Lifting Map Automorphisms and MacBeath’s Theorem

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Abstract

The purpose of the present note is essentially twofold. First we wish to indicate how coverings of maps can be obtained easily and in a direct way by using a cohomological construction. Secondly, we wish to apply this construction to obtain in an explicit way infinitely many examples of finite maps which are extremal in the sense of having $84(g - 1)$ automorphisms, where $g$ is the genus of the map. This latter result is MacBeath’s Theorem of the title.

1. Introduction

We begin by recalling the definition of a map, cf. Biggs-White [1, Chapter 5]. Let $\Gamma = (V, E)$ be a graph, with vertices $V$ and edges $E$. If $\{v, w\} \in E$, we write $v \sim w$. For each $v \in V$ let $\Gamma (v)$ consist of the vertices adjacent to $v$. To obtain a map structure, we assume that for each vertex $v \in V$ we are given a cyclic permutation (or rotation) $\rho_v : \Gamma (v) \rightarrow \Gamma (v)$. Then $\rho = (\rho_v)_{v \in V}$ can be regarded as a permutation on the set $\overrightarrow{E} = \{(v, w) \in V \times V | v \sim w\}$ by setting $\rho(v, w) = (w, \rho_w(v))$. The pair $(\Gamma, \rho)$ is called a map.

The map $(\Gamma, \rho)$ can be thought of determining an embedding of the underlying topological space $|\Gamma|$ into the compact orientable surface $X = X(\Gamma, \rho)$ constructed as follows. If $((v_1, w_1), \ldots, (v_r, w_r))$ is a cycle in $\overrightarrow{E}$ under $\rho$, and if $e_i = (v_i, w_i)$, $i = 1, \ldots, r$, iterate the construction which attaches

to \(|(v_1, w_1)| \cup \cdots \cup |(v_r, w_r)|\) in the obvious way. In turn, the map has genus \(g\) which is defined by the Euler formula \(2 - 2g = |V| - |E| + |F|\) where \(F\) is the set of cycles of \(\rho\) on the set \(\overrightarrow{E}\).

If \(G = \text{Aut} (\Gamma)\) is the automorphism group of the graph \(\Gamma\), and if \(x \in G\), we say that \(x\) is an automorphism of \(M = M(\Gamma, \rho)\), and write \(x \in \text{Aut} (\Gamma)\) if the diagram below commutes:

\[
\begin{array}{ccc}
\overrightarrow{E} & \xrightarrow{x} & \overrightarrow{E} \\
\rho \downarrow & & \downarrow \rho \\
\overrightarrow{E} & \xrightarrow{x} & \overrightarrow{E}
\end{array}
\]

where \(x\) acts on \(\overrightarrow{E}\) in the obvious way. Thus we get an injection of groups \(\text{Aut} (M) \hookrightarrow \text{Aut} (\Gamma)\).

The map \(M = (\Gamma, \rho)\) is said to be symmetrical if \(G = \text{Aut} (M)\) acts transitively on the vertices of \(\Gamma\) and satisfies \(|G_v| = k\), for each \(v \in V(\Gamma)\), where \(k\) is the valence of \(\Gamma\) (which exists by vertex transitivity), and where \(G_v\) is the stabilizer of the vertex \(v\). Note that this is equivalent to the ostensibly stronger condition that \(G\) acts vertex transitively and, for any vertex \(v\), there is an element \(x \in G\) such that \(x|_{\Gamma(v)} = \rho_v\).

One has the following map version of Hurwitz’s Theorem cf. [op. cit., page 132].

**Theorem 1.** If \(M\) is a map, of genus \(g > 1\), then \(|\text{Aut} (M)| \leq 84(g - 1)\).

### 2. Conjugacy Maps

That there are plenty of symmetrical maps is seen by the following construction; c.f. [7]. Let \(G\) be a finite group and let \(\mathcal{C}, \mathcal{C}'\) be two conjugacy classes in \(G\). We form the graph \(\Gamma\) as that having vertex set \(\mathcal{C}\) and edges \(\{x, y\}\) where \(xy \in \mathcal{C}'\). For convenience, call \(\mathcal{C}\) the object class and call \(\mathcal{C}'\) the target class. Assume the following.

(i) The centralizer of each element of \(\mathcal{C}\) is cyclic.

(ii) If \(x \in \mathcal{C}\) then \(C_G(x)\) acts regularly on \(\Gamma(x)\).

Now fix \(x \in \mathcal{C}\) and fix a generator \(\rho_x \in C_G(x)\). If \(y \in \mathcal{C}\), define \(\rho_y \in C_G(y)\) by setting \(\rho_y = z\rho_x z^{-1}\), where \(y = zxz^{-1}\). Clearly \(\rho_y\) is well-defined and so we obtain the mapping \(\rho = (\rho_y)_y : \overrightarrow{E} \to \overrightarrow{E}\), giving rise to a map \(M = M(G, \mathcal{C}, \mathcal{C}')\) as above.
We shall call such maps \textit{group maps}. Furthermore, it is clear that each element of $G$ commutes with the action of $\rho$ on $\overrightarrow{E}$, and so $G \leq \text{Aut}(M)$. (In fact equality must hold; \cite[page 114]{op. cit.}) As a result we see that conjugacy maps are symmetrical maps.

