \in [\beta]_V} w(\beta')
\]

\[
= 0.
\]

Thus, \((C_*(M), \partial_*)\) is a chain complex over \( M \). As for oriented maps, this assignment is functorial in that mappings \( M \to M' \) induce mappings \( C_*(M) \to C_*(M') \) entirely analogously with the above. That is, assume that \( M = (B, a, c, b) \), \( M' = (B', a', b', c') \) are maps and that \( \phi : M \to M' \) is a morphism (applied on the right, so that \( \beta \mapsto \beta \phi \in B' \), \( \beta \in B \)). If \( W, W' \) have the obvious definitions, then we have the induced mapping \( \phi_* : W \to W' \), where if \( w \in W \), then

\[ w\phi_*(\beta') = \sum_{\beta \in [\beta]_V} \phi^{-1}(w(\beta)). \]

Arguing as in the oriented case, one checks that \( \phi_* : W \to W' \) determines a morphism \( \phi_* = (\phi_*)_i : C_*(M) \to C_*(M') \). Therefore, we
obtain a functor from the category of maps to the category of chain complexes of abelian groups.

The corresponding homology groups of \( \mathcal{M} \) are defined from the chain complex in the usual way:

\[
H_0(\mathcal{M}) = \text{coker}(\partial_1), \quad H_1(\mathcal{M}) = \ker(\partial_1)/\text{im}(\partial_2), \quad H_2(\mathcal{M}) = \ker(\partial_2).
\]

Again, the functoriality of \( \mathcal{M} \mapsto H_i(\mathcal{M}) \) is easily established. As for oriented maps, if \( \phi : \mathcal{M} \to \mathcal{M}' \) is a morphism of maps, then the induced mappings in homology are frequently denoted \( H_i(\phi) : H_i(\mathcal{M}) \to H_i(\mathcal{M}') \), \( i = 0, 1, 2 \).

When \( \mathcal{M} = (B, a, b, c) \) is orientable, we can effect a comparison between the homology of \( \mathcal{M} \) and the homology of the oriented map \( \mathcal{M}^+ = (B^+, bc, ac) \), where \( B^+ \subseteq B \) is one of the \( G^+ = \langle bc, ac \rangle \)-orbits in \( B \), as follows. Define \( W^+ \) to be the finitely-supported functions \( B^+ \to \mathbb{Z} \) and define the chain complex \( C_i^+(\mathcal{M}) = C_i^+(\mathcal{M}^+) \) (constructed as above); thus,

\[
C_0^+(\mathcal{M}) = W^+\langle bc \rangle, \quad C_1^+(\mathcal{M}) = W^+/W^+\langle ac \rangle, \quad C_2(\mathcal{M}) = W^+\langle ab \rangle.
\]

We regard \( W^+ \) as a subgroup of \( W \) via “extension by zero;” thus, if \( w \in W^+, \beta \in B \), we have

\[
w(\beta) = \begin{cases} 
w(\beta) & \text{if } \beta \in B^+ \\
0 & \text{if } \beta \notin B^+.
\end{cases}
\]

Next define mappings \( \theta_i : C_i^+(\mathcal{M}) \to C_i(\mathcal{M}) \) as follows:

\( \theta_0 : C_0^+(\mathcal{M}) \to C_0(\mathcal{M}) \): If \( w \in W^+ \), define \( \theta(w) = w + wc \in W^\langle c \rangle \). Note that the restriction \( \theta_0 \) of \( \theta \) to \( W^+\langle bc \rangle \) maps \( W^+\langle bc \rangle \to W^\langle b,c \rangle = W^\langle b,c \rangle \), giving \( \theta_0 : C_0^+(\mathcal{M}) = W^+\langle bc \rangle \to W^\langle b,c \rangle = C_0(\mathcal{M}) \).

\( \theta_1 : C_1^+(\mathcal{M}) \to C_1(\mathcal{M}) \): Note that \( W^+\langle ac \rangle \) is in the kernel of the composition

\[
W^+ \xrightarrow{\theta} W^\langle c \rangle \to W^\langle c \rangle /W^\langle a,c \rangle,
\]
giving the mapping $\theta_1 : C_1^+(\mathcal{M}) = W^+/W^{+(ae)} \to W^{(c)}/W^{(a,c)} = C_1(\mathcal{M})$.

$\theta_2 : C_2^+(\mathcal{M}) \to C_2(\mathcal{M})$: This is just the mapping

$$W^+(ab) \to W^{(ab)} \to W^{(ab)}/W^{(a,b)},$$

where the inclusion is determined as above.

Since the vertices, edges, and faces of $\mathcal{M}^+$ are in bijective correspondence with the vertices, edges and faces of $\mathcal{M}$, we have the following:

**Lemma 7.1.7.** The mappings $\theta_i : C_i^+(\mathcal{M}) \to C_i(\mathcal{M})$, $i = 0, 1, 2$, are isomorphisms.

In fact,

**Lemma 7.1.8.** The mappings $\theta_i : C_i^+(\mathcal{M}) \to C_i(\mathcal{M})$, $i = 0, 1, 2$, collectively define an isomorphism of chain complexes $\theta_\ast : C_\ast^+(\mathcal{M}) \to C_\ast(\mathcal{M})$.

**Proof.** We need only check that $\theta_\ast$ intertwines the boundary maps of $C_\ast^+(\mathcal{M})$ and $C_\ast(\mathcal{M})$. Thus, we denote by $\partial_i^+ : C_i^+(\mathcal{M}) \to C_{i-1}^+(\mathcal{M})$, $i = 1, 2$, the boundary mappings of the chain complex $C_\ast^+(\mathcal{M})$.

$\theta_0\partial_1^+ = \partial_1 \theta_1$: If $\beta \in B$, we have

$$\theta_0\partial_1^+ w(\beta) = \partial_1^+ w(\beta) + (\partial_1^+ w)c(\beta) = \sum_{\beta' \in [\beta]_{V^+}} (w(\beta') - wa(\beta') + wc(\beta') - wc(\beta')) = \partial_1 \theta_1 w(\beta).$$

$\theta_1\partial_2^+ = \partial_2 \theta_2$: This is clear, for if $w \in C_2^+(\mathcal{M}) = W^{+(ab)}$, then

$$\theta_1\partial_2^+ w = w + wc + W^{(a,c)} = \partial_2 \theta_2.$$
Corollary 7.1.8.1. If $\mathcal{M}$ is orientable, then $H_i(\mathcal{M}) \cong H_i(\mathcal{M}^+)$, $i=0,1,2$.

The following summarizes the homology of an orientable map $\mathcal{M}$:

**Theorem 7.1.9.** Let $\mathcal{M}$ be a finite, connected, orientable map.

(1) $H_0(\mathcal{M}) \cong H_2(\mathcal{M}) \cong \mathbb{Z}$ and $H_1(\mathcal{M})$ is free abelian of rank $2g$, where $g$ is the genus of $\mathcal{M}$.

(2) If $\mathcal{M}$ has orientations $B^+$ and $B^-$, define the orientation classes $[B^+]$ and $[B^-]$ by setting

$$[B^+] = \sum_{\beta \in B^+} \chi_\beta, \quad [B^-] = \sum_{\beta \in B^-} \chi_\beta.$$ 

Then

(a) $[B^+], [B^-]$ both generate $H_2(\mathcal{M})$,

(b) $[B^+] = -[B^-]$, and

(c) If $\sigma \in \text{Aut}(\mathcal{M})$ is an orientation reversing automorphism, then $H_2(\sigma)[B^+] = [B^-]$, and so $H_2(\sigma)$ acts as $-1$ on $H_2(\mathcal{M})$.

**Proof.** Part (1) follows immediately from Lemma 7.1.3, Proposition 7.1.5, and Corollary 7.1.8.1, together with the fact that the genus of $\mathcal{M}$ is the same as the genus of the oriented map $\mathcal{M}^+$. That $[B^+]$ generates $H_2(\mathcal{M})$ follows from the fact that the same element generates $H_2^+(\mathcal{M})$, together with Corollary 7.1.8.1. The rest is trivial.

Note that if $\mathcal{M}$ is infinite, then $H_2(\mathcal{M}) = 0$ because $H_2(\mathcal{M}^+) = 0$. If $\mathcal{M}$ is orientable, infinite but with finite faces, we still have, from Proposition 7.1.5, together with Corollary 7.1.8.1 that $H_1(\mathcal{M})$ is free.

In case $\mathcal{M}$ is non-orientable, we have the following result.

**Theorem 7.1.10.** If $\mathcal{M}$ is connected and non-orientable, then $H_0(\mathcal{M}) \cong \mathbb{Z}$ and $H_2(\mathcal{M}) = 0$. 
Proof. As for oriented maps, we have the augmentation map \( C_0(M) \to \mathbb{Z} \), and just as in the oriented case, one shows that the sequence

\[
C_1(M) \xrightarrow{\partial} C_0(M) \xrightarrow{\varepsilon} \mathbb{Z} \to 0
\]

is exact, proving that \( H_0(M) \cong \mathbb{Z} \).

We now consider \( H_2(M) \). Let \( w + W^{(a,b)} \in \ker \partial_2 \). Then \( w \in W^{(ab)} \) and \( w + wc \in W^{(a,c)} \), which forces

\[
w + wc = wa + wac. \tag{\ast}
\]

We shall now show that, in fact, \( w \in W^{(a,b)} \), and so \( w \) represents the 0-coset in \( C_2(M) \). To this end, let \( f \) be a face of \( M \) and let \( \beta, \alpha \beta \in f \). By non-orientability, there exists \( g \in G^+ \) such that \( g \beta = a \beta \). We may factor \( g \) as

\[
g = x_1 y x_{r-1} y \cdots x_2 y x_1,
\]

where each \( x_i \in \langle ab \rangle \) and where \( y = ac \). We set \( n_0 = w(\beta) \), \( n'_0 = w(a\beta) \).

Our goal is to show that \( n_0 = n'_0 \).

For \( i = 1, 2, \ldots, r \), set

\[
n_i = w(y x_i y x_{i-1} \cdots x_2 y x_1 \beta), \quad n'_i = w(a y x_i y x_{i-1} \cdots x_2 y x_1 \beta),
\]

\[
m_i = w(c y x_i y x_{i-1} \cdots x_2 y x_1 \beta), \quad m'_i = w(ac y x_i y x_{i-1} \cdots x_2 y x_1 \beta).
\]

As a result of Equation (\( \ast \)), we have

\[
n_i - n'_i = m'_i - m_i,
\]

for \( i = 1, 2, \ldots, r \). Note also that for \( i = 1, 2, \ldots, r \), we have

\[
m_i = n'_{i-1}, \quad m'_i = n_{i-1}.
\]

To see this, note that

\[
m_i = w(c y x_i y x_{i-1} \cdots x_2 y x_1 \beta)
\]

\[
= w(a x_i y x_{i-1} \cdots x_2 y x_1 \beta) \quad \text{(since} \ y = ac \text{)}
\]

\[
= w(x_i^{-1} a y x_{i-1} \cdots x_2 y x_1 \beta)
\]

\[
= w x_i^{-1} (a y x_{i-1} \cdots x_2 y x_1 \beta)
\]

\[
= w(a y x_{i-1} \cdots x_2 y x_1 \beta) \quad \text{(since} \ w \in W^{(ab)} \text{)}
\]

\[
= n'_{i-1}
\]
Similarly, \( m_i' = n_{i-1} \). Therefore,

\[
n_i - n_i' = m_i' - m_i = n_{i-1} - n_{i-1}', \quad i = 1, 2, \ldots, r.
\]

Next, we have

\[
n_r = w(y x_r y x_{r-1} \cdots x_2 y x_1 \beta) = w(y g \beta) = w(y a \beta) = w(c \beta) = wc(\beta);
\]
similarly, \( n_r' = wac(\beta) \). Therefore, using (*) we get

\[
n_r - n_r' = wc(\beta) - wac(\beta) = wa(\beta) - w(\beta) = n'_0 - n_0,
\]
and so

\[
n'_0 - n_0 = n_r - n_r' = n_{r-1} - n_{r-1}' = \cdots = n_0 - n'_0.
\]

This implies \( 2n_0 = 2n'_0 \), which forces \( n_0 = n'_0 \). Therefore, \( w \in W^{(a,b)} \), as claimed, proving the theorem.

**Proposition 7.1.11.** Let \( p : \mathcal{M} \to \mathcal{M}' \) be a morphism of connected orientable maps. If \( \beta' \) is a blade in \( \mathcal{M}' \), and if \( n = |\beta' p^{-1}| \), then \( p_2 : H_2(\mathcal{M}) \to H_2(\mathcal{M}') \) is multiplication by \( n \), i.e., \( p_2[B^+'] = n[B'^+] \), where \( B, B' \) are the sets of blades in \( \mathcal{M}, \mathcal{M}' \), respectively.

**Proof.** Recall that if \( p_s : \text{Mon}(\mathcal{M}) \to \text{Mon}(\mathcal{M}') \) is induced by \( p \), and if \( g \in \text{Mon}(\mathcal{M}) \), then \( g \) maps the elements of \( \beta' p^{-1} \) bijectively to \( (g' \beta') p^{-1} \), where \( g' = p_s(g) \), and so all fibers have the same cardinality. Therefore, it is clear that \( p_2[B^+] = n[B'^+] \).

The integer \( n \) above is called the **degree** of the morphism \( p : \mathcal{M} \to \mathcal{M}' \).

### 7.2 Homology with Coefficients

If \( \mathcal{M} = (B, a, b, c) \) is a map, and if \( A \) is an abelian group, we define the **chain complex of \( \mathcal{M} \) with coefficients in \( A \)** by setting \( C_s(\mathcal{M}; A) = \)
\((C_i(\mathcal{M}; A), \partial_i)\), where \(C_i(\mathcal{M}; A) = C_i(\mathcal{M}) \otimes A\) and \(\partial_i = \partial_i \otimes 1 : C_i(\mathcal{M}; A) = C_i(\mathcal{M}) \otimes A \to C_{i-1}(\mathcal{M}; A) = C_{i-1}(\mathcal{M}) \otimes A\). This is still a chain complex, whose homology groups \(H_i(\mathcal{M}; A)\) are defined as the homology of this chain complex. These homology groups are functorial in both arguments in that if \(\phi : \mathcal{M} \to \mathcal{M}'\) is a morphism, and if \(\alpha : A \to A'\) is an abelian group homomorphism, then one has induced homomorphisms \(H_i(\phi, \alpha) : H_i(\mathcal{M}, A) \to H_i(\mathcal{M}', A')\), \(i = 0, 1, 2\) satifying the usual properties.

The relationship between the integral homology and the homology with coefficients in the abelian group \(A\) of the map \(\mathcal{M}\) is summarized via the *Universal Coefficient Theorem*,

**Theorem 7.2.1.** Let \(\mathcal{M}\) be a map, and let \(A\) be an abelian group.

1. If \(i = 1, 2\), then there is a natural split short exact sequence
   \[
   0 \longrightarrow H_i(\mathcal{M}) \otimes_\mathbb{Z} A \longrightarrow H_i(\mathcal{M}; A) \longrightarrow \text{Tor}(H_{i-1}(\mathcal{M}), A) \longrightarrow 0,
   \]
   and,

2. \(H_0(\mathcal{M}; A) \cong H_0(\mathcal{M}) \otimes_\mathbb{Z} A\).

The fact that the above short exact sequence is *natural* means the following. If \(\phi : \mathcal{M} \to \mathcal{M}'\) is a morphism and if \(\alpha : A \to A'\) is a homomorphism of abelian groups, then the following ladder commutes:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_i(\mathcal{M}) & \otimes_\mathbb{Z} A & \longrightarrow & H_i(\mathcal{M}; A) & \longrightarrow & \text{Tor}(H_{i-1}(\mathcal{M}), A) & \longrightarrow & 0 \\
& & H_i(\phi) \otimes \alpha & & H_i(\phi, \alpha) & & \text{Tor}(H_{i-1}(\phi), \alpha) & & \\
0 & \longrightarrow & H_i(\mathcal{M}') & \otimes_\mathbb{Z} A' & \longrightarrow & H_i(\mathcal{M}'; A') & \longrightarrow & \text{Tor}(H_{i-1}(\mathcal{M}'), A') & \longrightarrow & 0
\end{array}
\]

Since \(H_0(\mathcal{M}) \cong \mathbb{Z}\) (hence is torsion free), we have the following:

---

Corollary 7.2.1.1. Let $\mathcal{M}$ be a map, and let $A$ be an abelian group. Then

1. $H_i(\mathcal{M}; A) \cong H_i(\mathcal{M}) \otimes \mathbb{Z} A$, $i = 0, 1$, (natural isomorphisms) and
2. $H_2(\mathcal{M}; A) \cong H_2(\mathcal{M}) \otimes \mathbb{Z} A \oplus \text{Tor}(H_1(\mathcal{M}); A)$.

If $\mathcal{M}$ is orientable, then the homology groups $H_i(\mathcal{M})$ are all torsion free, and so we have the following:

Corollary 7.2.1.2. Let $\mathcal{M}$ be an orientable map, and let $A$ be an abelian group. Then $H_i(\mathcal{M}; A) \cong H_i(\mathcal{M}) \otimes \mathbb{Z} A$, $i = 0, 1, 2$.

In order to compute $H_2(\mathcal{M})$, where $\mathcal{M}$ is non-orientable, we first prove the following:

Proposition 7.2.2. Let $\mathcal{M}$ be finite, connected and non-orientable. Then $H_2(\mathcal{M}; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, and $H_2(\mathcal{M}; \mathbb{Z}/p\mathbb{Z}) = 0$, for any odd prime $p$.

Proof. We denote by 1 the nonzero element in $\mathbb{Z}/2\mathbb{Z}$. Let $f_1, f_2, \ldots, f_r$ be the faces in $\mathcal{M}$, fix blades $\beta_i \in f_i$, $i = 1, 2, \ldots, r$, and let $f_i^+$ be the $\langle ab \rangle$-orbit of $\beta_i \in f_i$. Note that if $w \in W^{(a,b)}$ and if $w$ is nowhere vanishing on $f_i$, then either $w$ is identically 1 on $f_i^+$, or is identically 1 on $f_i^-$ (where $f_i^- = f_i \setminus f_i^+$), or $w$ is identically 1 on all of $f_i$. Note that modulo $W^{(a,b)}$, we may take $w$ to be identically 0 on $f_i$ or to be identically 1 on $f_i^+$ and identically 0 on $f_i^-$. If the latter happens, we say that the element $w + W^{(a,b)}$ is supported on $f_i$, or that $f_i$ is in the support of $w + W^{(a,b)}$.

We claim that if $w + W^{(a,b)}$ is chosen so that every face is in its support, then $0 \neq w + W^{(a,b)}$ and $w + W^{(a,b)} \in \ker \partial_2 : C_2(\mathcal{M}; \mathbb{Z}/2\mathbb{Z}) \to C_1(\mathcal{M}; \mathbb{Z}/2\mathbb{Z})$. The first assertion is clear, so we turn to the second. Thus, let $e$ be an edge in $\mathcal{M}$, say $e$ is the $\langle a, c \rangle$-orbit of the blade $\beta \in B$. Let $f$ be the face containing $\beta$ and let $f'$ be the face containing $c\beta$. By the above, we may assume that $w(\beta) = 1$ and that $w(a\beta) = 0$. By the same token, we may assume that $w(c\beta) = 1$ and that $w(ac\beta) = 0$. 


Therefore, we see that $w + wc$ vanishes on the edge $e$. Since $e$ was arbitrary, we see that $\partial_2(w + W^{(a, b)}) = 0 \in C_1(M; \mathbb{Z}/2\mathbb{Z})$.

Next, we show that $w + W^{(a, b)}$ is the unique nonzero element of $C_2(M; \mathbb{Z}/2\mathbb{Z})$ in $\ker \partial_2$. This time, let $u + W^{(a, b)} \in \ker \partial_2$, but where there exists a face $f$ not in the support of $u$. If $\beta \in f$, and if $f'$ is the face containing $c\beta$, then we conclude that $u(c\beta) = c(ac\beta) = 0$, as well, and so $f'$ is not in the support of $u$, either. By connectivity, we infer that $u$ must be identically zero, proving that $H_2(M; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

If $p$ is an odd prime, one can argue exactly as in the proof of Theorem 7.1.10.

Applying Corollary 7.2.1.1, we deduce the following:

**Corollary 7.2.2.1.** If $M$ is non-orientable, then $\text{Tor}(H_1(M); \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$; if $p$ is an odd prime, $\text{Tor}(H_1(M); \mathbb{Z}/p\mathbb{Z}) = 0$.

Finally, we have the following absolute (i.e., with coefficients in $\mathbb{Z}$) and mod-2 homology of the non-orientable map $M$:

**Corollary 7.2.2.2.** If $M$ is non-orientable of genus $g$, then

1. $H_1(M; \mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^g$, and
2. $H_1(M) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}^{g-1}$, where $\mathbb{Z}^{g-1}$ is free abelian of rank $g - 1$.

**Proof.** The first statement follows immediately from the Hopf Trace Formula. The second statement follows from the fact that $H_1(M)$ is a finitely-generated abelian group and

$$\text{Tor}(H_1(M); \mathbb{Z}/p\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2 \\ 0 & \text{if } p \text{ is odd.} \end{cases}$$

Finally if $M$ is a map and if $\epsilon : A \to B$ is a homomorphism of abelian groups, then there are induced homomorphisms in homology:

$$\epsilon_{i,*} : H_i(M; A) \to H_i(M; B), \ i = 0, 1, 2.$$
We shall require the following result:

**Proposition 7.2.3.** Let $\mathcal{M}$ be a map and let $\epsilon : A \to B$ be a surjective homomorphism. Then $\epsilon_* : H_1(\mathcal{M}; A) \to H_1(\mathcal{M}; B)$ is surjective.

**Proof.** This follows from the corresponding long exact sequence in homology. Namely, if $K = \ker(\epsilon : A \to B)$, then the sequence below is exact:

$$\cdots \to H_1(\mathcal{M}; A) \xrightarrow{\epsilon_*} H_1(\mathcal{M}; B) \to H_0(\mathcal{M}; K) \to H_0(\mathcal{M}; A) \to \cdots .$$

Since $H_0(\mathcal{M}; K) \to H_0(\mathcal{M}; A)$ is clearly injective, the result follows.

### 7.3 Cohomology

In this section we define cohomology with coefficients and relate this to the construction based on voltages modulo equivalence as used in the theory of principal derived maps.

If $\mathcal{M}$ is a map, and if $A$ is any abelian group, define the abelian groups

$$C^i(\mathcal{M}; A) = \text{Hom}_\mathbb{Z}(C_i(\mathcal{M}), A), \ i = 0, 1, 2.$$ 

We thus obtain the “cochain complex”

$$C^0(\mathcal{H}; A) \xrightarrow{\delta^0} C^1(\mathcal{H}; A) \xrightarrow{\delta^1} C^2(\mathcal{H}; A),$$

where $\delta^0 = \text{Hom}(\partial_1, 1_A) : \phi \mapsto \phi \circ \partial_1$ and $\delta^1 = \text{Hom}(\partial_2, 1_A) : \phi \mapsto \phi \circ \partial_2$, in terms of which one can define the cohomology of $\mathcal{M}$ with coefficients in $A$: $H^0(\mathcal{M}; A) = \ker(\delta^0)$, $H^1(\mathcal{M}; A) = \ker(\delta^1)/\text{im}(\delta^0)$, $H^2(\mathcal{M}; A) = \text{coker}(\delta^1)$. Here, the functoriality is as follows: if $\phi : \mathcal{M}' \to \mathcal{M}$ is a morphism, and if $\alpha : A \to A'$ is a homomorphism of abelian groups, then one has the associated homomorphisms $H^i(\phi, \alpha) : H^i(\mathcal{M}, A) \to H^i(\mathcal{M}', A')$, $i = 0, 1, 2$.

Let $\mathcal{M}$ be a map, and let $A$ be an abelian group. Define the group $C(\mathcal{M}; A)$ of voltages on $\mathcal{M}$ as in Section 5.1 with the usual definition of equivalence “$\sim$” of voltages, and set $D(\mathcal{M}; A) = C(\mathcal{M}; A)/\sim$, where
we can think of both $C(M; A)$ and $D(M; A)$ as abelian groups. Finally, recall from Subsection 5.4.1 the cohomology group $H(M; A)$ of $M$ with coefficients in $A$; this is clearly a subgroup of $D(M; A)$.

A close scrutiny of the definitions reveals the following:

**Proposition 7.3.1.** If $M$ is a map and $A$ is an abelian group, we have $D(M; A) \cong C^1(M; A)/\delta^0 C^0(M; A)$, and that $H^1(M; A) \cong H(M; A)$.

**Proof.** Note first that we may identify $\text{Hom}(W^{(c)}/W^{(a,c)}, A)$ with the subgroup of all homomorphisms $f : W^{(c)} \to A$ such that $f$ kills $W^{(a,c)}$. The homomorphism $f : W^{(c)} \to A$ determines a mapping $f' : B \to A$ via

$$f'(\beta) = f(\chi_{\{\beta, c\beta\}}),$$

where $\chi_{\{\beta, c\beta\}} \in \text{Hom}(W^{(c)}, A)$ is the characteristic function on the set $\{\beta, c\beta\}$, $\beta \in B$. As such, it’s obvious that $f'$ satisfies $f'(c\beta) = f'(\beta)$.

Next, if $f$ kills $W^{(a,c)}$ then

$$f' (\beta) + f' (a\beta) = f(\chi_{\{\beta, c\beta\}}) + f(\chi_{\{a\beta, ac\beta\}})$$

$$= f(\chi_{\{\beta, c\beta\}} + \chi_{\{a\beta, ac\beta\}})$$

$$= 0,$$

since $\chi_{\{\beta, c\beta\}} + \chi_{\{a\beta, ac\beta\}} \in W^{(a,c)}$. In other words, we get a homomorphism

$$\text{Hom}(W^{(c)}/W^{(a,c)}, A) \to D(M; A)$$

given by $f \mapsto f'$. Since it is clear that this mapping is bijective, the result follows.

Next, we write $C^i(M) = C^i(M; \mathbb{Z})$, $D(M) = D(M; \mathbb{Z})$.

**Lemma 7.3.2.** Let $M$ be a finite connected map. If $A$ is an abelian group, then

1. $C^i(M; A) \cong C^i(M) \otimes \mathbb{Z} A$, $i = 0, 1, 2$;

2. $D(M; A) \cong D(M) \otimes \mathbb{Z} A$;
(3) $D(\mathcal{M}) \cong C^1(\mathcal{M})/\delta^0C^0(\mathcal{M})$ is free of rank $|E| - |V| + 1$.

Proof. (1) For any abelian group $A'$, there is a natural homomorphism

$$\text{Hom}_\mathbb{Z}(A', \mathbb{Z}) \otimes A \rightarrow \text{Hom}_\mathbb{Z}(A', A),$$

given by $f \otimes a \mapsto (a' \mapsto f(a')a \in A)$, where $f \in \text{Hom}_\mathbb{Z}(A', \mathbb{Z})$, $a \in A$ and $a' \in A'$. If $A'$ is finitely generated and free, this is easily seen to be an isomorphism, so we obtain (1) by taking $A' = C_i(\mathcal{M})$, $i = 0, 1, 2$.

(2) follows from Proposition 7.3.1, part (1) of the above, together with the fact that tensor product commutes with cokernels.

To prove (3), note first that we have the exact sequence

$$C_1(\mathcal{M}) \overset{\partial_1}{\rightarrow} C_0(\mathcal{M}) \overset{\epsilon^*}{\rightarrow} \mathbb{Z} \rightarrow 0;$$
as the functor $\text{Hom}_\mathbb{Z}(-; A)$ is left exact, we obtain the exact sequence

$$0 \rightarrow A \overset{\epsilon}{\rightarrow} C^0(\mathcal{M}; A) \overset{\delta^0}{\rightarrow} C^1(\mathcal{M}; A),$$

where $\epsilon^* = \text{Hom}(\epsilon, 1_A) : A \cong \text{Hom}_\mathbb{Z}(\mathbb{Z}, A) \rightarrow C^0(\mathcal{M}; A)$. In particular, taking $A = \mathbb{Z}$ we see that $\text{rank } \delta^0C^0(\mathcal{M}) = |V| - 1$. Therefore the free part of $C^1(\mathcal{M})/\delta^0C^0(\mathcal{M})$ has rank

$$k = \text{rank } C_1(\mathcal{M}) - \text{rank } C_0(\mathcal{M}) + 1 = |E| - |V| + 1.$$

If $C^1(\mathcal{M})/\delta^0C^0(\mathcal{M})$ has $p$-torsion for some prime $p$, and if $Z_p$ is a cyclic group of order $p$, then

$$|C^1(\mathcal{M})/\delta^0C^0(\mathcal{M}) \otimes Z_p| > p^k.$$

On the other hand, by (2), together with Proposition 7.3.1, we have

$$C^1(\mathcal{M})/\delta^0C^0(\mathcal{M}) \otimes Z_p \cong C^1(\mathcal{M}; Z_p)/\delta^0C^0(\mathcal{M}; Z_p),$$

and the right hand side has order $p^k$ as $\ker(\delta^0 : C^0(\mathcal{M}; Z_p) \rightarrow C^1(\mathcal{M}; Z_p)) \cong Z_p$. Thus $C^1(\mathcal{M})/\delta^0C^0(\mathcal{M})$ is torsion-free; being finitely generated it must be free.

As a result of Lemma 7.3.2, part (1), we may deduce the following analog of Theorem 7.2.1, giving the corresponding coefficient theorem for cohomology.\(^5\)

Theorem 7.3.3. Let $M$ be a finite connected map, and let $A$ be an abelian group.

1. If $i = 0, 1$, then there is a natural split short exact sequence:

$$0 \to H^i(M; \mathbb{Z}) \otimes A \to H^i(M; A) \to \operatorname{Tor}(H^{i+1}(M; \mathbb{Z}), A) \to 0,$$

and,

2. $H^2(M; A) \cong H^2(M; \mathbb{Z}) \otimes A$.

The naturality of the above short exact sequence is analogous with the corresponding naturality statement of Theorem 7.2.1. Thus, if $\phi : M' \to M$ is a morphism of maps, and if $\alpha : A \to A'$ is a homomorphism of abelian groups, then one obtains the following commutative ladder:

$$
\begin{array}{ccc}
0 & \to & H^i(M; \mathbb{Z}) \otimes A \\
& \downarrow & \downarrow \\
H^i(\phi) \otimes \alpha & \to & H^i(M'; A') \\
& \downarrow & \downarrow \\
0 & \to & \operatorname{Tor}(H^{i+1}(M; \mathbb{Z}), A) \\
\end{array}
$$

We should point out that the cohomology groups $H^i(M; A)$ can be deduced from the homology groups via the following Universal Coefficient Theorem\(^6\), as follows:

Theorem 7.3.4. Let $M$ be a map and let $A$ be an abelian group.

1. If $i = 1, 2$, then there is a natural split short exact sequence:

$$0 \to \operatorname{Ext}(H_{i-1}(M), A) \to H^i(M; A) \to \operatorname{Hom}(H_i(M), A) \to 0.$$

2. $H^0(M; A) \cong \operatorname{Hom}(H_0(M); A)$.

Thus, the above tells us to what extent cohomology is the dual of homology.

\(^6\)See op. cit., Theorem 3.3, page 179.
Corollary 7.3.4.1. Let \( \mathcal{M} \) be a connected map and let \( A \) be an abelian group.

1. If \( \mathcal{M} \) is orientable, \( H^i(\mathcal{M}; A) \cong \text{Hom}(H_i(\mathcal{M}), A), i = 0, 1, 2, \) and

2. if \( \mathcal{M} \) is nonorientable, then \( H^i(\mathcal{M}; A) \cong \text{Hom}(H_i(\mathcal{M}), A), \) if \( i = 0, 1, \) and \( H^2(\mathcal{M}; A) \cong \text{Ext}(\mathbb{Z}_2, A). \)

Finally, we can now affect the classical comparison between the fundamental group of a finite uniform map and its homology.

Corollary 7.3.4.2. Let \( \mathcal{M} \) be a finite connected uniform map. Then

\[
H_1(\mathcal{M}) \cong \frac{\pi_1(\mathcal{M}, \beta)}{\pi_1(\mathcal{M}, \beta)'}.
\]

Proof. First of all, note that since \( \mathcal{M} \) is finite (i.e., \( B \) is finite), then the fundamental group \( \pi_1(\mathcal{M}, \beta) \) has finite index in the Coxeter group \( \Delta = \Delta(k, l) \), where \( k, l \) the face and vertex valencies of \( \mathcal{M} \), respectively. Next, since \( \Delta \) is finitely generated, so is the fundamental group \( \pi_1(\mathcal{M}, \beta). \)

Next, by Theorem 5.4.2, together with Proposition 7.3.1, and Corollary 7.3.4.1, we infer that for any abelian group \( A, \)

\[
\text{Hom}(H_1(\mathcal{M}), A) \cong H^1(\mathcal{M}; A) \cong \text{Hom}(\pi_1(\mathcal{M}, \beta), A) \\
\cong \text{Hom}(\pi_1(\mathcal{M}, \beta)/\pi_1(\mathcal{M}, \beta)', A);
\]

since both \( H_1(\mathcal{M}) \) and \( \pi_1(\mathcal{M}, \beta)/\pi_1(\mathcal{M}, \beta)' \) are finitely-generated abelian groups, this is clearly enough.

We can improve on Corollary 7.3.4.2 so as to include infinite orientable uniform maps and at the same time render the isomorphic explicit. In fact, for any uniform map \( \mathcal{M}, \) we shall construct a homomorphism \( h: \pi_1(\mathcal{M}, \beta_0) \to H_1(\mathcal{M}) \) such that the following triangle commutes for any abelian group \( A: \)

\footnote{This follows, e.g., from Theorem 11.24, p. 258 of J. Rotman, The Theory of Groups, An Introduction, Allyn and Bacon, Inc., Boston, 1973.}
\[ H^1(\mathcal{M}; A) \xrightarrow{\delta} \text{Hom}(H_1(\mathcal{M}), A) \xrightarrow{} \text{Hom}(\pi_1(\mathcal{M}, \beta_0), A). \]

In the above, \( \delta : H^1(\mathcal{M}; A) \rightarrow \text{Hom}(H_1(\mathcal{M}), A) \) is the “evaluation map” of Theorem 7.3.4, \( \chi \) is the characteristic map, and \( \text{Hom}(h, 1_A) : \text{Hom}(H_1(\mathcal{M}), A) \rightarrow \text{Hom}(\pi_1(\mathcal{M}, \beta_0), A) \) is induced by \( h : \pi_1(\mathcal{M}, \beta_0) \rightarrow H_1(\mathcal{M}) \).

