THE UNIQUENESS OF STRONG ROW ECHelon FORM

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Definitions. Let $A = [a_{ij}]$ be an $n \times m$ matrix over a field $k$. Assume that
(a) The first non-zero entry of each row is 1.
(b) Rows 1, 2, ..., $r$ are the non-zero rows and rows $r + 1, r + 2, \ldots, n$ are zero rows.
(c) If $a_{1,i_1}, a_{2,i_2}, \ldots, a_{r,i_r}$ are the first non-zero entries in rows 1, 2, ..., $r$, respectively, then $i_1 < i_2 < \cdots < i_r$. Then we say that the matrix $A$ is in row echelon form.
If, in addition, $A$ also satisfies
(d) For all $j = 1, 2, \ldots, r$ and for all $k < j$, we have $a_{k,i_j} = 0$, then we say that the matrix $A$ is in strong row echelon form.

Next, assume that $A = [a_{ij}]$ is an $n \times n$ matrix, i.e., is a square matrix. Assume that
(1) The first non-zero entry of each row is 1.
(2) The first non-zero entry of each row is on the diagonal.
(3) If $a_{jj} \neq 0$, then for all $k < j$ we have $a_{kj} = 0$.
(4) A is upper triangular.
Then we say that $A$ is in Hermite normal form. Note that unlike a matrix in strong row echelon form, a matrix in Hermite normal form may have zero rows interspersed among the non-zero rows. However, note that by a permutation of rows, a matrix in Hermite normal form can be transformed into a matrix in strong row echelon form.

Lemma 1. Let $A, A'$ be matrices in Hermite normal form having the same diagonal elements. Then $AA' = A'$.

Proof. Let $A = [\alpha_{ij}], A' = [\alpha'_{ij}], AA' = [\alpha''_{ij}]$; since $A, A'$ are upper-triangular, so is $AA'$, and so if $i < j$, $\alpha''_{ij} = 0$. If $i \leq j$, $\alpha''_{ij} = \sum_{k=i}^{j} \alpha_{ik} \alpha_{kj}$. If $\alpha_{ii} = 0$, then $\alpha_{ik} = 0$, for all $k$. Therefore $\alpha''_{ij} = 0$ for all $j \geq i$. However, since $\alpha_{ii} = 0$, we
have $\alpha'_{ii} = 0$ and so $\alpha'_{ij} = 0 = \alpha''_{ij}$. If $\alpha_{ii} = 1$ then $\alpha_{ik} \neq 0$ implies that $\alpha_{kk} = 0$. Therefore, $\alpha'_{kk} = 0$ and so $\alpha_{kj} = 0$ for all $j \geq k$. Therefore

$$\alpha''_{ij} = \sum_{k=1}^{j} \alpha_{ik} \alpha_{kj} = \alpha_{ii} \alpha'_{ij} = \alpha'_{ij}.$$ 

The proof follows.

The following is immediate.

**Corollary.** If $A$ is in Hermite normal form, then $A$ is idempotent, i.e., $A^2 = A$.

Let $A, B$ be $n \times m$ matrices over the field $k$. Recall that $A$ and $B$ are said to be row equivalent if there is a nonsingular matrix $C$ such that $A = CB$. When this happens, we write $A \equiv B$. Clearly $\equiv$ is an equivalence relation.

It is a very easy induction argument to prove that any square matrix $A$ is row equivalent to a matrix in Hermite normal form. Similarly, any matrix (not necessarily square) is equivalent to a matrix in strong row echelon form. Proving uniqueness can also be accomplished also by induction, but this is rather more difficult. Below, we give an alternate approach which is considerably simpler.

**Lemma 2.** Assume that the square matrices $A_1, A_2$ are in Hermite normal form and are row equivalent. Then $A_1 A_2 = A_1$.

**Proof.** We have $A_2 = PA_1$ for some nonsingular matrix $P$. Therefore $PA_1 PA_1 = PA_1$; since $P$ is nonsingular we have $A_1 PA_1 = A_1$. It follows immediately that $A_1 A_2 = A_1$.

**Corollary.** Let $A_1, A_2$ be as in Lemma 2, above. Then $A_1, A_2$ have the same diagonal elements.

**Proof.** We have $(A_1 - A_2)^2 = A_1^2 - A_1 A_2 - A_2 A_1 + A_2^2 = A_1 - A_2 + A_2 = 0$. Thus $A_1 - A_2$ is nilpotent; since $A_1 - A_2$ is upper triangular, the diagonal elements are zero. The result follows.

**Corollary.** If $A$ is a square matrix, then $A$ has a unique Hermite normal form.

**Proof.** If $A$ has Hermite normal forms $A_1, A_2$, then $A_1, A_2$ are row equivalent and so

$$A_1 = A_1 A_2 = A_2.$$ 

We turn now to the related question of uniqueness of strong row echelon form. Given an $n \times m$ matrix $A$, we set $r = \max \{m, n\}$, and let $A_H$ be the $r \times r$ matrix by adding rows (or columns) of 0’s to make $A$ into an $r \times r$ matrix. Thus, if

$$A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix} \text{ then } A_H = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}. $$
Similarly, if
\[
A = \begin{bmatrix}
* & * \\
* & * \\
* & *
\end{bmatrix}
\] then \(A_H = \begin{bmatrix}
* & * & 0 \\
* & 0 & 0 \\
* & 0 & 0
\end{bmatrix}\).

**Theorem.** Let \(A\) be an \(n \times m\) matrix. Then \(A\) is row equivalent to a unique matrix in strong row echelon form.

**Proof.** Let \(A\) be row equivalent to matrices \(A_1, A_2\), where \(A_1, A_2\) are in strong row echelon form. Then \(A_1 \sim R A_2\) implies that \((A_1)_H \sim R (A_2)_H\). Furthermore, by simple permutations of the rows of \((A_1)_H\) and \((A_2)_H\), we may obtain matrices in Hermite normal form, which must coincide. This clearly implies that \((A_1)_H = (A_2)_H\), and so \(A_1 = A_2\).

**References**


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