Suppose we have a group $G$ and conjugacy classes $\mathcal{C}, \mathcal{C}'$, with graph $\Gamma$ defined as above. Assume further that if $\{x, x'\}$ is an edge in $\Gamma$, then $C_G(x) \cap C_G(x')$ is trivial. Thus if $k = |C_G(x)|, x \in \mathcal{C}$, then condition (ii) above obtains precisely when the valence of $\Gamma$ is $k$. This latter condition can be checked via character theory since

$$\text{valence} = \frac{|\mathcal{C}| \cdot |\mathcal{C}'|}{|G|} \sum_{\chi} \chi(x)^2 \chi(x') \chi(1),$$

where $x \in \mathcal{C}, x' \in \mathcal{C}'$, and where the summation is over the irreducible complex characters of $G$. (See Dornhoff \cite[(19.2)]{3}.) Thus the verification that the data $G, \mathcal{C}, \mathcal{C}'$ produce a group map can be obtained by a character calculation.

We shall call a map which arises from a group as above a \textit{group map}. A natural question to ask is whether every symmetrical map is a group map. That this isn’t so can be seen, e.g., from the following general construction. Suppose that $G$ is a finite group generated by elements $x, y \in G$, such that $x^k = y^l = (xy)^2 = 1$. Then $G$ is a homomorphic to the \textit{Von Dyke} group $D(k, l, 2)$ with presentation as above, which acts on an infinite map $M$ which is tessilated by $l$-gons. If $K$ is the kernel of $D(k, l, 2) \to G$, then $G$ acts on the symmetrical map $M/K$. As a very specific example, consider the group $G = PSL(2, 29)$. One has that $G$ is generated as above with $k = 7$ and $l = 3$. As a result, $G$ acts on a map with $\frac{1}{2}|G|$ vertices, and with vertex stabilizer cyclic of order 7. On the other hand, elements of order 7 in $G$ are centralized by elements of order 14, and so the vertices of the above map cannot be obtained as a conjugacy class of elements of order 7. Nonetheless, the conjugacy map construction accounts for a large number of symmetrical maps; c.f. \cite{7}. We shall itemize a few important classes of such examples.

1. \textit{The Platonic Solids}. These are, of course, the only symmetrical maps of genus 0. In each case we can obtain the map within the appropriate automorphism group. Describing how two of these occur should be sufficiently convincing. To get the map of the “cube,” with automorphism group $S_4$, we take as object class the elements of order 4, and as target class the class of double 2-cycles. That the valence of the graph is correct, either a direct argument or the character calculation (*) will do. Finally, for each $x$ in the object class take $\rho_x = x$; one then argues that the mapping $\rho$ on $\overrightarrow{V}$ has order 4. As another example, consider the icosahedral map, with automorphism group $A_5$. To obtain the “internal” construction, take as object class one of the two classes of elements of order 5, and take as target class the class of elements of order 3. As above, for each $x$ in the object class take $\rho_x = x$ and discover that $\rho$ operates in cycles of size 3, as required.

2. \textit{The Cayley and Paley Maps} (see \cite[Chapter 5]{1}). In these cases one has an automorphism group having the structure of a cyclic group of order $n$ acting
on an elementary abelian $p$–group. In each case one needs only to take an appropriate class of elements outside the normal $p$–group. For example, to construct the Paley map of [op. cit., page 134], take as object class the class of involutions in the group $G$ (which has the structure of a cyclic group of order 4 acting on an elementary abelian group of order 9). As target class take either of the two classes of elements of order 3. This gives the graph. Now construct the map $\rho$ determined by setting $\rho_x = y$, where $x$ is a fixed element of the object class, and where $y$ is an element of order 4 centralizing $x$.

3. Maps Arising from the Groups $PSL(2,7)$ and $SL(2,8)$. Here we describe how two of the best-known examples of maps which are extremal in the sense of Theorem 1 occur as conjugacy maps. In both cases we take as object class one of the classes of elements of order 7, and as target class the unique class of elements of order 3. The valence calculation (*) shows that we obtain a group map. Furthermore one calculates the genus $g$ to be 3 in the first case and 7 in the second case. Thus, we get $|G| = 84(g – 1)$ in each case. On the other hand the map arising from the group $G = PSL(2,29)$ described above is easily seen to be extremal; thus not all extremal maps are conjugacy maps. (For a classification of extremal maps having simple automorphism group of order less than one million, see Conder [2].)

3. The Functors $\pi_1$, $H_1$, and $H^1$

In this section we recall the elementary functors $\pi_1$, $H_1$ and $H^1$, adapted to the map context, and discuss some of their important relationships. As these discussions will be familiar to anyone with some exposure to algebraic combinatorial topology, proofs will be omitted.