We now proceed to define \( h : \pi_1(\mathcal{M}, \beta_0) \rightarrow H_1(\mathcal{M}) \). Thus, we continue to assume that \( \mathcal{M} \) is uniform with vertex valency \( l \) and face valency \( k \), and set \( \Delta = \Delta (k, l) \). If \( \gamma \in \pi_1(\mathcal{M}, \beta_0) \), we write \( \gamma \) as we did in Section 5.3, viz., as \( \gamma = wu_1 \cdots u_2 u_1 \), where \( w, s_1 u_j \in \langle s_2, s_3 \rangle, j = 1, 2, \ldots, r \), and set

\[ h(\gamma) = \sum_{j=1}^{r} \left( \chi_{u_1 \cdots u_2 u_1 \beta_0} + \chi_{s_1 u_1 \cdots u_2 u_1 \beta_0} \right) + W^{(a,c)} \in W^{(c)} / W^{(a,c)}. \]

Note first that the above determines an element of \( H_1(\mathcal{M}) \), for

\[ \partial_{1} h(\gamma) = \partial_{1} \left( \sum_{j=1}^{r} \left( \chi_{u_1 \cdots u_2 u_1 \beta_0} + \chi_{s_1 u_1 \cdots u_2 u_1 \beta_0} \right) \right) \]

\[ = \sum_{j=1}^{r} \left( \chi_{[u_1 \cdots u_2 u_1 \beta_0]} - \chi_{[s_1 u_1 \cdots u_2 u_1 \beta_0]} + \chi_{[s_2 u_1 \cdots u_2 u_1 \beta_0]} - \chi_{[s_3 s_1 u_1 \cdots u_2 u_1 \beta_0]} \right) \]

\[ = \sum_{j=1}^{r} \left( 2 \chi_{[u_1 \cdots u_2 u_1 \beta_0]} - 2 \chi_{[s_1 u_1 \cdots u_2 u_1 \beta_0]} \right) \]

\[ = 2 \sum_{j=1}^{r} \left( \chi_{[u_1 \cdots u_2 u_1 \beta_0]} - \chi_{[s_1 u_1 \cdots u_2 u_1 \beta_0]} \right) = 0. \]

To show that the above expression is independent of the particular factorization of \( \gamma \), it suffices to show that if also

\[ \gamma' = wu_r \cdots u_t w_{t-1} \cdots u_2 u_1, \]
where \( u = (s_is_j)^{m_{ij}} \), and

\[
m_{ij} = \begin{cases} 
2 & \text{if } i = j \text{ or if } \{i, j\} = \{1, 3\} \\
k & \text{if } \{i, j\} = \{1, 2\} \\
l & \text{if } \{i, j\} = \{2, 3\}, 
\end{cases}
\]

then \( h(\gamma') = h(\gamma) \). Clearly, we have \( h(\gamma') = h(\gamma) \) whenever \( \{i, j\} \subseteq \{2, 3\} \). Next, if

\[
\gamma' = u_{r_1} \cdots u_{t_1} (s_1 s_3)^{2} u_{t-1} \cdots u_2 u_1,
\]

we have

\[
h(\gamma') = \sum_{j=1}^{t-1} \left( X_{u_j \cdots u_2 u_1 \beta_0} + X_{s_1 u_j \cdots u_2 u_1 \beta_0} \right) \\
+ X_{s_1 s_3 u_{t-1} \cdots u_2 u_1 \beta_0} + X_{s_1 u_{t-1} \cdots u_2 u_1 \beta_0} \\
+ X_{(s_1 s_3)^2 u_{t-1} \cdots u_2 u_1 \beta_0} + X_{s_3 s_3 u_{t-1} \cdots u_2 u_1 \beta_0} \\
+ \sum_{j=t}^{r} \left( X_{u_j \cdots u_2 u_1 \beta_0} + X_{s_3 u_j \cdots u_2 u_1 \beta_0} (s_1 s_3)^2 u_{t-1} \cdots u_2 u_1 \beta_0 \right) \\
= \sum_{j=1}^{r} \left( X_{u_j \cdots u_2 u_1 \beta_0} + X_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) \\
+ X_{s_1 u_{t-1} \cdots u_2 u_1 \beta_0} + X_{s_3 s_1 u_{t-1} \cdots u_2 u_1 \beta_0} + X_{u_{t-1} \cdots u_2 u_1 \beta_0} + X_{s_3 u_{t-1} \cdots u_2 u_1 \beta_0} \\
= \sum_{j=1}^{r} \left( X_{u_j \cdots u_2 u_1 \beta_0} + X_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) + \text{an element of } W^{(a,c)},
\]

and so it follows that \( h(\gamma') = h(\gamma) \). Finally, if \( u = (s_1 s_2)^k \), then

\[
\gamma' = u_{r_1} \cdots u_{t_1} (s_1 s_2)^{k} u_{t-1} \cdots u_2 u_1;
\]
this gives
\[ h(\gamma') = \sum_{j=1}^{t-1} \left( \chi_{u_j \cdots u_2 u_1 \beta_0} + \chi_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) \]
\[ + \sum_{m=1}^{k} \left( \chi_{(s_1 s_2)^m u_{t-1} \cdots u_2 u_1 \beta_0} + \chi_{s_3 (s_1 s_2)^m u_{t-1} \cdots u_2 u_1 \beta_0} \right) \]
\[ + \sum_{j=t}^{r} \left( \chi_{u_j \cdots u_2 u_1 \beta_0} + \chi_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) \]
\[ = \sum_{j=1}^{k} \left( \chi_{u_j \cdots u_2 u_1 \beta_0} + \chi_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) \]
\[ + \sum_{m=1}^{k} \left( \chi_{(s_1 s_2)^m \beta} + \chi_{s_3 (s_1 s_2)^m \beta} \right) \quad \text{(where } \beta = u_{t-1} \cdots u_2 u_1 \beta_0) \]
\[ = \sum_{j=1}^{k} \left( \chi_{u_j \cdots u_2 u_1 \beta_0} + \chi_{s_3 u_j \cdots u_2 u_1 \beta_0} \right) + \partial_2 \chi_{[\beta]_{p+1}} \]

so, again, we have that \( h(\gamma') = h(\gamma) \). This completes the proof that \( h : \pi_1(M, \beta_0) \to H_1(M) \) is well-defined.

Next, we prove that \( h \) is a homomorphism. Thus, let
\[ \gamma = w u_r \cdots u_2 u_1, \quad \gamma' = w'u_s' \cdots u'_2 u'_1, \]
where \( w, w', s_i u_i, s'_i u'_j \in \langle s_2, s_3 \rangle, \ i = 1, 2, \ldots, r, \ j = 1, 2, \ldots s \). We have
\[ h(\gamma \gamma') = \sum_{j=1}^{s} \left( \chi_{u'_j \cdots u'_2 u'_1 \beta_0} + \chi_{s_3 u'_j \cdots u'_2 u'_1 \beta_0} \right) \]
\[ + \sum_{i=1}^{t} \left( \chi_{u_i \cdots u_2 u_1 \gamma' \beta_0} + \chi_{s_3 u_i \cdots u_2 u_1 \gamma' \beta_0} \right) \]
\[ = \sum_{j=1}^{s} \left( \chi_{u'_j \cdots u'_2 u'_1 \beta_0} + \chi_{s_3 u'_j \cdots u'_2 u'_1 \beta_0} \right) \]
\[ + \sum_{i=1}^{t} \left( \chi_{u_i \cdots u_2 u_1 \beta_0} + \chi_{s_3 u_i \cdots u_2 u_1 \beta_0} \right) \]
\[ = h(\gamma) + h(\gamma') \]
We now have the following fundamental result.

**Theorem 7.3.5.** Let $\mathcal{M}$ be a uniform map with vertex valency $l$ and face valency $k$. Fix a blade $\beta_0$ in $\mathcal{M}$ and set $\pi_1(\mathcal{M}, \beta_0) = \pi_1^{(k,l)}(\mathcal{M}, \beta_0)$. Let $h : \pi_1(\mathcal{M}, \beta_0) \rightarrow H_1(\mathcal{M})$ be the above homomorphism.

1. $h$ is natural in that given a morphism $\phi : \mathcal{M} \rightarrow \mathcal{M}'$ of maps, where $\mathcal{M}'$ uniform and where $\beta'_0 = \beta_0\phi$, then the following diagram commutes:

$$\begin{array}{ccc}
\pi_1(\mathcal{M}, \beta_0) & \longrightarrow & H_1(\mathcal{M}) \\
\downarrow & & \downarrow \\
\pi_1(\mathcal{M}', \beta'_0) & \longrightarrow & H_1(\mathcal{M}')
\end{array}$$

2. For any abelian group $A$, we have a commutative triangle

$$\begin{array}{ccc}
H^1(\mathcal{M}; A) & \xrightarrow{\delta} & \text{Hom}(H_1(\mathcal{M}), A) \\
\downarrow{\alpha} & & \downarrow{\text{Hom}(h_1, 1_A)} \\
\text{Hom}(\pi_1(\mathcal{M}, \beta_0), A).
\end{array}$$

**Proof.** (1) is obvious. For (2), note that any element of $H_1(\mathcal{M}) \subseteq W^{(c)}/W^{(a,c)}$ is of the form

$$\sum_{\beta \in B} n_{\beta} (\chi_{\beta} + \chi_{c\beta}) + W^{(a,c)},$$

where each $n_{\beta} \in \mathbb{Z}$ and only finitely many $n_{\beta} \neq 0$. If $\zeta = [z] \in H^1(\mathcal{M}; A)$, then the evaluation map $\delta : H^1(\mathcal{M}; A) \rightarrow \text{Hom}(H_1(\mathcal{M}), A)$
is given by
\[
\delta(\zeta) \left( \sum_{\beta \in B} n_\beta (\chi_\beta + \chi_\alpha) \right) = \sum_{\beta \in B} n_\beta z_\beta \in A.
\]

Therefore, if \( \gamma = wu_r \ldots u_2 u_1 \in \pi_1(\mathcal{M}, \beta_0), w, s_1 u_j \in \langle s_2, s_3 \rangle, j = 1, 2, \ldots, r \), then,
\[
\text{Hom}(h, 1_A) \circ \delta(\zeta)(\gamma) = \delta(\zeta)(h(\gamma)) = \sum_{j=1}^r z_{u_j \ldots u_2 u_1 \beta_0} = \chi_z(\gamma),
\]
proving the theorem.

**Corollary 7.3.5.1.** If \( \mathcal{M} \) is an orientable uniform map, of finite face valency \( k \), then \( \pi_1(\mathcal{M}, \beta_0)/\pi_1(\mathcal{M}, \beta_0)' \cong H_1(\mathcal{M}) \).

**Proof.** In the triangle of Theorem 7.3.5, part (2), we have that \( \chi \) and \( \delta \) are isomorphisms for any abelian group \( A \). Next, let \( X \) be the commutator factor group of \( \pi_1(\mathcal{M}, \beta_0) \), i.e., \( X = \pi_1(\mathcal{M}, \beta_0)/\pi_1(\mathcal{M}, \beta_0)' \); therefore, for any abelian group \( A \), we have \( \text{Hom}(\pi_1(\mathcal{M}, \beta_0), A) = \text{Hom}(X, A) \). Note that the homomorphism \( h : \pi_1(\mathcal{M}, \beta_0) \to H_1(\mathcal{M}) \) induces the homomorphism \( \overline{h} : X \to H_1(\mathcal{M}) \), which gives rise to the exact sequence
\[
X \overset{\overline{h}}{\to} H_1(\mathcal{M}) \to \text{coker}(\overline{h}) \to 0,
\]
from which one develops the exact sequence
\[
0 \to \text{Hom}(\text{coker}(\overline{h}), A) \to \text{Hom}(H_1(\mathcal{M}), A) \overset{\cong}{\to} \text{Hom}(X, A).
\]
Therefore, it follows that for all abelian groups \( A \), \( \text{Hom}(\text{coker}(\overline{h}), A) = 0 \), which clearly implies that \( \text{coker}(\overline{h}) = 0 \), i.e., that \( \overline{h} : X \to H_1(\mathcal{M}) \) is surjective. Next, if \( K = \ker(\overline{h} : X \to H_1(\mathcal{M})) \), then we have a short exact sequence
\[
0 \to K \to X \overset{\overline{h}}{\to} H_1(\mathcal{M}) \to 0;
\]
we already noted on page 106 that \( H_1(\mathcal{M}) \) is a free abelian group, and so the above short exact sequence splits. Therefore, for any abelian group \( A \), the following sequence is exact:
\[
0 \to \text{Hom}(H_1(\mathcal{M}), A) \overset{\overline{h}}{\to} \text{Hom}(X, A) \to \text{Hom}(K, A) \to 0.
\]
Since $\overline{h} : \text{Hom}(H_1(\mathcal{M}), A) \cong \text{Hom}(X, A)$, we infer that $\text{Hom}(K, A) = 0$ for all abelian groups $A$. Clearly this implies that $K = 0$ and so $\overline{h} : X \to H_1(\mathcal{M})$ is an isomorphism.

**Remark.** We would still like to prove the above isomorphism for any map and using the combinatorial fundamental group.

The following is now immediate, in view of Theorem 2.1.4:

**Corollary 7.3.5.2.** Let $p : \mathcal{M}' \to \mathcal{M}$ be an unramified covering of uniform maps, and assume that the group $A(\mathcal{M}'/\mathcal{M})$ of covering transformations is abelian. Then

$$\text{coker}(H_1(p) : H_1(\mathcal{M}') \to H_1(\mathcal{M})) \cong A(\mathcal{M}'/\mathcal{M}).$$
Chapter 8
Applications to Principal Derived Maps

(Much of this chapter represents joint work with Chris Schroeder)

In this chapter we shall apply the results of the previous chapter to
the study of principal derived maps $\mathcal{M}_z$, where $\mathcal{M}$ is a finite orientable
map (usually assumed to be regular), where $z \in C(\mathcal{M}; A)$, and where
$A$ is an abelian group (usually assumed to be finite). The two central
questions of this chapter are

(i) the connectivity of $\mathcal{M}_z$, and

(ii) the regularity of $\mathcal{M}_z$ when $\mathcal{M}$ is also regular.

The first section, which can be thought of as a “warm up,” shows
the relative ease with which one can construct principal derived maps
$\mathcal{M}_z$ that are connected, regular, and such that the covering $\mathcal{M}_z \rightarrow \mathcal{M}$
is unramified.

8.1 Application: Macbeath’s Theorem

In this section, we use the results of the preceding sections to explicitly
construct connected principal derived maps of a given map $\mathcal{M}$, regular
when $\mathcal{M}$ is. We shall, throughout, assume that $\mathcal{M}$ is a finite orientable
map, and so $H_1(\mathcal{M}; \mathbb{Z})$ is free abelian of rank $2g$, where $g$ is the genus
of $\mathcal{M}$. We apply the fundamental commutative triangle of page 120, using as group of coefficients $H_1(\mathcal{M}; A)$, where $A$ is, for the moment, an arbitrary abelian group:

$H^1(\mathcal{M}; H_1(\mathcal{M}; A)) \xrightarrow{\delta} \text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}; A)) \xrightarrow{\chi} \text{Hom}(\pi_1(\mathcal{M}, \beta_0), H_1(\mathcal{M}; A)).$

We note that by Corollary 7.3.4.1, part (1), $\delta$ is an isomorphism. Next, applying Theorem 7.3.3, part (1), we have $H^1(\mathcal{M}; H_1(\mathcal{M}; A)) \cong H^1(\mathcal{M}; \mathbb{Z}) \otimes H_1(\mathcal{M}; A)$, and so we can express the above commutative triangle as follows:

$H^1(\mathcal{M}; \mathbb{Z}) \otimes H_1(\mathcal{M}; A) \xrightarrow{\delta} \text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}; A)) \xrightarrow{\chi} \text{Hom}(\pi_1(\mathcal{M}, \beta_0), H_1(\mathcal{M}; A)).$

Therefore, if $\eta \in H^1(\mathcal{M}; \mathbb{Z})$, $c_A \in H_1(\mathcal{M}; A)$, we have

$$\delta(\eta \otimes c_A) = (c \mapsto \eta(c)c_A \in A), \quad c \in H_1(\mathcal{M}).$$

Since $\delta$ is an isomorphism, we may identify $H^1(\mathcal{M}; \mathbb{Z}) \otimes H_1(\mathcal{M}; A)$ with $\text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}; A))$ via $\delta$ and write

$$(\eta \otimes c_A)(c) = \eta(c)c_A \in H_1(\mathcal{M}; A), \quad c_A \in H_1(\mathcal{M}; A), \quad c \in H_1(\mathcal{M}).$$

Now let $A = \mathbb{Z}_n$ with quotient map $\epsilon : \mathbb{Z} \rightarrow \mathbb{Z}_n$ and corresponding element

$$\epsilon_* \in \text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}; \mathbb{Z}_n)).$$

Note that by Proposition 7.2.3, $\epsilon_* : H_1(\mathcal{M}) \rightarrow H_1(\mathcal{M}; \mathbb{Z}_n)$ is surjective. Applying again Corollary 7.3.4.1, part (1), we have that $H^1(\mathcal{M}; \mathbb{Z}) \cong$
\[ \text{Hom}(H_1(\mathcal{M}), \mathbb{Z}) \cong \mathbb{Z}^{2g}; \text{ therefore, there exist dual } \mathbb{Z}-\text{bases } \{ \eta_1, \ldots, \eta_{2g} \}, \text{ and } \{ c_1, \ldots, c_{2g} \} \text{ of } H^1(\mathcal{M}; \mathbb{Z}) \text{ and } H_1(\mathcal{M}), \text{ respectively, and satisfying } \eta_i(c_j) = \delta_{ij} \text{ (Kronecker } \delta). \text{ We now set} \]

\[ \zeta = \sum_{i=1}^{2g} \eta_i \otimes \epsilon_s(c_i) \in H^1(\mathcal{M}; \mathbb{Z}) \otimes H_1(\mathcal{M}; \mathbb{Z}_n). \]

Therefore,

\[ \delta(\zeta)(c_j) = \sum_{i=1}^{2g} \eta_i(c_j)\epsilon_s(c_j) = \epsilon_s(c_j), \]

which is to say that \( \delta(\zeta) = \epsilon_s \). Therefore, \( \chi(\zeta) = \epsilon_s \circ h \), which is a surjective mapping \( \pi_1(\mathcal{M}, \beta_0) \to H_1(\mathcal{M}; \mathbb{Z}_n) \). As a result, applying Proposition 5.3.2, we see that if \( \zeta = [z] \in H^1(\mathcal{M}, H_1(\mathcal{M}; \mathbb{Z}_n)) \), then the principal derived map \( \mathcal{M}_z \) is connected.

Finally, we shall show that every automorphism of \( \mathcal{M} \) lifts to the connected map \( \mathcal{M}_z \). Thus, let \( \phi \in \text{Aut}(\mathcal{M}) \). We set \( \phi^*_s = H_1(\phi) : H_1(\mathcal{M}) \to H_1(\mathcal{M}) \); by Corollary 7.2.1.1 we may identify \( H_1(\mathcal{M}; \mathbb{Z}_n) \) with \( H_1(\mathcal{M}) \otimes \mathbb{Z}_n \), producing the automorphism \( \phi^*_s \otimes 1_{\mathbb{Z}_n} : H_1(\mathcal{M}; \mathbb{Z}_n) \to H_1(\mathcal{M}; \mathbb{Z}_n) \). Furthermore, the mapping \( \phi \mapsto \phi^*_s \otimes 1_{\mathbb{Z}_n} \) defines a homomorphism \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(H_1(\mathcal{M}; \mathbb{Z}_n)) \). Therefore, the lifting criterion \( \phi^*\zeta = \zeta \alpha(\phi)_* \) can be expressed (in analogy with the discussion on page 66) by saying that \( \zeta \) is a fixed point of \( H^1(\phi, \alpha(\phi)^{-1}) \). However, from the natural isomorphisms

\[ H^1(\mathcal{M}; H_1(\mathcal{M}; \mathbb{Z}_n)) \xrightarrow{\delta} \text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}; \mathbb{Z}_n)) \]

\[ \to \text{Hom}(H_1(\mathcal{M}), H_1(\mathcal{M}) \otimes \mathbb{Z}_n), \]

we see that \( \zeta \) being a fixed point of \( H^1(\phi, \alpha(\phi)^{-1}) \) is equivalent with \( \delta(\zeta) \) being a fixed point of \( \text{Hom}(H_1(\phi), H_1(\phi^{-1}) \otimes 1_{\mathbb{Z}_n}) = \text{Hom}(\phi^*_s, \phi^{-1}_s \otimes 1_{\mathbb{Z}_n}) \). In turn, as \( \delta(\zeta) = \epsilon_s \), we see that this latter condition translates into the condition that \( \epsilon_s \phi^*_s = (\phi^*_s \otimes 1_{\mathbb{Z}_n})\epsilon_s \). But

\[ \epsilon_s \phi^*_s = (1_{H_1(\mathcal{M})} \otimes \epsilon) \phi^*_s = \phi^*_s \otimes \epsilon = (\phi^*_s \otimes 1_{\mathbb{Z}_n})(1_{H_1(\mathcal{M})} \otimes \epsilon) = (\phi^*_s \otimes 1_{\mathbb{Z}_n})\epsilon_s, \]

and so the result follows.
From the above work we extract a few results of interest. Note first that given that the orientable map \( \mathcal{M} \) has Euler characteristic \( 2 - 2g \), then the connected principal derived map \( \mathcal{M}_z \) constructed above has Euler characteristic \( 2gn(2 - 2g) \). Furthermore, this map is regular:

\[ \text{Theorem 8.1.1. Let } \mathcal{M} \text{ be a regular orientable map of Euler characteristic } 2 - 2g \text{ and genus } g. \text{ Then for each positive integer } n, \text{ there is a regular connected map of Euler characteristic } 2gn(2 - 2g) \text{ covering } \mathcal{M}. \]

Recall from Theorem 4.2.2 that every orientable map of genus \( g \geq 2 \) has at most \( 168(g - 1) \) automorphisms. Maps realizing this upper bound are called \textit{extremal}. A very important extremal regular map is the one having genus 3 and automorphism group \( \text{PGL}_2(7) \). Because of the above, we conclude easily that if \( \mathcal{M} \) is extremal, so is \( \mathcal{M}_z \), where \( \zeta = [\zeta] \in H^1(\mathcal{M}; H_1(\mathcal{M}; \mathbb{Z}_n)) \) as constructed above. Thus, we have Macbeath’s Theorem: \(^1\)

\[ \text{Theorem 8.1.2. There are infinitely many extremal maps.} \]

\section{8.2 Connectivity}

In this section we shall relax the requirement that the covering \( \mathcal{M}_z \to \mathcal{M} \) of \( \mathcal{M} \) by the principal derived map \( \mathcal{M}_z, z \in D(\mathcal{M}; A) \) be an unramified covering, where \( A \) is an abelian group. In this case we use Proposition 7.3.1 to identify \( D(\mathcal{M}; A) \) with \( C^1(\mathcal{M}; A)/\delta^0(\mathcal{M}; A) \) and thereby obtain the mapping

\[ D(\mathcal{M}; A) \xrightarrow{\delta^1} C^2(\mathcal{M}; A) = \text{Hom}(C_2(\mathcal{M}), A); \]

since \( \delta^1D(\mathcal{M}; A) \) kills \( H_2(\mathcal{M}) \subseteq C_2(\mathcal{M}) \), then \( \delta^1 \) induces a mapping

\[ D(\mathcal{M}; A) \longrightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A). \]

We shall continue to denote the above mapping by $\delta^1 : D(M; A) \to \text{Hom}(C_2(M)/H_2(M), A)$.

**Theorem 8.2.1.** Assume that $M$ is a connected map, and let $A$ be an abelian group.

1. If $M$ is orientable, then $\delta^1 : D(M; A) \to \text{Hom}(C_2(M)/H_2(M), A)$ is surjective.

2. If $M$ is nonorientable, then

$$\text{coker}(\delta^1 : D(M; A) \to \text{Hom}(C_2(M)/H_2(M), A)) \cong \text{Ext}(\mathbb{Z}_2, A).$$

**Proof.** Assume first that $M$ is orientable. We have the short exact sequence

$$0 \to C_2(M)/H_2(M) \to C_1(M) \to C_1(M)/\partial_1 C_2(M) \to 0.$$

From the chain isomorphism $C^+_s(M) \cong C_8(M)$, we infer that

$$C_1(M)/\partial_2 C_2(M) \cong C^+_1(M)/\partial^+_2 C^+_2(M);$$

from Proposition 7.1.5, it follows that $C_1(M)/\partial_2 C_2(M)$ is free. It follows that the above short exact sequence splits, from which it follows that

$$\text{Hom}(C_1(M), A) \to \text{Hom}(C_2(M)/H_2(M), A)$$

is surjective, proving (1). If $M$ is nonorientable, then $H_2(M) = 0$ and so, by definition,

$$\text{coker}(\eta_A : D(M; A) \to \text{Hom}(C_2(M)/H_2(M), A)) \cong H^2(M; A).$$

Now apply part (2) of Corollary 7.3.4.1 to finish the proof.

For the remainder of this section, we shall assume that the base map $M$ is regular and orientable, with vertex valency $l$ and face valency $k$. Inside $\Delta(\infty, l)$ we have the orbifold fundamental group $\pi_1^{(\infty, l)}(M, \beta_0)$ relative to the fixed blade $\beta_0$. Since $M$ is regular, we infer that for any blade $\beta$ of $M$, $\pi_1^{(\infty, l)}(M, \beta_0) = \pi_1^{(\infty, l)}(M, \beta)$. If $R_k$ is the normal
closure in $\Delta(\infty, l)$ of the element $(s_1 s_2)^k$, then it is clear that $R_k \leq \pi_1^{(\infty,l)}(\mathcal{M}, \beta_0)$; furthermore, there is a short exact sequence of groups

$$1 \longrightarrow R_k \longrightarrow \pi_1^{(\infty,l)}(\mathcal{M}, \beta_0) \longrightarrow \pi_1(\mathcal{M}, \beta_0) \longrightarrow 1.$$ 

**Lemma 8.2.2.** Let $\zeta = [z] \in D(\mathcal{M}; A)$, where $A$ is an abelian group. If $\chi_z : \pi_1^{(\infty,l)}(\mathcal{M}, \beta) \rightarrow A$ is the characteristic homomorphism, then $\text{im} \delta^1(\zeta) = \chi_z(R_k) \subseteq A$.

**Proof.** First of all, one computes that if $\chi_{[\beta_0]_{F^+}}$ is the characteristic function on the oriented face in $\mathcal{M}$ containing $\beta_0$, then

$$\delta^1(\zeta)(\chi_{[\beta_0]_{F^+}}) = \sum_{\beta \in [\beta_0]_{F^+}} z_\beta = \sum_{j=1}^k z_{(ab)^j\beta_0}.$$

On the other hand, from page 69, we see that $\chi_z((s_1 s_2)^k)$ is given by

$$\chi_z((s_1 s_2)^k) = z_{a_b\beta_0} + z_{(ab)^2\beta_0} + \cdots + z_{(ab)^k\beta_0} \in A.$$

Therefore, we see that

$$\delta^1(\zeta)(\chi_{[\beta_0]_{F^+}}) = \chi_z((s_1 s_2)^k).$$

Note that if $\beta \in B$ is any other blade in $\mathcal{M}$, then by regularity there exists $\phi \in \text{Aut}(\mathcal{M})$ such that $\beta_0 \phi = \beta$. Therefore,

$$\delta^1(\zeta)(\chi_{[\beta]_{F^+}}) = \delta^1(\zeta)(\chi_{[\beta_0]_{F^+} \phi}) = \delta^1(\zeta)(\chi_{[\beta_0]_{F^+} \phi^*}) = \delta^1(\phi^* \zeta)(\chi_{[\beta_0]_{F^+}}),$$

from which we conclude that $\text{im}(\delta^1(\zeta))$ is generated by the images in $A$ of $\delta^1(\phi^* \zeta)(\chi_{[\beta_0]_{F^+}})$, where $\phi$ ranges over $\text{Aut}(\mathcal{M})$. On the other hand, again using regularity, for each $g \in G = \text{Mon}(\mathcal{M})$, there exists a unique $g' \in \text{Aut}(\mathcal{M})$ such that $g\beta_0 = \beta_0 g'$. Next, let $\gamma = wu_r \cdots u_2u_1 \in \text{Aut}(\mathcal{M})$.
$\pi_1^{(\infty,1)}(\mathcal{M}, \beta_0)$, where $w, s_1 u_j \in \langle s_2, s_3 \rangle$, $j = 1, 2, \ldots, k$, and let $s \in \{s_1, s_2, s_3\}$. We shall show that $\chi_z(s\gamma s) = \chi_{s^{*}z}(\gamma)$, which will clearly prove the lemma. Note first that if $s = s_2, s_3$ then one has that

\[
\chi_z(s\gamma s) = z_{u_1 s_2 \beta_0} + z_{u_2 u_1 s_2 \beta_0} + \cdots + z_{u_r \cdots u_2 u_1 s_2 \beta_0} = z_{u_1 \beta_0} + z_{u_2 u_1 \beta_0} + \cdots + z_{u_r \cdots u_2 u_1 \beta_0} = (s^{*}z)_{u_1 \beta_0} + (s^{*}z)_{u_2 u_1 \beta_0} + \cdots + (s^{*}z)_{u_r \cdots u_2 u_1 \beta_0} = \chi_{s^{*}z}(\gamma).
\]

If $s = s_1$, we have

\[
\chi_z(s\gamma s) = z_{s_1 \beta_0} + z_{u_1 s_1 \beta_0} + z_{u_2 u_1 s_1 \beta_0} + \cdots + z_{u_r \cdots u_2 u_1 s_1 \beta_0} = (s^{*}z)_{\beta_0} + (s^{*}z)_{u_1 \beta_0} + (s^{*}z)_{u_2 u_1 \beta_0} + \cdots + (s^{*}z)_{u_r \cdots u_2 u_1 \beta_0} = (s^{*}z)_{\beta_0} + (s^{*}z)_{u_1 \beta_0} + (s^{*}z)_{u_2 u_1 \beta_0} + \cdots + (s^{*}z)_{u_r \cdots u_2 u_1 \beta_0} = \chi_{s^{*}z}(\gamma).
\]

Now let $\zeta = [z] \in D(\mathcal{M}; A)$ and set $A_0 = \text{im} \delta^1(\zeta) \subseteq A$. If $\epsilon_0 : A \to A/A_0$, then $\delta^1(\zeta_{\epsilon_0}) = 0 \in \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A/A_0)$; since $H^1(\mathcal{M}; A/A_0) = \text{ker}(\delta^2 : D(\mathcal{M}; A/A_0) \to \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A/A_0))$, we infer that $\zeta_{\epsilon_0} \in H^1(\mathcal{M}; A/A_0)$. By duality, this gives the homomorphism $\delta(\zeta_{\epsilon_0}) : H_1(\mathcal{M}) \to A/A_0$.

The following is our main connectivity criterion for regular orientable maps.

**Theorem 8.2.3.** Let $\mathcal{M}$ be orientable and regular, let $A$ be an abelian group, and let $\zeta = [z] \in D(\mathcal{M}; A)$. If $A_0 = \text{im} \delta^1(\zeta) \subseteq A$, and if $\epsilon_0 : A \to A/A_0$ is the projection map, then $\mathcal{M}_\zeta$ is connected if and only if the homomorphism

\[
\delta(\zeta_{\epsilon_0}) : H_1(\mathcal{M}) \longrightarrow A/A_0
\]
is surjective.

**Proof.** We have a commutative diagram (with short exact rows) of the form:

\[
\begin{array}{cccccc}
1 & \longrightarrow & R_k & \longrightarrow & \pi_1^{(\infty, l)} (\mathcal{M}, \beta_0) & \longrightarrow & \pi_1 (\mathcal{M}, \beta_0) & \longrightarrow & 1 \\
\downarrow \chi_z & & \downarrow \chi_z & & \downarrow \chi_{z_0} & \\
0 & \longrightarrow & A_0 & \longrightarrow & A & \longrightarrow & A/A_0 & \longrightarrow & 0
\end{array}
\]

By Lemma 8.2.2, \( \chi_z : R_k \to A_0 \) is surjective; one easily checks that \( \chi_z : \pi_1^{(\infty, l)} (\mathcal{M}, \beta_0) \to A \) is surjective if and only if \( \chi_{z_0} : \pi_1 (\mathcal{M}, \beta_0) \to A/A_0 \) is surjective.\(^2\) Using the commutative triangle of Theorem 7.3.5, part (2), we see that \( \chi_{z_0} : \pi_1 (\mathcal{M}, \beta_0) \to A/A_0 \) is surjective if and only if \( \delta (\zeta e_0) : H_1 (\mathcal{M}) \to A/A_0 \) is surjective. Now apply Proposition 5.3.2.

The following lemma does not make use of regularity.

**Lemma 8.2.4.** Let \( \mathcal{M} \) be an orientable map, let \( A \) be an abelian group, and let \( A_0 \) be a subgroup of \( A \). Then the diagram below has exact rows

\[\text{...}\]

\(^2\)This is a direct verification; however a lazy approach would be to apply the so-called “Snake Lemma.”
and columns:

\[
0 \rightarrow H^1(\mathcal{M}; A_0) \rightarrow D(\mathcal{M}; A) \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A_0) \rightarrow 0
\]

\[
0 \rightarrow H^1(\mathcal{M}; A) \rightarrow D(\mathcal{M}; A) \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A) \rightarrow 0
\]

\[
0 \rightarrow H^1(\mathcal{M}; A/A_0) \rightarrow D(\mathcal{M}; A/A_0) \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A/A_0) \rightarrow 0
\]

**Proof.** Note first of all that since \( \mathcal{M} \) is orientable, then by Theorem 8.2.1, \( \delta^1 : D(\mathcal{M}; B) \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), B) \) is surjective for any abelian group \( B \). By definition, \( H^1(\mathcal{M}; B) = \ker(\delta^1) \), proving that the rows of the above diagram are exact. Next, since the abelian group \( C_2(\mathcal{M})/H_2(\mathcal{M}) \) is free (hence projective), then the functor \( \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), \cdot) \) is exact; thus, the exact sequence \( 0 \rightarrow A_0 \rightarrow A \rightarrow A/A_0 \rightarrow 0 \) induces the exact sequence

\[
0 \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A_0) \rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A)
\]

\[
\rightarrow \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A/A_0) \rightarrow 0,
\]

proving that the right column above is exact.

Using the fact that \( \mathcal{M} \) is orientable, we apply Theorem 7.1.9 to conclude that \( H_1(\mathcal{M}) \) is free (of rank \( 2g \), where \( g \) is the genus of \( \mathcal{M} \)).
Therefore, the functor \(\text{Hom}(H_1(\mathcal{M}), \cdot)\) is exact and so the short exact sequence \(0 \to A_0 \to A \to A/A_0 \to 0\) induces the short exact sequence

\[
0 \to \text{Hom}(H_1(\mathcal{M}), A_0) \to \text{Hom}(H_1(\mathcal{M}), A) \to \text{Hom}(H_1(\mathcal{M}), A/A_0).
\]

By Corollary 7.3.4.1, part (1) we have that (in the orientable case) cohomology and homology are in perfect duality and so the above short exact sequence translates into the exact sequence

\[
0 \to H^1(\mathcal{M}; A_0) \to H^1(\mathcal{M}; A) \to H^1(\mathcal{M}; A/A_0) \to 0.
\]

Therefore the left-hand column in the above diagram is exact. The exactness of the middle column is obtained through Lemma 7.3.2, parts (2) and (3), together with the fact that the freeness of \(D(\mathcal{M})\) implies that the functor \(D(\mathcal{M}) \otimes \cdot\) is exact.\(^3\)

**Lemma 8.2.5.** Consider the following commutative diagram of abelian

\(^3\)Alternatively, the exactness of the middle column can be inferred from the Snake Lemma.
groups with exact rows and columns:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\mu' & & \epsilon' \\
\mu_X & & \mu_Y \\
Y' & & Z' \\
\epsilon_X & & \epsilon_Z \\
X & & Y \\
\mu'' & & \epsilon'' \\
X'' & & Y'' \\
0 & 0 & 0
\end{array}
\]

Set \( Y_0 = \{ y \in Y \mid \epsilon(y) \in \mu_Z(Z') \} \). Then

\[
Y_0 / \ker \mu \mu_X \cong X'' \oplus Z'.
\]

**Proof.** By hypothesis, there exist homomorphisms \( \mu_0 : Y_0 \to X'' \) and \( \epsilon_0 : Y_0 \to Z' \) making the following diagram commute:
Denote by \( \{\mu_0, \epsilon_0\} : Y_0 \to X'' \oplus Z' \) the induced homomorphism. If \( K_\mu = \ker(\mu_0 : Y_0 \to X'') \), \( K_\epsilon = \ker(\epsilon_0 : Y_0 \to Z') \), then obviously \( \ker \{\mu_0, \epsilon_0\} = K_\mu \cap K_\epsilon \). We have

\[
K_\mu = \ker \mu_0 = Y_0 \cap \ker (\epsilon_Y : Y \to Y'') = \mu_Y(Y') \subseteq Y_0
\]

\[
K_\epsilon = \ker \epsilon_0 = Y_0 \cap \ker (\epsilon : Y \to Z) = \mu(X) \subseteq Y_0;
\]

thus,

\[
K_\mu \cap K_\epsilon = \mu_Y(Y') \cap \mu(X) = \mu_\mu_Y(X) = \mu_Y \mu'(X') \subseteq Y_0.
\]

Finally, we show that \( \{\mu_0, \epsilon_0\} : Y_0 \to X'' \oplus Z' \) is surjective. If \( x'' \in X'' \), \( z' \in Z' \), then choose preimages \( x \in X \), \( y' \in Y' \) such that \( \epsilon_X(x) = x'' \), \( \epsilon'(y') = z' \). Then

\[
\mu_0(\mu_Y(y') + \mu(x)) = \mu_0 \mu(x) = \epsilon(x) = x'',
\]

and

\[
\epsilon_0(\mu_Y(y') + \mu(x)) = \epsilon_0 \mu_Y(y') = \epsilon'(y') = z',
\]

and so

\[
\{\mu_0, \epsilon_0\}(\mu_y(y') + \mu(x)) = (x'', z') \in X'' \oplus Z',
\]

which finishes the proof.

If \( A \) is an abelian group with subgroup \( A_0 \subseteq A \), we set

\[
D(\mathcal{M}; A)_{A_0} = \{\zeta \in D(\mathcal{M}; A) \mid \delta^1(\zeta)(C_2(\mathcal{M})) \subseteq A_0\}.
\]

As immediate consequence of Lemma 8.2.5 is the following:

**Corollary 8.2.5.1.** If \( \mathcal{M} \) is orientable, \( A \) is an abelian group, and \( A_0 \) is a subgroup, then

\[
D(\mathcal{M}; A)_{A_0}/H^1(\mathcal{M}; A_0) \cong H^1(\mathcal{M}; A/A_0) \oplus \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A_0).
\]
If $X, Y$ are abelian groups, we denote by $\text{SHom}(X, Y)$ the surjective homomorphisms $X \to Y$. As an immediate consequence of Theorem 8.2.3 together with Corollary 8.2.5.1, we infer the following parametrization of connected principal derived maps over the regular orientable map $\mathcal{M}$.

**Corollary 8.2.5.2.** Let $\mathcal{M}$ be a regular orientable map, and let $A$ be an abelian group. The following sets are in bijective correspondence:

(i) $\{\zeta = [z] \in D(\mathcal{M}; A) \mid \mathcal{M}_z$ is connected $\};$

(ii) $\bigcup_{A_0 \subseteq A} H^1(\mathcal{M}; A_0) \times \text{SHom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A_0) \times \text{SHom}(H_1(\mathcal{M}), A/A_0)$.

### 8.2.1 Totally Ramified Coverings

Recall that a covering $\mathcal{M}' \to \mathcal{M}$ is called *totally ramified* if it cannot be factored nontrivially as $\mathcal{M}' \to \mathcal{M}_{\text{un}} \to \mathcal{M}$, where $\mathcal{M}_{\text{un}} \to \mathcal{M}$ is an unramified covering.

**Lemma 8.2.6.** Let $\zeta = [z] \in D(\mathcal{M}; A)$, where $A$ is an abelian group. Then $\mathcal{M}_\zeta$ is connected and the covering $\mathcal{M}_\zeta \to \mathcal{M}$ is totally ramified if and only if $\delta^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \to A$ is surjective.

**Proof.** Assume that $\mathcal{M}_\zeta$ is connected and that the covering $\mathcal{M}_\zeta \to \mathcal{M}$ is totally ramified. Let $A_0 = \text{im} \delta^1(\zeta) \subseteq A$ and let $\epsilon : A \to A/A_0$ be the projection. Note that $\zeta_{\epsilon^*} \in \ker (\delta^1 : D(\mathcal{M}; A/A_0) \to \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A/A_0)) = H^1(\mathcal{M}; A/A_0)$ and so $\mathcal{M}_{\zeta_{\epsilon^*}} \to \mathcal{M}$ is unramified. Since we have a factorization $\mathcal{M}_\zeta \to \mathcal{M}_{\zeta_{\epsilon^*}} \to \mathcal{M}$ we see that if $\mathcal{M}_\zeta \to \mathcal{M}$ is totally ramified, then we must have $A_0 = A$, i.e., that $\delta^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \to A$ is surjective. Conversely, assume that $\delta^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \to A$ is surjective. Then by Theorem 8.2.3 we already know that $\mathcal{M}_\zeta$ is connected. Next note that a factorization of $\mathcal{M}_\zeta \to \mathcal{M}$ must be of the form $\mathcal{M}_\zeta \to \mathcal{M}_{\zeta_{\epsilon_0}} \to \mathcal{M}$ for some epimorphism of the form $\epsilon_0 : A \to A/A_0$. If $\mathcal{M}_{\zeta_{\epsilon_0}} \to \mathcal{M}$ is unramified, then $\zeta_{\epsilon^*_0} \in H^1(\mathcal{M}; A/A_0)$; since $\mathcal{M}_{\zeta_{\epsilon^*_0}}$ is connected (being mapped surjectively onto by $\mathcal{M}_\zeta$), we infer that $\chi_{\zeta_{\epsilon^*_0}} : \pi_1(\mathcal{M}, \beta_0) \to A/A_0$ is surjective,
which by the above commutative ladder together with Lemma 8.2.2 shows that $\text{im} \delta^1(\zeta) \subseteq A_0$. Thus, $A_0 = 0$ and so $M_z \to M$ is totally ramified.

8.2.2 Case Study: Cyclic Principal Derived Coverings

In this subsection, we shall give a very specific parametrization of connected principal derived maps $M_z$, up to $\cong M$, where the group $A$ of covering transformations is cyclic of finite order $n$. We continue to assume that $M$ is regular and finite, with $r$ faces. By Theorem 5.1.6 the $\cong M$-isomorphism classes of principal derived coverings with cyclic group $A = \mathbb{Z}_n$ is parametrized by $D(M; \mathbb{Z}_n)/\text{Aut}(\mathbb{Z}_n) = D(M; \mathbb{Z}_n)/\mathcal{U}(\mathbb{Z}_n)$. From Corollary 8.2.5.2 we know that $\{\zeta = [z] \in D(M; \mathbb{Z}_m) \mid M_z \text{ is connected} \}$ is in bijective correspondence with the set

$$
\bigcup_{m \mid n} H^1(M; \mathbb{Z}_m) \times \text{SHom}(C_2(M)/H_2(M), \mathbb{Z}_m) \times \text{SHom}(H_1(M), \mathbb{Z}_{m/n}).
$$

Next, it is easy to show that for any abelian group $X$, then $f \in \text{SHom}(X, \mathbb{Z}_m)$ if and only if $f$ has order $m$ in the abelian group $\text{Hom}(X, \mathbb{Z}_m)$. Furthermore, if $X$ is free of finite rank $k$, then it follows that

$$
Y = \text{Hom}(X, \mathbb{Z}_m) \cong \mathbb{Z}_m^{(k)} = \mathbb{Z}_m \oplus \mathbb{Z}_m \oplus \cdots \oplus \mathbb{Z}_m, \text{ } k \text{ terms}
$$

If, for any divisor $h$ of $m$, we denote by $f(h)$ the number of elements of $Y$ whose order divides $h$, and denote by $g(h)$ the number of elements of $Y$ whose order equals $h$, then we have

$$
 f(h) = \sum_{l \mid h} g(l);
$$

by Möbius inversion, we have

$$
 g(m) = \sum_{l \mid m} \mu(m/l)f(l),
$$
where the Möbius function satisfies
\[
\mu(h) = \begin{cases} 
(-1)^t & \text{if } h \text{ factors into } t \text{ distinct primes,} \\
0 & \text{if not.}
\end{cases}
\]

Since the number of elements of \( Y \) having order dividing \( l \) is clearly \( l^k \), we conclude that the number of elements of order \( m \) in \( Y = \text{Hom}(X, \mathbb{Z}_m) \) is exactly \( \sum_{l|m} \mu(m/l)l^k \).

As a result, since \( C_2(\mathcal{M})/H_2(\mathcal{M}) \) is free of rank \( r - 1 \), and \( H_1(\mathcal{M}) \) is free of rank \( 2g \), we see that the number of elements \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_n) \) such that \( \mathcal{M}_z \) is connected is given by
\[
\sum_{m|n} |H^1(\mathcal{M}; \mathbb{Z}_m)| \sum_{l|m} \mu(m/l)l^{r-1} \sum_{h|(n/m)} \mu(n/hm)h^{2g}.
\]

Since \( |H^1(\mathcal{M}; \mathbb{Z}_m)| = m^{2g} \), we infer that the number of elements \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_n) \) such that \( \mathcal{M}_z \) is connected is
\[
\sum_{m|n} m^{2g} \sum_{l|m} \mu(m/l)l^{r-1} \sum_{h|(n/m)} \mu(n/hm)h^{2g}.
\]

Finally, if \( \zeta \mapsto (\theta^1, \theta) \in \text{SHom}(C_2(\mathcal{M})/H_2(\mathcal{M}), \mathbb{Z}_m) \times \text{SHom}(H_1(\mathcal{M}), \mathbb{Z}_{n/m}) \), then the order of \( \zeta \) in the abelian group \( D(\mathcal{M}, \mathbb{Z}_n) \) must be \( \text{lcm}(m, n/m) \).

Since the stabilizer in \( \text{Aut}(\mathbb{Z}_n) = \mathcal{U}(\mathbb{Z}_n) \) of such an element is easily checked to have order \( \phi(n)/\phi(\text{lcm}(m, n)) \) (where \( \phi \) is Euler’s totient function) we see that the size of the \( \text{Aut}(\mathbb{Z}_n) \)-orbits of such elements is exactly \( \phi(\text{lcm}(m, n)) \). Therefore, we have the following count:

**Theorem 8.2.7.** If \( \mathcal{M} \) is a finite, orientable, regular map, of genus \( g \), then the number of \( \cong_\mathcal{M} \)-classes of coverings \( \mathcal{M}' \to \mathcal{M} \) (unramified over vertices) by connected maps whose group of covering transformations of \( \mathcal{M}_z \to \mathcal{M} \) is cyclic of order \( n \) is given by
\[
\sum_{m|n} m^{2g} \sum_{l|m} \mu(m/l)l^{r-1} \sum_{h|(n/m)} m^{2g} \mu(m/l)l^{r-1} \mu(n/hm)h^{2g}/\phi(\text{lcm}(m, n)).
\]

Note in particular that the term in the above sum corresponding to \( m = 1 \) corresponds to the \( \cong_\mathcal{M} \)-classes of unramified coverings by
connected maps. At the other extreme, the term corresponding to \( m = n \) corresponds to the \( \cong_{\mathcal{M}} \)-classes of totally ramified coverings by connected maps:

**Theorem 8.2.8.** Let \( \mathcal{M} \) be a finite, orientable, regular map.

1. The number of \( \cong_{\mathcal{M}} \)-classes of unramified coverings \( \mathcal{M}' \to \mathcal{M} \) with \( \mathcal{M}' \) connected and whose group of covering transformations of \( \mathcal{M}' \to \mathcal{M} \) is cyclic of order \( n \) is given by

\[
\sum_{l|n} \mu(n/l) l^{2g} / \phi(n).
\]

2. The number of \( \cong_{\mathcal{M}} \)-classes of totally ramified coverings (unramified over vertices) \( \mathcal{M}' \to \mathcal{M} \) with \( \mathcal{M}' \) connected and whose group of covering transformations of \( \mathcal{M}' \to \mathcal{M} \) is cyclic of order \( n \) is given by

\[
n^{2g} \times \sum_{l|n} \mu(n/l) l^{r-1} / \phi(n).
\]

For example, a regular map \( \mathcal{M} \) of genus 1 and having four faces has 496 \( \cong_{\mathcal{M}} \)-classes of coverings \( \mathcal{M}' \to \mathcal{M} \) (unramified over vertices) by connected maps \( \mathcal{M}' \) and with cyclic group \( \mathbb{Z}_4 \) of covering transformations. Of these, six are unramified and 448 are totally ramified.

### 8.3 Bifunctors, Homology, and Representation Theory

In Chapter 5 we noted that the \( Z \)-valued voltages (modulo equivalence) on \( \mathcal{M} \) defined a bifunctor

\[
\text{D} : \text{Map} \times \text{Group} \longrightarrow \text{Set};
\]

furthermore, when \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(Z) \) is a homomorphism, then one has the \( \alpha \)-isotypical voltage classes \( \text{D}(\mathcal{M}; Z)_{\alpha} \) (see page 66), consisting of classes \( \zeta \in \text{D}(\mathcal{M}; Z) \) such that \( \text{D}(\phi, \alpha(\phi)^{-1})(\zeta) = \zeta \). If \( Z = A \)
is an abelian group and if $\alpha : \text{Aut}(M) \to \text{Aut}(A)$ determines an action of $\text{Aut}(M)$ on $A$, then $D(M; A)$ becomes a left $\text{Aut}(M)$-module via $\zeta \mapsto \phi^*\zeta\alpha(\phi)^{-1}$; thus the $\alpha$-isotypical voltage classes are the $\text{Aut}(M)$-invariants in the module $D(M; A)$.

In the preceding sections, we have seen important bifunctors $\text{Map} \times \text{AbGroup} \to \text{AbGroup}$, e.g.,

$$(M, A) \mapsto H^1(M; A), \quad (M, A) \mapsto \text{Hom}(H_1(M), A).$$

A few general remarks concerning this situation should help in the sequel. Let

$$\mathcal{H} : \text{Map} \times \text{Group} \to \text{Set}$$

be a bifunctor. If $Z$ is a group, and if $\alpha : \text{Aut}(M) \to \text{Aut}(Z)$ is a homomorphism, define the set of $\alpha$-isotypical $\mathcal{H}$-voltage classes by setting

$$\mathcal{H}(M; Z)_\alpha = \{\zeta \in \mathcal{H}(M; Z) \mid \mathcal{H}(\phi, \alpha(\phi)^{-1})(\zeta) = \zeta \text{ for all } \phi \in \text{Aut}(M)\}.$$  

Note that via $\alpha$, $\mathcal{H}(M; Z)$ admits a left action by $\text{Aut}(M)$ via $\zeta \mapsto \mathcal{H}(\phi, \alpha(\phi)^{-1})(\zeta)$ and so $\mathcal{H}(M; Z)_\alpha$ is simply the set of $\text{Aut}(M)$-fixed points relative to this action.

The utility of this formalism is the following. If $\mathcal{H}' : \text{Map} \times \text{Group} \to \text{Set}$ is another bifunctor, and if $\eta : \mathcal{H} \to \mathcal{H}'$ is a natural transformation of bifunctors, then for any map $M$, any coefficient group $Z$, and any homomorphism $\alpha : \text{Aut}(M) \to \text{Aut}(Z)$, $\eta_{M, Z}$ maps $\mathcal{H}(M; Z)_\alpha$ to $\mathcal{H}'(M; Z)_\alpha$. If $\eta : \mathcal{H} \cong \mathcal{H}'$ is a natural equivalence of bifunctors, then one obtains an isomorphism

$$\eta_{M, Z} : \mathcal{H}(M; Z)_\alpha \cong \mathcal{H}'(M; Z)_\alpha.$$

In particular, if $M$ is orientable, then by Theorem 7.3.4 and Corollary 7.3.4.1, we have that for any homomorphism $\alpha : \text{Aut}(M) \to \text{Aut}(A)$, where $A$ is an abelian group, the above discussion guarantees that

$$\delta : H^1(M; A)_\alpha \cong \text{Hom}(H_1(M), A)_\alpha. \quad (* \text{ )}$$

Indeed, enroute to proving Macbeath’s theorem (Theorem 8.1.2), we already used the above isomorphism with $A = H_1(M; \mathbb{Z}_n)$ and where
\[ \delta(\zeta) = \epsilon_s \in \text{Hom}(H_1(M), H_1(M; \mathbb{Z}_n))_\alpha, \text{ where } \alpha \text{ is just the action of } \text{Aut}(M) \text{ induced in homology; i.e., } \alpha(\phi) = \phi_s \otimes 1_{\mathbb{Z}_n}. \]

Notice that if \( A \) is an \( \text{Aut}(M) \)-module via the homomorphism \( \alpha : \text{Aut}(M) \to \text{Aut}(A) \), an element \( \theta \in \text{Hom}(H_1(M), A)_\alpha \) is nothing more than an \( \text{Aut}(M) \)-module homomorphism \( H_1(M) \to A \).

This having been observed, we can summarize our findings as follows.

**Theorem 8.3.1.** Let \( M \) be a regular orientable map, and let \( A \) be an abelian group. Let \( \zeta = [z] \in H_1(M; A) \), and set \( \theta = \delta(\zeta) : H_1(M) \to A \). Then \( M_z \) is connected and regular if and only if \( \theta : H_1(M) \to A \) is a surjective \( \text{Aut}(M) \)-module homomorphism for some \( \text{Aut}(M) \)-module structure on \( A \).

**Proof.** Noting that \( \delta_1(z) = 0 \), we may apply Theorem 8.2.3 to infer that \( M_z \) is connected if and only if \( \theta \) is surjective. If \( M_z \) is connected, then by Corollary 5.2.2.1 \( M_z \) is regular if and only if \( \zeta \in H_1(M; A)_\alpha \), for some \( \alpha : \text{Aut}(M) \to \text{Aut}(A) \) (which induces an \( \text{Aut}(M) \)-module structure on \( A \)). Using \( (\ast) \), we see that this happens if and only if \( \theta \in \text{Hom}(H_1(M), A)_\alpha \), which is to say that \( \theta \) is a surjective \( \text{Aut}(M) \)-module homomorphism.

We are in a position to classify all unramified coverings \( M' \to M \) of \( M \) by connected regular maps \( M' \) and having abelian group of covering transformations.\(^4\) To wit:

**Theorem 8.3.2.** Let \( M \) be a regular orientable map.

1. Let \( A \) be an abelian group. Then the following set are in bijective correspondence:

   (i) the set of \( \cong_M \)-isomorphism classes of unramified coverings \( M' \to M \) by connected regular maps having group of covering transformations \( A(M'/M) \cong A \), and

\(^4\)A similar parametrization was obtained in the context of graphs in A. Malič, D. Marušič, and P. Potočnik, Elementary abelian covers of graphs, University of Ljubljana preprint series, Vol. 39 (2001), 787. (See the comments after Proposition 5.1.)
(ii) the set of \( \text{Aut}(M) \)-submodules \( M \) of the first homology group \( H_1(M) \) such that \( H_1(M)/M \cong A \).

2. There is a bijective correspondence between

(i) the set of \( \cong_M \)-isomorphism classes of unramified coverings \( M' \to M \) by connected regular maps and having abelian group of covering transformations, and

(ii) the set of \( \text{Aut}(M) \)-submodules of the first homology group \( H_1(M) \).

**Proof.** Note first of all that by *Theorem* 5.1.1 such covering maps are of the form \( M_z \) for some voltage \( z \in C(M; A) \). Next, as noted in *Subsection* 5.4.1 we must have \([z] \in H(M; A)\); by *Proposition* 7.3.1 we have \( H(M; A) \cong H^1(M; A) \) and so we must have \( M' \cong_M M_z \), where \([z] \in H^1(M; A)\). The correspondence in question is given by

\[
M_z \mapsto \ker(\delta(\zeta) : H_1(M) \to A).
\]

Note that every \( \text{Aut}(M) \)-submodule of \( H_1(M) \) corresponds to some principal derived map \( M_z \); if \( M \subseteq H_1(M) \) is an \( \text{Aut}(M) \)-submodule, let \( A = H_1(M)/M \), with projection map \( \theta \in \text{Hom}(H_1(M); A) \) and take \( \zeta = \delta^{-1}(\theta) \). Since \( \theta \) is an \( \text{Aut}(M) \)-module homomorphism, \( \theta \in \text{Hom}(H_1(M); A)_{\alpha} \) for some \( \alpha : \text{Aut}(M) \to \text{Aut}(A) \), forcing \( \zeta \in \text{Hom}(H_1(M); A)_{\alpha} \), as well. Thus, \( M_z \) so constructed is connected and regular.

Assume now that \( \zeta = [z], \zeta' = [z'] \in H^1(M; A) \) and that \( M_z \cong_M M_{z'} \). As \( M_z, M_{z'} \) are connected, apply *Proposition* 5.1.4 and obtain that for some automorphism \( \gamma \in \text{Aut}(A) \), \( \zeta' = \zeta \gamma \). But then it is clear that \( \delta(\zeta') = \delta(\zeta) \gamma_s : H_1(M) \to A \); this obviously has the same kernel as \( \delta(\zeta) \).

Conversely, let \( \zeta = [z], \zeta = [z'] \in H^1(M; Z) \), let \( \theta = \delta(\zeta), \theta' = \delta(\zeta') : H_1(M) \to A \), such that \( \ker \theta = \ker \theta' \). Then one easily constructs an \( \text{Aut}(M) \)-automorphism \( \gamma \) of \( A \) such that \( \theta \gamma = \theta' \).

Therefore, it follows immediately that \( \zeta' = \zeta \gamma_s \) from which it follows (*Theorem* 5.1.3) that \( M_z \cong_M M_{z'} \), which finishes the proof.

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\(^5\)Indeed, define \( \gamma : A \to A \) by setting \( \theta(c) \gamma = \theta'(c) \), where \( c \in H_1(M) \); it is routine to verify that \( \gamma \) is a well-defined \( \text{Aut}(M) \)-module automorphism.
Therefore we see that the classification of the unramified coverings \( \mathcal{M}' \to \mathcal{M} \) of the regular orientable map \( \mathcal{M} \) by regular maps \( \mathcal{M}' \) is intimately related to the representation theory of \( \text{Aut}(\mathcal{M}) \) on the integral first homology group of \( \mathcal{M} \).

At the other extreme, one might enquire as to the existence of a similar parametrization of totally ramified coverings by regular maps of the regular orientable map \( \mathcal{M} \), based on the surjection

\[
\delta^1 : D(\mathcal{M}; A) \to \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A).
\]

However, this is quite a bit more subtle; whereas for any action \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(A) \) we have the induced homomorphism of \( \alpha \)-invariants

\[
\delta^1 : D(\mathcal{M}; A)_{\alpha} \to \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), A)_{\alpha},
\]

*it need not follow that this homomorphism is surjective.* The complete investigation of this question would involve the group cohomology of the automorphism group \( \text{Aut}(\mathcal{M}) \) (with coefficients in \( H^1(\mathcal{M}; A) \)\(^6\)), taking us well outside the scope of the present analysis.

The issue here is whether, for a given action of \( \mathcal{M} \) on the coefficient group \( A \), an \( \text{Aut}(\mathcal{M}) \)-module homomorphism \( \theta : C_2(\mathcal{M})/H_2(\mathcal{M}) \to A \) is covered by an isotypical voltage class in \( D(\mathcal{M}; A) \). In the companion paper, we shall show that there are two particularly commonly-occurring choices for \( \theta \) (the so-called *Steinberg* and *Accola* homomorphisms), and the question becomes that of finding all \( \alpha \)-isotypical preimages of \( \theta \).

### 8.4 Application: Regular Cyclic Coverings of Platonic Maps

This section applies the material of this chapter, together with work on the characteristic homomorphism to obtain the regular cyclic covering of the Platonic maps. This was carried out originally in my papers with

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Gareth Jones;\(^7\) however the present treatment is entirely self contained.

Since any Platonic map \(M\) is simply connected, we see that any covering of of \(M\) must be totally ramified. If we seek regular such coverings of \(M\) that are unramified over the vertices, then such a covering \(M' \to M\) by a connected map \(M'\) must be of the form \(M_z \to M\) for some \(z \in C(M; Z)\). If we assume that the group of covering transformations is an abelian group \(A\), then of course we may take \(z \in C(M; A)\).

From Lemma 8.2.6 we see that if \(M\) is a map and if \(M_z \to M\) is a totally ramified covering with \(M_z\) connected, then

\[
\theta^1 = \delta^1(\zeta) : C_2(M)/H_2(M) \to A
\]

is surjective, where \(\zeta = [z] \in D(M; A)\).

If in addition we assume that \(M_z\) is regular, then we know that if \(\alpha : \text{Aut}(M) \to \text{Aut}(A)\) is a homomorphism inducing an \(\text{Aut}(M)\)-module structure on \(A\), it follows that \(\zeta \in D(M; A)_\alpha\), which implies in particular that \(\theta^1 = \delta^1(\zeta) : C_2(M)/H_2(M) \to A\) is an \(\text{Aut}(M)\)-module epimorphism.

When \(M\) is a Platonic map, then \(H^1(M; A) = 0\), and so we have an isomorphism \(\delta^1 : D(M; A) \xrightarrow{\cong} \text{Hom}(C_2(M)/H_2(M), A)\). In this case, for any \(\text{Aut}(M)\)-structure \(\alpha : \text{Aut}(M) \to \text{Aut}(A)\) on \(A\), we conclude that

\[
\delta^1 : D(M; A)_\alpha \xrightarrow{\cong} \text{Hom}_{\text{Aut}(M)}(C_2(M)/H_2(M), A).
\]

This gives the following analog of Theorem 8.3.2:

**Theorem 8.4.1.** Let \(M\) be a Platonic map.

1. Let \(A\) be an abelian group. Then the following sets are in bijective correspondence:

   (i) the set of \(\cong_M\)-isomorphism classes of coverings (unramified over vertices) \(M' \to M\) by connected regular maps having group of covering transformations \(A(M'/M) \cong A\), and

(ii) the set of Aut(\( \mathcal{M} \))-submodules \( C \subseteq C_2(\mathcal{M}) \) with \( H_2(\mathcal{M}) \subseteq C \) and where \( C_2(\mathcal{M})/C \cong A \).

2. There is a bijective correspondence between

(i) the set of \( \cong_{\mathcal{M}} \)-isomorphism classes of coverings (unramified over vertices) \( \mathcal{M}' \to \mathcal{M} \) by connected regular maps and having abelian group of covering transformations, and

(ii) the set of Aut(\( \mathcal{M} \))-submodules \( C \subseteq C_2(\mathcal{M}) \) with \( H_2(\mathcal{M}) \subseteq C \).

We shall investigate the above in the particular case when \( A \) is cyclic, say \( A = \mathbb{Z}_d \), where if \( d = 0 \), write \( \mathbb{Z}_0 = \mathbb{Z} \), the additive group of the integers. Assume, for the moment that \( \mathcal{M} \) is an arbitrary finite regular connected orientable map having face valency \( k \) and vertex valency \( l \). We can classify the Aut(\( \mathcal{M} \))-submodules of \( C_2(\mathcal{M})/H_2(\mathcal{M}) \) having quotient isomorphic to \( \mathbb{Z}_d \), as follows.

Note first of all that an Aut(\( \mathcal{M} \))-module structure on \( \mathbb{Z}_d \) is tantamount to giving a homomorphism \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \cong \mathcal{U}(\mathbb{Z}_d) \), where \( \mathcal{U}(\mathbb{Z}_d) \) is the group of units of \( \mathbb{Z}_d \). Given that \( \mathcal{M} \) is assumed to be regular, we have \( \text{Aut}(\mathcal{M}) \cong G \), where \( G = \langle a, b, c \rangle \) is the monodromy group of \( \mathcal{M} \). Thus, if we fix a blade \( \beta \) of \( \mathcal{M} \), we may express \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \), where \( \langle a', b' \rangle \), \( \langle a', c' \rangle \), \( \langle b', c' \rangle \) are, respectively, the stabilizers of the face, edge, and vertex containing \( \beta \).

If \( u \in \mathcal{U}(\mathbb{Z}_d) \cong \text{Aut}(\mathbb{Z}_d) \) is an involutary unit, and if the assignment \( \text{Aut}(\mathcal{M}) \to \mathcal{U} \) determined by

\[
    a' \mapsto -1, \ b' \mapsto -1, \ c' \mapsto -u,
\]

is a homomorphism, denote the corresponding homomorphism by \( \alpha_{(u)} : \text{Aut}(\mathcal{M}) \to \mathcal{U} \). Note that since \( \mathcal{M} \) is orientable, \( \alpha_{(1)} : \text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d) \) is always a homomorphism, which we call the Steinberg homomorphism.\(^8\) Note that if \( \mathcal{M} \) is regular with odd vertex valency then

\(^8\)The reason for calling this the Steinberg homomorphism is because of the analogy of the monodromy group of an algebraic map with the so-called Hecke algebra of a chamber system. In case the chamber system is a building of Lie type, the Hecke algebra satisfies the same braid relations as the extended triangle group \( \Delta(k, l) \). Furthermore, mapping the generators of the Hecke algebra to \( -1 \) determines the corresponding Steinberg representation of the corresponding finite group of Lie type.
\( \alpha_{(1)} \) is the only possible homomorphism \( \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \) mapping both \( a', b' \mapsto -1 \). At the other extreme is the Accola homomorphism \( \alpha_{(-1)} : \text{Aut}(\mathcal{M}) \to U(\mathbb{Z}_d) \).

In the sequel, we shall denote the Steinberg and Accola homomorphisms by \( \alpha_{St} \) and \( \alpha_{Acc} \), respectively.

**Lemma 8.4.2.** Let \( \mathcal{M} \) be a finite regular orientable map. Assume that \( C \subseteq C_2(\mathcal{M}) \) is an \( \text{Aut}(\mathcal{M}) \)-submodule of \( C_2(\mathcal{M}) \) and that \( C_2(\mathcal{M})/C \cong \mathbb{Z}_d \). Then the induced \( \text{Aut}(\mathcal{M}) \)-module structure on \( \mathbb{Z}_d \) is induced by one of the homomorphisms \( \alpha(u) \), where \( u \) is an involutory unit of \( \mathbb{Z}_d \).

**Proof.** Note first of all that if \( \theta, \theta' : C_2(\mathcal{M}) \to \mathbb{Z}_d \) are surjective homomorphisms, both having kernel \( C \), then \( \theta' = \theta \mu \) for some \( \mu \in U(\mathbb{Z}_d) \). If \( \mathcal{M} \) has \( r \) faces, then there exist \( r \) blades \( \beta = \beta_1, \beta_2, \ldots, \beta_r \), all in the same \( G^+ \)-orbit and such that \( \{X_{[\beta_i]F^+} + W^{(a,b)}, X_{[\beta_i]F^+} + W^{(a,b)}, \ldots, X_{[\beta_i]F^+} + W^{(a,b)} \} \) is a \( \mathbb{Z} \)-basis of \( C_2(\mathcal{M}) \). We write \( m_i = X_{[\beta_i]F^+} + W^{(a,b)}, i = 1, 2, \ldots, r \). As above, we set \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \), where \( \langle a', b' \rangle, \langle a', c' \rangle, \langle b', c' \rangle \) are, respectively, the stabilizers of the face, edge, and vertex containing \( \beta \). If \( \text{Aut}(\mathcal{M})^+ = \langle a'b', a'c' \rangle \), then since \( \text{Aut}(\mathcal{M})^+ \) acts transitively on \( m_1, m_2, \ldots, m_r \), we see that the \( \text{Aut}(\mathcal{M}) \)-module homomorphism is uniquely determined by \( \theta(m_1) \); furthermore, since \( \theta \) is surjective, we may assume (by the first remark above) that \( \theta(m_1) = 1 \). Next, if \( \alpha : \text{Aut}(\mathcal{M}) \to U(\mathbb{Z}_d) \) is the corresponding homomorphism, then note that since \( m_1a' = -m_1 = m_1b' \), we infer that

\[
-1 = -\theta(m_1) = \theta(-m_1) = \theta(m_1a') = \theta(m_1)\alpha(a') = \alpha(a'),
\]

and so \( \alpha(a') = -1 \). Similarly, \( \alpha(b') = -1 \). Since clearly \( \alpha(c') = -u \), for some involutory unit in \( \mathbb{Z}_d \), we conclude that \( \alpha = \alpha_{(u)} \).

Note that if \( \alpha_{St} = \alpha_{(1)} \) is the Steinberg homomorphism, then \( \text{Aut}(\mathcal{M})^+ \subseteq \ker \alpha \) and it follows that \( \theta(m_i) = \theta(m_j) \) for all \( i, j \); up to unit multiple this means that we may assume that \( \theta(m_i) = 1, i = 1, 2, \ldots, r \). If \( \alpha = \alpha_{(u)} \) where \( u \neq 1 \), then \( K^+ = \ker (\alpha : \text{Aut}(\mathcal{M})^+ \to U(\mathbb{Z}_d)) \) has index 2 in \( \text{Aut}(\mathcal{M})^+ \), from which it follows easily that \( r \) is

\[\text{Even when } \mathcal{M} \text{ is orientable, this homomorphism need not exist. Indeed, the automorphism group of the regular type-(4, 4) affine map in Corollary 9.3.10.1 (1) on page 233 clearly does not admit such a homomorphism when the parameter } m \text{ is odd.}\]
even. Therefore, we may re-index the basis elements of $C_2(\mathcal{M})$ if necessary so that $m_2, m_4, \ldots m_r$ and $m_1, m_3, \ldots m_{r-1}$ are the two $K^+$-orbits. From this it follows that $\theta(m_i) = u^i$, $i = 1, 2, \ldots, r$.

Conversely, for the homomorphisms $\alpha_u : \text{Aut}(\mathcal{M}) \to U(\mathbb{Z}_d)$, we see that the corresponding recipies for $\theta = \theta_u : C_2(\mathcal{M}) \to \mathbb{Z}_d$ do, in fact, give $\text{Aut}(\mathcal{M})$-module surjections, proving the lemma.

We shall denote $\ker(\theta_u) : C_2(\mathcal{M}) \to \mathbb{Z}_d = C_u$ Note that $H_2(\mathcal{M}) \subseteq C_u$ if and only if

$$\theta_u(m_1 + m_2 + \cdots + m_r) = 0 \in \mathbb{Z}_d,$$

i.e., if and only if

$$u + u^2 + \cdots + u^r = 0 \in \mathbb{Z}_d.$$

Note that if $\alpha = \alpha_{St}$, i.e., if $u = 1$, then the above happens precisely when $d|r$. On the other hand, if $\alpha = \alpha_{Acc}$, and so $u = -1$, then $r$ is even and $H_2(\mathcal{M})$ is always contained in the kernel of $\theta_{(-1)} = \theta_{Acc}$.

We shall write $C_{St} = C_{(1)}$ and $C_{Acc} = C_{(-1)}$. If $\zeta = [z]$ and if $\delta^1(\zeta) = \theta_u : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d$ (surjective), we call $\mathcal{M}_z \to \mathcal{M}$

(St) a Steinberg covering if the action if $u = 1$ , and

(Acc) an Accola covering if $u = -1$.

Note that by Lemma 8.2.6, the Steinberg and Accola coverings are totally ramified coverings.

We now apply Theorem 8.4.1 (1) to obtain the following classification of regular cyclic coverings of Platonic maps. Thus, assume that $\mathcal{M}$ is a tetrahedron, octahedron, cube, icosahedron or a dodecahedron,\(^{10}\) and hence can be realized as $\mathcal{M}_G$, where $G$ is a Coxeter group with presentation

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^k = (ac)^2 = (bc)^l \rangle,$$

\(^{10}\)I've excluded the so-called *hosohedron*, which is a somewhat degenerate map of genus 0. This is the regular map on the sphere with two antipodal vertices, $t$ edges and $t$ faces (“tunes”).
and where \((k, l) = (3, 3), (3, 4), (4, 3), (3, 5)\) or \((5, 3)\). Note in particular that Accola homomorphisms \(G \to \mathcal{U}(\mathbb{Z}_d)\) (and hence \(\text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d)\)) exists only for the octahedron.

The following theorem gives the Steinberg coverings of the Platonic maps.

**Theorem 8.4.3.** Let \(\mathcal{M}\) be a Platonic map, let \(d\) be a fixed nongative integer, and let \(r\) be the number of faces in \(\mathcal{M}\). If \(d|r\), define \(\zeta = [z] \in \text{D}(\mathcal{M}; \mathbb{Z}_d)\) by setting

\[\delta^1(\zeta) = \theta_{(1)} \in \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), \mathbb{Z}_d),\]

where \(\theta_{(1)}(m_i) = 1, i = 1, 2, \ldots, r\), and where \(\{m_1, m_2, \ldots, m_r\}\) is the basis of \(C_2(\mathcal{M})\) given above. Then \(\mathcal{M}_z \to \mathcal{M}\) satisfies

(i) \(\mathcal{M}_z\) is connected and regular;

(ii) \(\mathcal{M}_z \to \mathcal{M}\) is unramified over vertices; and

(iii) \(\mathcal{M}_z \to \mathcal{M}\) has group of covering transformations isomorphic with \(\mathbb{Z}_d\).

Assuming that \(\mathcal{M}\) is not the octahedron, then up to \(\cong_{\mathcal{M}}\)-isomorphism, there are no other coverings \(\mathcal{M}' \to \mathcal{M}\) satisfying (i), (ii), and (iii), and if \(d \nmid r\) there are no coverings \(\mathcal{M}' \to \mathcal{M}\) satisfying (i), (ii), and (iii).

When \(\mathcal{M}\) is the octahedron, we have additional coverings corresponding to involutory units \(u \neq 1\) in \(\mathcal{U}(\mathbb{Z}_d)\), such that \(\sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d\) as follows. In this case, \(\text{Aut}(\mathcal{M})^+ \cong S_4\) (the symmetric group on 4 symbols), and has a unique subgroup \(K^+\) of index 2. Assume that the basis elements \(m_1, \ldots, m_8 \in C_2(\mathcal{M})\) have been indexed so that \(m_{2i}, i = 1, 2, 3, 4\) and \(m_{2i-1}, i = 1, 2, 3, 4\) are the two \(K^+\)-orbits. Then the mapping \(\theta_{(u)} : C_2(\mathcal{M}) \to \mathbb{Z}_d\) given by \(\theta_{(u)}(m_i) = (-1)^{i-1}, i = 1, 2, \ldots, 8\) is surjective and its kernel is an \(\text{Aut}(\mathcal{M})\)-submodule of \(C_2(\mathcal{M})\) containing \(H_2(\mathcal{M})\) precisely when \(\sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d\). Note that the Accola homomorphism \(\alpha_{\text{Acc}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\) exists for the automorphism group of
the octahedron $\mathcal{M}$, and so $\mathcal{M}$ admits Accola coverings. Thus, we have the following:

**Theorem 8.4.4.** For the octahedron $\mathcal{M}$, the $\cong_{\mathcal{M}}$ classes of coverings $\mathcal{M}' \to \mathcal{M}$ satisfying (i), (ii), and (iii) in Theorem 8.4.3 are in bijective correspondence with the involutory units $u \in \mathbb{Z}_d$ such that $\sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d$. For each such involutory unit $u$, the corresponding covering map has the form $\mathcal{M}' = \mathcal{M}_z$, where if $\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)$ and if $\theta(u) : C_2(\mathcal{M}) \to \mathbb{Z}_d$ is as above, then $\delta^1(\zeta) = \theta(u)$.

Note that if the connected map $\mathcal{M}_z$ is connected and regular, then its automorphism group $\text{Aut}(\mathcal{M}_z)$ is an extension of the cyclic group $\mathbb{Z}_d$ by the group $\text{Aut}(\mathcal{M})$. Appealing to the discussion on page 66, we see that the action of $\text{Aut}(\mathcal{M})$ on $\mathbb{Z}_d$ is given by $\phi \zeta \phi^{-1} = \zeta^{\alpha(\phi)}$, and so $\phi \mapsto \alpha(\phi)$ is the associated action of $\text{Aut}(\mathcal{M})$ on $\mathbb{Z}_d$. This is enough to give us a presentation of $\text{Aut}(\mathcal{M}_z)$, where $\mathcal{M}_z$ is the associated principal derived map parametrized by $u \in \mathcal{U}(\mathbb{Z}_d)$.

**Theorem 8.4.5.** If $\mathcal{M}$ is a Platonic map with vertex valency $l$ and face valency $k$, and if the principal derived map $\mathcal{M}_z$ is parametrized by $u \in \mathbb{Z}_d$ as above, then

$$\text{Aut}(\mathcal{M}_z) \cong \langle x_1, x_2, x_3, \mid x_1^2 = x_2^2 = x_3^2 = (x_1 x_3)^2 = (x_1 x_2)^{dk}, (x_2 x_3)^l = 1, x_3 (x_1 x_2)^k x_3 = (x_1 x_2)^{-uk} \rangle.$$  

**Proof.** If we let $A$ be the above presented group, then the discussion above guarantees that $\text{Aut}(\mathcal{M}_z)$ satisfies all of the above relations. However, one checks that $A/\langle (x_1 x_2)^k \rangle \cong \text{Aut}(\mathcal{M})$, and so $|A| = |\text{Aut}(\mathcal{M}_z)|$, completing the proof.