Let $M = M(\Gamma, \rho)$ be a map. If $\sigma \in \overrightarrow{E}$ with $\sigma = (v_1, v_2)$, we call $v_1$ the initial vertex and $v_2$ the terminal vertex, and write $i(\sigma) = v_1$, $t(\sigma) = v_2$. A path in $M$ is a (possibly empty) sequence $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $t(\sigma_i) = i(\sigma_{i+1})$, $i = 1, \ldots, r – 1$. We write $i(\gamma) = i(\sigma_1)$ and $t(\gamma) = t(\sigma_r)$. (If $\emptyset$ denotes the empty sequence, there is no need to define $i(\emptyset)$, $t(\emptyset)$.) For a sequence $\gamma$ denote by $\text{vert}(\gamma)$ the set of vertices of the constituent edges of $\gamma$. If $\sigma = (v_1, v_2) \in \overrightarrow{E}$ we write $\sigma^{-1} = (v_2, v_1) \in \overrightarrow{E}$. More generally, if $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ is a path, write $\gamma^{-1} = (\sigma_r^{-1}, \sigma_{r-1}^{-1}, \ldots, \sigma_1^{-1})$. If $\gamma, \gamma'$ are paths, we write $\gamma \sim \gamma'$ if $\gamma, \gamma'$ can be expressed as juxtapositions $\gamma = \alpha \delta \beta, \gamma' = \alpha \delta' \beta$ where $i(\delta) = i(\delta'), t(\delta) = t(\delta')$, and where $\text{vert}(\delta) \cup \text{vert}(\delta') \subseteq \text{vert}(\nu)$, where $\nu$ is a cycle in $\overrightarrow{E}$ under $\rho$. Write $\simeq$ for the transitive closure of $\sim$ and call the resulting equivalence relation homotopy.

If $\nu \in M$ we define the fundamental group $\pi_1(M, \nu)$ to be the group whose underlying set consists of the homotopy classes of paths whose initial and terminal vertex is $\nu$ and whose operation is juxtaposition. Note that the identity of $\pi_1(M, \nu)$ is the class of the empty path $\emptyset$. Finally, the construction $\pi_1$ is em functorial in the following sense. If $f : M \rightarrow M'$ is a morphism of maps, then there is a group homomorphism $f_* : \pi_1(M, \nu) \rightarrow \pi_1(M', f(\nu))$ given by $f[\gamma] = [f(\gamma)]$.

Next, we define the (integral) homology functor $H_1$, as follows. Define $C_1(M)$ to be the free abelian group on $\overrightarrow{E}$, and define $C_0(M)$ to be the free abelian group.
on $V$. Define $\partial : C_1(M) \rightarrow C_0(M)$ by setting $\partial(v_1, v_2) = v_2 - v_1$ and extending by $\mathbb{Z}$-linearity, and set $Z_1(M) = \ker \partial$. Let $B_1(M)$ denote the subgroup of $C_1(M)$ generated by elements of the form $\sigma_1 + \sigma_2 + \cdots + \sigma_r$ where $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ is a cycle in $\overrightarrow{E}$, and by elements of the form $\sigma + \sigma^{-1}$, $\sigma \in \overrightarrow{E}$. Note that $B_1(M) \subseteq Z_1(M)$; set $H_1(M) = Z_1(M)/B_1(M)$, the 1-homology group of $M$.

Note that if $v \in V$, there is an obvious group homomorphism $\pi_1(M, v) \rightarrow H_1(M)$. Since $H_1(M)$ is an abelian group this map factors through the commutator factor group $\pi_1(M, v)/\pi_1(M, v)'$. We have the following well-known result.

**Theorem 2.** If $M$ is connected, then $H_1(M) \cong \pi_1(M, v)/\pi_1(M, v)'$ for any vertex $v$.

Furthermore, if $g$ is the genus of $M$ as defined in Section 1, we have

**Theorem 3.** $H_1(M) \cong \mathbb{Z}^{(2g)}$, where $\mathbb{Z}^{(2g)}$ is the direct sum of $2g$ copies of $\mathbb{Z}$.

The homology construction $H_1$ is functorial in essentially the same way as is $\pi_1$. Thus a morphism $f : M \rightarrow M'$ of maps induces an abelian group homomorphism $f_* : H_1(M) \rightarrow H_1(M')$. In one important sense, however, $H_1$ has better functorial properties (as does the cohomology functor $H^1$, defined below) than does $\pi_1$. Indeed, witness the fact that if $G = \text{Aut}(M)$ is the group of automorphisms of the map $M$, then $G$ will act on $H_1(M)$, whereas only the stabilizer $G_v$ of the vertex $v$ will act on the group $\pi_1(M, v)$.

It will be useful to introduce coefficients into $H_1$, as follows. Given a map $M$, and given an abelian group $A$, define the groups $C_1(M; A) = C_1(M) \otimes A$, $C_0(M; A) = C_0(M) \otimes A$. Set $Z_1(M; A) = \ker \partial \otimes 1_A$, and set $B_1(M; A) = B_1(M) \otimes A \subseteq C_1(M; A)$; since $C_1(M)$ is a free abelian group, this latter definition really does give a subgroup of $C_1(M; A)$. Again, one has $B_1(M; A) \subseteq Z_1(M; A)$; set $H_1(M; A)$, the 1-homology group of $M$, with coefficients in $A$. Note that $H_1(M; \mathbb{Z})$ can be identified with $H_1(M)$.

The functor $H_1$ is functorial in the coefficients, as follows. If $A, A'$ are abelian groups, and if $\alpha : A \rightarrow A'$ is a homomorphism, then we get a homomorphism $1 \otimes \alpha : C_1(M; A) \rightarrow C_1(M; A')$ which maps $Z_1(M; A)$ to $Z_1(M; A')$. In turn this map factors through $H_1(M; A)$, producing a map $\alpha_* : H_1(M; A) \rightarrow H_1(M; A')$.