Note that $\text{Aut}(\mathcal{M}_z)^+$ is a central extension of $\mathbb{Z}_d$ if and only if $u = 1 \in \mathbb{Z}_d$. In turn, this possibility happens precisely when $d|r$, where $r$ is the number of faces of the map $\mathcal{M}$.

We itemize the possibilities corresponding to $u = 1$; at the same time we give the structure of $\text{Aut}(\mathcal{M}_z)$, as determined in my paper
with Gareth Jones. Note that in this case the automorphism groups of the covering maps have presentation

\[
\text{Aut}(\mathcal{M}_z) \cong \langle x_1, x_2, x_3, \mid x_1^2 = x_2^2 = x_3^2 = (x_1 x_2)^2 = (x_1 x_3)^2 = (x_2 x_3)^4 = 1, x_3 (x_1 x_2)^k x_3 = (x_1 x_2)^{-k} \rangle
\]

Tetrahedron. We have \( r = 4 \) and so there are three corresponding coverings of \( \mathcal{M}_z \mathcal{M} \) by regular maps, corresponding to \( d = 1, 2, \) and 4. These maps can be denoted \( \mathcal{M}_d \) and have genera as listed below. Also, we have given the structure of \( \text{Aut}(\mathcal{M}_z) \):

\[
\begin{align*}
\text{Aut}(\mathcal{M}_z) &\cong \text{PGL}_2(3) \cong S_4; \\
\text{Aut}(\mathcal{M}_z) &\cong \text{Z}_2 \times \text{PGL}_2(3) \cong \text{Z}_2 \times S_4; \\
\text{Aut}(\mathcal{M}_z) &\cong \text{Z}_4 \ast \text{Z}_2 \text{GL}_2(3) \text{ (This is an amalgamated product of \text{GL}_2(3) by a cyclic group of order 4 over the center of \text{GL}_2(3); each element of \text{GL}_2(3) of determinant \(-1\) inverts the factor \text{Z}_4).}
\end{align*}
\]

Cube. In this case, \( r = 6 \) with 4 divisors, \( d = 1, 2, 3, \text{ and } 6.\)

\[
\begin{align*}
\text{Aut}(\mathcal{M}_z) &\cong \text{Z}_2 \times \text{PGL}_2(3) \cong \text{Z}_2 \times S_4 \\
\text{Aut}(\mathcal{M}_z) &\cong \text{Z}_4 \ast \text{Z}_2 \text{GL}_2(3) \text{ (This is an amalgamated product of \text{GL}_2(3) by a cyclic group of order 4 over the center of \text{GL}_2(3); each element of \text{GL}_2(3) of determinant \(-1\) inverts the factor \text{Z}_4).} \\
\text{Aut}(\mathcal{M}_z) &\cong \text{S}_3 \times S_4 \text{ (The action of \text{S}_4 on the normal \text{S}_3 is as follows. First, note that \text{Aut}(\text{S}_3) \cong \text{S}_3; thus map \text{S}_4 \to \text{S}_3 \cong \text{Aut}(\text{S}_3); the kernel of this homomorphism is the normal Klein four-group in \text{S}_4.)} \\
\text{Aut}(\mathcal{M}_z) &\cong \text{S}_3 \times \text{GL}_2(3) \text{ (The action of \text{GL}_2(3) on the normal \text{S}_3 is essentially as in the case } d = 3, \text{ where we recall that \text{GL}_2(3)/\text{Z(\text{GL}_2(3))} \cong \text{S}_4.)}
\end{align*}
\]

\[11\text{Regular cyclic coverings of the Platonic maps, Europ. J. Combinatorics (2000) 21, 333–345: see especially pages 337–338. Note, however, that in this article, since only oriented maps (and hypermaps) were considered, the groups calculated are } \text{Aut}(\mathcal{M}_z)^+ \text{ and not } \text{Aut}(\mathcal{M}_z).\]
Octahedron. Here, \( r = 8 \), with divisors \( d = 1, 2, 4, 8 \).

\[
\begin{align*}
  d = 1 : g &= 0, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times \text{PGL}_2(3) \cong Z_2 \times S_4 \\
  d = 2 : g &= 3, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times Z_2 \times \text{PGL}_2(3) \cong Z_2 \times Z_2 \times S_4. \\
  d = 4 : g &= 9, \quad \text{Aut}(\mathcal{M}_z) \cong D_8 \times \text{PGL}_2(3) \cong D_8 \times S_4. \\
  d = 8 : g &= 21, \quad \text{Aut}(\mathcal{M}_z) \cong D_{16} \circ Z_2 \text{ GL}_2(3) \text{ (This is a central product of the dihedral group of order 16 with GL}_2(3) \text{ over their centers.)}
\end{align*}
\]

Dodecahedron. \( r = 12 \), and \( d = 1, 2, 3, 4, 6, 12 \).

\[
\begin{align*}
  d = 1 : g &= 0, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times A_5. \\
  d = 2 : g &= 5, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times Z_2 \times A_5. \\
  d = 3 : g &= 10, \quad \text{Aut}(\mathcal{M}_z) \cong S_3 \times A_5. \\
  d = 4 : g &= 15, \quad \text{Aut}(\mathcal{M}_z) \cong D_8 \circ Z_2 \text{ SL}_2(5). \\
  d = 6 : g &= 25, \quad \text{Aut}(\mathcal{M}_z) \cong D_{12} \times A_5. \\
  d = 12 : g &= 55, \quad \text{Aut}(\mathcal{M}_z) \cong D_{24} \circ Z_2 \text{ SL}_2(5).
\end{align*}
\]

Icosahedron. \( r = 12 \) and \( d = 1, 2, 4, 5, 10, 20 \).

\[
\begin{align*}
  d = 1 : g &= 0, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times A_5. \\
  d = 2 : g &= 9, \quad \text{Aut}(\mathcal{M}_z) \cong Z_2 \times Z_2 \times A_5. \\
  d = 4 : g &= 27, \quad \text{Aut}(\mathcal{M}_z) \cong D_8 \circ Z_2 \text{ SL}_2(5). \\
  d = 5 : g &= 36, \quad \text{Aut}(\mathcal{M}_z) \cong D_{10} \times A_5. \\
  d = 10 : g &= 81, \quad \text{Aut}(\mathcal{M}_z) \cong D_{20} \times A_5. \\
  d = 20 : g &= 171, \quad \text{Aut}(\mathcal{M}_z) \cong D_{40} \circ Z_2 \text{ SL}_2(5). 
\end{align*}
\]

The only cyclic coverings corresponding to \( u = -1 \) occur for the octahedron, and we have one such covering for each positive integer \( d \). These maps are the Accola maps\(^{12}\) and have genus \( 3(d - 1) \) and automorphism group with presentation

---

\(^{12}\) These were first constructed as Riemann surfaces by R. D. M. Accola, On the number of automorphisms of a closed Riemann surface, *Trans. Amer. Math. Soc.*, 131 (1968), 308–408.
\[
\text{Aut}(\mathcal{M}_z) \cong \langle x_1, x_2, x_3, \mid x_1^2 = x_2^2 = x_3^2 = (x_1 x_3)^2 = (x_1 x_2)^{d_k} = (x_2 x_3)^l = 1, x_3(x_1 x_2)^k x_3 = (x_1 x_2)^k \rangle.
\]

### 8.5 Cyclic Coverings of Regular Affine Maps

In the preceding section, the cyclic coverings of the Platonic maps by regular maps were classified. Because the Platonic maps are simply connected, these coverings are totally ramified. In general, if \( \mathcal{M} \) is a regular orientable map, and if \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d) \), where \( \alpha : \text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d) \), then by Corollary 5.2.1.2 together with Lemma 8.2.6 the principal covering \( \mathcal{M}_z \to \mathcal{M} \) is regular, connected, and totally ramified precisely when

\[
\delta^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d
\]

is a surjective homomorphism.

As in the preceding discussion, we write \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \), where \( \langle a', b' \rangle, \langle a', c' \rangle, \) and \( \langle b', c' \rangle \) are, respectively, the stabilizers of the face, edge and vertex of a fixed blade \( \beta \). From Lemma 8.4.2, the actions of \( \text{Aut}(\mathcal{M}) \) on \( \mathbb{Z}_d \) must be of the form \( \alpha = \alpha_{(u)} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), where \( u \) is an involutory unit in \( \mathbb{Z}_d \). We recall:

**The Steinberg homomorphism.** If \( u = 1 \) we get the Steinberg homomorphism, given by

\[
\alpha_{\text{St}}(a') = \alpha_{\text{St}}(b') = \alpha_{\text{St}}(c') = -1.
\]

**The Accola homomorphism.** This is the homomorphism determined by

\[
\alpha_{\text{Acc}}(a') = \alpha_{\text{Acc}}(b') = -1, \ \alpha_{\text{Acc}}(c') = -u.
\]

As we’ll see shortly, the Accola homomorphism need not exist for the automorphism group of an affine map. Recall that if \( \zeta \in D(\mathcal{M}; \mathbb{Z}_d)_{\text{asst}} \), then \( \delta^1(\zeta) \) kills \( H_2(\mathcal{M}) \) if and only if \( d|r \), where \( r \) is the number of faces of \( \mathcal{M} \).
We shall investigate in detail the unramified, Steinberg, and Accola coverings of the regular affine maps. First of all, the regular affine maps are of the form\(^\S13\)

\[ M = M(\Delta(k, l)/K, s_1, s_2, s_3), \]

where

\[ K \leq \Delta(k, l) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_3)^2 = (s_1s_2)^k = (s_2s_3)^l = 1 \rangle, \]

and where \((k, l) = (4, 4)\) or \((3, 6)\).

### 8.5.1 The Regular Affine Maps of Type \((4, 4)\)

The focus of this section will be on the regular affine maps of type \((4, 4)\).

As proved in Chapter 9 (see Proposition 9.3.8), the group \(\Delta(4, 4)\) has the following structure.

**Proposition 8.5.1.** There is an isomorphism

\[ \Delta(4, 4) \cong \langle i, \tau \rangle \rtimes \mathbb{Z}[i], \]

(where \(i\) acts by right multiplication on the Gaussian integers \(\mathbb{Z}[i]\) and \(\tau\) acts as complex conjugation) given by

\[ s_1 \mapsto (\tau, 0), \ s_2 \mapsto (-i\tau, 0), \ s_3 \mapsto (-\tau, 1). \]

(Note that \(\langle i, \tau \rangle \cong D_8\).)

From Proposition 8.5.1, the normal subgroups of \(\Delta(4, 4)\) by which the quotients are nondegenerate maps can be identified with ideals \(I \subseteq \mathbb{Z}[i]\) which are invariant under complex conjugation. Since \(\mathbb{Z}[i]\) is a principal ideal domain, we may write any ideal as \(I = (a + bi)\mathbb{Z}[i]\); however, for \(I\) to be invariant under complex conjugation, there are only two possibilities:

1. \(I = m\mathbb{Z}[i]\), for some integer \(m\);

\(^{13}\text{A more complete treatment of these maps is given in Chapter 9.}\)
(2) \( I = m(1 + i)\mathbb{Z}[i] \), for some integer \( m \).

Furthermore, note that under the above isomorphism

\[ s_2s_1s_2s_3 \mapsto (1, 1), \quad s_1s_2s_3s_2 \mapsto (1, i); \]

also

\[ s_1s_2s_1s_3s_2s_3 \mapsto (1, 1 + i), \quad s_2s_1s_3s_2s_3s_1 \mapsto (1, 1 - i). \]

Therefore, we conclude that in terms of the generators \( s_1, s_2, s_3 \in \Delta(4, 4) \), the ideals are generated as follows:

\[ m\mathbb{Z}[i] \leftrightarrow \langle (s_2s_1s_2s_3)^m, (s_1s_2s_3s_2)^m \rangle, \]

\[ m(1 + i)\mathbb{Z}[i] \leftrightarrow \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m \rangle. \]

Corresponding to these two possibilities, we have the following; see Corollary 9.3.10.1 on page 233:

**Corollary 8.5.1.1.** The monodromy group of a finite regular affine map of type \((4, 4)\) has one of the following presentations:

1. \( G = \langle a_1, b_1, c_1 | a_1^2 = b_1^2 = c_1^2 = (a_1c_1)^2 = (a_1b_1)^4 = (b_1c_1)^4 = (b_1a_1b_1c_1)^m = 1 \rangle \), for some positive integer \( m \);

2. \( G = \langle a_2, b_2, c_2 | a_2^2 = b_2^2 = c_2^2 = (a_2c_2)^2 = (a_2b_2)^4 = (b_2c_2)^4 = (a_2b_2a_2c_2b_2c_2)^m = 1 \rangle \), for some positive integer \( m \).

Note first of all that the automorphism group of an affine map of type (1) above does not admit an Accola homomorphism unless \( m \) is even. The automorphism group of type (2) always admits an Accola homomorphism. Next, notice that the affine map of type (1) has a “fundamental domain” in the shape of a square in the complex plane; that corresponding to type (2) has the shape of a diamond. Furthermore, if we denote the groups in family (1) above by \( G(1, m) \) and those in family (2) by \( G(2, m) \), where \( m \) is a positive integer, then there is a two-fold covering \( G(2, m) \rightarrow G(1, m) \), given by \( a_2 \mapsto a_1, \quad b_2 \mapsto b_1, \quad c_2 \mapsto c_1 \). Note that the kernel of the above mapping is the normal closure in \( G(2, m) \) of \((b_2c_2b_2a_2)^m \). Similarly, there is a two-fold covering \( G(1, 2m) \rightarrow G(2, m) \)
given by \(a_1 \mapsto a_2, b_1 \mapsto b_2, c_1 \mapsto c_2\), whose kernel is the normal closure in \(G(1, 2m)\) of \((c_1b_1c_1a_1b_1a_1)^m\). Therefore, if \(\mathcal{M}(1, m)\), \(\mathcal{M}(2, m)\) are the corresponding maps, then there are unramified two-fold coverings

\[
\mathcal{M}(2, m) \to \mathcal{M}(1, m), \quad \text{and} \quad \mathcal{M}(1, 2m) \to \mathcal{M}(2, m).
\]

Let \(\mathcal{M}\) be one of the affine maps \(\mathcal{M}(1, m)\) or \(\mathcal{M}(2, m)\). We may identify the blades of \(\mathcal{M}\) with the group \(G = \langle i, \tau \rangle \times \mathbb{Z}[i]/I\). In turn the monodromy involutions of the monodromy group \(G = \langle a, b, c \rangle\) are given by left multiplications by \(a = (\tau, 0), b = (-i \tau, 0), c = (-\tau, 1)\), respectively. Similarly, the automorphism group \(\text{Aut}(\mathcal{M})\) is generated by involutions \(a', b', c'\) given by right multiplications by \((\tau, 0), (-i \tau, 0), (-\tau, 1)\), respectively. If we set \(D = \langle i, \tau \rangle \cong D_8\) (dihedral of order 8), we see that the faces of \(\mathcal{M}\) are of the form \(\{(x, w) \mid x \in D\}\), where \(w\) ranges over the cosets of \(I\) in \(\mathbb{Z}[i]\). As a result, we see that the normal subgroup \(N = \{(1, w) \mid w \in \mathbb{Z}[i]/I\}\) acts regularly by right multiplication on the faces of \(\mathcal{M}\). In other words, given the face \(f_w = \{(x, w) \mid x \in D\}\) and the element \(w' \in \mathbb{Z}[i]/I\), we have that

\[
f_w \cdot (1, w') = \{(x, w + w') \mid x \in D\}.
\]

Finally, from our observations above, we see that

\[
\pi_1(\mathcal{M}(1, m), (1, 0)) = \langle (s_2s_1s_2s_3)^m, (s_1s_2s_3s_2)^m \rangle,
\]

\[
\pi_1(\mathcal{M}(2, m), (1, 0)) = \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m \rangle.
\]

In case \(m = 2\) we can picture the blades of \(\mathcal{M} = \mathcal{M}(1, 2)\) as below. The darkened points represent the lattice points in the complex plane (and therefore parametrize the Gaussian integers \(\mathbb{Z}[i]\)). The blade \(\beta_0 = (1, 0)\) is indicated, as are the monodromy images \(a\beta_0, b\beta_0,\) and \(c\beta_0\):
The Regular Maps $\mathcal{M}(1, m)$—Unramified Coverings

In this subsection, we continue to assume that $\mathcal{M} = \mathcal{M}(1, m)$ and shall classify the unramified coverings $\mathcal{M}' \to \mathcal{M}$ by regular connected maps and having a cyclic group of covering transformations.

We have $\pi_1(\mathcal{M}(1, m), (1, 0)) = \langle (s_2 s_1 s_2 s_3)^m, (s_1 s_2 s_3 s_2)^m \rangle$, which is an abelian group. Furthermore, by regularity, $\pi_1(\mathcal{M}(1, m), (1, 0)) \leq G$, and so the homomorphism

$$h : \pi_1(\mathcal{M}(1, m), (1, 0)) \to H_1(\mathcal{M}(1, m))$$

of page 117 is an $\text{Aut}(\mathcal{M})$-equivariant isomorphism, where the action of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ on $\pi_1(\mathcal{M}(1, m), (1, 0))$ is simply conjugation by the elements $s_1, s_2, s_3$. Therefore, if we set $\gamma_x = s_2 s_1 s_2 s_3$, $\gamma_y = s_1 s_2 s_3 s_2$, and set

$$\frac{\partial}{\partial x} = \epsilon_s h(\gamma_x^m), \quad \frac{\partial}{\partial y} = \epsilon_s h(\gamma_y^m) \in H_1(\mathcal{M}; \mathbb{Z}_d),$$

then we know that $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ is a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$. We denote by $dx, dy$ the corresponding dual $\mathbb{Z}_d$-basis elements of $H^1(\mathcal{M}; \mathbb{Z}_d)$. Furthermore, by equivariance, we infer that

$$\frac{\partial}{\partial x}(a') = \epsilon_s h(s_1 \gamma_x^m s_1), \quad \frac{\partial}{\partial y}(a') = \epsilon_s h(s_1 \gamma_y^m s_1)$$

with similar statements for the generators $b', c' \in \text{Aut}(\mathcal{M})$. Since $s_1 \gamma_x s_1 = \gamma_x$, and since $s_1 \gamma_y s_1 = \gamma_y^{-1}$ we conclude immediately that

$$\frac{\partial}{\partial x}(a') = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}(a') = -\frac{\partial}{\partial y}.$$
Again, one likewise computes the effects of $b'$ and $c'$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$.

One has the following result.

**Proposition 8.5.2.** Assume that $\mathcal{M} = \mathcal{M}(1, m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

$$(a')^* dx = dx, \quad (a')_* dy = -dy,$$

$$(b')^* dx = dy, \quad (b')^* dy = dx.$$

$$(c')^* dx = -dx, \quad (c')^* dy = dy.$$

**Proof.** Very easy calculations show that that

$$\partial/\partial x(a')_* = \partial/\partial x, \quad \partial/\partial y(a')^* = -\partial/\partial y,$$

$$\partial/\partial x(b')_* = \partial/\partial y, \quad \partial/\partial y(b')^* = \partial/\partial x,$$

$$\partial/\partial x(c')_* = -\partial/\partial x, \quad \partial/\partial y(c')^* = \partial/\partial y.$$

The corresponding representation of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is adjoint to the above (in terms of matrices over $\mathbb{Z}_d$, the matrix representation is given by the transpose of the matrix representation on homology). The result follows.

In terms of the ordered $\mathbb{Z}_d$-basis $(dx, dy)$ of $H^1(\mathcal{M}; \mathbb{Z}_d)$, we have the matrix representation of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ below:

$$a' \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore, an $\alpha$-isotypical class in $H^1(\mathcal{M}; \mathbb{Z}_d)$ is realized by a “simultaneous eigenvector” for the above matrices. Thus, if $\zeta = xd x + yd y$ is an $\alpha$-isotypical class, $x, y \in \mathbb{Z}_d$, then the homomorphism $\theta = \delta(\zeta) : H_1(\mathcal{M}) \to \mathbb{Z}_d$ is determined by $\theta(\partial/\partial x) = x$, $\theta(\partial/\partial y) = y$. Therefore, we see that the principal derived map $\mathcal{M}_\zeta$ is connected precisely when $x, y$ together generate $\mathbb{Z}_d$. However, note that $\alpha(b')\zeta = (b')^*(\zeta) =$
\( y \, dx + x \, dy \), from which we conclude that \( \alpha(b')x = y \), \( \alpha(b')y = x \). Therefore, we conclude that
\[
\text{im}(\theta: H_1(M) \to \mathbb{Z}_d) = \langle x \rangle = \langle y \rangle \subseteq \mathbb{Z}_d,
\]
from which it follows that \( x, y \) are units in \( \mathbb{Z}_d \).

Next applying the condition \( \alpha(a')\zeta = (a')^*\zeta = x \, dx - y \, dy \), we obtain the conditions
\[
\alpha(a')x = x, \quad \alpha(a')y = -y.
\]
Since \( x, y \in \mathbb{Z}_d \) are units, we infer that \( 1 = \alpha(a') = -1 \) in \( \mathbb{Z}_d \), which forces \( d = 2 \). It is easy to check that in this case, there is a unique \( \alpha \)-isotypical class, viz.,
\[
\zeta = dx + dy \in H^1(M; \mathbb{Z}_2).
\]

Note that we have already observed above that \( \mathcal{M}(2, m) \to \mathcal{M}(1, m) \) is unramified and \( \mathcal{M}(2, m) \) is regular. We have, therefore, the following classification theorem.

**Theorem 8.5.3.** Up to \( \cong_{\mathcal{M}(1, m)} \), there is a unique nontrivial unramified covering \( \mathcal{M}' \to \mathcal{M}(1, m) \) by a regular connected map \( \mathcal{M}' \) and having cyclic group of covering transformations, namely, the 2-fold covering \( \mathcal{M}(2, m) \to \mathcal{M}(1, m) \).

**The Regular Maps \( \mathcal{M}(1, m) \)—The Steinberg Coverings**

In this subsection we determine the Steinberg coverings of the affine map \( \mathcal{M} = \mathcal{M}(1, m) \), and having cyclic group \( \mathbb{Z}_d \) of covering transformations. Recall that since \( \mathcal{M} \) has \( m^2 \) faces, then \( d \) must satisfy the condition \( d|m^2 \).

Let \( \beta_1, \beta_2, \ldots, \beta_{m^2} \) be blades in the same \( G^+ \)-orbit generating the \( m^2 \) faces of \( \mathcal{M} \), and set \( m_i = \chi[\beta_i, p^+] + W^{(a, b)} \in C_2(\mathcal{M}), i = 1, 2, \ldots, m^2 \). Then, up to a unit multiple in \( \mathbb{Z}_d \), the \( \text{Aut}(\mathcal{M}) \)-module homomorphism \( \theta = \theta_{\text{St}} : C_2(\mathcal{M}) \to \mathbb{Z}_d \) satisfies \( \theta(m_i) = 1, i = 1, 2, \ldots, m^2 \).
We shall begin by explicitly constructing an element $z \in (\delta^1)^{-1}(\theta)$ as follows. Start with the blade $\beta_0 = (1, 0)$, and define the blade $\beta_1 = b(1, 0) = (i\tau, 0)(1, 0) = (i\tau, 0)$. Define the voltages $z_1, z_2$ via the assignments

\[ z_1 : \beta_1 \mapsto -1, \ c\beta_1 \mapsto -1, \ a\beta_1 \mapsto 1, \ ac\beta_1 \mapsto 1, \]

\[ z_2 : \beta_0 \mapsto 1, \ c\beta_0 \mapsto 1, \ a\beta_0 \mapsto -1, \ ac\beta_0 \mapsto -1, \]

and mapping all remaining blades to 0. Next, define the automorphisms $\sigma_1, \sigma_2 \in \text{Aut}(\mathcal{M})$ to be translations one unit in the horizontal and vertical directions, respectively. Therefore,

\[ \beta = (x, w) \mapsto (x, w + 1) = \beta\sigma_1, \text{ and } \beta = (x, w) \mapsto (x, w + i) = \beta\sigma_2. \]

Therefore, $\sigma_1$ and $\sigma_2$ represent right multiplications in $\langle i, \tau \rangle \ltimes \mathbb{Z}[i]$ by $(1, 1)$ and $(1, i)$, respectively.

In terms of the translations $\sigma_1, \sigma_2 \in \text{Aut}(\mathcal{M})$ defined above, we define

\[ z'_1 = \sum_{k=0}^{m-2} \sum_{j=0}^{m-1} (k + 1)(\sigma_2^{-k})(\sigma_1^{-j})z_1. \]

We indicate the values of $z'_1$ for $m = 4$ below:
To the voltage \( z'_1 \) we add the voltage \( z'_2 \) given by

\[
z'_2 = \sum_{k=0}^{m-2} (k+1)m(\sigma_1^{-k})^* \sigma_2^* z_2,
\]

obtaining the voltage \( z_{St} = z'_1 + z'_2 \). In what follows, we shall identify the voltage \( z_{St} \) with the voltage class it determines in \( D(M; \mathbb{Z}_d) \).

If \( m = 4 \), \( z_{St} \) is as follows:
One checks that if $d|m^2$, then $\delta^1(z_{\text{St}}) = \theta$. Note, however, that even though $\theta$ is a surjective $\text{Aut}(\mathcal{M})$-module homomorphism $C_2(\mathcal{M})/H_2(\mathcal{M}) \rightarrow \mathbb{Z}_d$, one cannot conclude that $z_{\text{St}} \in \text{D}(\mathcal{M}; \mathbb{Z}_d)[\alpha]$. This will be addressed momentarily.

We set $D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d[\theta]) = \mathbb{Z}_d\langle [z_{\text{St}}] \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$; the action of $\text{Aut}(\mathcal{M})$ on $D_\theta$ is given below:

**Proposition 8.5.4.** Assume that $\mathcal{M} = \mathcal{M}(1,m)$, and that $d$ is a positive integer dividing $m^2$. Then the action of $\text{Aut}(\mathcal{M})$ on $D_\theta = \mathbb{Z}_d\langle [z_{\text{St}}] \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

$$(a')^*dx = dx, \quad (a')^*dy = -dy,$$

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\( (b')^* dx = dy, \ (b')^* dy = dx, \)
\( (c')^* dx = -dx, \ (c')^* dy = dy, \)
\( (a')^* z_{St} = -z_{St} - m dx, \ (b')^* z_{St} = -z_{St}, \ (c')^* z_{St} = -z_{St} + 2m dy. \)

**Proof.** Note first that if \( g' \in \{a', b', c'\} \), then since \( \alpha_{St}(g') = -1 \) we infer that \( g' z_{St} \equiv -z_{St} \mod H_1(M; \mathbb{Z}_d) \). Therefore, if \( (g')^* z_{St} = -z_{St} + x dx + y dy \), then the coefficients \( x, y \) can be computed via

\[
  x = \partial/\partial x ((g')^* z_{St} + z_{St}), \quad y = \partial/\partial y ((g')^* z_{St} + z_{St}).
\]

For example, if \( m = 4 \), we illustrate \((c')^* z_{St} + z_{St}\), together with \( \partial/\partial x, \partial/\partial y \) below:

\[
\begin{array}{cccccccc}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
\end{array}
\]

where in the above, a subdiagram of the form

\[
\begin{array}{cccc}
& & & \\
& \beta_0 & & \\
& & & \\
\end{array}
\]
\[ \chi_{c\beta} + \chi_{c\beta} + W^{(a,c)}. \]

Thus in the above we see that \((d')^*z_{St} + z_{St} = 8dy\); in general it is easy to see that \((d')^*z_{St} + z_{St} = 2m dy\). The remaining calculations are entirely similar.

In terms of matrices over \(\mathbb{Z}_d\), and relative to the ordered basis \((z_{St}, dx, dy)\) of \(\mathbb{Z}_d \langle z_{St} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\), we have the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\) given by

\[
\begin{align*}
    a' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ -m & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \\
    b' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\
    c' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2m & 0 & 1 \end{bmatrix}.
\end{align*}
\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map \(\mathcal{M}(1, m)\). Note that if \(\zeta \in D_\theta\) is \(\alpha\)-isotypical for some \(\alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), then we may assume that \(\zeta\) is of the form \(\zeta = z_{St} + xdx + ydy\) for suitable \(x, y \in \mathbb{Z}_d\). We have the following.

**Theorem 8.5.5.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{St : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)}\), by a regular connected map. Then \(d|m\); furthermore, up to \(\cong_{\mathcal{M}(1,m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)\) as below:

1. If \(d\) is odd, then up to unit multiples in \(\mathbb{Z}_d\), \(\zeta = z_{St} \in D(\mathcal{M}; \mathbb{Z}_d)\) is the unique such class;
(2) If \( d = 2d_0 \), then up to unit multiples in \( \mathbb{Z}_d \), there are two possibilities for \( \zeta \), namely, \( \zeta = z_{St} \) and \( \zeta = z_{St} + d_0dx + d_0dy \).

**Proof.** Note first that since \( \alpha_{St}(a') = \alpha_{St}(b') = \alpha_{St}(c') = -1 \), then an \( \alpha_{St} \)-isotypical voltage \( \zeta = z_{St} + xdx + ydy \), must satisfy \(- (z_{St} + xdx + ydy) = -\zeta = \alpha_{St}(a')\zeta = (a')^*\zeta = -z_{St} - (m - x)dx - ydy \), from which we conclude that \( 2x = m \). Next, \(-\zeta = \alpha_{St}(b')\zeta = (b')^*\zeta = -z_{St} + ydx + xdy \), and so \( y = -x \). Finally, we have \(- (z_{St} + xdx + ydy) = -\zeta = \alpha_{St}(c')\zeta = (c')^*\zeta = -z_{St} - xdx + (2m + y)dy \), and so \(-2y = 2m \). Therefore, \( m = 2x = -2y = 2m \) and so \( m = 0 \), i.e., \( d|m \), as claimed. From this the remaining assertions are entirely routine.

**The Regular Maps \( \mathcal{M}(1, m) \)—The Accola Coverings**

We turn now to the Accola coverings of \( \mathcal{M}(1, m) \). Therefore, we must assume that \( m \) is even. Note first that in terms of the identification of \( \Delta(4, 4) \) with \( \langle i, \tau \rangle \cong \mathbb{Z}[i] \), then \( \Delta^+(4, 4) = \langle i \rangle \cong \mathbb{Z}[i] \). If \( \text{Aut}(\mathcal{M}) \) is the automorphism group of \( \mathcal{M}(1, m) \), with subgroup \( \mathcal{M}(1, m)^+ \), then it is routine to check that

\[
K^+ = \ker(\alpha_{\text{Acc}} : \text{Aut}(\mathcal{M})^+ \to \{ \pm 1 \}) = \langle i \rangle \cong (1 + i)\mathbb{Z}[i].
\]

Therefore, we see that the faces in a \( K^+ \)-orbit form a “checkerboard” pattern inasmuch as no two faces sharing an edge can be in the same \( K^+ \)-orbit. From this, it is easy to determine an element \( z_{\text{Acc}} \) such that \( \delta^1(z_{\text{Acc}}) = \theta_{\text{Acc}} : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d \). To do this, let \( \beta_1 = b\beta_0 \) and recall the voltage \( z_1 \) given on page 158:

\[
z_1 : \beta_1 \mapsto 1, \; c\beta_1 \mapsto 1, \; a\beta_1 \mapsto -1, \; ac\beta_1 \mapsto -1.
\]

Let \( m = 2n \) and set

\[
z_{\text{Acc}} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} (-1)^j (\sigma_{j}^{-2k})^* (\sigma_{1}^{-j})^* z_1.
\]

As above, we shall not draw a distinction between the voltage \( z_{\text{Acc}} \) and the class it determines in \( D(\mathcal{M}; \mathbb{Z}_d) \).
If $m = 4$, then $z_{\text{Acc}}$ is depicted as below:

\[
\begin{array}{cccc}
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 \\
\end{array}
\]

We set $\theta = \theta_{\text{Acc}}$ and set $D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d\langle z_{\text{Acc}} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$. Note that

\[(a')^*z_{\text{Acc}} + z_{\text{Acc}}, (b')^*z_{\text{Acc}} + z_{\text{Acc}}, (c')^*z_{\text{Acc}} - z_{\text{Acc}} \in H^1(\mathcal{M}; \mathbb{Z}_d).\]

Arguing as with the Steinberg voltage (the present case is even easier), one obtains

**Proposition 8.5.6.** Assume that $m$ is an even integer and set $\mathcal{M} = \mathcal{M}(1, m)$. If $\theta = \theta_{\text{Acc}}$, then the action of $\text{Aut}(\mathcal{M})$ on $D_\theta = \mathbb{Z}_d\langle z_{\text{Acc}} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

\[
\begin{align*}
(a')^*dx &= dx, & (a')^*dy &= -dy, \\
(b')^*dx &= dy, & (b')^*dy &= dx, \\
(c')^*dx &= -dx, & (c')^*dy &= dy,
\end{align*}
\]
\((a')^\ast z_{\text{Acc}} = -z_{\text{Acc}},\)

\((b')^\ast z_{\text{Acc}} = -z_{\text{Acc}},\)

\((c')^\ast z_{\text{Acc}} = z_{\text{Acc}},\)

In terms of matrices over \(\mathbb{Z}_d\), and relative to the ordered basis 
\((z_{\text{Acc}}, dx, dy)\) of \(\mathbb{Z}_d\langle z_{\text{Acc}} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\) the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \) given by

\[
a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

The above representation easily implies the following classification of the Accola coverings of \(\mathcal{M}(1, m)\).

**Theorem 8.5.7.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{Acc}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then up to \(\cong_{\mathcal{M}(1, m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)\) as below:

1. If \(d\) is odd, then \(\zeta = z_{\text{Acc}} \in D(\mathcal{M}; \mathbb{Z}_d)\) is the unique such class;

2. If \(d = 2d_0\) is even, there there are two possibilities for \(\zeta\), namely, \(\zeta = z_{\text{Acc}}\) and \(\zeta = z_{\text{Acc}} + d_0dx + d_0dy\).

**Remark:** Note that if \(d = 2\) then the Steinberg and Accola coverings of \(\mathcal{M}(1, m)\) coincide (up to \(\cong_{\mathcal{M}(1, m)}\); there are two such).

**The Regular Maps \(\mathcal{M}(2, m)\)—Unramified Coverings**

We turn now to the maps \(\mathcal{M} = \mathcal{M}(2, m)\) and start by determining the unramified coverings by connected regular maps. We shall first determine \(\mathbb{Z}_d\)-bases for \(H_1(\mathcal{M}; \mathbb{Z}_d)\) and \(H^1(\mathcal{M}; \mathbb{Z}_d)\). From page 154, the fundamental group of \(\mathcal{M}\) is given by \(\pi_1(\mathcal{M}; \beta_0) = \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m \rangle\),
where $\beta_0 = (1, 0)$. Set $\gamma_1 = (s_1 s_2 s_1 s_3 s_2 s_3), \gamma_2 = (s_2 s_1 s_3 s_2 s_3 s_1)$ and define the cycles $\partial / \partial \alpha_1 = \epsilon_1 h(\gamma_1^m), \partial / \partial \alpha_2 = \epsilon_1 h(\gamma_2^m) \in H_1(\mathcal{M}; \mathbb{Z}_d)$, where $\epsilon : \mathbb{Z} \rightarrow \mathbb{Z}_d$ is the projection map. As above, we know that \{\partial / \partial \alpha_1, \partial / \partial \alpha_2\} is a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$. We denote by \{d\alpha_1, d\alpha_2\} the corresponding $\mathbb{Z}_d$-dual basis of $H^1(\mathcal{M}; \mathbb{Z}_d)$.

The action of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$, for any non-negative integer $d$, is determined below.

\begin{proposition}
Assume that $\mathcal{M} = \mathcal{M}(2, m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

\begin{align*}
(a')^*(d\alpha_1) &= d\alpha_2, \\
(b')^*(d\alpha_1) &= d\alpha_1, \\
(b')^*(d\alpha_2) &= -d\alpha_2, \\
(c')^*(d\alpha_1) &= -d\alpha_2.
\end{align*}
\end{proposition}

\begin{proof}
The above result is obtained by first computing the effects of the generators $a', b', c' \in \text{Aut}(\mathcal{M})$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$, which are computed by conjugating the elements $\gamma_1^m$ and $\gamma_2^m$ by the elements $s_1, s_2, s_3$, respectively. The action on the $\mathbb{Z}_d$-dual $H^1(\mathcal{M}; \mathbb{Z}_d)$ is obtained by dualizing.

In analogy with the work on page 66, one easily concludes that the only isotypical class in $H^1(\mathcal{M}; \mathbb{Z}_d)$ giving a connected principal derived map is $\zeta = d\alpha_1 + d\alpha_2 \in H^1(\mathcal{M}; \mathbb{Z}_2)$. The following is now immediate:

\begin{theorem}
Up to $\cong_{\mathcal{M}(2, m)}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(2, m)$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 2-fold covering $\mathcal{M}(1, 2m) \to \mathcal{M}(2, m)$.
\end{theorem}

We turn now to the classification of the Steinberg and Accola coverings of $\mathcal{M}(2, m)$. 
The Regular Maps $M(2, m)$—The Steinberg Coverings

In order to compute the totally ramified coverings of $M(2, m)$ it shall be convenient to capitalize on the 2-fold covering $p : M(2, m) \rightarrow M(1, m)$. To this end, note first that the calculations

$s_1 s_2 s_3 s_2 s_3 = s_2 s_1 s_3 s_2 s_3 s_2 s_3 s_1 = s_2 s_1 s_2 s_3 s_1 (s_1 s_2 s_3 s_2)^{-1}$

imply that under the induced mapping in homology $p_* : H_1(M(2, m); \mathbb{Z}_d) \rightarrow H_1(M(1, m); \mathbb{Z}_d)$ we have

$$p_* : \partial / \partial \alpha_1 \mapsto \partial / \partial x + \partial / \partial y, \quad \partial / \partial \alpha_2 \mapsto \partial / \partial x - \partial / \partial y.$$

Therefore, if $p^*(dx) = a_1 d\alpha_1 + a_2 d\alpha_2$, then

$$a_1 = \partial / \partial \alpha_1 (a_1 d\alpha_1 + a_2 d\alpha_2) = \partial / \partial \alpha_1 (p^*(dx)) = (\partial / \partial \alpha_1 p_*)(dx) = (\partial / \partial x + \partial / \partial y)(dx) = 1.$$  

Likewise, one obtains $a_2 = 1$, forcing $p^*(dx) = d\alpha_1 + d\alpha_2$. Similarly, one obtains $p^*(dy) = d\alpha_1 - d\alpha_2$.

We summarize:

**Lemma 8.5.10.** Relative to the covering $p : M(2, m) \rightarrow M(1, m)$, we have

$$p^*(dx) = d\alpha_1 + d\alpha_2, \quad p^*(dy) = d\alpha_1 - d\alpha_2.$$  

We have $\text{Aut}(M(1, m)) = \langle a', b', c' \rangle$; since $M(2, m)$ is regular, then $a', b', c'$ lift to generators of $\text{Aut}(M(2, m))$; we shall, for convenience (and simplicity) continue to denote these lifted automorphisms of $M(2, m)$ also as $a', b', c'$. As a result, we see that if $\zeta \in D(M(1, m); \mathbb{Z}_d)$, then

$$(a')^* p^*(\zeta) = p^*((a')^* \zeta)$$

with similar results for $b'$, $c'$.

In analyzing the Steinberg coverings of $M(2, m)$ we start by recalling the voltage $z_{St}^1 := z_{St} \in D(M(1, m); \mathbb{Z}_d)$ on page 159; in terms of this, define

$$z_{St}^2 = p^*(z_{St}^1) \in D(M(2, m); \mathbb{Z}_d).$$
It is clear that \( \delta^1(z_{St}^2) = \theta_{St} : C_2(\mathcal{M}(2,m)) / H_2(\mathcal{M}(2,m)) \to \mathbb{Z}_d \). Therefore, if \( \theta = \theta_{St} \) then \( (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(z_{St}^2) \oplus H^1(\mathcal{M}(2,m); \mathbb{Z}_d) \). We already have enough information to determine the action of Aut(\( \mathcal{M}(2,m) \)) on \( D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) \). Using the results of Proposition 8.5.4, we compute:

\[
(a')^*(z_{St}^2) = (a')^*p^*(z_{St}^1)
= p^*((a')^*(z_{St}^1))
= p^*(-z_{St}^1 - mdx)
= -z_{St}^2 - md\alpha_1 - md\alpha_2.
\]

Next,

\[
(b')^*(z_{St}^2) = (b')^*p^*(z_{St}^1)
= p^*((b')^*(z_{St}^1))
= p^*(-z_{St}^1)
= -z_{St}^2.
\]

\[
(c')^*(z_{St}^2) = (c')^*p^*(z_{St}^1)
= p^*((c')^*(z_{St}^1))
= p^*(-z_{St}^1 + 2mdy)
= -z_{St}^2 + 2md\alpha_1 - 2md\alpha_2.
\]

We summarize below the action of Aut(\( \mathcal{M}(2,m) \)) on \( D_\theta \):

**Proposition 8.5.11.** Assume that \( \mathcal{M} = \mathcal{M}(2,m) \) and that \( d \) is a positive integer dividing \( 2m^2 \). Then the action of Aut(\( \mathcal{M} \)) on \( D_\theta = \mathbb{Z}_d(z_{St}^2) \oplus H^1(\mathcal{M}; \mathbb{Z}_d) \) is determined by

\[
(a')^*(d\alpha_1) = d\alpha_2,
(b')^*(d\alpha_1) = d\alpha_1, \quad (b')^*(d\alpha_2) = -d\alpha_2,
(c')^*(d\alpha_1) = -d\alpha_2,
(a')^*(z_{St}^2) = -z_{St}^2 - md\alpha_1 - md\alpha_2,
\]
(b')^*(z^2_{St}) = -z^2_{St},
(c')^*(z^2_{St}) = -z^2_{St} + 2md\alpha_1 - 2md\alpha_2.

In terms of the ordered \( \mathbb{Z}_d \)-basis \((z^2_{St}, d\alpha_1, d\alpha_2)\) of \( D_\theta \), we have the matrix representation of \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \):

\[
\begin{align*}
a' & \mapsto \begin{bmatrix} -1 & 0 & 0 \\
-m & 0 & 1 \\
-m & 1 & 0 \end{bmatrix}, & b' & \mapsto \begin{bmatrix} -1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \end{bmatrix}, & c' & \mapsto \begin{bmatrix} -1 & 0 & 0 \\
2m & 0 & -1 \\
-2m & -1 & 0 \end{bmatrix}.
\end{align*}
\]

The above provides the following classification of the Steinberg coverings of the regular affine map \( \mathcal{M}(2, m) \).

**Theorem 8.5.12.** Let \( \mathcal{M}' \to \mathcal{M} = \mathcal{M}(2, m) \) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \( \mathbb{Z}_d \) and inducing the action \( \alpha = \alpha_{St} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), by a regular connected map. Then \( d|m \); furthermore, up to \( \cong_{\mathcal{M}(2, m)} \), \( \mathcal{M}' = \mathcal{M}_z \), where \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d) \) as below:

1. If \( d \) is odd, then up to unit multiples in \( \mathbb{Z}_d \), \( \zeta = z^2_{St} \in D(\mathcal{M}; \mathbb{Z}_d) \) is the unique such class;

2. If \( d = 2d_0 \), then up to unit multiples in \( \mathbb{Z}_d \), there are two possibilities for \( \zeta \), namely, \( \zeta = z^2_{St} \) and \( \zeta = z^2_{St} + d_0d\alpha_1 + d_0d\alpha_2 \).

**Proof.** We let \( \zeta = z_{St} + a_1d\alpha_1 + a_2d\alpha_2 \in D(\mathcal{M}; \mathbb{Z}_d) \) be an \( \alpha_{St} \)-isotypical vector. From \( (a')^*\zeta = -\zeta \) we obtain the condition \( a_1 + a_2 = m \). From \( (b')^*\zeta = -\zeta \) we learn that \( 2a_1 = 0 \in \mathbb{Z}_d \). Finally from \( (c')^*\zeta = -\zeta \) we have \( a_2 - a_1 = 2m \). These conditions jointly imply that \( m = 0 \in \mathbb{Z}_d \) (and so \( d|m \) and so \( a_2 = -a_1 = a_1 \)). Everything follows.

**The Regular Maps \( \mathcal{M}(2, m) \)—The Accola Coverings**

Finally we turn to the classification of the Accola coverings of the maps \( \mathcal{M}(2, m) \). As in the case \( \mathcal{M} = \mathcal{M}(1, m) \) we construct an element
\( z_{\Delta^{2}}^{2} \in (\delta^{1})^{-1}(\theta) \), where \( \theta = \theta_{\Delta^{2}} \), as follows. We set

\[
    z_{\Delta^{2}}^{2} = p^{*}(z_{\Delta^{2}}^{1}) \in D(M(2, m); \mathbb{Z}_{d}),
\]

where \( z_{\Delta^{2}}^{1} \) is as on page 163; note that \( \delta^{1}(z_{\Delta^{2}}^{2}) = \theta_{\Delta^{2}} \).

Using the results of Proposition 8.5.6 together with Lemma 8.5.10 we can compute the effect of \( a', b', \) and \( c' \) on the \( \mathbb{Z}_{d} \)-basis elements \( z_{\Delta^{2}}^{2}, \mathcal{d}\alpha_{1}, \mathcal{d}\alpha_{2} \) of \( (\delta^{1})^{-1}(\theta) = \mathbb{Z}_{d}(z_{\Delta^{2}}^{2}) \oplus H^{1}(M(2, m); \mathbb{Z}_{d}) \). First, we have

\[
    (a')^{*}(z_{\Delta^{2}}^{2}) = (a')^{*}p^{*}(z_{\Delta^{2}}^{1}) = p^{*}((a')^{*}(z_{\Delta^{2}}^{1})) = p^{*}(-z_{\Delta^{2}}^{1}) = -z_{\Delta^{2}}^{2}.
\]

Next,

\[
    (b')^{*}(z_{\Delta^{2}}^{2}) = (b')^{*}p^{*}(z_{\Delta^{2}}^{1}) = p^{*}((b')^{*}(z_{\Delta^{2}}^{1})) = p^{*}(-z_{\Delta^{2}}^{1}) = -z_{\Delta^{2}}^{2}.
\]

\[
    (c')^{*}(z_{\Delta^{2}}^{2}) = (c')^{*}p^{*}(z_{\Delta^{2}}^{1}) = p^{*}((c')^{*}(z_{\Delta^{2}}^{1})) = p^{*}(z_{\Delta^{2}}^{1}) = z_{\Delta^{2}}^{2}.
\]

One therefore has the following matrix representation of \( \text{Aut}(M(2, m)) = \langle a', b', c' \rangle \) on \( \mathbb{Z}_{d}(z_{\Delta^{2}}^{2}) \oplus H^{1}(M(2, m); \mathbb{Z}_{d}) \) relative to the \( \mathbb{Z}_{d} \)-basis \( (z_{\Delta^{2}}^{2}, \mathcal{d}\alpha_{1}, \mathcal{d}\alpha_{2}) \):

\[
    a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \ b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ c' \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]
From the above, we may extract the classification of the Accola coverings of the map $\mathcal{M}(2, m)$:

**Theorem 8.5.13.** Let $\mathcal{M}' \to \mathcal{M} = \mathcal{M}(2, m)$ be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with $\mathbb{Z}_d$ and inducing the action $\alpha = \alpha_{\text{Acc}} : \mathcal{M} \to \text{Aut}(\mathbb{Z}_d)$, by a regular connected map. Then up to $\cong_{\mathcal{M}(2, m)}$, $\mathcal{M}' = \mathcal{M}_{z}$, where $\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)$ as below:

1. If $d$ is odd, then $\zeta = z_{\text{Acc}}^2 \in D(\mathcal{M}; \mathbb{Z}_d)$ is the unique such class;
2. If $d = 2d_0$ is even, there are two possibilities for $\zeta$, namely, $\zeta = z_{\text{Acc}}^2$ and $\zeta = z_{\text{Acc}}^2 + d_0d\alpha_1 + d_0d\alpha_2$.

**Remark:** Note that if $d = 2$ then the Steinberg and Accola coverings of $\mathcal{M}(2, m)$ coincide (up to $\cong_{\mathcal{M}(1, m)}$, there are two such).

### 8.5.2 The Regular Affine Maps of Type $(3, 6)$

In this subsection we consider the regular affine maps of type $(3, 6)$. These maps are represented in the form

$$\mathcal{M} = \mathcal{M}(\Delta/K, s_1, s_2, s_3),$$

where $\Delta = \Delta(3, 6)$, and

$$\Delta(3, 6) = \langle s_1, s_2, s_3 | s_i^2 = (s_1s_3)^2 = (s_1s_2)^3 = (s_2s_3)^6 = 1 \rangle,$$

and where $K \leq \Delta$.

**Proposition 8.5.14.** There is an isomorphism

$$\Delta(3, 6) \cong \langle \omega, \tau \rangle \ltimes \mathbb{Z}[\omega],$$

where $\omega$ acts by right multiplication on $\mathbb{Z}[\omega]$ and $\tau$ acts as complex conjugation. An isomorphism is given by

$$s_1 \mapsto (-\tau, 1), \ s_2 \mapsto (\tau\omega, 0), \ s_3 \mapsto (\tau, 0).$$

(Note that $\langle \omega, \tau \rangle \cong D_{12}$.)
With the above isomorphism, we may view the blades of an affine map of type \( \{3, 6\} \) as below. The bullets (●) indicate the vertices of an affine map having vertex valency 6 and face valency 3; in viewing the affine maps of vertex valency 3 and face valency 6, the bullets identify the faces. Under the above isomorphism \( \Delta \cong \langle \omega, \tau \rangle \ltimes \mathbb{Z}[\omega] \), if \( \beta_0 = (1, 0) \), then \( \beta_0, a\beta_0, b\beta_0, \) and \( c\beta_0 \) are marked below.

The following is routine. Note first that it is well known that \( \mathbb{Z}[\omega] \) is a principal ideal domain.

**Lemma 8.5.15.** The only ideals of \( \mathbb{Z}[\omega] \) invariant under complex conjugation are of the form \( a\mathbb{Z}[\omega] \) and \( (a + a\omega)\mathbb{Z}[\omega] \), where \( a \in \mathbb{Z} \).
We shall denote by $\mathcal{M}(1, m)$ the regular maps corresponding to the ideal $a\mathbb{Z}[\omega]$ and by $\mathcal{M}(3, m)$ the regular maps corresponding to the ideal $(a + a\omega)\mathbb{Z}[\omega]$.\(^{14}\) Note that $[a\mathbb{Z}[\omega] : (a + a\omega)\mathbb{Z}[\omega]] = 3 = [(a + a\omega)\mathbb{Z}[\omega] : 3a\mathbb{Z}[\omega]]$. Therefore, we have three-fold coverings $\mathcal{M}(3, m) \to \mathcal{M}(1, m)$ and $\mathcal{M}(1, 3m) \to \mathcal{M}(3, m)$.

Note that under the above isomorphism $\Delta \cong \langle \omega, \tau \rangle \ltimes \mathbb{Z}[\omega]$, we have that $s_2 s_3 s_2 s_3 s_2 s_1 \mapsto (1, 1)$ and $s_3 s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_2 s_1 \mapsto (1, 1 + \omega)$.

Therefore, the monodromy groups and fundamental groups of the maps $\mathcal{M}(1, m)$ and $\mathcal{M}(3, m)$ are as given below:

Type $\mathcal{M}(1, m)$:

$$G = \langle a_1, b_1, c_1 \mid a_1^2 = b_1^2 = c_1^2 = (a_1 c_1)^2 = (a_1 b_1)^3 = (b_1 c_1)^6 = (b_1 c_1 b_1 c_1 b_1 a_1)^m = 1 \rangle,$$

for some positive integer $m$, with fundamental group

$$\pi_1(\mathcal{M}(1, m), (1, 0)) = \langle (s_2 s_3 s_2 s_3 s_2 s_1)^m, (s_3 s_2 s_3 s_2 s_1 s_2)^m \rangle.$$

Type $\mathcal{M}(3, m)$:

$$G = \langle a_3, b_3, c_3 \mid a_3^2 = b_3^2 = c_3^2 = (a_3 c_3)^2 = (a_3 b_3)^3 = (b_3 c_3)^6 = (c_3 b_3 c_3 a_3 b_3 c_3 b_3 a_3)^m = 1 \rangle,$$

for some positive integer $m$ with fundamental group

$$\pi_1(\mathcal{M}(1, 3m), (1, 0)) = \langle (s_3 s_2 s_3 s_2 s_3 s_1 s_2 s_3 s_2 s_1)^m, (s_3 s_1 s_2 s_3 s_2 s_1 s_2 s_3 s_2 s_1)^m \rangle;$$

(note that $s_3 s_1 s_2 s_3 s_2 s_1 s_3 s_2 s_3 \mapsto (1, -1 + \omega^2)$.

Thus, denoting the monodromy groups of the maps $\mathcal{M}(1, m)$ by $G(1, m)$ and denoting the monodromy groups of the maps $\mathcal{M}(3, m)$ by $G(3, m)$ we see that $a_3 \mapsto a_1$, $b_3 \mapsto b_1$ and $c_3 \mapsto c_1$ gives a three-fold covering $G(3, m) \to G(1, m)$ whose kernel is the normal closure in $G(3, m)$ of $(b_3 c_3 b_3 c_3 b_3 a_3)^m$. Likewise, there is a three-fold covering

\(^{14}\)Note that for each fixed positive integer $m$, there are two mutually dual maps corresponding to $\mathcal{M}(1, m)$: one with face valency 3 and one with face valency 6. Similar comments apply, of course, to $\mathcal{M}(3, m)$.
\( G(1, 3m) \rightarrow G(3, m) \) given by \( a_1 \mapsto a_3, \ b_1 \mapsto b_3 \) and \( c_1 \mapsto c_3 \) and whose kernel is the normal closure in \( G(1, 3m) \) of \((c_1 b_1 c_1 c_1 a_1 b_1 c_1 b_1 a_1)^m\).

We turn now to the cyclic coverings of the regular maps of type \((3, 6)\).

**The Regular Maps \( \mathcal{M}(1, m) \)—Unramified Coverings**

As in the previous sections, if we denote \( \gamma_x = s_2 s_3 s_2 s_3 s_2 s_1, \gamma_\omega = s_3 s_2 s_3 s_2 s_1 s_2 \), then the fundamental group of the affine map \( \mathcal{M} = \mathcal{M}(1, m) \) is generated by \( \gamma_x^m, \gamma_\omega^m \). Therefore, the elements

\[
\frac{\partial}{\partial x} = \epsilon_s h(\gamma_x), \quad \frac{\partial}{\partial \omega} = \epsilon_s h(\gamma_\omega) \in H_1(\mathcal{M}; \mathbb{Z}_d),
\]

together form a \( \mathbb{Z}_d \)-basis of \( H_1(\mathcal{M}; \mathbb{Z}_d) \). We denote by \( \{dx, d\omega\} \) the corresponding dual basis of \( H^1(\mathcal{M}; \mathbb{Z}_d) \). The action of \( \mathcal{M} = \langle a', b', c' \rangle \) on \( H_1(\mathcal{M}; \mathbb{Z}_d) \) is given by conjugation by \( s_1, s_2, s_3 \) on the elements \( \gamma_x, \gamma_\omega \).

Since \( s_1 \gamma_x s_1 = \gamma_x^{-1}, s_1 \gamma_\omega s_1 = \gamma_x^{-1} \gamma_\omega \) we infer that

\[
\frac{\partial}{\partial x(a')} = -\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y(a')} = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}.
\]

Likewise, one computes that

\[
\frac{\partial}{\partial x(b')} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial x(c')} = \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y(c')} = \frac{\partial}{\partial x} - \frac{\partial}{\partial x}.
\]

Taking adjoints, one immediately obtains the following result:

**Proposition 8.5.16.** Assume that \( \mathcal{M} = \mathcal{M}(1, m) \), and that \( d \) is a non-negative integer. Then the action of \( \text{Aut}(\mathcal{M}) \) on \( H^1(\mathcal{M}; \mathbb{Z}_d) \) is given by

\[
(a')^*(dx) = -dx - d\omega, \quad (a')(d\omega) = d\omega,
\]

\[
(b')^*(dx) = d\omega,
\]

\[
(c')^*(dx) = dx + d\omega, \quad (c')^*(d\omega) = -d\omega.
\]

In turn, from the above, one concludes that if \( \zeta \in H^1(\mathcal{M}; \mathbb{Z}_d) \) is an \( \alpha \)-isotypical class yielding a regular, connected, nontrivial unramified
covering of $\mathcal{M}$ having cyclic group $\mathbb{Z}_d$ of covering transformations, then $d = 3$ and (up to unit multiple) $\zeta = dx - dw$.

Note that we have already observed above that $\mathcal{M}(3, m) \to \mathcal{M}(1, m)$ is unramified and $\mathcal{M}(2, m)$ is regular. We have, therefore, the following classification theorem.

**Theorem 8.5.17.** Up to $\cong_{\mathcal{M}(1, m)}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(1, m)$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 3-fold covering $\mathcal{M}(3, m) \to \mathcal{M}(1, m)$.

We now consider the Steinberg and Accola coverings of $\mathcal{M}(1, m)$.

**The Regular Maps $\mathcal{M}(1, m)$—The Steinberg Coverings**

In this subsection we determine the Steinberg coverings of the affine map $\mathcal{M} = \mathcal{M}(1, m)$, and having cyclic group $\mathbb{Z}_d$ of covering transformations. Since it is easy to verify that $\mathcal{M}$ has $2m^2$ faces, it follows that $d$ must satisfy the condition $d|2m^2$.

Let $\beta_1, \beta_2, \ldots, \beta_{2m^2}$ be blades in the same $G^+$-orbit generating the $2m^2$ faces of $\mathcal{M}$, and set $m_i = \chi_{[\beta_i, \zeta]_{G^+}} + W^{(a,b)} \in C_2(\mathcal{M}), i = 1, 2, \ldots, 2m^2$. Then, up to a unit multiple in $\mathbb{Z}_d$, the $\text{Aut}(\mathcal{M})$-module homomorphism $\theta = \theta_{\text{St}} : C_2(\mathcal{M}) \to \mathbb{Z}_d$ satisfies $\theta(m_i) = 1, i = 1, 2, \ldots, 2m^2$.

We shall begin by explicitly constructing an element $z \in (\delta^1)^{-1}(\theta)$ as follows. Start with the blade $\beta_0 = (1, 0)$, and define the blade $\beta_1 = b(1, 0) = (0, i\tau)(0, 1) = (0, i\tau)$. Define the voltages $z_0, z_1$ via the assignments

\[ z_0 : a \beta_0, ac \beta_0 \mapsto 1, \quad \beta_0, c \beta_0 \mapsto -1, \]
\[ z_1 : ba \beta_0, cba \beta_0 \mapsto 1, \quad aba \beta_0, acba \beta_0 \mapsto -1, \] and
\[ z_2 : b \beta_0, c \beta_0 \mapsto 1, \quad aba \beta_0, acb \beta_0 \mapsto -1, \]

with all remaining blades mapped to zero.
Next, define the automorphisms $\sigma_1, \sigma_\omega \in \text{Aut}(\mathcal{M})$, $\mathcal{M} = \mathcal{M}(1, m)$ to be the translations one unit in the direction of the real unit 1 and the complex unit $\omega$. Therefore,

$$\sigma_1 : (g, \zeta) \mapsto (g, \zeta + 1) = (g, \zeta)(1, 1), \text{ and } \sigma_\omega : (g, \zeta) \mapsto (g, \zeta + \omega) = (g, \zeta)(1, \omega).$$

In terms of the translations $\sigma_1, \sigma_\omega$ defined above, we now define the voltages

$$z'_0 = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} 2k(\sigma_\omega^{-k})^*(\sigma_1^{-j})^*z_0,$$

$$z'_1 = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (2k + 1)(\sigma_\omega^{-k})^*(\sigma_1^{-j})^*z_1,$$

and set $z' = z'_0 + z'_1$.

The voltage $z'$, together with $\partial/\partial x$ and $\partial/\partial \omega$ are depicted below, again for $m = 3$.\dagger
† Again, the convention is that for each blade $\beta$ labeled with value $z'_\beta$, then the blade $a\beta$ carries the value $-z'_\beta$.

Next, define voltages

$$z''_1 = \sum_{k=0}^{m-1} 2mk(\sigma_{1}^{-k})^*\sigma_{\omega}^*z_1, \quad z''_2 = \sum_{k=0}^{m-1} 2mk(\sigma_{1}^{-k})^*\sigma_{\omega}^*z_2.$$ 

Finally, set $z_{st} = z' + z''_1 + z''_2$. This voltage is illustrated below for $m = 3$ (with the usual conventions):
One checks that if \(d \mid 2m^2\), then \(\delta^1(z_{S_0}) = \theta\). As before, even though \(\theta\) is a surjective \(\text{Aut}(\mathcal{M})\)-module homomorphism \(C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d\), one cannot conclude that \(z_{S_i} \in D(\mathcal{M}; \mathbb{Z}_d)_\alpha\) for any homomorphism \(\alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\). This is remedied through the action of the automorphisms \(\alpha', \beta', \) and \(\gamma'\) on \(D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d\langle [z_{S_i}] \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\). Routine calculations reveal that this action is as below:

**Proposition 8.5.18.** Assume that \(\mathcal{M} = \mathcal{M}(1, m)\), and that \(d\) is a positive integer dividing \(2m^2\). Then the action of \(\text{Aut}(\mathcal{M})\) on \(D_\theta = \mathbb{Z}_d\langle [z_{S_i}] \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\) is given by

\[
(a')^*dx = -dx - d\omega, \quad (a')^*d\omega = d\omega,
\]

\[
(b')^*dx = d\omega,
\]
\[(c')^* dx = dx + d\omega, \quad (c')^* d\omega = -d\omega,\]
\[(a')^* z_{St} = -z_{St} - m(m + 2)d\omega,\]
\[(b')^* z_{St} = -z_{St},\]
\[(c')^* z_{St} = -z_{St} + m^2 d\omega.\]

In terms of matrices over \(\mathbb{Z}_d\), we have the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\) given by
\[
a' \mapsto \begin{bmatrix}
  -1 & 0 & 0 \\
  0 & -1 & 0 \\
  -m(m + 2) & -1 & 1
\end{bmatrix}, \quad b' \mapsto \begin{bmatrix}
  -1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
\end{bmatrix}, \quad c' \mapsto \begin{bmatrix}
  -1 & 0 & 0 \\
  0 & 1 & 0 \\
  m^2 & 1 & -1
\end{bmatrix}.
\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map of type \((3, 6)\). Note that if \(\zeta \in D_\theta\) is \(\alpha\)-isotypical for \(\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), then we may assume that \(\zeta\) is of the form \(\zeta = z_{St} + xdx + yd\omega\) for suitable \(x, y \in \mathbb{Z}_d\). We have the following:

**THEOREM 8.5.19.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then \(d|2m\); furthermore, up to \(\cong_{\mathcal{M}(1, m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = -z_{St} - m^2 dx + m^2 d\omega \in D(\mathcal{M}; \mathbb{Z}_d)\).

**PROOF.** We set \(\zeta = z_{St} + xdx + yd\omega\); since \(\alpha_{St}(a') = \alpha_{St}(b') = \alpha_{St}(c') = -1\), then the condition
\[-z_{St} - xdx - yd\omega = -\zeta = \alpha_{St}(a')\zeta = (a')^* \zeta \]
\[-z_{St} - xdx + (-m(m + 2) - x + y)d\omega\]
yields \(2y - x = m(m + 2)\). Next, the condition \(-\zeta = \alpha_{St}(b')\zeta = (b')^* \zeta\) clearly yields \(x = -y\). Therefore, it follows that \(3x = m(m + 2)\). From
\[-z_{St} - xdx - yd\omega = -\zeta = \alpha_{St}(c')\zeta = (c')^* \zeta = -z_{St} + xdx + (m^2 + x - y)d\omega,\]
we obtain $2x = 0$ and $x = -m^2$. Thus,

$$m(m + 2) = 3y = -3x = 3m^2;$$

since we know a priori that $2m^2 = 0 \in \mathbb{Z}_d$, we conclude that $2m = 0 \in \mathbb{Z}_d$, proving that $d|2m$. Everything follows.

The Regular Maps $\mathcal{M}(1, m)$—The Accola Coverings

Note that the Accola homomorphism exists for both both the maps $\mathcal{M}(1, m)$ and $\mathcal{M}(3, m)$, for all positive integers $m$. Furthermore a construction of $z_{\text{Acc}} \in D(\mathcal{M}; \mathbb{Z}_d)$, $\mathcal{M} = \mathcal{M}(1, m)$ is obtained very easily, as follows. If we set $\beta = ba\beta_0$, then we define the voltage $z$ by setting

$$z : \beta, c\beta \mapsto 1, \ a\beta, ac\beta \mapsto -1.$$  

Set

$$z_{\text{Acc}} = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (\sigma_1^k)^* (\sigma_\omega^j)^* z.$$  

For $m = 3$, this voltage is illustrated below (with usual conventions):
It is clear that $z_{\text{Acc}}$ induces the Accola homomorphism, i.e., that $\delta^1(z_{\text{Acc}}) = \theta_{\text{Acc}} : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d$. It is obvious that $(b')^*z_{\text{Acc}} = -z_{\text{Acc}}$; routine calculations reveal that $(a')^*z_{\text{Acc}} + z_{\text{Acc}} + md\omega = (c')^*z_{\text{Acc}} - z_{\text{Acc}} + md\omega = 0$, from which we conclude that $(a')^*z_{\text{Acc}} = -z_{\text{Acc}} - md\omega$ and that $(c')^*z_{\text{Acc}} = z_{\text{Acc}} - md\omega$.

We have, therefore, the following:

**Proposition 8.5.20.** If $\theta = \theta_{\text{Acc}} : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d$, then the action of $\text{Aut}(\mathcal{M})$ on $D_\theta = \mathbb{Z}_d\langle [z_{\text{Acc}}]\rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$ ($\mathcal{M} = \mathcal{M}(1, m)$) is given by

$$(a')^*dx = -dx - d\omega, \quad (a')^*d\omega = d\omega,$$
(b')^*dx = d\omega, \\
(c')^*dx = dx + d\omega, \ (c')^*d\omega = -d\omega, \\
(a')^*z_{Acc} = -z_{Acc} - md\omega, \\
(b')^*z_{Acc} = -z_{Acc}, \\
(c')^*z_{Acc} = z_{Acc} + md\omega,

In terms of matrices over \(\mathbb{Z}_d\), and relative to the ordered basis \((z_{Acc}, dx, d\omega)\) of \(\mathbb{Z}_d\langle z_{Acc}\rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\), the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c'\rangle\) is given by

\[
\begin{align*}
a' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -m & -1 & 1 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 1 & -1 \end{bmatrix}.
\end{align*}
\]

From the above calculations, we may infer the following classification of the Accola coverings of the affine \((3, 6)\)-map \(\mathcal{M}(1, m)\).

**Theorem 8.5.21.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{z_{Acc}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then we must have \(d \mid 2m\). Furthermore, up to \(\cong_{\mathcal{M}(1, m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = [z] = z_{Acc} - xdx + xd\omega\) and where \(\in \mathbb{Z}_d\) is any element satisfying \(3x = m\).

**The Regular Maps** \(\mathcal{M}(3, m)\)—**Unramified Coverings**

We turn now to the maps \(\mathcal{M} = \mathcal{M}(3, m)\) and start by determining the unramified coverings by connected regular maps. We shall start by
determining $\mathbb{Z}_d$-bases for $H_1(\mathcal{M}; \mathbb{Z}_d)$ and $H^1(\mathcal{M}; \mathbb{Z}_d)$. From page 173, the fundamental group of $\mathcal{M}$ is given by

$$\pi_1(\mathcal{M}; \beta_0) = \langle (s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1)^m, (s_3s_1s_2s_3s_2s_3s_2s_3s_2s_3s_2s_2s_1)^m \rangle,$$

where $\beta_0 = (1, 0)$. Set

$$\gamma_1 = (s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1), \quad \gamma_2 = (s_3s_1s_2s_3s_2s_3s_2s_3s_2s_2s_3s_2),$$

and define the cycles $\partial/\partial \tau_1 = \epsilon_1 h(\gamma_1^m)$, $\partial/\partial \tau_2 = \epsilon_2 h(\gamma_2^m) \in H_1(\mathcal{M}; \mathbb{Z}_d)$, where $\epsilon : \mathbb{Z} \to \mathbb{Z}_d$ is the projection map. As above, we know that

$$\{\partial/\partial \tau_1, \partial/\partial \tau_2\}$$

is a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$; denote by $\{d\tau_1, d\tau_2\}$ the corresponding $\mathbb{Z}_d$-dual basis of $H^1(\mathcal{M}; \mathbb{Z}_d)$. By conjugating $\gamma_1^m$, $\gamma_2^m$ by the elements $s_1$, $s_2$, $s_3$ we may compute the effect of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$. In turn, the representation of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is obtained by dualizing (i.e., by taking adjoints). The results are summarized below:

**Proposition 8.5.22.** Assume that $\mathcal{M} = \mathcal{M}(3, m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

$$(a')^*(d\tau_1) = d\tau_2,$$

$$(b')^*(d\tau_1) = d\tau_1 - d\tau_2, \quad (b')^*(d\tau_2) = -d\tau_2,$$

$$(c')^*(d\tau_1) = -d\tau_2.$$

In turn, from the above, one easily determines that if $\zeta \in H^1(\mathcal{M}; \mathbb{Z}_d)$ is an $\alpha$-isotypical class yielding a regular, connected, unramified covering of $\mathcal{M}$ having cyclic group $\mathbb{Z}_d$ of covering transformations, then $d = 3$ and (up to unit multiple) $\zeta = d\tau_1 + d\tau_2$.

We have already observed above that $\mathcal{M}(1, 3m) \to \mathcal{M}(3, m)$ is unramified and $\mathcal{M}(1, 3m)$ is regular. We have, therefore, the following classification theorem.
Theorem 8.5.23. Up to $\cong_{\mathcal{M}(3,m)}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(3,m)$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 3-fold covering $\mathcal{M}(1,3m) \to \mathcal{M}(3,m)$.

We turn now to the classification of the Steinberg and Accola coverings of $\mathcal{M}(3,m)$.

The Regular Maps $\mathcal{M}(3,m)$—The Steinberg Coverings

In order to compute the totally ramified coverings of $\mathcal{M}(3,m)$, we proceed as on page 167 and capitalize on the 3-fold covering $p: \mathcal{M}(3,m) \to \mathcal{M}(1,m)$. In this case, we have that

$$s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1 = s_2s_3s_2s_3s_2s_1 \cdot s_3s_2s_3s_2s_1s_2,$$

and so it follows that under the induced mapping $p_*: H_1(\mathcal{M}(3,m); \mathbb{Z}_d) \to H_1(\mathcal{M}(1,m); \mathbb{Z}_d)$, we have $\partial/\partial \tau_1 p_* = \partial/\partial x + \partial/\partial \omega$. Also

$$s_3s_1s_2s_3s_2s_3s_2s_1s_2 = (s_2s_3s_2s_3s_1s_2)^{-2} \cdot s_3s_2s_3s_2s_1s_2,$$

from which we conclude that $\partial/\partial \tau_2 p_* = -2\partial/\partial x + \partial/\partial \omega$.

The lemma below gives the pull-backs of the basis elements $dx$ and $d\omega$ of $H^1(\mathcal{M}(1,m); \mathbb{Z}_d)$ in terms of the basis elements $d\tau_1$ and $d\tau_2$ of $H^1(\mathcal{M}(3,m); \mathbb{Z}_d)$.

Lemma 8.5.24. Relative to the covering $p: \mathcal{M}(3,m) \to \mathcal{M}(1,m)$, we have

$$p^*(dx) = d\tau_1 - 2d\tau_2, \quad p^*(d\omega) = d\tau_1 + d\tau_2.$$

Proof. As computed above,

$$\partial/\partial \tau_1 p_* = \partial/\partial x + \partial/\partial \omega, \quad \partial/\partial \tau_2 p_* = -2\partial/\partial x + \partial/\partial \omega.$$

Therefore, it follows that if $p^*(dx) = \tau_1 d\tau_1 + \tau_2 d\tau_2$, then

$$\tau_1 = \partial/\partial \tau_1 (p^* dx) = (\partial/\partial \tau_1 p_*) dx = (\partial/\partial x + \partial/\partial \omega)(dx) = 1.$$
Also,
\[\tau_2 = \frac{\partial}{\partial \tau_2} (p^* \mathrm{d}x) = (\frac{\partial}{\partial \tau_2} p_\ast) \mathrm{d}x = (-2 \frac{\partial}{\partial x} + \frac{\partial}{\partial \omega})(\mathrm{d}x) = -2;\]

One similarly verifies the corresponding result for \(p^*(\mathrm{d}\omega)\).

We have \(\mathrm{Aut}(\mathcal{M}(1, m)) = \langle a', b', c' \rangle\); since \(\mathcal{M}(3, m)\) is regular, then \(a', b', c'\) lift to generators of \(\mathrm{Aut}(\mathcal{M}(3, m))\); we shall, for convenience (and simplicity) continue to denote these lifted automorphisms of \(\mathcal{M}(3, m)\) as \(a', b', c'\). As a result, we see that if \(\zeta \in D(\mathcal{M}(1, m); \mathbb{Z}_d)\), then
\[(a')^* p^*(\zeta) = p^*((a')^* \zeta),\]
with similar results for \(b', c'\).