We have the following very simple relationship between $H_1(M)$ and $H_1(M; A)$. (The obvious analogue is false in the general topological context.)

**Theorem 4.** For the map $M$, one has $H_1(M; A) \cong H_1(M) \otimes A$, for any coefficient group $A$.

We turn now to the definition of our cohomological functor $H^1$, which for our purposes is the most important construction. The functors $\pi_1, H_1$ merely play supporting roles. Let $M = M(\Gamma, \rho)$ be a map and let $\pi$ be a group. Define

$$C^1(M; \pi) = \prod \pi_\sigma,$$

where each $\pi_\sigma = \pi$, and where the elements $\sigma$ range over the elements of $\overrightarrow{E}$. Elements of $C^1(M; \pi)$ are called 1-cochains with values in $\pi$. If $c = (c_\sigma) \in C^1(M; \pi)$ and if $\sigma = (v, w)$, we shall usually write $c_{vw}$ in place of $c_{(v, w)}$.

Let $Z^1(M; \pi)$ consist of those $z = (z_\sigma) \in C^1(M; \pi)$ satisfying
(i) \( z_{uv} = z_{vu}^{-1} \) when \( u \sim v \), and

(ii) if \((\sigma_1, \sigma_2, \ldots, \sigma_r)\) is a cycle in \( \vec{E} \) under \( \rho \), then \( z_{\sigma_1}z_{\sigma_2} \cdots z_{\sigma_r} = 1 \).

Elements of \( Z^1(M; \pi) \) are called 1-cocycles with values in \( \pi \). Note that an element \( z \in Z^1(M; \pi) \) can be thought of as an analogue for maps of the notion of voltage assignment for graphs; see [4, Section 4]. Now set

\[
C^0(M; \pi) = \prod_v \pi_v,
\]

where each \( \pi_v = \pi \) and where \( v \) ranges over the vertices of \( M \). We call elements of \( C^0(M; \pi) \) 0-cochains with values in \( \pi \). The cocycles \( z = (z_{\sigma}) \), \( z' = (z'_{\sigma}) \) are equivalent if there exists a 0-cochain \( \phi = (\phi_v) \) such that whenever \((v, w) \in \vec{E}\), then

\[
z'_{vw} = \phi_v^{-1}z_{vw}\phi_w.
\]

This induces an equivalence relation on \( Z^1(M; \pi) \); the set of equivalence classes is denoted by \( H^1(M; \pi) \), called the 1-cohomology set of \( M \), with values in \( \pi \). Note that if \( \pi \) is an abelian group, then \( H^1(M; \pi) \) inherits the structure of an abelin group, and can be identified with the classical singular cohomology group, with values in \( \pi \), of the surface \(|M|\).

The functorial property of \( H^1 \) is as follows. Let \( f : M \to M' \) be a morphism of maps. And let \( \pi \) be a group. Then \( f \) determines a mapping \( f^* : C^1(M'; \pi) \to C^1(M; \pi) \), as follows. If \( z' = (z'_{\sigma}) \in C^1(M'; \pi) \), set \( f^*(z') = z = (z_{\sigma}) \) where \( z_{\sigma} = z'_{f(\sigma)} \). It is immediate that \( f^* \) carries \( Z^1(M'; \pi) \) into \( Z^1(M; \pi) \) and preserves equivalence of cocycles. Thus \( f^* \) determines a map (which we still denote \( f^* \))

\[
f^* : H^1(M'; \pi) \to H^1(M; \pi).
\]

Next let \( \alpha : \pi \to \pi' \) be a homomorphism of groups. Then \( \alpha \) is easily seen to induce a mapping of sets

\[
\alpha_* : H^1(M; \pi) \to H^1(M; \pi').
\]

It is clear that if \( \pi, \pi' \) are abelian groups, then \( \alpha_* \) is a homomorphism of abelian groups.

If \( A \) is an abelian group, we have an obvious homomorphism \( H^1(M; \mathbf{Z}) \otimes A \to H^1(M; A) \). That this is an isomorphism is the following coefficient theorem.

**Theorem 5.** For any map \( M \), the map \( H^1(M; \mathbf{Z}) \otimes A \to H^1(M; A) \) is an isomorphism for any coefficient group \( A \).

We have a map \( H^1(M; \mathbf{Z}) \to H_1(M)^\mathbf{Z} := \text{Hom}(H_1(M), \mathbf{Z}) \) as follows. If \( z \in Z^1(M; \mathbf{Z}) \) and \( a = \sum m_{\sigma} \sigma \in Z_1(M) \) simply evaluate:

\[
z \mapsto (a \mapsto \sum m_{\sigma}z_{\sigma}).
\]

One checks that this gives the desired homomorphism. Furthermore,

**Theorem 6.** The above map \( H^1(M; \mathbf{Z}) \to H_1(M)^\mathbf{Z} \) is an isomorphism.
The fundamental relationship between $\pi_1(M, v)$ and $H^1(M; \pi)$, where $\pi$ is an arbitrary group is best expressed through the so-called characteristic map, which we now set out to define; (cf. e.g., [6]). Let $z \in Z^1(M; \pi)$ and let $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ be a closed path in $M$, based at the vertex $v$. If $[\gamma]$ is the homotopy class of $\gamma$ in $\pi_1(M, v)$ we set $\chi_z[\gamma] = z_{\sigma_1} z_{\sigma_2} \cdots z_{\sigma_r} \in \pi$.