In analyzing the Steinberg covering of \(\mathcal{M}(3, m)\) we proceed as in the case of the type \((4, 4)\) affine maps, recalling the voltage \(z_{\text{St}}^1 = z_{\text{St}} \in D(\mathcal{M}(1, m); \mathbb{Z}_d)\) on page 177. We now define
\[z_{\text{St}}^2 = p^*(z_{\text{St}}^1) \in D(\mathcal{M}(3, m); \mathbb{Z}_d).\]

It is clear that \(\delta^1(z_{\text{St}}^2) = \theta_{\text{St}} : C_2(\mathcal{M}(3, m))/H_2(\mathcal{M}(3, m)) \to \mathbb{Z}_d\). Therefore, if \(\theta = \theta_{\text{St}}\) then \((\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(z_{\text{St}}^2) \oplus H^1(\mathcal{M}(3, m); \mathbb{Z}_d)\). We already have enough information to determine the action of \(\mathrm{Aut}(\mathcal{M}(3, m))\) on \(D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta))\). Using the results of Proposition 8.5.18, we compute:

\[(a')^*(z_{\text{St}}^2) = (a')^* p^*(z_{\text{St}}^1) = p^*((a')^*(z_{\text{St}}^1)) = p^*(-z_{\text{St}}^1 - m(m + 2)\mathrm{d}\omega) = -z_{\text{St}}^2 - m(m + 2)d\tau_1 - m(m + 2)d\tau_2.\]

Next,

\[(b')^*(z_{\text{St}}^2) = (b')^* p^*(z_{\text{St}}^1) = p^*((b')^*(z_{\text{St}}^1)) = p^*(-z_{\text{St}}^1) = -z_{\text{St}}^2.\]
\[(c')^*(z_{St}^2) = (c')^*p^*(z_{St}^1)
= p^*((c')^*(z_{St}^1))
= p^*(-z_{St}^1 + m^2d\omega)
= -z_{St}^2 + m^2d\tau_1 + m^2d\tau_2.\]

We summarize below the action of \(\text{Aut}(\mathcal{M}(3, m))\) on \(D_\theta\):

**Proposition 8.5.25.** Assume that \(\mathcal{M} = \mathcal{M}(3, m)\) and that \(d\) is a positive integer dividing \(6m^2\). Then the action of \(\text{Aut}(\mathcal{M})\) on \(D_\theta = \mathbb{Z}_d \langle z_{St}^2, h \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\) is determined by

\[
(a')^*(d\tau_1) = d\tau_2, \\
(b')^*(d\tau_1) = d\tau_1 - d\tau_2, \quad (b')^*(d\tau_2) = -d\tau_2, \\
(c')^*(d\tau_1) = -d\tau_2, \\
(a')^*(z_{St}^2) = -z_{St}^2 - m(m + 2)d\tau_1 - m(m + 2)d\tau_2, \\
(b')^*(z_{St}^2) = -z_{St}^2, \\
(c')^*(z_{St}^2) = -z_{St}^2 + m^2d\tau_1 + m^2d\tau_2.
\]

In terms of the ordered \(\mathbb{Z}_d\)-basis \((z_{St}^2, d\alpha_1, d\alpha_2)\) of \(D_\theta\), we have the matrix representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\):

\[
a' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
-m(m + 2) & 0 & 1 \\
-m(m + 2) & 1 & 0
\end{bmatrix}, \quad
b' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & -1
\end{bmatrix}, \quad
c' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
m^2 & 0 & -1 \\
m^2 & -1 & 0
\end{bmatrix}.
\]

The above provides the following classification of the Steinberg coverings of the regular affine map \(\mathcal{M}(3, m)\).

**Theorem 8.5.26.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(3, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{St} : \)
$\text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)$, by a regular connected map. Then $d \mid 2m$; furthermore, up to $\cong_{\mathcal{M}(3,m)}$, $\mathcal{M}' = \mathcal{M}_2$, where $\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)$ is given uniquely up to unit multiples in $\mathbb{Z}_d$ by $\zeta = z^2_{\text{St}} + m^2 d\tau_2 \in D(\mathcal{M}; \mathbb{Z}_d)$.

**Proof.** If one writes the $\alpha_{\text{St}}$-isotypical class as $\zeta = z^2_{\text{St}} + \tau_1 d\tau_1 + \tau_2 d\tau_2$, then $-\zeta = \alpha_{\text{St}}(a') \zeta = (a')^*(\zeta)$ implies that

$$-z^2_{\text{St}} - \tau_1 d\tau_1 - \tau_2 d\tau_2 = -z^2_{\text{St}} + (\tau_2 - m(m+2)) d\tau_1 + (\tau_1 - m(m+2)) d\tau_2.$$ 

Therefore, it follows that $\tau_1 + \tau_2 = m(m+2)$. Next, the equation $-\zeta = \alpha_{\text{St}}(b') \zeta = (b')^*(\zeta)$ clearly implies that $\tau_2 = \tau_1 + \tau_2$, and so $\tau_1 = 0$. Finally, $-\zeta = \alpha_{\text{St}}(c') \zeta = (c')^*(\zeta)$ implies that $\tau_2 = m^2$; in turn this implies that $m^2 = \tau_1 + \tau_2 = m(m+2)$, and so $2m = 0 \in \mathbb{Z}_d$, as claimed.

**The Regular Maps $\mathcal{M}(3,m)$—The Accola Coverings**

Finally we turn to the classification of the Accola coverings of the maps $\mathcal{M}(3,m)$. As in the case $\mathcal{M} = \mathcal{M}(1,m)$ we construct an element $z^2_{\text{Acc}} \in (\delta^1)^{-1}(\theta)$, where $\theta = \theta_{\text{Acc}}$, as follows. We set

$$z^2_{\text{Acc}} = p^*(z^1_{\text{Acc}}) \in D(\mathcal{M}(3,m); \mathbb{Z}_d),$$

where $z^1_{\text{Acc}} = z_{\text{Acc}}$ is as on page 180; note that $\delta^1(z^2_{\text{Acc}}) = \theta_{\text{Acc}}$.

Using the results of Proposition 8.5.20 together with pull backs under $p : \mathcal{M}(3,m) \to \mathcal{M}(1,m)$ of $d\mathbf{x}$, $d\mathbf{w}$ to $\mathbb{Z}_d$-linear combinations of $d\tau_1$, $d\tau_2$, we can compute the effect of $a'$, $b'$, and $c'$ on the $\mathbb{Z}_d$-basis elements $z^2_{\text{Acc}}$, $d\tau_1$, $d\tau_2$ of $(\delta^1)^{-1}(\theta) = \mathbb{Z}_d\langle z^2_{\text{Acc}} \rangle \oplus H^1(\mathcal{M}(3,m); \mathbb{Z}_d)$. First, we have

$$(a')^*(z^2_{\text{Acc}}) = (a')^*p^*(z^1_{\text{Acc}}) = p^*((a')^*(z^1_{\text{Acc}})) = p^*(-z^1_{\text{Acc}} - m d\omega) = -z^2_{\text{Acc}} - m d\tau_1 - m d\tau_2.$$
Next,
\[
(b')^* (z_{\text{Acc}}^2) = (b')^* p^* (z_{\text{Acc}}^1) \\
= p^* ((b')^* (z_{\text{Acc}}^1)) \\
= p^* (-z_{\text{Acc}}^1) \\
= -z_{\text{Acc}}^2.
\]

\[
(c')^* (z_{\text{Acc}}^2) = (c')^* p^* (z_{\text{Acc}}^1) \\
= p^* ((c')^* (z_{\text{Acc}}^1)) \\
= p^* (z_{\text{Acc}}^1 - m d\omega) \\
= z_{\text{Acc}}^2 - m d\tau_1 - m d\tau_2.
\]

One therefore has the following matrix representation of \( \text{Aut}(\mathcal{M}(3, m)) = \langle a', b', c' \rangle \) on \( \mathbb{Z}_d (z_{\text{Acc}}^2) \oplus H^1(\mathcal{M}(3, m); \mathbb{Z}_d) \) relative to the \( \mathbb{Z}_d \)-basis \( (z_{\text{Acc}}^2, d\tau_1, d\tau_2) \):

\[
a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\
-m & 0 & 1 \\
-m & 1 & 0 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\
0 & 1 & 0 \\
0 & -1 & -1 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} 1 & 0 & 0 \\
-m & 0 & -1 \\
-m & -1 & 0 \end{bmatrix}.
\]

From the above calculations, we may infer the following classification of the Accola coverings of the affine \((3, 6)\)-map \( \mathcal{M}(3, m) \).

**Theorem 8.5.27.** Let \( \mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m) \) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \( \mathbb{Z}_d \) and inducing the action \( \alpha = \alpha_{\text{Acc}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), by a regular connected map. Then we must have \( d \mid 2m \). Furthermore, up to \( \cong_{\mathcal{M}(3, m)} \), \( \mathcal{M}' = \mathcal{M}_z \), where \( \zeta = [z] = z_{\text{Acc}} + m d\tau_2 \).

**Remark:** Note that if \( d = 2 \) then the Steinberg and Accola coverings of \( \mathcal{M}(3, m) \) coincide (up to \( \cong_{\mathcal{M}(3, m)} \)).
8.5.3 The Regular Affine Maps of Type (6, 3)

Since the affine maps of type (6, 3) are the duals of the corresponding maps of type (3, 6), much of the work in the preceding subsection can be applied. Note, in particular, that an unramified covering of an affine map of type (3, 6) gives an unramified covering of the corresponding dual map of type (6, 3). We retain the notation of Subsection 8.5.2 and denote by $\mathcal{M}(1, m)$ and $\mathcal{M}(3, m)$ the regular affine maps of type (3, 6) containing $2m^2$ and $6m^2$ faces respectively, and by $\mathcal{M}(1, m)^*$ and $\mathcal{M}(3, m)^*$ their duals, thereby interchanging the vertices and faces. Therefore, there are the 3-fold unramified coverings $\mathcal{M}(1, 3m)^* \to \mathcal{M}(3, m)^* \to \mathcal{M}(1, m)^*$. Furthermore, the above remarks guarantee the following analog of Theorems 8.5.17 and 8.5.23:

**Theorem 8.5.28.** The unramified coverings of the regular affine maps of type (6, 3) are as follows.

(i) Up to $\cong_{\mathcal{M}(1, m)^*}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(1, m)^*$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 3-fold covering $\mathcal{M}(3, m)^* \to \mathcal{M}(1, m)^*$.

(ii) Up to $\cong_{\mathcal{M}(3, m)^*}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(3, m)^*$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 3-fold covering $\mathcal{M}(1, 3m)^* \to \mathcal{M}(3, m)^*$.

Note in addition that since the face valency of the regular maps of type (6, 3) is odd (= 3), there are no Accola coverings of the maps $\mathcal{M}(1, m)^*$, $\mathcal{M}(3, m)^*$. Therefore we need only classify the Steinberg coverings. Note that in the present case, if $\mathcal{M}$ is a map of type (3, 6), and if $\mathcal{M}^*$ is the corresponding dual map, voltages $z : B \to \mathbb{Z}_4$, where $B$ is the blade set of $\mathcal{M}^*$ (= blade set of $\mathcal{M}$) will satisfy

$$z(a\beta) = z(\beta) = -z(c\beta),$$

for all $\beta \in B$. This is obvious, since the duality operation $\mathcal{M} \to \mathcal{M}^*$ simply interchanges the roles of the monodromy involutions $a$ and $c$. 
The Regular Maps $\mathcal{M}(1,m)^*$ and $\mathcal{M}(3,m)^*$—The Steinberg Coverings

We note first of all that the fundamental groups of both $\mathcal{M}(1,m)^*$ and $\mathcal{M}(3,m)^*$ are the same as that of $\mathcal{M}(1,m)$ and $\mathcal{M}(3,m)$, respectively. However, the differences are in the definitions of the mapping $h : \pi_1(\mathcal{M}^*, \beta_0) \to H_1(\mathcal{M}^*; \mathbb{Z}) (\mathcal{M} = \mathcal{M}(1,m) \text{ or } \mathcal{M}(3,m))$ given on page 117 in that the roles of $s_1$ and $s_3$ need to be reversed. The actions of the automorphism group on $H_1(\mathcal{M}^*; \mathbb{Z})$ can, exactly as before, computed via conjugation on the fundamental group. In particular, the results will be exactly as in the previous subsection. Therefore, if $d$ is a positive integer, we obtain $\mathbb{Z}_d$-bases $\{\partial/\partial x^*, \partial/\partial \omega^*\}$ and $\{\partial/\partial \tau_1^*, \partial/\partial \tau_2^*\}$ of $H_1(\mathcal{M}(1,m)^*; \mathbb{Z}_d)$ and $H_1(\mathcal{M}(3,m)^*; \mathbb{Z}_d)$, respectively, via (cp. page 173)

$$\mathcal{M}(1,m)^* : \partial/\partial x^* = \epsilon_* h((s_2 s_1 s_2 s_1 s_2 s_3)^m), \partial/\partial \omega^* = \epsilon_* h((s_1 s_2 s_1 s_2 s_3 s_2)^m);$$

$$\mathcal{M}(3,m)^* : \partial/\partial \tau_1^* = \epsilon_* h((s_1 s_2 s_1 s_2 s_3 s_1 s_2 s_3)^m), \partial/\partial \tau_2^* = \epsilon_* h((s_1 s_3 s_2 s_1 s_2 s_3 s_2 s_1 s_2)^m).$$

We denote by $\{dx^*, d\omega^*\}$ and $\{d\tau_1^*, d\tau_2^*\}$ the corresponding dual $\mathbb{Z}_d$-bases of $H^1(\mathcal{M}(1,m)^*; \mathbb{Z}_d)$ and $H^1(\mathcal{M}(3,m)^*; \mathbb{Z}_d)$, respectively. We have the following analogues of Proposition 8.5.16 and Proposition 8.5.22 below:

**Proposition 8.5.29.** The action of $\text{Aut}(\mathcal{M})^*$ on $H^1(\mathcal{M}^*; \mathbb{Z}_d)$ is given below:

$\mathcal{M}(1,m)^* : \text{Aut}(\mathcal{M}(1,m)^*) = \langle a', b', c' \rangle$ and

$$(a')^*(dx^*) = dx^* + d\omega^*, \quad (a')(d\omega^*) = -d\omega^*,$$

$$(b')^*(dx^*) = d\omega^*,$$

$$(c')^*(dx^*) = -dx^* - d\omega^*, \quad (c')(d\omega^*) = d\omega^*.$$

$\mathcal{M}(3,m)^* : \text{Aut}(\mathcal{M}(3,m)^*) = \langle a', b', c' \rangle$ and

$$(a')^*(d\tau_1^*) = -d\tau_2^*,$$

$$(b')^*(d\tau_1^*) = d\tau_1^* - d\tau_2^*, \quad (b')(d\tau_2^*) = -d\tau_2^*,$$
(c')^*(d\tau_1^*) = d\tau_2^*.

Furthermore, if \( p : \mathcal{M}(3,m)^* \rightarrow \mathcal{M}(1,m)^* \) is the 3-fold covering, then the pull back \( p^* : H^1(\mathcal{M}(1,m)^*; \mathbb{Z}_d) \rightarrow H^1(\mathcal{M}(3,m)^*; \mathbb{Z}_d) \) is exactly as in Lemma 8.5.24:

**Lemma 8.5.30.** With respect to the 3-fold covering mapping \( p : \mathcal{M}(3,m)^* \rightarrow \mathcal{M}(1,m)^* \), the pull back \( p^* : H^1(\mathcal{M}(1,m)^*; \mathbb{Z}_d) \rightarrow H^1(\mathcal{M}(3,m)^*; \mathbb{Z}_d) \) is given by

\[
p^*(dx^*) = d\tau_1^* - 2d\tau_2^*, \quad p^*(d\omega^*) = d\tau_1^* + d\tau_2^*.
\]

Next, in determining the Steinberg coverings of the affine map \( \mathcal{M}^* = \mathcal{M}(1,m)^* \), and having cyclic group \( \mathbb{Z}_d \) of covering transformations, we recall that since \( \mathcal{M}^* \) has \( m^2 \) faces, then \( d|m^2 \).

As usual, let \( \beta_1, \beta_2, \ldots, \beta_{m^2} \) be blades in the same \( G^+ \)-orbit generating the \( m^2 \) faces of \( \mathcal{M}^* \), and set \( m_i = \chi_{[\beta_i F^+_m} + W^{(b,c)} \in C_2(\mathcal{M}), i = 1, 2, \ldots, m^2 \). Then, up to a unit multiple in \( \mathbb{Z}_d \), the \( \text{Aut}(\mathcal{M}^*) \)-module homomorphism \( \theta = \theta_{\text{St}} : C_2(\mathcal{M}^*) \rightarrow \mathbb{Z}_d \) satisfies \( \theta(m_i) = 1, i = 1, 2, \ldots, m^2 \).

In close analogy with the work on page 158, we begin by explicitly constructing an element \( z_{\text{St}} = z \in (\delta^1)^{-1}(\theta) \). Start with the blade \( \beta_0 = (1,0) \), and define the blade \( \beta_1 = b(1,0) = (\tau \omega, 0)(1,0) = (\tau \omega, 0) \). Define the voltages \( z_1, z_2 \) via the assignments

\[
\begin{align*}
z_1 &: \beta_1 \mapsto -1, \quad a\beta_1 \mapsto -1, \quad c\beta_1 \mapsto 1, \quad ac\beta_1 \mapsto 1, \\
z_2 &: \beta_0 \mapsto 1, \quad a\beta_0 \mapsto 1, \quad c\beta_0 \mapsto -1, \quad ac\beta_0 \mapsto -1,
\end{align*}
\]

with all remaining blades mapped to zero.

In terms of the translations \( \sigma_1, \sigma_\omega \) defined on page 176 above, we now define the voltage

\[
z_1' = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (k + 1)(\sigma_{\omega}^{-k})^*(\sigma_1^{-j})^*z_1.
\]

The voltage \( z_1' \), is depicted below for \( m = 3 \).
To the voltage $z_1'$ we add the voltage $z_2'$, defined by

$$z_2' = \sum_{k=0}^{m-1} m(k + 1)(\sigma_1^{-k})^* z_2,$$

and set $z_{St}' = z_1' + z_2'$, illustrated below for $m = 3$ (together with $\partial/\partial x^*, \partial/\partial \omega^* \in H_1(\mathcal{M}(1,m);\mathbb{Z}_d)$)
One checks that if \(d|m^2\), then \(\delta^1(z_{S_t}) = \theta\). As usual, even though \(\theta\) is a surjective \(\text{Aut}(\mathcal{M}^*)\)-module homomorphism \(C_2(\mathcal{M}^*)/H_2(\mathcal{M}^*) \to \mathbb{Z}_d\), one cannot conclude that \(z_{S_t}^* \in D(\mathcal{M}^*; \mathbb{Z}_d)\alpha\) for any homomorphism \(\alpha : \text{Aut}(\mathcal{M}^*) \to \text{Aut}(\mathbb{Z}_d)\). This is remedied through the action of the automorphisms \(a', b',\) and \(c'\) on \(D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d\langle \theta \rangle) = \mathbb{Z}_d\langle [z_{S_t}^*] \rangle \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d)\), given below:

**Proposition 8.5.31.** Assume that \(\mathcal{M} = \mathcal{M}(1, m)\), and that \(d\) is a positive integer dividing \(m^2\). Then the action of \(\text{Aut}(\mathcal{M}^*)\) on \(D_\theta = \mathbb{Z}_d\langle [z_{S_t}^*] \rangle \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d)\) is given by
\[(a')^*d\mathbf{x}^* = d\mathbf{x}^* + d\mathbf{\omega}^*, \quad (a')^*d\mathbf{\omega}^* = -d\mathbf{\omega}^*,\]

\[(b')^*d\mathbf{x}^* = d\mathbf{\omega}^*,\]

\[(c')^*d\mathbf{x}^* = -d\mathbf{x}^* - d\mathbf{\omega}^*, \quad (c')^*d\mathbf{\omega}^* = d\mathbf{\omega}^*,\]

\[(a')^*\mathbf{z}^*_\text{St} = -\mathbf{z}^*_\text{St} - m(m + 1)d\mathbf{x}^* - \frac{1}{2}m(m + 1)d\mathbf{\omega}^* = -\mathbf{z}^*_\text{St} - m d\mathbf{x}^* - \frac{1}{2}m(m + 1)d\mathbf{\omega}^* \quad \text{(since } m^2 = 0 \in \mathbb{Z}_d),\]

\[(b')^*\mathbf{z}^*_\text{St} = -\mathbf{z}^*_\text{St},\]

\[(c')^*\mathbf{z}^*_\text{St} = -\mathbf{z}^*_\text{St} + \frac{1}{2}m(m + 5)d\mathbf{\omega}^*.\]

In terms of matrices over \(\mathbb{Z}_d\), we have the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\) given by

\[a' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
-m & 1 & 0 \\
-\frac{1}{2}m(m + 1) & 1 & -1
\end{bmatrix}, \quad b' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},\]

\[c' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
\frac{1}{2}m(m + 5) & -1 & 1
\end{bmatrix}.\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map \(\mathcal{M}^* = \mathcal{M}(1, m)^*\) of type \((6, 3)\). Note that if \(\zeta \in \text{D}_\theta\) is \(\alpha\)-isotypical for \(\alpha = \alpha_{\text{St}} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)\), then we may assume that \(\zeta\) is of the form \(\zeta = \mathbf{z}^*_\text{St} + x d\mathbf{x}^* + y d\mathbf{\omega}^*\) for suitable \(x, y \in \mathbb{Z}_d\). We have the following.

**Theorem 8.5.32.** Let \(\mathcal{M}' \rightarrow \mathcal{M}^* = \mathcal{M}(1, m)^*\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{St}} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then \(d|m\); furthermore, up to \(\cong_{\mathcal{M}(1,m)^*}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = \mathbf{z}^*_\text{St} + \frac{1}{2}m(m + 1)d\mathbf{x}^* + \frac{1}{2}m(m + 1)d\mathbf{\omega}^* \in \text{D}(\mathcal{M}^*; \mathbb{Z}_d)\).
PROOF. Since \( \alpha_{\text{St}}(a') = \alpha_{\text{St}}(b') = \alpha_{\text{St}}(c') = -1 \), and if \( \zeta = z_{\text{St}}^* + xd^* + yd\omega^* \), then the condition
\[
-z_{\text{St}}^* - xd^* - yd\omega^* = -\zeta = \alpha_{\text{St}}(a')\zeta = (a')^*\zeta
\]
\[
= -z_{\text{St}}^* - (m - x)d^* - (\frac{1}{2}m(m + 1) - x + y)d\omega^*
\]
yields \( x = \frac{1}{2}m(m + 1) \). Next, the condition \( -\zeta = \alpha_{\text{St}}(b')\zeta = (b')^*\zeta \)
clearly yields \( x = -y \). From
\[
-z_{\text{St}}^* - xd^* - yd\omega^* = -\zeta = \alpha_{\text{St}}(c')\zeta = (c')^*\zeta
\]
\[
= -z_{\text{St}}^* - xd^* + (\frac{1}{2}m(m + 5) - x + y)d\omega^*,
\]
we obtain
\[
2x = -2y = \frac{1}{2}m(m + 5) - x = \frac{1}{2}m(m + 5) - \frac{1}{2}m(m + 1) = 2m.
\]
Therefore \( m(m + 1) = 2x = 2m \) and so it follows that \( m = 0 \in \mathbb{Z}_d \)
(since \( m^2 = 0 \)), i.e., that \( d | m \). Therefore, \( 2x = 2y = 0 \) and so we infer
that \( y = \frac{1}{2}m(m + 1) \).

Finally, we turn to the Steinberg coverings of \( \mathcal{M}(3, m)^* \).

We proceed as in the earlier subsections, starting with the Steinberg voltage \( z_{\text{St}}^1 = z_{\text{St}}^* \in D(\mathcal{M}(1, m^*; \mathbb{Z}_d), \) given on page 192. We now define
\[
z_{\text{St}}^2 = p^*(z_{\text{St}}^1) \in D(\mathcal{M}(3, m)^*; \mathbb{Z}_d).
\]
It is clear that \( \delta^1(z_{\text{St}}^2) = \theta_{\text{St}} : C_2(\mathcal{M}(3, m)^*)/H_2(\mathcal{M}(3, m)^*)) \rightarrow \mathbb{Z}_d \).
Therefore, if \( \theta = \theta_{\text{St}} \), then \( (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(z_{\text{St}}^2) \oplus H_1(\mathcal{M}(3, m)^*; \mathbb{Z}_d) \).
Using the results of Proposition 8.5.31, we have
\[
(a')^*(z_{\text{St}}^2) = (a')^*p^*(z_{\text{St}}^1)
\]
\[
= p^*((a')^*(z_{\text{St}}^1))
\]
\[
= p^*(-z_{\text{St}}^1 - md^* - \frac{1}{2}m(m + 1)d\omega^*)
\]
\[
= -z_{\text{St}}^2 - \frac{1}{2}m(m + 3)d\tau_1^* - \frac{1}{2}m(m - 3)d\tau_2^*
\]
Next,
\[
(b')^*(z_{\text{St}}^2) = (b')^*p^*(z_{\text{St}}^1)
\]
\[
= p^*((b')^*(z_{\text{St}}^1))
\]
\[
= p^*(-z_{\text{St}}^1)
\]
\[
= -z_{\text{St}}^2.
\]
\[(c')^*(z^*_{St}) = (c')^*p^*(z^*_{St}) = p^*((c')^*(z^*_{St})) = p^*(-z^*_{St} + \frac{1}{2}m(m+5)d\omega^*) = -z^*_{St} + \frac{1}{2}m(m+5)d\tau^*_1 + \frac{1}{2}m(m+5)d\tau^*_2.\]

Therefore, the action of \( Aut(\mathcal{M}(3,m)^*) \) on \( D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) \) is as given below:

**Proposition 8.5.33.** Assume that \( \mathcal{M} = \mathcal{M}(1,m) \), and that \( d \) is a positive integer dividing \( 3m^2 \). Then the action of \( Aut(\mathcal{M}^*) \) on \( D_\theta = \mathbb{Z}_d([z_{St}]) \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d) \) is given by

\[
(a'^*)d\tau^*_1 = -d\tau^*_2, \\
(b'^*)d\tau^*_1 = d\tau^*_1 - d\tau^*_2, \ (b'^*)d\tau^*_2 = -d\tau^*_2, \\
(c'^*)d\tau^*_1 = d\tau^*_2; \\
(a'^*)z^*_{St} = -z^*_{St} - \frac{1}{2}m(m + 3)d\tau^*_1 - \frac{1}{2}m(m - 3)d\tau^*_2; \\
(b'^*)z^*_{St} = -z^*_{St}, \\
(c'^*)z^*_{St} = -z^*_{St} + \frac{1}{2}m(m + 5)d\tau^*_1 + \frac{1}{2}m(m + 5)d\tau^*_2.
\]

In terms of matrices over \( \mathbb{Z}_d \), we have the representation of \( Aut(\mathcal{M}) = \langle a', b', c' \rangle \) given by

\[
a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ -\frac{1}{2}m(m+3) & 0 & -1 \\ -\frac{1}{2}m(m-3) & -1 & 0 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix},
\]

\[
c' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{2}m(m+5) & 0 & 1 \\ \frac{1}{2}m(m+5) & 1 & 0 \end{bmatrix}.
\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map \( \mathcal{M}^* = \mathcal{M}(3,m)^* \) of type \( (6,3) \).
Again, if $\zeta \in D_0$ is $\alpha$-isotypical for $\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \to \text{Aut}(\mathbb{Z}_d)$, then we may assume that $\zeta$ is of the form $\zeta = z^s_{St} + \tau_1d\tau_1^s + \tau_2d\tau_2^s$ for suitable $\tau_1, \tau_2 \in \mathbb{Z}_d$. We have the following; the proof in entirely routine.

**Theorem 8.5.34.** Let $\mathcal{M}' \to \mathcal{M}^* = \mathcal{M}(3, m)^*$ be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with $\mathbb{Z}_d$ and inducing the action $\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \to \text{Aut}(\mathbb{Z}_d)$, by a regular connected map. Then $d|m$; furthermore, up to $\cong_{\mathcal{M}(3, m)^*}$, $\mathcal{M}' = \mathcal{M}_z$, where $\zeta = z^s_{St} + \frac{1}{2}m(m + 3)d\tau_2^s \in D(\mathcal{M}^*; \mathbb{Z}_d)$.

**Remark:** If $d$ is odd then it follows easily that $\frac{1}{2}m(m + 3) = 0 \in \mathbb{Z}_d$. Therefore, the unique isotypical class in this case is $\zeta = z^s_{St} \in D(\mathcal{M}(3, m)^*; \mathbb{Z}_d)$. 
Chapter 9

Duality and the Coxeter-Petrie Complex

9.1 Duality of Maps

Let $\mathcal{M} = (B, a, b, c)$ be a connected map, and let $H_\Delta = \pi_1^\infty(\mathcal{M}, \beta) \leq \Delta = \Delta(\infty, \infty)$ be the orbifold fundamental group. Thus, we may identify $\mathcal{M}$ with the map $\mathcal{M}^\infty/H_\Delta = (\Delta/H_\Delta, s_1, s_2, s_3)$, where $\mathcal{M}^\infty$ is the regular map $\mathcal{M}^\infty = (\Delta, s_1, s_2, s_3)$. Furthermore, if $H'_\Delta \leq \Delta$ and is conjugate to $H_\Delta$, say $H'_\Delta = \tau^{-1}H_\Delta\tau$, then the mapping $\Delta/H_\Delta \rightarrow \Delta/H'_\Delta$ given by $xH_\Delta \mapsto x\tau H'_\Delta$ is a well-defined isomorphism of maps. Conversely, if $H'_\Delta \leq \Delta$ is a subgroup, and if there is a bijection $\Delta/H_\Delta \rightarrow \Delta/H'_\Delta$ realizing an isomorphism of maps (i.e., is a bijection that commutes with left multiplication by elements of $\Delta$), then $H'_\Delta$ and $H_\Delta$ are conjugate in $\Delta$. Therefore, one seeks to produce new maps from old by the process $(\Delta/H_\Delta, s_1, s_2, s_3) \mapsto (\Delta/H'_\Delta, s_1, s_2, s_3)$, where $\sigma : \Delta \rightarrow \Delta$ is an outer automorphism.

Descriptions of the above duality operations can be found in S. E. Wilson’s article\textsuperscript{1} and again in a paper by S. Lins.\textsuperscript{2} These results were described more algebraically by Jones and Thornton\textsuperscript{3} where by showing that the group of outer automorphisms of $\Delta$ is isomorphic with the symmetric group $S_3$, the above duality operations are the only operations

\textsuperscript{1}Operators over regular maps, Pacific J. Math. 81 (1979), 559–568.
\textsuperscript{2}Graph-encoded maps, JCT (B) 32 (1982), 171–181.
available. In fact, in Theorem 1 of this paper, they showed that any permutation of \(s_1, s_3, s_1s_3\) uniquely determines an outer automorphism of \(\Delta\) that fixes \(s_2\).

Thus, let \(\mathcal{M}\) be a connected map, represented as \(\mathcal{M} = (\Delta/H_\Delta, s_1, s_2, s_3)\), let \(\sigma \in \text{Out}(\Delta)\), and form the new map \(\mathcal{M}^\sigma = (\Delta/H_\Delta^\sigma, s_1, s_2, s_3)\). Note first that \(\mathcal{M}^\sigma \cong (\Delta/H_\Delta, s_1, s_2, s_3)\), where \(\sigma\gamma = \sigma\gamma\sigma^{-1}, \gamma \in \Delta\), via the mapping \(\gamma H_\Delta \mapsto \sigma\gamma H_\Delta\). The map \(\mathcal{M}^\sigma\) is called the \(\sigma\)-dual of \(\mathcal{M}\). If \(\mathcal{M}^\sigma = \mathcal{M}\) we say that \(\mathcal{M}\) is \(\sigma\) self-dual.

In particular, if \(\sigma\) is the transposition \(s_1 \leftrightarrow s_3\), then \(\mathcal{M}^\sigma\) is called the classical dual of \(\mathcal{M}\). If \(\sigma\) is the transposition \(s_1 \leftrightarrow s_1s_3\), then \(\mathcal{M}\) is called the Petrie dual of \(\mathcal{M}\).

Therefore, we see that the various dual forms of the given map \(\mathcal{M} = (B, a, b, c)\) can be exhibited as follows:

\[
\mathcal{M} = (B, a, b, c);
\mathcal{M}^* = (B, c, b, a) \text{ (Classical Dual)};
p(\mathcal{M}) = (B, ac, b, c) \text{ (Petrie Dual)};
p(\mathcal{M}^*) = (B, ac, b, a);
(p(\mathcal{M}))^* = (B, c, b, ac);
p((p(\mathcal{M}))^*) = (B, a, b, ac) = (p(\mathcal{M}^*))^*;
\]

**Exercise 1.** Let \(\mathcal{M} = (B, a, b, c)\) be a regular map and assume that the order of \(ab\) is odd. Prove that \(p(\mathcal{M})\) is not orientable.

**Exercise 2.** Let \(\mathcal{M} = (B, a, b, c)\) be an orientable regular map, and assume that \(G^+\) is a simple group. Prove that of the six dual forms of \(\mathcal{M}\), only \(\mathcal{M}\) itself and \(\mathcal{M}^*\) are orientable.

**Example.** The pictures below depict the map of the cube and its various duals. Note that the cube and its classical dual, the octahedron are, of course, orientable of genus 0. If \(\mathcal{M}\) denotes the map of the cube and \(p(\mathcal{M})\) its Petrie dual, then \(p(\mathcal{M})\) and its classical dual \(p(\mathcal{M})^*\), are orientable of genus 1. The Petrie dual \(p(\mathcal{M}^*)\) and its classical dual \((p(\mathcal{M}^*))^*\) are non-orientable of genus 4.
Note that the monodromy group of each of the dual forms of $M$ agrees with that of $M$. As a result, we see that $M$ is regular if and only if each of its duals is regular. As the above exercises show, the same result does not apply to orientability; see also the lemma below.
Note that the classical dual of $\mathcal{M} = (B, a, b, c)$, obtained by interchanging the involutory generators $a$ and $c$, is tantamount to reversing the roles of vertices and faces in $\mathcal{M}$. In this way, it can be seen that the tetrahedron can be seen as self-dual and that the octahedron and cube (hexahedron) are dual to each other. However, there is an additional species of objects in the map $\mathcal{M} = (B, a, b, c)$, the so-called Petrie polygons, which can be thought of as the orbits in $B$ of the subgroup $G_p = \langle b, ac \rangle$. Again, using as example the tetrahedron, we see that as the element $bac$ has order 4, then $[G : G_p] = 3$, i.e., in $\mathcal{M}$ there are three Petrie polygons. Thus, the Petrie dual of $\mathcal{M}$ has as its varieties the vertices, edges and Petrie polygons, which collectively tessellate a surface. A computation of the Euler characteristic of this surface results in $\chi = 1$, and so the surface cannot be orientable (instead, it’s homeomorphic with the real projective plane). It is evident, therefore, that duality in the present sense need not preserve orientability.

The following was observed in Nedela-Škoviera\textsuperscript{4} but not proved. It is a nice exercise.\textsuperscript{5}

\textbf{Proposition 9.1.1.} Let $\mathcal{M}$ be an orientable map. Then the Petrie dual $p(\mathcal{M})$ is orientable if and only if the underlying graph $\Gamma$ of $\mathcal{M}$ is bipartite.

Note that both $\mathcal{M}$ and $p(\mathcal{M})$ have the same vertices and edges, and hence have the same underlying graph.

\section*{9.2 Geometries, Chamber Systems, and Coxeter Groups}

The purpose of this section is to define two related concepts, that of an $M$-geometry and that of an $M$-chamber system. We shall show that for


\textsuperscript{5}This same result also appears in S. Wilson’s paper, Operators over regular maps, Pacific J. Math. \textbf{81}, no. 2 (1979), page 568.
a particular class of chamber systems, the so-called Coxeter complexes, these notions are essentially equivalent. However, for maps in general, they are not.

The notion of chamber system was introduced in Section 1.1, and serves to generalize the notion of a building (whose precise definition we don’t go into here), which proved to be the relevant object in classifying groups of Lie type.

Recall that the chamber system $C = (C, \sim, i \in I)$ is called quasithin if every $\sim_i$-equivalence class in $C$ has at most two chambers. Otherwise $C$ is called thick. From the point of geometry, most of the interesting chamber systems, including those arising from the groups of Lie type, are all thick. However, from the point of view of map theory, the quasithin complexes are the relevant objects of study, as it is clear that a map regarded as a chamber system is quasithin. In fact, we shall be exclusively concerned with thin chamber systems and morphisms (i.e., 1-morphisms) thereof.

A second object of relevance here is that of a geometry over $I$; we follow Francis Buekenhout’s treatment. Such an object is a quadruple $G = (V, \sim, I, t)$, where $V$ is a set (of varieties), $\sim$ is a symmetric relation on $V$, $I$ is a set, and $t : V \to I$ is a mapping. If $i \in I$, $v \in V$ and $t(v) = i$, we say that $v$ has type $i$. We assume that if $v, v' \in V$ are such that $v \sim v'$ and $t(v) = t(v')$, then $v = v'$. In this case the automorphism group of the geometry $G$ is the group of permutations of $V$ preserving incidence $\sim$ and type, that is, $\sigma : V \to V$ defines an automorphism of $G$ if and only for all $v \in V$ we have $t(v) = t(v\sigma)$ and whenever $v \sim v'$ in $V$ we have $v\sigma \sim v'\sigma$.

Note that in the obvious way, the varieties of a geometry form the vertices of a multipartite graph. Thus, we shall say that the geometry $G$ is connected if and only if the corresponding multipartite graph is connected.

A flag in the geometry $G$ is a set of pairwise incident varieties. If $F$ is a flag, then the type of $F$ is the subset $t(F) \subseteq I$, and the cotype

---


of \( F \) is \( I - t(F) \). If \( F \) is a flag, then the residue \( \text{Res}(F) \) of \( F \) is the set of all varieties incident with every variety of \( F \) and whose types are contained in \( I - t(F) \). Thus, if \( \sim_F, t_F \) denote the restrictions of \( \sim \) and \( t \) to \( \text{Res}(F) \), then \( (\text{Res}(F), \sim_F, I - t(F), t_F) \) is again a geometry.

As with chamber systems, we define the rank of the geometry \( G = (V, \sim, I, t) \) is the cardinality of \( I \). The corank of the flag \( F \) is the cardinality of \( I - t(F) \).

Therefore, we see that if \( \mathcal{M} = (B, a, b, c) \) is a map, then we may regard \( \mathcal{M} \) as a rank-3 geometry whose varieties are the vertices, edges and faces (as defined above) (orbits in \( G \) of the appropriate “parabolic subgroups” of \( G = \langle a, b, c \rangle \)), and whose incidences are given by non-trivial intersections of the orbits.