It is clear that $\chi_z : \pi_1(M, v) \to \pi$ is a group homomorphism, sometimes called the characteristic class of the cocycle $\gamma \in Z^1(M; \pi)$.

The dependence of $\chi_z$ on $z \in z \in H^1(M; \pi)$ is as follows. Assume that $z' \in Z^1(M; \pi)$ with $[z'] = [z]$; thus there is an element $\phi = (\phi_u) \in C^0(M; \pi)$ satisfying $z'_uw = \phi_u^{-1}z_{uw}\phi_w$ whenever $(u, w) \in E$. A direct calculation shows that $\chi_{z'}[\gamma] = \phi_u^{-1} \chi_z[\gamma] \phi_u$.

Let $G_1, G_2$ be two groups and let $f, f' : G_1 \to G_2$ be homomorphisms. Say that $f, f'$ are equivalent if they differ by an inner automorphism of $G_2$, i.e., if there exists $g_2 \in G_2$ such that $f(g_1) = g_2^{-1}f'(g_1)g_2$ for all $g_1 \in G_1$. Denote by $[G_1, G_2]$ the set of homomorphisms $G_1 \to G_2$ modulo this equivalence relation. Therefore, by the above paragraph, we have a map

$$\chi : H^1(M; \pi) \longrightarrow [\pi_1(M, v), \pi], \ [z] \longmapsto \chi_z.$$ 

The map $\chi$ is called the characteristic map, relative to the coefficient group $\pi$. One has the following (see, e.g., [op. cit.]).

**Theorem 7.** If $M$ is connected, then the characteristic map is an isomorphism for any coefficient group $\pi$.

4. Coverings of Maps and $H^1$

Let $M = M(\Gamma, \rho), M' = M(\Gamma', \rho')$ be maps, and let $f : \Gamma \to \Gamma'$ be a graph morphism. We say that $f$ is a morphism of maps is the following diagram commutes:

$$\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\rho \downarrow & & \downarrow \rho' \\
E & \xrightarrow{f} & E' 
\end{array}$$

If $f : M \to M'$ is a surjective morphism such that $f|_{\Gamma(x)} : \Gamma(x) \to \Gamma(f(x))$ is a bijection for each $x \in V(\Gamma)$, we call $f$ a covering of maps. Note that if $|f| : |M| \to |M'|$ is the induced map of topological spaces, then $|f|$ is a covering projection in the classical sense.

We now show how elements of $H^1(M; \pi)$ induce map coverings of $M = M(\Gamma, \rho)$. Let $\pi$ be again arbitrary, and let $z \in Z^1(M; \pi)$. Define the graph $\Gamma_z$ as that having
vertex set $V \times \pi$, where $V = V(\Gamma)$, and edges $(v, \gamma) \sim (v', \gamma')$ if and only if $v \sim v'$ and $\gamma' = z_{v'v}\gamma$. Thus we have a morphism of graphs

$$pr_1 : \Gamma_z \rightarrow \Gamma,$$

where $pr_1|\Gamma_z(v, \gamma) : \Gamma_z(v, \gamma) \rightarrow \Gamma(v)$ is a bijection for each $(v, \sigma) \in V(\Gamma_z)$. As a result, each rotation $\rho_v : \Gamma(v) \rightarrow \Gamma(v)$ can be lifted to one of $\Gamma_z(v, \gamma)$, giving rise to a map $M_z = (\Gamma_z, \rho_z)$. Explicitly, if $(v, \gamma) \sim (v', \gamma')$ in $\Gamma_z$, if $\hat{v} = (v, \gamma)$, and if $v'' = \rho_v(v')$, then

$$(\rho_z)_{\hat{v}}(v', \gamma') = (v'', z_{v'\hat{v}}\gamma).$$

Therefore we get a morphism of maps

$$pr_1 : M_z \rightarrow M$$

which is obviously a covering of maps.

The dependence of $M_z$ on $z$ is as follows. Let $z, z' \in Z^1(M; \pi)$ and let $[z], [z']$ be the corresponding classes in $H^1(M; \pi)$. If $[z] = [z']$, then there is an element $\phi = (\phi_v) \in C^0(M; \pi)$ with $z_{vw} = \phi_v^{-1}z_{vw}\phi_w$ whenever $v \sim w$. Define $\eta : M_z \rightarrow M_{z'}$ by setting $\eta(v, \gamma) = (v, \gamma\phi_v)$ and check that $\eta$ is an isomorphism making the diagram below commute:

$$\begin{array}{ccc}
M_z & \xrightarrow{\eta} & M_{z'} \\
\downarrow & & \downarrow \\
M & & M
\end{array}$$

When the above happens, we write $M_z \cong_M M_{z'}$. Therefore we have

**Proposition 8.** If $z, z' \in Z^1(M; \pi)$ satisfy $[z] = [z'] \in H^1(M; \pi)$, then $M_z \cong_M M_{z'}$.

**Remark.** If $\pi$ acts on a set $X$, and if $z \in Z^1(M; \pi)$, then one can define a covering $M_z(X) \rightarrow M$ in more-or less the obvious way; see [6] for details.

If $N \rightarrow M$ is a covering of maps and if $\eta : N \rightarrow N$ is an isomorphism such that

$$\begin{array}{ccc}
N & \xrightarrow{\eta} & N \\
\downarrow & & \downarrow \\
M & & M
\end{array}$$

commutes, then $\eta$ is called a covering transformation of $N$ (over $M$). Let $\pi$ be a group and let $z \in Z^1(M; \pi)$; we have the covering $pr_1 : M_z \rightarrow M$ as above. If $g \in \pi$ then the transformation $\eta : M \rightarrow M$ defined by $\eta(v, h) = (v, hg)$, $h \in \pi$, is easily checked to be a covering transformation. We’ll use the following in Section 6.

**Proposition 9.** If $z \in Z^1(M; \pi)$ is such that $M_z$ is connected, then every covering transformation of $M_z$ over $M$ is of the form $(v, h) \mapsto (v, hg)$ for some $g \in \pi$. 

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**Proof.** Let $\eta : M_z \to M_{\hat{z}}$ be a covering transformation. We may assume that $\eta(v, 1) = (v, 1)$ for some $v \in V(M)$. Assume that $\eta(w, h) \neq (w, h)$ for some $w \in V(M)$ and some $h \in \pi$. Then by connectivity we may join $(v, 1)$ to $\eta(w, h)$ by a path $\gamma_1$, and we may join $(v, 1)$ to $(w, h)$ by a path $\gamma_2$. But then $\text{pr}_1(\gamma_1) = \text{pr}_1(\gamma_2)$, contradicting the fact that paths in $M$ have unique lifts into any covering space. (See [op. cit. (3.1)]).

5. A Lifting Criterion

Let $p : M' \to M$ be a covering of maps and let $g \in \text{Aut}(M)$. We say that $g$ lifts to an automorphism of $M'$ (or simply that $g$ lifts to $M'$) if there exists $g' \in \text{Aut}(M')$ such that the diagram below commutes:

$$
\begin{array}{ccc}
M' & \xrightarrow{g'} & M' \\
p & & p \\
M & \xrightarrow{g'} & M.
\end{array}
$$

Our main lifting criterion is as follows.

**Theorem 10.** Let $M$ be a map, let $\pi$ be a group, and let $[z] = \zeta \in H^1(M; \pi)$. Assume that $g \in \text{Aut}(M)$, $\alpha \in \text{Aut}(\pi)$ and are related by $g^*(\zeta) = \alpha_*(\zeta)$. Then $g$ lifts to $M_z$.

**Proof.** Let $\phi = (\phi_v) \in C^0(M; \pi)$ satisfy $\alpha(z_\sigma) = \phi_u^{-1}z_{g(\sigma)}\phi_v$, where $\sigma = (u, v)$. Define $\hat{g} : V(M_z) \to V(M_z)$ by setting $\hat{g}(v, \gamma) = (g(v), \phi_v(\gamma))$. Note that if $(v, \gamma) \sim (v', \gamma')$ then $\gamma = z_\sigma\gamma'$, where $\sigma = (v, v')$. Thus, $z_{g(\sigma)}\phi_v(\gamma') = \phi_v(\alpha(z_\sigma)\alpha(\gamma') = \phi_v(\alpha(\gamma))$ and so $\hat{g}(v, \gamma) \sim \hat{g}(v', \gamma')$. Thus $\hat{g} \in \text{Aut}(\Gamma_z)$. Finally, we show that

$$
\begin{array}{ccc}
\hat{V}_z & \xrightarrow{\hat{g}} & \hat{V}_z \\
\rho_z & & \rho_z \\
\hat{V}_z & \xrightarrow{\hat{g}} & \hat{V}_z
\end{array}
$$

commutes. If $(v, \gamma) \sim (v', \gamma')$ and if $\hat{\sigma} = ((v, \gamma), (v', \gamma'))$, then

$$
\rho_z\hat{g}\hat{\sigma} = \rho_z((g(v), \phi_v(\alpha(\gamma))), (g(v'), \phi_v(\alpha(\gamma'))))
= ((g(v'), \phi_v(\alpha(\gamma'))), (\rho_{g(v')}g(v'), z_{\rho_{g(v')}g(v')}\phi_v(\alpha(\gamma'))))
= ((g(v'), \phi_v(\alpha(\gamma'))), (g(\rho_{v'}(v)), z_{g(\rho_{v'}(v))g(v')}\phi_v(\alpha(\gamma'))))
= (\hat{g}(v', \gamma'), (g(\rho_{v'}(v)), \phi_v(\alpha(\gamma'))))
= \hat{g}(v', \gamma'), (\rho_{v'}(v), z_{\rho_{v'}(v)}\alpha(\gamma'))
= \hat{g}\rho_z\hat{\sigma}.
$$

Thus the theorem is proved.
As a simple illustration of the above theorem, let M be a map, acted on by a p-group P. Thus P acts on the vector space V := \( H_1(M; \mathbb{Z}/(p)) \). It is well-known that in this situation, there is a vector \( 0 \neq \xi \in V \) with \( g^*(\xi) = \xi \) for all \( g \in P \). Therefore, by Theorem 10 (with \( \alpha = 1 \)) we see that every automorphism of in P lifts to an automorphism of \( M_z \), where \( [z] = \xi \).