Note that there is a functor from the category of chamber systems over \( I \) to geometries over \( I \), as follows. First, let \( \mathcal{C} = (C, \sim_i, i \in I) \) be a chamber system over \( I \). If \( J \subseteq I \), we define a residue of type \( J \) to consist of the \( \sim_J \)-equivalence classes in \( C \), where \( \sim_J \) is the transitive closure of the equivalence relations \( \sim_j \), \( j \in J \). Such a residue is also said to have cotype \( I - J \). From the chamber system \( \mathcal{C} \) we define a geometry \( \mathcal{G}(\mathcal{C}) = (V, \sim, I, t) \) where \( V \) consists of the residues of cotype \( i \), \( i \in I \), and where \( R \sim R' \) if and only if \( R \cap R' \neq \emptyset \). Conversely, given a geometry \( \mathcal{G} = (V, \sim, I, t) \), we can form the associated chamber system \( \mathcal{C}(\mathcal{G}) = (C, \sim_i, i \in I) \), where \( C \) is the set of flags of cotype \( i \in I \), and where \( F \sim_i F' \) if and only if \( v \in F, v' \in F', t(v), t(v') = j \neq i \) implies that \( v = v' \).

For a particular class of chamber systems, viz., the so-called buildings, the chamber system viewpoint is equivalent with the geometry viewpoint inasmuch as for such a chamber system \( \mathcal{C} \) one has \( \mathcal{G}(\mathcal{C}) \cong \mathcal{C} \). We shall prove this for a relevant special class of thin buildings, namely the Coxeter complexes of Subsection 9.2.3. For maps in general, the chamber system viewpoint can differ considerably from the geometry viewpoint.
Example. Let \( W = \langle a, b, c \rangle \) be a rank 3 elementary abelian subgroup and define the regular map \( \mathcal{M} = (W, a, b, c) \). As a chamber system \( \mathcal{M} \) has eight chambers, acted regularly on by \( W \). As a geometry, there are two vertices, two edges, two faces and eight maximal flags, again acted on regularly by \( W \). This map can be depicted as a cellular decomposition of the 2-sphere, as indicated to the right:

However, if \( A = \langle abc \rangle \subseteq W \), and if we form the orbit map \( \mathcal{M}/A = (W/A, \bar{a}, \bar{b}, \bar{c}) \), where \( \bar{a}, \bar{b}, \bar{c} \in W/A \) are the images of \( a, b, c \), respectively, then \( \mathcal{M}/A \) has four chambers, acted on regularly by \( W/A \). This map can be viewed as decomposing the real projective plane. However, one easily verifies that as a geometry, \( \mathcal{M}/A \) has one vertex, one edge and one face, and hence only one maximal flag. As a result, \( G(C) \) is trivial, even though \( \text{Aut}(\mathcal{C}) \) is a Klein four-group.

### 9.2.1 Generalized Polygons

In order to assign diagrams to geometries and chambers systems (in much that same way that geometries, buildings and groups of Lie type correspond to Dynkin or Coxeter diagrams), we need the concept of a generalized polygon, which is a special type of rank 2 chamber system (or polygon).

First, we do this for rank 2 chamber systems.

**Definition 9.2.1.** Let \( \mathcal{C} = (C, \sim_i, \ i \in I = \{1, 2\}) \) be a connected rank 2 chamber system. We call \( \mathcal{C} \) a generalized \( m \)-gon, \( m \in \mathbb{N} \cup \{\infty\} \), if the following conditions hold

\[(CS_0) \text{ If } c, c' \in C, \text{ then } c \sim_i c' \text{ for at most one } i \in I. \]
(CS\textsubscript{1}) For any $c \in C$ and $i \in I$, there exists $c' \in C$ with $c \sim_i c'$.

(CS\textsubscript{2}) If $c = c_0, c_1, \ldots, c_k \in C$ are distinct, and if there exists a “closed path”

$$c = c_0 \sim_{i_1} c_1 \sim_{i_2} c_2 \sim \cdots \sim_{i_k} c_k \sim_{i_0} c_0,$$

then $k + 1 \geq 2m$.

(CS\textsubscript{3}) Assume that $c, c' \in C$ and that there exists a path from $c$ to $c'$ of the form

$$c = c_0 \sim_{i_1} c_1 \sim_{i_2} c_2 \sim_{i_3} \cdots \sim_{i_m} c_m = c',$$

then there exists a path from $c$ to $c'$ of the form

$$c = d_0 \sim_{i_2} d_1 \sim_{i_1} d_2 \sim_{i_2} \cdots \sim_{i_m} d_m = c'.$$

For rank 2 geometries, the relevant definition is as below. Recall that the varieties of a rank 2 geometry form the vertices of a bipartite graph.

**Definition 9.2.2.** Let $G = (V, \sim, I = \{1, 2\}, t)$ be a connected rank 2 geometry. We call $G$ a *generalized $m$-gon*, $m \in \mathbb{N} \cup \{\infty\}$ if

(G\textsubscript{1}) As a bipartite graph, $G$ has diameter $m$.

(G\textsubscript{2}) As a bipartite graph, $G$ has girth $2m$.

(G\textsubscript{3}) If $m$ is infinite, then every variety is incident with at least two other varieties.

### 9.2.2 Diagrams, $M$-Chamber Systems and $M$-Geometries

The vast majority of chamber systems and geometries, including those arising from maps, can be associated with diagrams, in the same way that the groups (and related structures) of Lie type correspond to Dynkin (or Coxeter) diagrams. Thus, we recall that a (finite rank)
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diagram $\Delta = \Delta(M)$ of type $M = [m_{ij}]$ has nodes corresponding to elements of $I$ (a set of cardinality $k$, and edges labelled $m_{ij} \in \mathbb{Z} \cup \{\infty\}$. We adopt the common convention that $m_{ii} = 1$, $i \in I$, and that if no edge is drawn between $i$ and $j$, then it is understood that $m_{ij} = 2$. Therefore, the diagram $\Delta$ below

![Diagram](attachment:diagram.png)

represents a diagram with $I = \{1, 2, 3\}$ and and $m_{12} = 3$, $m_{23} = 6$ and all other $m_{ij} = 2, i \neq j$.

The diagram below

![Diagram](attachment:diagram.png)

similarly represents a diagram with $I = \{1, 2, 3, 4\}$ and and $m_{12} = 3$, $m_{23} = 6$, $m_{24} = \infty$ and all other $m_{ij} = 2, i \neq j$.

We are finally in a position to define $M$-chamber systems and $M$-geometries. Thus, let $M = [m_{ij}]$, $i, j \in I$ and let $C$ be a chamber system over $I$. We say that $C$ is an $M$-chamber system (or a chamber system of type $M$), if for any $\{i, j\} \subseteq I$, $i \neq j$ and any residue $R$ of cotype $\{i, j\}$, then as a chamber system over $\{i, j\}$, $R$ is a generalized $m_{ij}$-gon. In exactly the analogous fashion, we define what it means for the geometry $G$ to be an $M$-geometry.

We note that the underlying maps of the regular Platonic solids, namely the tetrahedron, cube (hexahedron), octahedron, dodecahedron and the icosahedron can be regarded as $M$-geometries belonging to the diagrams below, if the nodes are marked with the following variety types:

![Tetrahedron](attachment:tetrahedron.png)

Tetrahedron: vertices edges faces
Cube:

\[
\begin{array}{ccc}
\circ & 4 & \circ \\
\circ & \text{vertices} & \circ \\
\circ & \text{edges} & \circ \\
\circ & \text{faces} & \circ \\
\end{array}
\]

Octahedron:

\[
\begin{array}{ccc}
\circ & 3 & \circ \\
\circ & \text{vertices} & \circ \\
\circ & \text{edges} & \circ \\
\circ & \text{faces} & \circ \\
\end{array}
\]

Dodecahedron:

\[
\begin{array}{ccc}
\circ & 5 & \circ \\
\circ & \text{vertices} & \circ \\
\circ & \text{edges} & \circ \\
\circ & \text{faces} & \circ \\
\end{array}
\]

Icosahedron:

\[
\begin{array}{ccc}
\circ & 3 & \circ \\
\circ & \text{vertices} & \circ \\
\circ & \text{edges} & \circ \\
\circ & \text{faces} & \circ \\
\end{array}
\]

As a somewhat more interesting example, the genus three map underlying the so-called “Klein’s quartic” belongs to the diagram

\[
\begin{array}{ccc}
\circ & 3 & \circ \\
\circ & \text{vertices} & \circ \\
\circ & \text{edges} & \circ \\
\circ & \text{faces} & \circ \\
\end{array}
\]

9.2.3 Coxeter Groups and the Coxeter Complex

The Coxeter group of type \( M = [m_{ij}] \) is the presented group:

\[ W = \langle s_i \mid (s_is_j)^{m_{ij}} = 1, \ i, j \in I \rangle. \]

The cardinality of \( I \) is called the rank of \( W \).

If \( J \subseteq I \), set \( W_J = \langle s_j \mid j \in J \rangle \). The groups \( W_J, \ J \subseteq I \) are called parabolic subgroups of \( W \).

We summarize some important facts about Coxeter groups, all of which can be found in Mark Ronan’s book, Chapter 2.\(^8\)

Theorem 9.2.1. Let $W$ be a Coxeter group of type $M$.

1. If $J \subseteq I$, then the parabolic subgroup $W_J$ is a Coxeter group of type $M_J = [m_{ij}], i, j \in J$.

2. If $J, K \subseteq I$, then $W_J \cap W_K = W_{J \cap K}$.

3. If $J_1, J_2, \ldots, J_r \subseteq I$, and if $W_{J_1} w_1, W_{J_2} w_2, \ldots, W_{J_r} w_r$ are pairwise nondisjoint, then

$$W_{J_1} w_1 \cap W_{J_2} w_2 \cap \cdots \cap W_{J_r} w_r = W_{J_1 \cap \cdots \cap J_r} w,$$

for some $w \in W$.

The third assertion above probably needs some elaboration. First of all, as elements of the Coxeter group $W$ are products of the generating involutions $s_i, i \in I$, then we have the notion of the length of an element of $W$. In turn, this defines a distance function on $W$, with the distance between elements $w, w' \in W$ being the length of $w^{-1} w'$. At the same time, we may view $W$ as a graph, with elements $w, w'$ being incident precisely when $w' = s_i w$, for some $i \in I$. Thus, $W$ is endowed with the graph distance, which agrees with the above.\(^9\) In this setting, perhaps the most important property of the Coxeter group $W$ is its so-called strong gatedness. In the present notation, this says that for any $w \in W$ and any coset $W_J w'$, $J \subseteq I$, then\(^10\)

(i) there is a unique element $v' \in W_J w'$, $J \subseteq I$ of minimal distance from $w$,

(ii) for any $u' \in W_J w'$, $d(w, u') = d(w, v') + d(v', u')$, and

(iii) any minimal length path in $W$ from $v'$ to $u'$ is wholly contained inside $W_J w'$.

Using the above, we can give a proof of the third part of the above theorem. Note that using induction we clearly may assume that $r = 3$. Thus, let $w \in W_{J_1} w_1 \cap W_{J_2} w_2$ and let $w' \in W_{J_1} w_1 \cap W_{J_3} w_3$. By (i) above there is a unique element $v' \in W_{J_3} w_3$ of minimal distance from $w$, and

\(^9\)This follows from Ronan, op. cit., Theorem 2.11.

\(^10\)Ronan, op. cit., Theorems 2.9 and 2.10.
such that \( d(w, w') = d(w, v') + d(v', w') \). Also, by (iii) above, since \( w, w' \in W_{j_1}w_1 \), it follows that \( v' \in W_{j_1}w_1 \), and so \( v' \in W_{j_1}w_1 \cap W_{j_3}w_3 \). By assumption, there exists \( w'' \in W_{j_2}w_2 \cap W_{j_3}w_3 \), and so \( d(w, w'') = d(w, v') + d(v', w'') \); since \( w, w'' \in W_{j_2}w_2 \) we must have \( v' \in W_{j_2}w_2 \), as well. Therefore, \( v' \in W_{j_1}w_1 \cap W_{j_2}w_2 \cap W_{j_3}w_3 \), and so

\[
W_{j_1}w_1 \cap W_{j_2}w_2 \cap W_{j_3}w_3 = W_{j_1}v' \cap W_{j_2}v' \cap W_{j_3}v' = (W_{j_1} \cap W_{j_2} \cap W_{j_3})v' = W_{j_1 \cap j_2 \cap j_3}v'.
\]

**Definition 9.2.3.** The Coxeter complex corresponding to the Coxeter group \( W = \langle s_i \mid (s_is_j)^{m_{ij}} = 1, \ i, j \in I \rangle \) is the thin chamber complex \( \mathcal{C}(W) = (W, \sim_i, \ i \in I) \), where \( w \sim_i w' \) if and only if \( w' = s_iw \). Note that \( W \cong \text{Aut}(\mathcal{C}(W)) \), via right multiplication. If \( M = [m_{ij}] \), it is clear that \( \mathcal{C}(W) \) is an \( M \)-chamber system, as follows from the fact that the rank 2 parabolic subgroups of \( W \) are dihedral groups.

It is immediate that the Coxeter complex of a rank 3 Coxeter group is a map in the sense of Section 1.2. More generally, any map \( \mathcal{M} = (B, a, b, c) \) is the orbit map of \( \mathcal{C}(W) \), where

\[
W = \langle s_i \mid (s_is_j)^{m_{ij}} = 1, \ i, j \in I \rangle,
\]
and where \( m_{12} = o(ab), m_{23} = o(bc), m_{13} = 2 \). Indeed, let \( G = \langle a, b, c \rangle \) be the monodromy group of \( \mathcal{M} \) and let \( \pi : W \to G \) be the natural projection defined by \( s_1 \mapsto a, s_2 \mapsto b, s_3 \mapsto c \). If \( H \leq G \) is the stabilizer of some blade \( b \in B \), and if \( M = \pi^{-1}(H) \leq W \), then we may identify \( B \) with the left cosets \( W/M \) of \( M \) in \( W \), and the actions of \( a, b, c \) are seen to correspond to the actions of \( s_1, s_2, s_3 \) on \( W/M \). Thus, we see that \( \mathcal{M} \) is exhibited as the \( M \)-orbit map of \( \mathcal{C}(W) \).

**Definition 9.2.4.** The Coxeter geometry corresponding to the Coxeter group \( W = \langle s_i \mid (s_is_j)^{m_{ij}} = 1, \ i, j \in I \rangle \) is the geometry \( \mathcal{G}(W) = (V, \sim_i, I, t) \), where \( V \) consists of all cosets \( W(i)w, \ w \in W, \ i \in I \), where \( W(i) = W_{i-\{i\}} \), where \( W(i)w_i \sim W(j)w_j \) if and only if \( W(i)w_i \cap W(j)w_j \neq \emptyset \), and where \( t : V \to I \) is the map \( t(W(i)w) = i \in I \). Again, note that
W acts by right multiplication as a group of automorphisms of \( G(W) \). As above, since the rank 2 parabolic subgroups of \( W \) are dihedral, it follows easily that \( G(W) \) is an \( M \)-geometry, where \( M = [m_{ij}] \).

**Theorem 9.2.2.** Let \( W = \langle s_i \mid i \in I \rangle \) be a finite-rank Coxeter group. Then \( CG(C(W)) \cong C(W) \), and \( CGG(W) \cong G(W) \).

**Proof.** If \( F \) is the set of maximal flags in \( G(W) \), then the above really boils down to showing that the mapping \( W \to F, w \mapsto \{ W_{(i)}w \mid i \in I \} \) is bijective. It is injective since by **Theorem 9.2.1, part 2**, if \( w, w' \) map to the same maximal flag, then \( w^{-1}w' \in \bigcap_{i \in I} W_{(i)} = \{ 1 \} \). This mapping is surjective by **Theorem 9.2.1, part 3**.

As an immediate corollary, we see that \( W \) (acting by right multiplication) is the full automorphism group of the geometry \( G(W) \).

### 9.3 The Coxeter-Petrie Complex

In this section we shall define the **Coxeter-Petrie complex** of a map \( M \), which as a thin chamber system shall be an orbit complex of a rank 4 Coxeter complex. Furthermore, this complex shall have the property that the rank 3 residues are chamber systems defining the various duals of \( M \), including the so-called Petrie dual of \( M \).

Thus, let \( M = (B, a, b, c) \) be a map; thus the underlying varieties of \( M \) are the vertices, edges and faces. In addition, there are Petrie polygons, which are “zig-zag” paths of edges, such that any two, but not three consecutive edges bound a common face. Equivalently, such a Petrie polygon is obtained from a fixed initial edge \( e \) by applying the “glide reflection” \( bac \) to \( e \). The Petrie dual \( p(M) \) of \( M \) is the map whose underlying varieties are vertices, edges and Petrie polygons of \( M \). The automorphism group of the Petrie dual \( p(M) \) is isomorphic with the automorphism group of \( M \), as will be made clear below. In addition
to the Petrie dual, there is the classical dual of $\mathcal{M}$, which is simply obtained by interchanging the vertices and faces of $\mathcal{M}$. Thus, one intuitively sees that the map of the tetrahedron is self-dual whereas the cube/octahedron and dodecahedron/icosahedron pairs are in duality relative to classical duality.

The formal definitions of these notions of dual have already been given in Section 9.1. Thus, starting with a map $\mathcal{M} = (B, a, b, c)$, with vertices, edges and faces identified as in Section 1.2, we see that the classical dual of $\mathcal{M}$ is given by changing the "presentation" of $\mathcal{M}$, that is by setting $\mathcal{M}^* = (B, c, b, a)$. In so doing, we see that the roles of the vertices and faces have been reversed, while retaining the automorphism group. Likewise we represent the Petrie dual via the presentation $\mathfrak{p}(\mathcal{M}) = (B, ac, b, c)$. Thus, we see that the vertices and edges remain unchanged (inasmuch as the relevant rank 2 residues do), but that there is a new variety introduced through the orbits of the dihedral subgroup $G_p = \langle ac, b \rangle$ of the monodromy group $G = \langle a, b, c \rangle$. Since the monodromy groups of $\mathcal{M}$ and $\mathfrak{p}(\mathcal{M})$ are clearly the same, so are their automorphism groups.

The various duals of the map $\mathcal{M}$ were shown\textsuperscript{11} to correspond to the outer automorphisms of

$$\Gamma = \Gamma(\infty, \infty) = \langle s_1, s_2, s_3 | s_1^2 = s_2^2 = s_3^2 = (s_1s_3)^2 = 1 \rangle.$$  

Furthermore, it was shown\textsuperscript{12} that $\text{Out}(\Gamma) \cong S_3$, and that these outer automorphisms correspond precisely to the six permutations of the involutions $s_1, s_3$ and $s_1s_3$ of $\Gamma$. Thus the classical dual corresponds to the outer automorphism $s_1 \leftrightarrow s_3$ of $\Gamma$ and the Petrie dual corresponds to the automorphism $s_1 \leftrightarrow s_1s_3$. As a result of their parametrization of the map duals, Jones and Thornton were able to construct, for any map $\mathcal{M}$ a reflexible covering map $\hat{\mathcal{M}}$ that is self-dual with respect to each of the notions of duality.

The present discussion is inspired by Jones and Thornton’s work but differs in that a construction is given of a rank 4 chamber system that contains the various duals of $\mathcal{M}$ as rank 3 residues. To this end, assume that $\mathcal{M} = (B, a, b, c)$ is a map with monodromy group $G = \langle a, b, c \rangle$.\textsuperscript{11,12}


\textsuperscript{12}op. cit.
We shall assume that $\mathcal{M}$ is connected, i.e., $G$ acts transitively on the set of blades $B$, but not necessarily that $\mathcal{M}$ is regular. Therefore, the stabilizer $H$ of a given blade $b \in B$ need not be trivial. If we set $k = o(ab)$, $l = o(bc)$, the orders of the elements $ab, bc \in G$, then $G$ is a homomorphic image of the Coxeter group $W = \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^k = (s_2s_3)^l = (s_1s_3)^2 = 1 \rangle$, which corresponds to the Coxeter diagram

$$\begin{array}{ccc}
\circ & k & \circ \\
\circ & l & \circ \\
\circ & m & \circ \\
\circ & s_1 & \circ \\
\circ & s_2 & \circ \\
\circ & s_3 & \circ \\
\circ & s_4 & \circ \\
\end{array}$$

vertices edges faces

Thus, if $\theta : W \to G$ is the obvious projection, and if $M = \theta^{-1}(H) \leq W$, then $\mathcal{M} \cong \mathcal{M}(W)/M$, thereby representing $\mathcal{M}$ as an orbit map of the Coxeter complex $\mathcal{M}(W)$. Next, set $m = o(bac)$, the order of the “glide reflection” $bac \in G$. From this, we construct a rank 4 Coxeter group with presentation $W = \langle s_1, s_2, s_3, s_4 \mid C \rangle$, where $C$ is the set of Coxeter relations deduced from the diagram:

$$\begin{array}{ccc}
\circ & k & \circ \\
\circ & l & \circ \\
\circ & m & \circ \\
\circ & s_1 & \circ \\
\circ & s_2 & \circ \\
\circ & s_3 & \circ \\
\circ & s_4 & \circ \\
\end{array}$$

Inside $W$ are the rank 3 parabolic subgroups $W_1, W_3, W_4$, where $W_i = \langle s_j \mid j \neq i \rangle$, $i = 1, 3, 4$; furthermore, these groups have relations given by the Coxeter diagram of $W$. In particular, we have $W \cong W_4$. We have surjective mappings $W_i \to G$, given as follows:

$W_1 \to G$, $s_2 \mapsto b, s_3 \mapsto c, s_4 \mapsto ac$,

$W_3 \to G$, $s_1 \mapsto a, s_2 \mapsto b, s_4 \mapsto ac$.

and

$W_4 \to G$, $s_1 \mapsto a, s_2 \mapsto b, s_3 \mapsto c$.

\footnote{This follows from Ronan, op. cit., Corollary 2.14.}
We set $K_i = \ker(W_i \to G)$, with respective generating sets $R_i$, $i = 1, 3, 4$. Set $R = R_1 \cup R_3 \cup R_4$ and define the Coxeter-Petrie group by the presentation

$$G = G(M) = \langle s_1, s_2, s_3, s_4 \mid C \cup R \rangle,$$

where $C$ is the set of Coxeter relations. Correspondingly, we define the Coxeter-Petrie complex $c(M)$ to be the corresponding rank 4 (thin) chamber system. Note that because of the possible additional relations, a Coxeter-Petrie chamber system need not be a Coxeter complex.

For convenience, we record the following result.

**Lemma 9.3.1.** Let $M$ be a map with monodromy group $G = \langle a, b, c \rangle$, where $o(ab) = k$, $o(bc) = l$, and let $m = o(ac)$. If $G(M) = \langle s_1, s_2, s_3, s_4 \mid C \cup R \rangle$, where $R = R_1 \cup R_3 \cup R_4$, then $R_1$ contains the relation $(s_2s_3s_4)^k = 1$, and $R_3$ contains the relation $(s_2s_1s_4)^l = 1$.

**Proof.** This is a direct and easy calculation.

### 9.3.1 The Coxeter-Petrie Complex of the Platonic Solids.

In this subsection, we shall consider in more detail the structure of the Coxeter-Petrie group of a Platonic Solid. Thus, the map in question is $M = (B, a, b, c)$, where $a, b, c$ satisfying the defining relations given by the Coxeter diagram:

![Coxeter diagram](image)

and where $\{k, l\} = \{3, 3\}, \{3, 4\}, \{3, 5\}$, i.e., $M$ is the tetrahedron, octahedron (cube), and icosahedron (dodecahedron), respectively. In turn, the Coxeter-Petrie group has generators $s_1, s_2, s_3, s_4$ satisfying the Coxeter relations depicted in the diagram
together with the additional relations $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ described in the subsection above.

**Theorem 9.3.2.** Let $\mathcal{M}$ be a Platonic solid with monodromy group $G = \langle a, b, c \rangle$, and let $m$ be the length of its Petrie polygons. Thus $m = o(bac)$ and the corresponding Coxeter-Petrie group has the presentation

$$G = \langle s_1, s_2, s_3, s_4 | C \cup R \rangle,$$

where $R$ consists precisely of the two relations $(s_2 s_1 s_4)^l = (s_2 s_3 s_4)^k = 1$.

**Proof.** We must determine the kernels of the surjections of the parabolic subgroups $W_i \to \langle a, b, c \rangle$, given on page 213. Since $\langle s_1, s_2, s_3 \rangle \to \langle a, b, c \rangle$ is an isomorphism, we see that $\mathcal{R}_4 = \emptyset$. Furthermore, from Lemma 9.3.1, we have that $\mathcal{R}_1$ contains the relation $(s_2 s_3 s_4)^k = 1$ and $\mathcal{R}_3$ contains the relation $(s_2 s_1 s_4)^l = 1$. We shall show that there are no others. Let $N_i$ denote the normal closure in $W_1$ of the relation $(s_2 s_3 s_4)^k = 1$; similarly let $N_3$ denote the normal closure in $W_3$ of the relation $(s_2 s_3 s_4)^l = 1$. Thus we have surjective mappings $W_i / N_i \to G (\cong W \cong W_4)$, $i = 1, 3$. However, the mappings $W_4 \to W_i / N_i$, $i = 1, 3$ given by

\[
s_1 \mapsto s_3 s_4, \quad s_2 \mapsto s_2, \quad s_3 \mapsto s_3,
\]

\[
s_1 \mapsto s_1, \quad s_2 \mapsto s_2, \quad s_3 \mapsto s_1 s_4,
\]

are easily checked to determine well-defined homomorphisms, and are inverse to the homomorphisms $W_i / N_i \to W_4$. The result follows.

Next, we calculate upper bounds on the orders of the Coxeter Petrie groups corresponding to the Platonic Solids.
Theorem 9.3.3. Let $\mathcal{M}$ be a Platonic solid, and let $G = G(\mathcal{M})$ be its Coxeter-Petrie group. Then

1. $|G| \leq 96$ if $\mathcal{M}$ is the tetrahedron;

2. $|G| \leq 384$ if $\mathcal{M}$ is the octahedron;

3. $|G| \leq 3840$ if $\mathcal{M}$ is the icosahedron.

Proof. We consider the above results in turn.

Tetrahedron. Proof (1). In this case $G = \langle s_1, s_2, s_3, s_4 \rangle$ with relations as described above. Define the elements $x, y \in G$ by setting $x = s_1s_3s_4, y = s_2s_4s_2$; note that $s_1, s_3, s_4$ commute, $x^2 = y^2 = 1$ in $G$. Define the subgroup $N$ of $G$ by setting $N = \langle x, y \rangle$. We state and prove a number of claims; bear in mind that if $i, j, k$ is any permutation of 1, 2, 4 or of 2, 3, 4, then we have that $(s_is_js_k)^3 = 1$.

(1) $N$ is elementary abelian. Indeed, note that

$$
xy = s_1s_3s_4s_2(s_1s_3s_4)s_2 \\
= s_1s_3s_4s_2(s_3s_4s_1)s_2 \quad (\text{since } s_1, s_3, s_4 \text{ commute}) \\
= s_1(s_3s_4s_2s_3s_4)s_1s_2 \\
= s_1(s_2s_4s_3s_2)s_1s_2 \quad (\text{since } (s_3s_4)^3 = 1) \\
= s_1s_2s_3s_1(s_2s_1s_2s_1) \\
= s_1s_2s_3s_1(s_2s_1s_2s_1) \quad (\text{since } (s_2s_1)^3 = 1) \\
= s_1s_2(s_4s_3s_1)s_2s_1 \\
= s_1s_2(s_1s_3s_4)s_2s_1 \quad (\text{since } s_1, s_3, s_4 \text{ commute}) \\
= (s_1s_2s_1)s_3s_4s_2s_1 \\
= (s_2s_1s_2)s_3s_4s_2s_1 \quad (\text{since } (s_1s_2)^3 = 1) \\
= s_2s_1(s_2s_3s_4s_2)s_1 \\
= s_2s_1(s_4s_3s_2s_4s_3)s_1 \quad (\text{since } (s_2s_3s_4)^3 = 1) \\
= s_2s_1(s_4s_3s_2s_4s_3)s_1 \\
= s_2s_1(s_3s_4)s_2(s_4s_3s_1) \quad (\text{since } s_3, s_4 \text{ commute}) \\
= s_2s_1s_3s_4s_2(s_1s_3s_4) \quad (\text{since } s_1, s_3, s_4 \text{ commute}) \\
= yx,
$$

proving the result.
(2) \( N \trianglelefteq G \). This is carried out in steps.

(a) \( s_1xs_1 = x, s_1ys_1 = xy \). The first statement is, of course, obvious. As for the second,

\[
\begin{align*}
  s_1ys_1 &= (s_1s_2s_1)s_3s_4s_2s_1 \\
          &= (s_2s_1s_2)s_3s_4s_2s_1 \\
          &= s_2s_1(s_2s_3s_4s_2)s_1 \\
          &= s_2s_1(s_4s_3s_2s_4s_2)s_1 \\
          &= s_2s_1(s_3s_4)s_2(s_4s_3s_1) \\
          &= s_2s_1s_3s_4s_2(s_1s_3s_4) \\
          &= yx = xy.
\end{align*}
\]

(b) \( s_2xs_2 = y, s_2ys_2 = x \). This follows from the definitions of \( x, y \).

(c) \( s_3xs_3 = x, s_3ys_3 = xy \). The first statement is obvious. For the second,

\[
\begin{align*}
  s_3ys_3 &= s_3s_2s_1(s_3s_4)s_2s_3 \\
          &= s_3s_2s_1(s_4s_3)s_2s_3 \\
          &= s_3s_2s_1s_3s_4(s_3s_2s_3) \\
          &= s_3s_2s_1s_4(s_2s_3s_2) \\
          &= s_3(s_2s_1s_4s_2)s_3s_2 \\
          &= s_3(s_4s_1s_2s_4s_1)s_3s_2 \\
          &= (s_3s_4s_1)s_2(s_4s_1s_3)s_2 \\
          &= (s_1s_3s_4)s_2(s_1s_3s_4)s_2 \\
          &= xy.
\end{align*}
\]

(d) As a result of the above, we see that the elements \( s_1, s_2, s_3 \) normalize \( N \). However, since \( s_1s_3s_4 = x \in N \), we conclude that \( s_4 \) also normalizes \( N \), and so \( N \trianglelefteq G \).

Furthermore, since \( s_4 \equiv s_1s_3 \) (modulo \( N \)), we see that \( G/N \) is a homomorphic image of the Coxeter group of order 24 corresponding to the diagram

```
  3 -- 3
```

Therefore, we conclude that $|G : N| \leq 24$; since $N$ is elementary abelian and generated by $x, y$, we have that $|N| \leq 4$. From this it follows that $|G| \leq 24 \cdot 4 = 96$.

**Tetrahedron, Proof (2).** Here, we consider the parabolic subgroup $H = W_4 = \langle s_1, s_2, s_3 \rangle \leq G$, and consider possible cosets $wH, w \in G$. We shall denote by $l(w)$, the length of $w$, i.e., the minimal number of factors when $w$ is written as a product of the generators $s_1, s_2, s_3, s_4$. Thus, if $l(w) = 1$, then $wH \neq H$ precisely when $w = s_4$. Equally clear is the fact that if $l(w) = 2$, then $wH \neq H$, $s_4H$ only when $w = s_2s_4$.

**Claim:** $s_1s_2s_4H = s_3s_2s_4H = s_4s_2s_4H$. Indeed, since $(s_1s_2s_4)^3 = 1$, we have $s_1s_2s_4H = s_4s_2s_1s_4s_2s_1H = s_4s_2s_1s_4H = s_4s_2s_4H$.

Similarly, $s_3s_2s_4H = s_4s_2s_4H$.

It is clear that multiplying any of the cosets $H, s_4H, s_2s_4H, s_4s_2s_4H$ by any of the generators $s_1, s_2, s_3, s_4$ will again yield one of these cosets. It follows, therefore, that $|G : H| \leq 4$; since $|H| \leq 24$, the result follows.

**Octahedron.** Here, the Coxeter-Petrie group $G = \langle s_1, s_2, s_3, s_4 \rangle$ has Coxeter relations taken from the diagram

![Diagram](image)

together with the Petrie relations $(s_2s_3s_4)^3 = (s_2s_1s_4)^4 = 1$.

**Lemma 9.3.4.** The element $(s_2s_4)^3$ is in the center of $G$.\(^{14}\)

\(^{14}\)In fact, it’s also true that $(s_1s_2s_3)^3$ is in the center of $G$ but we don’t need this fact.
PROOF. We set $G_i, i = 1, 3, 4$ to be the obviously defined rank-3 parabolic subgroups in $G$; as already noted above, these are isomorphic with the corresponding parabolic subgroups of the Coxeter group $W$: $G_i \cong W_i, i = 1, 3, 4$. We take as known the result that $(s_2 s_1 s_3)^3$ is the involution that generates the center of $G_4 \cong W_4 \cong W(B_3) \cong S_4 \times Z_2$, where $W(B_3)$ is the Coxeter group of type $B_3$. Next we have an isomorphism $G_4 \rightarrow G_3$ given by

$$s_1 \mapsto s_1, \ s_2 \mapsto s_2, \ s_3 \mapsto s_1s_4,$$

under which $(s_2 s_1 s_3)^3 \mapsto (s_2 s_4)^3$. Therefore, it follows that $(s_2 s_4)^3$ is central in $G_3$. Similarly, we have an isomorphism $G_4 \rightarrow G_1$, given by

$$s_1 \mapsto s_3s_4, \ s_2 \mapsto s_2, \ s_3 \mapsto s_3;$$

under this isomorphism $(s_2 s_1 s_3)^3 \mapsto (s_2 s_4)^3$. Therefore, $(s_2 s_4)^3$ is also central in $G_1$, which implies that $(s_2 s_4)^3$ is central in $G$.

As in the case of the tetrahedron, we present two proofs of the upper bound on $G$.

Octahedron, Proof (1). By analogy with the previous case, we define $x, y, z \in G$ by setting $x = s_1s_3s_4, y = s_2xs_2$, and $z = s_4ys_4$. Obviously $x^2 = y^2 = z^2 = 1$; and define the subgroup $N$ of $G$ by setting $N = \langle x, y, z \rangle$. We shall show that $N$ is a normal elementary abelian subgroup of $G$; since $G = G_4N$ and $|G_4| = 48$, this is enough.

(1) $N$ is elementary abelian. First, note that in proving that $xy = yx$ in the tetrahedral case, the only identities that were used were $s_1s_2s_1 = s_2s_1s_2$ and $(s_2s_4s_3)^3 = 1$, which continue to be valid in both the octahedral and icosahedral cases. Next, that $x$ commutes with $z$ is clear since $z = s_4yx_4$ and $s_4$ commutes with $x$. The proof will be complete if we can show that $s_2zs_2 = z$ for then $yz = zy$ follows by conjugating $xz = zx$ by $s_2$. We have
\[
s_2 z s_2 = s_2 s_4 s_2 (s_1 s_3 s_4) s_2 s_4 s_2 \\
= s_2 s_4 s_2 (s_4 s_1 s_3) s_2 s_4 s_2 \\
= (s_2 s_4 s_2 s_4) s_1 s_3 s_2 s_4 s_2 \\
= (s_2 s_4)^3 s_4 s_2 s_1 s_3 s_2 s_4 s_2 \\
= s_4 s_2 s_1 s_3 s_2 s_4 s_2 (s_2 s_4)^3 \quad \text{(since } (s_2 s_4)^3 \text{ is central)} \\
= s_4 s_2 s_1 s_3 s_4 s_2 s_4 \\
= z.
\]

Therefore, \(N\) is elementary abelian, as claimed.

(2) \(N \trianglelefteq G\). In steps:

(a) \(s_1 x s_1 = x, s_1 y s_1 = xy, s_1 z s_1 = xz\). The first statement is trivial. Next,

\[
s_1 y s_1 = s_1 s_2 (s_1 s_3 s_4) s_2 s_1 \\
= s_1 s_2 (s_4 s_3 s_1) s_2 s_1 \\
= s_1 s_2 s_4 s_3 (s_1 s_2 s_1) \\
= s_1 s_2 s_4 s_3 (s_2 s_1 s_2) \\
= s_1 (s_2 s_4 s_3 s_2) s_1 s_2 \\
= s_1 (s_3 s_4 s_2 s_3 s_4) s_1 s_2 \quad \text{(since } (s_2 s_4 s_3)^3 = 1) \\
= s_1 s_3 s_4 s_2 (s_3 s_4 s_1) s_2 \\
= s_1 s_3 s_4 s_2 (s_1 s_3 s_4) s_2 \\
= x y.
\]

Finally,

\[
s_1 z s_1 = s_1 s_4 s_2 s_1 s_3 s_4 s_2 (s_4 s_1) \\
= s_1 s_4 s_2 (s_1 s_3 s_4) s_2 (s_1 s_4) \\
= s_1 s_4 s_2 (s_3 s_4 s_1) s_2 s_1 s_4 \\
= s_1 s_4 s_2 s_3 s_4 (s_1 s_2 s_1) s_4 \\
= s_1 s_4 s_2 s_3 s_4 (s_2 s_1 s_2) s_4 \\
= s_1 (s_4 s_2 s_3 s_4 s_2) s_1 s_2 s_4 \\
= s_1 (s_3 s_2 s_4 s_3) s_1 s_2 s_4 \quad \text{(since } (s_4 s_2 s_3)^3 = 1) \\
= s_1 s_3 s_2 (s_4 s_3 s_1) s_2 s_4 \\
= s_1 s_3 s_2 (s_1 s_3 s_4) s_2 s_4 \\
= s_1 s_3 (s_4 s_4) s_2 s_1 s_3 s_4 s_2 s_4 \\
= x z.
\]
(b) \( s_2 s_2 = y, s_2 y s_2 = x, s_2 z s_2 = z \). The first two statements follow by definition of \( x, y \). That \( s_2 z s_2 = z \) was already proved above.