6. Connected Coverings and MacBeath’s Theorem

In the present section we shall construct, for any connected map, an infinite family of finite connected maps which cover the given map. This construction will have the property that if \( M \) is symmetrical and if \( M' \to M \) is a covering constructed as below, then \( M' \) is symmetrical.

Let \( M \) be a connected map, let \( v \) be a vertex of \( M \), and let \( A \) be an arbitrary abelian group. We have the following commutative triangle:

\[
\begin{array}{ccc}
H^1(M; H_1(M; A)) & \xrightarrow{\chi} & \text{Hom}(\pi_1(M, v), H_1(M; A)) \\
\chi' \downarrow & & \downarrow j \\
\text{Hom}(H_1(M), H_1(M; A)) & & 
\end{array}
\]

where if \( \zeta = \eta \otimes \bar{a} \in H^1(M; H_1(M; A)) \cong H^1(M; \mathbb{Z}) \otimes H_1(M; A) \) and if \( a \in H_1(M) \), then \( \chi'(\zeta)(a) = \eta(a)\bar{a} \in H_1(M; A) \). The mapping \( j \) is that induced by the isomorphism of Theorem 2 of Section 3.

Now let \( A = \mathbb{Z}/(n) \) and let \( \epsilon: \mathbb{Z} \to \mathbb{Z}/(n) \) be the quotient map. Therefore we have the induced homomorphism

\[
\epsilon_* \in \text{Hom}(H_1(M), H_1(M; A)).
\]

Let \( g \) be the genus of \( M \) and let \( \{\eta_1, \ldots, \eta_{2g}\} \), \( \{a_1, \ldots, a_{2g}\} \) be dual \( \mathbb{Z} \)-bases of \( H^1(M; \mathbb{Z}) \) and \( H_1(M) \), whose existence is guaranteed by Theorem 6 of Section 3. Now set

\[
\zeta = \sum_{i=1}^{2g} \eta_i \otimes \epsilon_*(a_i) \in H^1(M; H_1(M; \mathbb{Z}/(n))) \cong H^1(M; \mathbb{Z}) \otimes H_1(M; \mathbb{Z}/(n)).
\]

Thus we have

\[
\chi'(\zeta)(a_j) = \sum_{i=1}^{2g} \eta_i(a_j)\epsilon_*(a_i) = \epsilon_*(a_j),
\]

\( j = 1, \ldots, 2g \), and so \( \chi'(\zeta) = \epsilon_* \). This implies that \( \chi(\zeta): \pi_1(M, v) \to H_1(M; \mathbb{Z}/(n)) \) is surjective; by [6, Theorem 4.8] the map \( M_z, [z] = \zeta \) is connected. Since \( |H_1(M; \mathbb{Z}/(n))| = \)
We may apply Proposition 9, to see that the map $M_z$ admits exactly $n^{2g}$ covering transformations over $M$.

Finally we wish to show that every automorphism $g \in G = \text{Aut}(M)$ lifts to $M_z$. For $j = 1, \ldots, 2g$, let $g_*(a_j) = \sum_{k=1}^{2g} m_{kj} a_k, \ m_{kj} \in \mathbb{Z}$. Then for each $j = 1, \ldots, 2g$,

$$
\chi'(g^* \zeta)(a_j) = \chi'(g^* \sum_{i=1}^{2g} \eta_i \otimes \epsilon_*(a_i))(a_j)
= \sum_{i=1}^{2g} g^* \eta_i(a_j) \epsilon_*(a_i)
= \sum_{i=1}^{2g} \eta_i(g_*(a_j)) \epsilon_*(a_i)
= \sum_{i=1}^{2g} \sum_{k=1}^{2g} \eta_i(m_{kj} a_j) \epsilon_*(a_i)
= \sum_{i=1}^{2g} m_{ij} \epsilon_*(a_i)
= \epsilon_*(g_*(a_j)) = g_* \epsilon_*(a_j).
$$

Therefore we see that $g^*(\zeta) = g_*(\zeta)$; apply Theorem 10 of Section 5 to conclude that every automorphism of $M$ lifts to one of $M_z$.

In particular, it follows that if $M$ is symmetrical and if $z = [\zeta]$ is constructed as above, then $M_z$ is a symmetrical map, with automorphism group of size $n^{2g}|G|$, where $G = \text{Aut}(M)$, and where $g$ is the genus of $M$. Finally suppose that $M$ is extremal of genus $g > 1$, and let $G = \text{Aut}(M)$. For each integer $n \geq 1$, let $[z_n] = \zeta_n \in H^1(M; H_1(M; \mathbb{Z}/(n)))$ be constructed as above, and let $M_n = M_{z_n}$ be the corresponding connected map which covers $M$. If $g_n$ is the genus of $M_n$, then $g$ and $g_n$ are related by

$$
2 - 2g_n = n^{2g}(2 - 2g).
$$

If $G = \text{Aut}(M), \ G_n = \text{Aut}(M_n)$, then the equations $|G| = 84(g - 1), \ |G_n| = n^{2g}|G|$, together with the above equation relating the genera of the two surfaces, imply that $|G_n| = 84(g_n - 1)$. Owing to the existence of an extremal map (e.g., the group map constructed in Section 2 arising from $\text{PSL}(2,7)$) we get the following, cf. [5, Theorem 6].