(c) \( s_3 x s_3 = x, s_3 y s_3 = xz, s_3 z s_3 = xy \). Again, the first statement follows by the definition of \( x \). Next,

\[
\begin{align*}
\ s_3 y s_3 &= s_3 s_2 s_1(s_3 s_4 s_2 s_3) \\
&= s_3 s_2 s_1(s_2 s_4 s_3 s_2 s_4) \\
&= s_3(s_2 s_4 s_2) s_4 s_3 s_2 s_4 \\
&= s_3(s_1 s_2 s_1) s_4 s_3 s_2 s_4 \\
&= (s_3 s_1) s_2 s_1 s_4 s_2 s_4 \\
&= (s_1 s_3) s_4 s_2 s_1(s_4 s_3) s_2 s_4 \\
&= s_1 s_3 s_4 s_2 s_1(s_3 s_4) s_2 s_4 \\
&= xz.
\end{align*}
\]

Finally,

\[
\begin{align*}
\ s_3 z s_3 &= s_3 s_4 s_2 s_1 s_3 s_4 s_2 (s_4 s_3) \\
&= s_3 s_4 s_2 s_1 s_3 s_4 s_2 (s_3 s_4) \\
&= s_3 s_4 s_2 s_1 (s_3 s_4 s_2 s_3 s_4) \\
&= s_3 s_4 s_2 s_1 (s_2 s_3 s_4 s_2) \\
&= s_3 s_4 (s_2 s_1 s_2) s_3 s_4 s_2 \\
&= s_3 s_4 (s_1 s_2 s_1) s_3 s_4 s_2 \\
&= (s_3 s_4 s_1) s_2 s_1 s_3 s_4 s_2 \\
&= (s_1 s_3 s_4) s_2 s_1 s_3 s_4 s_2 = xy.
\end{align*}
\]

(d) Since \( s_4 \equiv s_1 s_3 \) (modulo \( N \)), we conclude that \( N \leq G \), as claimed.

**Octahedron, Proof (2).** We now set \( H = W_4 = \langle s_1, s_2, s_3 \rangle \) and obtain an upper bound on the number of left cosets of \( H \) in \( G \). Next, if \( w \in G \), define the length of \( w \) to be the smallest number integer \( l \) such that \( w \) can be written as

\[
w = s_{i_1} s_{i_2} \cdots s_{i_l}.
\]

Thus, it is clear that if \( l(w) \leq 2 \), then there are at most three distinct cosets of the form \( wH \), namely, \( H, s_4 H, s_2 s_4 H \). We shall
find it convenient to consider in turn the remaining cosets \( wH \), according to \( l(w) \).

\[ l(w) = 3: \] There are clearly three possible cosets \( wH \), where \( l(w) = 3 \), namely, \( s_1s_2s_4H, s_3s_2s_4H \) and \( s_4s_2s_4H \). However, by Claim 1 above, it follows that \( s_3s_2s_4H = s_4s_2s_4H \), and so there are at most two distinct cosets \( wH \), where \( l(w) = 3 \), namely,

\[ s_1s_2s_4H \text{ and } s_3s_2s_4H = s_4s_2s_4H. \]

\[ l(w) = 4: \] Note first that \( s_2s_1s_2s_4H = s_1s_2s_1s_4H = s_1s_2s_4H \). Also, since \( s_3s_2s_4H = s_4s_2s_4H \), it follows that \( s_3s_1s_2s_4H = s_1s_3s_2s_4H = s_4s_1s_2s_4H = s_1s_3s_2s_4H = s_1s_4s_2s_4H \). The only other coset \( wH \) with \( l(w) = 4 \) can be of the form \( s_2s_3s_2s_4H = s_2s_4s_2s_4H \). That is, there are at most two distinct cosets of the form \( wH, l(w) = 4 \):

\[ s_1s_3s_2s_4H = s_1s_4s_2s_4H \text{ and } s_2s_3s_2s_4H = s_2s_4s_2s_4H. \]

\[ l(w) = 5: \] First we show that \( s_1s_2s_4s_2s_4H = s_3s_2s_4s_2s_4H = s_2s_4s_2s_4H \). Indeed,

\[ s_1s_2s_4s_2s_4H = s_1(s_2s_4)^3s_4s_2H \]
\[ = (s_2s_4)^3s_1s_4s_2H \text{ since } (s_2s_4)^3 \text{ is central,} \]
\[ = (s_2s_4)^3s_4H \]
\[ = s_2s_4s_2s_4H. \]

Similarly, \( s_3s_2s_4s_2s_4H = s_2s_4s_2s_4H \).

Next, note that by Claim 2 above, \( s_2s_1s_4s_2s_4H = s_1s_4s_2s_4H \).

As a result, we conclude that there is at most one coset \( wH \), with \( l(w) = 5 \), viz., \( wH = s_4s_2s_4s_2s_4H \).

There are no further cosets: This is clear, for if \( i = 1, 2, 3 \), then

\[ s_is_4s_2s_4s_2s_4H = s_i(s_4s_2)^3H = (s_4s_2)^3s_iH = (s_4s_4)^3H = s_4s_2s_4s_2s_4H. \]

For convenience, we list the cosets obtained in the above analysis: \( H, s_4H, s_2s_4H, s_1s_2s_4H, s_3s_2s_4H = s_4s_2s_4H, s_2s_3s_2s_4H = s_2s_4s_2s_4H, s_1s_3s_2s_4H = s_1s_4s_2s_4H, s_4s_2s_4s_2s_4H. \)
Therefore, we conclude that \(|G : H| \leq 8\) and so \(|G| \leq 48 \cdot 8 = 384\).

**ICOSAHEDRON.** Here, the Coxeter-Petrie group \(G = \langle s_1, s_2, s_3, s_4 \rangle\) has Coxeter relations taken from the diagram

\[
\begin{array}{c}
s_1 \circ \\
\quad 3 \\
5 \\
\quad 10 \\
s_2 \\
\quad s_3 \\
\quad s_4
\end{array}
\]


together with the Petrie relations \((s_2s_3s_4)^3 = (s_2s_1s_4)^5 = 1\).

**Lemma 9.3.5.** The element \((s_2s_4)^5\) is in the center of \(G\).\(^{15}\)

**Proof.** To prove that \((s_2s_4)^5\) is central in \(G\), we use the known fact that \((s_2s_1s_3)^5\) is central in \(G_4 \cong W_4\); argue exactly as in **Lemma 9.3.4.**

By analogy with the previous cases, define \(x, y, z, w, u \in G\) by setting
\[
x = s_1s_3s_4, \quad y = s_2xs_2, \quad z = s_4ys_4, \quad w = s_2zs_2, \quad u = s_4ws_4;
\]
again, each of these elements is either an involution or is the identity. Define the subgroup \(N\) of \(G\) by setting \(N = \langle x, y, z, w, u \rangle\). We shall show that \(N\) is a normal elementary abelian subgroup of \(G\); since \(G = G_4N\), and \(|G_4| = 120\), this is enough.

(1) \(N\) is elementary abelian. The proof that \(xy = yx\) is proved exactly as in the tetrahedral and octahedral cases. Thus, upon conjugating by \(s_4\), it follows immediately that \(xz = zx\).

Next, if we can show that \(s_1ws_1 = xu\), then it will follow that \(xu\) is an involution and hence \(xu = ux\). Proving this assertion takes some work.

\(^{15}\)In analogy with the previous case, \((s_1s_2s_3)^5\) is also in the center of \(G\), but this fact is not required in the sequel.
\[ xu = s_1s_3s_2s_4s_2(s_1s_3s_4)s_2s_4s_2s_4 \]
\[ = s_1s_3s_2s_4s_2(s_3s_4s_1)s_2s_4s_2s_4 \]
\[ = s_1s_3s_2(s_4s_3s_4)s_1s_2s_4s_2s_4 \]
\[ = s_1s_3s_2(s_3s_2s_4s_3s_2)s_1s_2s_4s_2s_4 \] (since \((s_4s_2s_3)^3 = 1\))
\[ = s_1s_3s_2s_3s_2(s_4s_3)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_3s_2s_3s_2(s_3s_4)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_3s_2s_3s_2(s_4s_2s_3s_2s_3)s_2s_1s_2s_4s_2s_4 \] (since \((s_3s_2)^5 = 1\))
\[ = s_1(s_3s_2s_3s_2s_3s_2)s_4s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_3s_2(s_3s_2s_4s_2)s_2s_1s_2s_4s_2s_4 \] (since \((s_3s_2s_4)^3 = 1\))
\[ = s_1s_2s_3s_2s_4(s_3s_4s_3s_2s_3)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_3s_2s_4(s_3s_4s_3s_2s_3s_2)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_3s_2s_4(s_3s_4s_3s_2s_3s_2s_3)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_3s_2s_4(s_3s_4s_3s_2s_3s_2s_3s_2)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_3s_2s_4(s_3s_4s_3s_2s_3s_2s_3s_2s_3)s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2(s_3s_2s_3s_2s_3s_2)s_4s_2s_4s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2(s_4s_2s_3s_2s_3s_2)s_4s_2s_4s_2s_1s_2s_4s_2s_4 \]
\[ = s_1s_2s_4s_2s_3s_2s_4s_2s_1s_4s_2s_4s_2s_4(s_4s_2)^5 \] (since \((s_4s_2)^5\) is central)
\[ = (s_1s_2s_4s_2s_3s_2)s_2s_4s_2s_1s_2s_4s_2s_4s_2. \]

Since \(s_1w_1 = s_1s_2s_4s_2s_3s_1s_4s_2s_4s_2s_1\), we will have proved that \(s_1w_1 = xu\) as soon as we show that
\[
s_2s_4s_2s_1s_2s_4s_2s_4s_2 = s_1s_4s_2s_4s_2s_1. \]

Notice that this is an equation not involving the involution \(s_3\); that is, it suffices to prove this identity within the parabolic subgroup \(G_3\). However, we know that \(G_3\) is mapped isomorphically onto \(G_4\) via \(s_1 \mapsto s_1, \ s_2 \mapsto s_2, \ s_4 \mapsto s_1s_3\). Under this isomorphism, the above equation is equivalent to the equation
\[
s_2s_3s_2s_3s_2s_1s_3s_2 = s_3s_2s_1s_3s_2s_1 \]
in \(G_4\). Using the fact that \((s_2s_3)^5 = 1\) (and so \(s_2s_3s_2s_3s_2 = s_3s_2s_3s_2s_3\)), the above equation is seen to be valid, completing the verification that \(s_1w_1 = xu\), and hence the verification
that $xu = ux$. Since $u = s_4 w s_4$, and since $s_4$ commutes with $x$, we conclude that $xw = wx$. Therefore, $x$ is in the center of $N$.

Next, if we conjugate the equation $xw = wx$ by $s_2$, we get $yz = zy$. Likewise, if we conjugate $xz = xz$ by $s_2$, we get $yw = wy$. Proving that $yu = uy$ will follow from the assertion that $s_2 u s_2 = u$, for then $yu = uy$ is obtained by conjugating the equation $xu = ux$ by $s_2$. To this end, we have

$$s_2 u s_2 = s_2 s_4 s_2 s_4 s_2 (s_1 s_3 s_4) s_2 s_4 s_2 s_4 s_2$$

$$= s_2 s_4 s_2 s_4 s_2 (s_4 s_1 s_3) s_2 s_4 s_2 s_4 s_2$$

$$= (s_2 s_4)^5 s_4 s_2 s_4 s_2 s_1 s_3 s_2 s_4 s_2 s_4 s_2$$

$$= s_4 s_2 s_4 s_2 s_1 s_3 s_2 s_4 s_2 s_4 s_2 s_4 s_2 s_4 s_2$$

$$= u.$$  

Therefore, $y$ is also in the center of $N$.

Note that $zw = wz$ and $zu = uz$ follow by conjugating the equations $yu = uy$ and $yw = wy$ by $s_4$. Therefore $z$ is in the center of $N$. Finally, since $zu = uz$, and since $s_2 u s_2 = u$, we may conjugate $zu = uz$ by $s_2$ and get $wu = uw$. This concludes the proof that $N$ is abelian.

(2) $N \subseteq G$.

(a) $s_1 x s_1 = x$, $s_1 y s_1 = xy$, $s_1 z s_1 = xz$, $s_1 w s_1 = xu$, $s_1 u s_1 = xw$. In fact the first three equations follow exactly as in the octahedral case, since only relations that continue to be satisfied in the icosahedral case were used. Next, note that the relations $s_1 w s_1 = xu$, $s_1 u s_1 = xw$ are equivalent to each other in light of the fact that $s_1 x s_1 = x$. Since we have already noted that $s_1 w s_1 = xu$, the second relation is valid, as well.

(b) $s_2 x s_2 = y$, $s_2 y s_2 = x$, $s_2 z s_2 = w$, $s_2 w s_2 = z$, $s_2 u s_2 = u$.

We have already proved that $s_2 u s_2 = u$; the remaining
equations are obvious.
(c) \( s_3 x s_3 = x, \ s_3 y s_3 = xz, \ s_3 z s_3 = xy, \ s_3 w s_3 = xw, \ s_3 u s_3 = xu \). The first three relations were proved in the octahedral using identities valid in the icosahedral case. Next, we have \( xw = s_1 s_3 s_4 w = s_3 s_1 u s_4 = s_3 x w s_1 s_4 \). Therefore, \( w = s_3 w s_1 s_4 \) from which it follows that \( s_3 w s_3 = w s_1 s_3 s_4 = wx = xw \). The proof that \( s_3 u s_3 = xu \) follows by conjugating \( s_3 w s_3 = xw \) by \( s_4 \).
(d) Since \( s_4 \equiv s_1 s_3 \) (modulo \( N \)), we conclude that \( N \leq G \).

Next, we turn to the determination of the structure of \( G(\mathcal{M}) \), where \( \mathcal{M} \) is a Platonic solid. These will be taken up individually, for the tetrahedron, octahedron and icosahedron. Note first that the mapping \( G \to G \cong G_4 \) given by

\[
s_1 \mapsto a, \ s_2 \mapsto b, \ s_3 \mapsto c, \ s_4 \mapsto ac,
\]

determines a surjective homomorphism. Therefore, we see that \( G \) has the structure of a semidirect product \( N \rtimes W \), where \( W = G_4 = \langle s_1, s_2, s_3 \rangle \). In each case, \( N \) will be an elementary abelian 2-group, and hence can be regarded as a vector space over the binary field \( \mathbb{F}_2 \). Therefore, the structure of \( N \rtimes W \) is determined by a representation of \( W \) on \( N \), i.e., the multiplication in \( N \rtimes W \) is given by

\[
(n, w) \cdot (n', w') = (n + w(n'), ww'), \quad n, n' \in N, \ w, w' \in W;
\]

and where \( w(n') \) is in terms of the representation of \( W \) on \( N \).

We now itemize the three cases.

**Tetrahedron.** Here, take

\[
N = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{F}_2 \right\}.
\]
We take the matrix representation of $W$ on $N$ to be that determined by
\[
s_1 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
\]

Map $G \to N \rtimes W$ via
\[
s_i \mapsto \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, s_i \right), \quad i = 1, 2, 3, \quad s_4 \mapsto \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix}, s_1 s_3 \right).
\]

It is routine to check that this determines a surjective homomorphism. Since $|G| \leq 96$, we conclude that this must be an isomorphism.

**Octahedron.** In this case, set
\[
N = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{F}_2 \right\}.
\]

In this case the matrix representation is given by
\[
s_1 \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_2 \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_3 \mapsto \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Map $G \to N \rtimes W$ via
\[
s_i \mapsto \left( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, s_i \right), \quad i = 1, 2, 3, \quad s_4 \mapsto \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, s_1 s_3 \right).
\]

Again, this is checked to be a surjective homomorphism; comparing group orders shows it to be an isomorphism.

**Icosahedron.** In this case, set
\[
N = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} \mid a, b, c, d, e \in \mathbb{F}_2 \right\},
\]
on which $W$ is represented\(^{16}\) by

$$
s_1 \mapsto \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix},
\quad
s_2 \mapsto \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix},
\quad
s_3 \mapsto \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

We map $G \to N \rtimes W$ by

$$
s_i \mapsto \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\quad
s_i \mapsto \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix},
\quad
s_1s_3
$$

Again, this can be checked to be an isomorphism.

For the tetrahedron and octahedron, there are alternative descriptions of the Coxeter-Petrie groups.

**Theorem 9.3.6.** Let $G$ be the Coxeter-Petrie group of the tetrahedron $M$. Then $G \cong W(D_4)/Z$, where $W(D_4)$ is the Coxeter group of type $D_4$ and $Z$ is the center of $W(D_4)$.

**Proof.** First of all, the Coxeter group $W$ of type $D_4$ has generators $w_1, w_2, w_3, w_4$ and relations taken from the Coxeter diagram:

![Coxeter Diagram](attachment:image.png)

One has that $Z(W)$ cyclic of order 2; furthermore, if $i_1, i_2, i_3, i_4$ is any permutation of 1, 2, 3, 4, then the element $(w_{i_1}w_{i_2}w_{i_3}w_{i_4})^3$ is the central

---

\(^{16}\)This representation turns out to be a non-split extension of the trivial representation by an irreducible representation of degree 4.
involuion in $Z(W)$. Inside $W/Z$ define the elements $v_i = w_i$, $i = 1, 2, 3$, $v_4 = w_3 w_4$ (read modulo $Z$). Then one has that the elements $v_1, v_2, v_3, v_4$ satisfy the Coxeter relations depicted by

\[
\begin{array}{c}
\circ & 3 \\
\circ & 3 \\
\circ & 3 \\
\end{array}
\]

\[
\begin{array}{c}
v_1 \\
3 \\
v_4 \\
v_2 \\
v_3 \\
\end{array}
\]

together with the relations $(v_1 v_2 v_4)^3 = 1 = (v_3 v_2 v_4)^3$. Clearly $v_1, v_2, v_3, v_4$ generate $W/Z$ and so we get a surjective homomorphism $G \to W/Z$. Comparing the group orders finishes the job.

**Theorem 9.3.7.** Let $G$ be the Coxeter-Petrie group of the octahedron $M$. Then $G \cong W(D_4) \times Z$, where $Z$ is cyclic of order 2.

**Proof.** As above, let $W = W(D_4)$ with relations taken from

\[
\begin{array}{c}
\circ & 3 \\
\circ & 3 \\
\circ & 3 \\
\end{array}
\]

\[
\begin{array}{c}
w_1 \\
w_3 \\
w_2 \\
w_4 \\
\end{array}
\]

Let $Z = \langle z \rangle$ and let $z'$ be the nontrivial central element of $W$. In $W \times Z$, define the elements $v_1 = w_1, v_2 = w_2, v_3 = w_1 w_3 z, v_4 = w_4 zz'$. Then one shows that the elements $v_1, \ldots, v_4$ satisfy the Coxeter relations depicted by

\[
\begin{array}{c}
\circ & 3 \\
\circ & 3 \\
\circ & 3 \\
\end{array}
\]

\[
\begin{array}{c}
v_1 \\
v_4 \\
v_3 \\
v_2 \\
\end{array}
\]

and that $(v_1 v_2 v_4)^4 = (v_3 v_2 v_4)^3 = 1$. As $W \times Z = \langle v_1, v_2, v_3, v_4 \rangle$ we obtain a surjective homomorphism $G \to W \times Z$. A comparison of orders shows that this is an isomorphism.
9.3.2 Coxeter-Petrie Complexes Corresponding to Regular Affine Maps.

The affine Coxeter groups of rank 3 are defined by

\[ \Delta(k, l) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1 s_3)^r = (s_1 s_2)^k = (s_2 s_3)^l = 1 \rangle, \]

where \( \{k, l\} = \{3, 3\}, \) (in which case \( r = 3 \)), \( \{4, 4\}, \) or \( \{3, 6\} \) (in which case \( r = 2 \)). The corresponding hypermaps are called the universal affine hypermaps; note that except when \( \{k, l\} = \{3, 3\} \), these hypermaps are maps, the universal affine maps. We say that the map \( \mathcal{M} \) is a regular affine map if it is of the form \( \mathcal{M} \cong \mathcal{M}(\Delta(k, l)/K) \), where \( K \leq \Delta(k, l), \) \( \{k, l\} = \{4, 4\} \) or \( \{3, 6\} \), and where \( K \cap P = 1 \) for all rank 2 parabolic subgroups \( P \leq \Delta(k, l) \). This ensures that \( \mathcal{M} \) also has the same face and vertex valencies as \( \mathcal{M}(\Delta(k, l)) \); put differently, this says that the orbit mapping \( \mathcal{M}(\Delta(k, l)) \to \mathcal{M}(\Delta(k, l)/K) \) is unramified.

We begin by elucidating the structure of the above rank-3 affine Coxeter groups. First of all, we remark that in the context of Lie theory, the above groups are usually denoted as follows: \( \tilde{A}_2 = \Delta(3, 3), \) \( \tilde{B}_2 = \Delta(4, 4), \) \( \tilde{G}_2 = \Delta(3, 6). \) Furthermore, it is well known that the above groups can each be exhibited as the semidirect product of the corresponding rank-2 Coxeter group with the corresponding “coroot lattice;” see [11, (6.5)]. That the coroot lattice can be replaced by a suitable ring of algebraic integers of the form \( \mathbb{Z}[\omega], \) where \( \omega \) is a root of unity was already noticed by Jones and Singerman in [15, Section 7]. Here, we give here an approach that is both self-contained and more concrete.

**Proposition 9.3.8.** There is an isomorphism

\[ \Delta(k, l) \cong \mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle, \]

where \( (k, l) = (3, 3), \) \( (4, 4), \) or \( (3, 6), \) \( \omega = e^{2\pi i/l}, \) \( \omega \) acts by left multiplication on \( \mathbb{Z}[\omega], \) and where \( \tau \) acts as conjugation. (Note that \( \langle \omega, \tau \rangle \cong D_{2l}. \))

**Proof.** In each case above we have a semidirect product of an additive group \( (\mathbb{Z}[\tilde{i}], \) or \( \mathbb{Z}[\zeta] = \mathbb{Z}[\omega], \) \( \zeta = e^{2\pi i / 3}, \omega = e^{2\pi i / 6}) \) by a dihedral group.
We write elements of this group as ordered pairs \((a, g)\), remembering that the group multiplication is given by
\[
(a, g) \cdot (a', g') = (a + ga', gg').
\]

**Case 1.** \((k, l) = (4, 4)\). Inside the group \(W = \mathbb{Z}[i] \rtimes \langle i, \tau \rangle\), define the elements
\[
w_1 = (1, -\tau), \ w_2 = (0, \tau i), \ w_3 = (0, \tau).
\]

**Claim.** We have

(i) \(W = \langle w_1, w_2, w_3 \rangle\);

(ii) The mapping \(s_i \mapsto w_i, \ i = 1, 2, 3\), determines an isomorphism
\[
\alpha : \Delta(4, 4) \xrightarrow{\cong} W.
\]

**Proof.** (i) is entirely routine. For (ii), it is routine to check that \(w_1, w_2, w_3\) satisfy the Coxeter relations satisfied by \(s_1, s_2, s_3\) in \(\Delta(4, 4)\). Therefore, the assignment \(s_i \mapsto w_i, \ i = 1, 2, 3\), defines a surjective (by part (i)) homomorphism \(\Delta(4, 4) \rightarrow W\). Inside \(\Delta = \Delta(4, 4)\) is the subgroup \(K = \langle a_1 := s_1s_2s_3s_2, \ a_i := s_3s_2s_1s_2 \rangle\). Note that

(i) \(a_1, a_i\) have infinite orders since their homomorphic images in \(W\) do. Indeed, a direct calculation reveals that \(a_1 \mapsto (1, 1)\) and that \(a_i \mapsto (i, 1)\) which have infinite orders in \(\mathbb{Z}[i] \rtimes \langle i, \tau \rangle\).

(ii) \(a_1, a_i\) commute (direct verification);

(iii) \(\alpha(K) = \mathbb{Z}[i] \times \{1\} \subseteq W\);

(iv) \(K\) is a rank 2 free abelian group and the restriction of \(\alpha\) to \(K\) determines an isomorphism
\[
\alpha|_K : K \rightarrow \mathbb{Z}[i].
\]

(v) \(K \leq \Delta\). Indeed, easy calculations reveal that \(s_1a_1s_1 = a_1^{-1}, \ s_1a_is_1 = a_i, \ s_2a_1s_2 = a_i, \ s_2a_is_2 = a_1\). Since \(s_3 \in \langle s_1, s_2, K \rangle\), this is enough.
From the above, we get a commutative diagram

\[
1 \rightarrow K \rightarrow \Delta \rightarrow \Delta/K \cong \langle s_1, s_2 \rangle \rightarrow 1
\]

\[
1 \rightarrow \mathbb{Z}[i] \rightarrow W \rightarrow W/\mathbb{Z}[i] \cong \langle w_1, w_2 \rangle \rightarrow 1
\]

By the Five Lemma, the middle vertical arrow is an isomorphism, proving (ii).

**Case 2.** \((k, l) = (3, 3)\). Inside the group \(W = \mathbb{Z}[\zeta] \rtimes \langle i, \zeta \rangle\), \(\zeta = e^{2\pi i/3}\), define the elements

\[
w_1 = (1, \tau), \ w_2 = (0, \tau \zeta), \ w_3 = (0, \tau \zeta^{-1}).
\]

Set \(K = \langle a_1 = s_1s_2s_3s_2, \ a_\zeta = s_3s_1s_3s_2 \rangle\) and prove the obvious analogs of (i)–(v) of the above claim, proving the result in this case.

**Case 3.** \((k, l) = (3, 6)\). Inside the group \(W = \mathbb{Z}[\omega] \rtimes \langle i, \omega \rangle\), \(\omega = e^{2\pi i/6}\), define the elements

\[
w_1 = (1, -\tau), \ w_2 = (0, \tau \omega), \ w_3 = (0, \tau).
\]

Set \(K = \langle a_1 = s_1s_2s_3s_2s_3s_2, \ a_\omega = s_3s_2s_1s_2s_3s_2 \rangle\) and prove the obvious analogs of (i)–(v) of the above claim, proving the result in this case, as well.

Next, we shall determine those normal subgroups \(K \leq \Delta(k, l)\), such that \(K\) trivially intersects each proper parabolic subgroup of \(\Delta(k, l)\). The following was already pointed out in [15, p. 304].

**Lemma 9.3.9.** The normal subgroups \(K \leq \mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle, \ \omega \in \{e^{2\pi i/l}, \ l = 3, 4, 6\}, \) not meeting any proper parabolic subgroups are of the form \(I \times \{1\}\), where \(I \subseteq \mathbb{Z}[\omega]\) is an ideal of \(\mathbb{Z}[\omega]\) which is closed under complex conjugation.

**Proof.** We proof this for \(\omega = i\), leaving the remaining (similar) cases to the reader. Let \(Z\) be the normal subgroup \(Z = \mathbb{Z}[\bar{i}] \rtimes \{1\} \subseteq W =

\( Z[i] \bowtie \langle i, \tau \rangle \). If \( K \not\subseteq Z \), then \( K \) contains an element of the form \( \alpha = (a, g) \in Z[i] \bowtie \langle i, \tau \rangle \), where \( 1 \neq g \in \langle i, \tau \rangle \). Since \( K \subseteq W \), it follows easily that, in fact, \( K \) must contain an element of the form \( \alpha = (a, -1) \in W \). From this, we infer that \( Z^2 = 2Z[i] \bowtie \{1\} = [\alpha, Z] \subseteq K \), from which it follows that \( K \) contains one of the elements \((0, -1), (1, -1), (i, -1)\), or \((1 - i, -1)\). But \((0, -1) = w_2w_3w_2w_3, (0, i)^{-1}(i, -1)(0, i) = (1, -1) = w_1w_3\), and \((1 - i, -1) = w_1w_2w_1w_2\), proving the result.

Using the above, we can determine a presentation for the monodromy group of a regular affine map of type \((4, 4)\). In the oriented case, this is already contained in [15, p. 304]. Thus, let \( M \) be a regular affine map of type \((4, 4)\), i.e., one of the form \( M(\Delta(4, 4)/K) \), where as proved above, \( K \) can be identified with an ideal \( I \subseteq Z[i] \) invariant under complex conjugation. Since \( Z[i] \) is Euclidean with respect to the norm map, we have that \( Z[i] \) is a principal ideal domain. Therefore, we may write \( I \) in the form

\[
I = (x + yi)Z[i],
\]

where \( x, y \in \mathbb{Z} \).

The next two results are already contained in [??, Section 8.3].

**Lemma 9.3.10.** The ideal \( I = (x + yi)Z[i] \) is invariant under complex conjugation precisely when one of the following three conditions holds:

1. \( x = 0 \);
2. \( y = 0 \);
3. \( x = \pm y \).

As a result of Lemma 9.3.10, any ideal of \( Z[i] \) invariant under complex conjugation is of the form \( nZ[i] \) or \( n(1 + i)Z[i] \) for some non-negative integer \( n \).

**Corollary 9.3.10.1.** The monodromy group of a finite regular affine map of type \((4, 4)\) has one of the following presentations:
1. \( G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^4 = (bc)^4 = (abcb)^n = 1 \rangle \), for some positive integer \( n \);

2. \( G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^4 = (bc)^4 = (acba)^n = 1 \rangle \), for some positive integer \( n \).

**Proof.** The regular affine map \( \mathcal{M} \) has the form \( \mathcal{M}(\Delta(4, 4)/K) \), where, as shown above, \( K \) is one of the groups \( K_1 = \langle a_i^n = (s_1s_2s_3s_2)^n, a_i^n = (s_3s_2s_1s_2)^n \rangle \) or \( K_2 = \langle (a_1a_i)^n, (a_1^{-1} a_i)^n \rangle \). Therefore, \( K_1 \) is the normal closure in \( \Delta(4, 4) \) of \( a_i^n \) and \( K_2 \) is the normal closure in \( \Delta(4, 4) \) of \( (a_1a_i)^n = (s_1s_3s_2s_3s_1s_2)^n \).

We turn now to the Petrie polygons in \( \mathcal{M} \). Recall that the length \( m \) of the Petrie polygons in \( \mathcal{M} \) is the order of the glide reflection \( w_1w_2w_3 \) (modulo \( I \)).

**Proposition 9.3.11.** If \( I \) is either \( n\mathbb{Z}[i] \) or \( n(1 + i)\mathbb{Z}[i], n \in \mathbb{Z}, \) and if \( K = I \rtimes \langle i, \tau \rangle \leq W, \) then

\[
m = o(w_1w_2w_3) = 2n \pmod{K}.
\]

**Proof.** A direct calculation reveals that

\[
w_1w_2w_3 = (1, -i \tau).
\]

From this it follows that for any non-negative integer \( k, \)

\[
(w_1w_2w_3)^{2k} = (k(1 - i), 1), \quad (w_1w_2w_3)^{2k+1} = (k + 1 - ki, -i \tau) \notin K.
\]

The result follows easily.

As a result of the above, we can itemize the possible presentations for the Coxeter-Petrie group corresponding to a regular affine type \((4, 4)\) map. For any fixed positive integer \( n \), the Coxeter-Petrie group \( G \) has generators \( s_1, s_2, s_3, s_4 \) having defining relations deduced from the Coxeter diagram:
together with the additional relations

\[(1) \quad (s_2 s_1 s_4)^4 = (s_2 s_3 s_4)^4 = (s_1 s_2 s_3 s_2)^n = (s_1 s_2 s_1 s_4 s_2)^n = (s_3 s_4 s_2 s_3 s_2)^n = 1, \text{ or} \]

\[(2) \quad (s_2 s_1 s_4)^4 = (s_2 s_3 s_4)^4 = (s_1 s_2 s_3 s_1 s_2)^n = 1. \]

Note that in case (2) above, the remaining two relations, i.e., those obtained from \((s_1 s_3 s_2 s_3 s_1 s_2)^n = 1\) by replacing \(s_1\) by \(s_3\), \(s_4\) and by replacing \(s_3\) by \(s_1\), \(s_4\) both reduce to the relation \((s_4 s_2 s_4 s_2)^n = 1\), which is already specified among the Coxeter relations.

Finally, we summarize the relevant results for the finite regular affine maps of type \((3, 6)\). Namely, that these maps are organized into two families (corresponding to ideals of the form \(n\mathbb{Z}[\omega]\), and \(n(1 + \omega)\mathbb{Z}[\omega]\), where \(\omega = e^{2\pi i /6}\), and having monodromy groups

1. \(G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^3 = (bc)^6 = (abcacb)^n = 1 \rangle\), for some positive integer \(n\);

2. \(G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^3 = (bc)^6 = (acbcacbc)^n = 1 \rangle\), for some positive integer \(n\).

As for the Petrie polygons, we have that \(w_1 w_2 w_3 = (1, -\omega \tau), \quad (w_1 w_2 w_3)^2 = (1 - \omega, 1)\), from which it follows that if \(K = n\mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle\) or if \(K = n(1 + \omega)\mathbb{Z}[\omega] \rtimes \langle \omega, \tau \rangle\), then

\[o(w_1 w_2 w_3) = 2n \pmod{K}.\]

Finally, for any fixed positive integer \(n\), the Coxeter-Petrie group \(G\) has generators \(s_1, s_2, s_3, s_4\) having defining relations deduced from the Coxeter diagram:
together with the additional relations

\[(1) \quad (s_2s_1s_4)^6 = (s_2s_3s_4)^3 = (s_1s_2s_3s_2s_3s_2)^n = 1, \text{ or} \]

\[(2) \quad (s_2s_1s_4)^6 = (s_2s_3s_4)^3 = (s_1s_2s_3s_2s_3s_1s_2s_3s_2)^n = 1. \]

In case (1) above, there are, ostensibly, two additional relations needed, i.e., those obtained from \((s_1s_2s_3s_2s_3s_2)^n = 1\) by replacing \(s_1\) by \(s_3s_4\) and by replacing \(s_3\) by \(s_1s_4\) reduce to the relations \(1 = (s_1s_2s_3s_4s_2s_1s_4s_2)^n = (s_2s_1s_2s_4s_2s_4s_1s_2)^n, \) and \(1 = (s_4s_3s_2s_3s_2s_3s_2)^n = (s_2s_3s_4s_2s_4s_2s_3s_2)^n = 1. \) However, as \(s_2s_1s_2s_4s_2s_4s_1s_2\) is conjugate to \(s_2s_4s_2s_3,\) and as \(s_2s_3s_4s_2s_4s_2s_3s_2\) is conjugate to \(s_4s_2s_4s_2,\) we see that these relations are unnecessary. Case (2) is a bit more subtle, as follows. We have

\[
s_1s_3s_2s_3s_2s_3s_1s_2s_3s_2 = a_1a_\omega = (s_1s_2s_3s_2s_3s_2)(s_3s_2s_1s_2s_3s_2) = (s_1s_2s_3s_2s_3s_2)(s_3s_1s_2s_1s_3s_2) \\
= (s_1s_2s_3s_2s_3s_2)(s_4s_2s_4s_2s_3s_2) = (s_2s_3s_4s_2s_4s_2s_3s_2)(s_4s_2s_4s_2)
\]

Since \(a_1, a_\omega\) commute, so do the factors \((s_2s_3s_4s_2s_4s_2s_3s_2)\) and \((s_4s_2s_4s_2).\) As both are conjugates of \(s_4s_2s_4s_2,\) we see that the relation obtained from \((s_1s_3s_2s_3s_2s_3s_1s_2s_3s_2)^n = 1\) obtained via \(s_1 \mapsto s_3s_4\) is already implied by the Coxeter relation \((s_4s_2s_4s_2)^n = 1.\) Similarly, the relation obtained from \((s_1s_3s_2s_3s_2s_3s_1s_2s_3s_2)^n = 1\) by \(s_3 \mapsto s_1s_4\) already follows from the relation \((s_4s_2s_4s_2)^n = 1.\)

**Final Remark.** While the Coxeter-Petrie groups of the Platonic maps are finite, this question remains unsettled for the Coxeter-Petrie groups of the regular affine maps. For very small values of the parameter \(n,\)
the use of GAP gives finite group orders, but the results are far from being even suggestive.\footnote{At the SIGMAC '02 conference, held in Aveiro, Portugal, Marston Conder, using MAGMA, supplied even stronger evidence that, except for very small values of the parameter $n$, the affine Coxeter-Petrie groups are infinite.}

REFERENCES


12. A. Hurwitz, Algebraische Gebilde mit eindeutigen transformatio-


13. L. D. James, Imbeddings of the complete graph, Ars Combin. 16-


14. L. D. James and G. A. Jones, Regular orientable imbeddings of


16. G. A. Jones and D. B. Surowski, Regular cyclic coverings of the


17. G. A. Jones and D. B. Surowski, Cohomological constructions of

regular cyclic coverings of the Platonic maps, Europ. J. Combinator-


18. G. A. Jones and J. A. Thornton, Operations on maps, and outer

automorphisms, Journal of Combinatorial Theory, Ser. B 35,

1987, 93–103.

19. S. Lins, Graph-encoded maps, Journal of Combinatorial Theory,


(1985), 10–16.