**Theorem 11** (MacBeath’s Theorem). There are infinitely many finite extremal maps.

### 7. Coverings of the $\text{PSL}(2,7)$ Map

In this section we shall give an example of a family of symmetrical maps, none of which is a conjugacy map. Start with the group map constructed from the group
$G = \text{PSL}(2,7)$, defined as follows. Let $\mathcal{C}$ be a conjugacy class of elements of order 7, and let $\mathcal{C}'$ be the conjugacy class of elements of order 3. Let $\Gamma = \Gamma(G, \mathcal{C}, \mathcal{C}')$ be the graph constructed as in Section 2. This graph is easily checked to have valence 7, using the character calculation of Section 2. If $x \in \mathcal{C}$, we have $C_G(x) = \langle x \rangle$, which acts regularly on $\Gamma(x)$; thus we may simply set $\rho_x = x$ and thereby obtain the conjugacy map $M = M(G, \mathcal{C}, \mathcal{C}')$. One easily checks that there are 24 vertices, 82 edges and 56 faces, from which it follows that the genus of the map is 3. Consequently, by Theorem 1, it follows that $M$ is an extremal map. (This fact is, however, unimportant for what follows.)

Now for each integer $n > 1$, we may construct, as in Section 6, a cohomology class $\zeta = [z] \in H^1(M; N)$, where $N = H_1(M; \mathbb{Z}/(n))$, in such a way that the map $M_z$ is a symmetrical (and in fact extremal) map. We shall show that $M_z$ is not a conjugacy map.

First of all, set $M' = M_z$, and set $G' = \text{Aut}(M')$. We have an exact sequence

$$1 \rightarrow N \rightarrow G' \rightarrow G \rightarrow 1;$$

we shall regard $N$ as a subgroup of $G'$.

If $7 \mid n$ it is easily seen that $G'$ cannot have any self-centralizing elements of order 7. Thus we may assume that $7 \nmid n$.

Let $\mathcal{D}$ be a class of self-centralizing elements of order 7 in $G'$, and let $\mathcal{D}'$ be an arbitrary class in $G'$. Let $\Gamma' = \Gamma(G', \mathcal{D}, \mathcal{D}')$; we shall show that we can’t have both of the conditions

(i) $\Gamma'$ has valence 7, and

(ii) if $x \in \mathcal{D}$, then $C_{G'}(x)$ acts regularly on $\Gamma'(x)$.

Thus assume that conjugacy classes $\mathcal{D}, \mathcal{D}'$ exist in $G'$, satisfying the above two conditions. It is already obvious that $\mathcal{D} \cap N = \emptyset$. Let $x, y \in \mathcal{D}$ and assume that $xy = z \in \mathcal{D}'$. If $z \in N$, and if $\bar{x}, \bar{y}$ denote the images of $x, y$ in $G$ under the projection $G' \rightarrow G$, then it would follow that $\bar{x}$ would be an element of order 7 in $G$ which is conjugate to its inverse. Since no such elements exist, we infer that $z \notin N$, i.e., $\mathcal{D}' \cap N = \emptyset$.

Let $y_1 = y, y_2, \ldots, y_7$ be the conjugates of $y$ under $\langle x \rangle$. Since $G$ acts faithfully on $N$, we may find an element $1 \neq n \in N$ such that $[z, n] = z^{-1}n^{-1}zn \neq 1$. Set $m = [z, n]$, and set $w = ym$. Since $y$ is self centralizing, we infer that $y$ is conjugate to $w$. Also $xw = xym = zm = z[z, n] = n^{-1}zn$, and so $w \in \Gamma'(x)$. Since we have assumed that the valence of $\Gamma'$ is 7, we must have $w = y_i$ for some $i, 2 \leq i \leq 7$. But then $x$ centralizes $y (\text{Mod}(N))$, and so $y = x_i n_1$, for some $x_i \in \langle x \rangle$, $n_1 \in N$. This gives $z = xy = xx_1 n_1$. Since $z \notin N$, we have $x_1 \neq x^{-1}$, and so $xx_1$ is conjugate to $z = xx_1 n_1$. This puts $x_1 \in \Gamma'(x)$, clearly violating (ii). The proof is complete.

In closing, we wish to identify two questions suggested by the above work. Let $M$ be a symmetrical map and let $\zeta = [z] \in H^1(M; A)$ where $A \neq 1$ is an abelian group. Assume that every automorphism of $G$ lifts to one of $M' = M_z$. Thus, if $G' = \text{Aut}(M')$, $N = H^1(M; A)$, we have the exact sequence $1 \rightarrow N \rightarrow G' \rightarrow G \rightarrow 1$, as above.

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Question 1. Does $G'$ split over $N$?

Question 2. Is $M'$ ever a conjugacy map?

Obviously, the answer to Question 1 is in the affirmative whenever the Schur-Zaussenhaus theorem applies. As for Question 2, the arguments of the present section may well generalize to afford a negative answer.

References


