Regular Cyclic Coverings of Regular Affine Maps

W. Christopher Schroeder
Department of Mathematical Sciences
Morehead State University
150 University Blvd.
Morehead, KY, 40351 USA
c.schroeder@morehead-st.edu

and

David B. Surowski
Department of Mathematics, Kansas State University,
Manhattan, KS 66506-2602, USA
dbski@math.ksu.edu
RUNNING HEAD: Cyclic Coverings

SEND PROOFS TO:

Christopher Schroeder
Department of Mathematical Sciences
Morehead State University
150 University Blvd.
Morehead, KY, 40351 USA
c.schroeder@morehead-st.edu
Abstract

The regular coverings of regular affine algebraic maps are considered, and a large family of totally ramified coverings—the so-called Steinberg and Accola coverings—are fully classified.
1 Introduction.

The present paper builds on the preparatory results of [5] which in turn lead to the classification of two important families of totally ramified regular coverings of regular affine maps (regular maps of genus 1). This investigation can be also thought of as having been born out of the work of the second author with Gareth Jones in [3 and 4] wherein the regular coverings of the Platonic maps were considered. However, since the affine maps are not simply connected, there is a more conspicuous appearance of the homological methods introduced in [5], as well as of the representation theory of the automorphism groups of the regular affine maps.

In Section 2 we review the work of G. Jones and the second author in [4] but from a point of view more conspicuously in line with the development in [5]. In particular, two types of totally ramified coverings are introduced: the Steinberg and the Accola coverings, the classification of which shall occupy most of our efforts in this paper. In Section 3 the unramified, Steinberg, and Accola coverings by regular maps with cyclic group of covering transformations are classified for the regular affine maps of type \((4, 4)\), and in Section 4 the same are classified for the regular affine maps of types \((3, 6)\) and \((6, 3)\). Since any unramified covering of an affine map is also an affine map, and since any covering of a regular map can be factored into a totally ramified covering of an unramified covering (see [5, Theorem 4.1]), we see that all coverings of a regular affine map having a Steinberg or Accola totally ramified factor are classified.

We shall freely use concepts and notations introduced in the companion paper [5].

2 Recollections: Regular Cyclic Coverings of Platonic Maps

In this section, we recall some basic concepts, largely within the context of the second author’s joint work with G. Jones in [4], and which lead to the definition of the Steinberg and Accola coverings and the classification of the regular coverings of the Platonic maps.

Note first that since any Platonic map \(\mathcal{M}\) is simply connected, then any covering of \(\mathcal{M}\) must be totally ramified. If we seek regular such coverings of \(\mathcal{M}\) that are unramified over the vertices, then by [5, Theorem 3.1], such a covering \(\mathcal{M}' \to \mathcal{M}\) by a connected map \(\mathcal{M}'\) must be of the form \(\mathcal{M}_z \to \mathcal{M}\) for some \(z \in C(\mathcal{M}; Z)\), where \(C(\mathcal{M}; Z)\) is the set of \(Z\)-valued voltages on \(\mathcal{M}\), and where \(\mathcal{M}_z\) is the associated principal derived map.
From [5, Lemma 6.8] we see that if \( M \) is a map and if \( M_z \to M \) is a totally ramified covering with \( M_z \) connected and with abelian group \( A \) of covering transformations, then there is a surjective homomorphism

\[
\theta^1 := \delta^1(\zeta) : C_2(M)/H_2(M) \to A,
\]

where \( \zeta = [z] \in D(M; A) \); here \( D(M; A) \) is the group of voltages on \( M \) modulo voltage equivalence, and where \( \delta^1 : C^1(M; A) \to C^2(M; A) \) is the coboundary map. (Note that \( \delta^1(\zeta) \) kills \( H_2(M) \), and so we may factor \( \delta^1(\zeta) \) through the quotient group \( C_2(M)/H_2(M) \).)

If in addition we assume that \( M_z \) is regular, then we know that for some action \( \alpha : \text{Aut}(M) \to \text{Aut}(A) \) inducing an \( \text{Aut}(M) \)-module structure on \( A \), it follows that

\[
\zeta \in D(M; A)_\alpha = \{ \zeta \in D(M; A) | \phi^*(\zeta) = \zeta \alpha(\phi)_* \text{ for all } \phi \in \text{Aut}(M) \}.
\]

Therefore we conclude that relative to this action, \( \theta^1 = \delta^1(\zeta) : C_2(M)/H_2(M) \to A \) is a surjective \( \text{Aut}(M) \)-module epimorphism.

When \( M \) is a Platonic map, then the first cohomology group \( H^1(M; A) = 0 \), and so we have an isomorphism \( \delta^1 : D(M; A) \cong \text{Hom}(C_2(M)/H_2(M), A) \). In this case, for any \( \text{Aut}(M) \)-structure \( \alpha : \text{Aut}(M) \to \text{Aut}(A) \) on \( A \), we conclude that

\[
\delta^1 : D(M; A)_\alpha \cong \text{Hom}_{\text{Aut}(M)}(C_2(M)/H_2(M), A).
\]

Since all connected coverings of the Platonic Map \( M \) are totally ramified, we may exactly as in the proof of [5, Theorem 7.4] to conclude the following (see also [4, Theorem 1]):

**Theorem 2.1.** Let \( M \) be a Platonic Map.

1. Let \( A \) be an abelian group. Then the following sets are in bijective correspondence:

   (i) the set of \( \cong_{\text{Aut}(M)} \)-isomorphism classes of coverings (unramified over vertices) \( M' \to M \) by connected regular maps having group of covering transformations \( A(M'/M) \cong A \), and

   (ii) the set of \( \text{Aut}(M) \)-submodules \( C \subseteq C_2(M) \) with \( H_2(M) \subseteq C \) and where \( C_2(M)/C \cong A \).

2. There is a bijective correspondence between
(i) the set of $\cong_{\mathcal{M}}$-isomorphism classes of coverings (unramified over vertices) $\mathcal{M}' \to \mathcal{M}$ by connected regular maps and having abelian group of covering transformations, and

(ii) the set of $\text{Aut}(\mathcal{M})$-submodules $C \subseteq C_2(\mathcal{M})$ with $H_2(\mathcal{M}) \subseteq C$. ■

As in [4], we review the above in the particular case when $A$ is cyclic, say $A = \mathbb{Z}_d$, where if $d = 0$, we write $\mathbb{Z}_0 = \mathbb{Z}$, the additive group of the integers. Assume, for the moment, that $\mathcal{M}$ is an arbitrary finite regular connected orientable map having face valency $k$ and vertex valency $l$. We can classify the $\text{Aut}(\mathcal{M})$-submodules of $C_2(\mathcal{M})/H_2(\mathcal{M})$ having quotient isomorphic to $\mathbb{Z}_d$, as follows.

Note first of all that an $\text{Aut}(\mathcal{M})$-module structure on $\mathbb{Z}_d$ is tantamount to giving a homomorphism $\alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \cong \mathcal{U}(\mathbb{Z}_d)$, where $\mathcal{U}(\mathbb{Z}_d)$ is the group of units of (the ring) $\mathbb{Z}_d$. Given that $\mathcal{M}$ is assumed to be regular, we have $\text{Aut}(\mathcal{M}) \cong G$, where $G = \langle a, b, c \rangle$ is the monodromy group of $\mathcal{M}$. Thus, if we fix a blade $\beta$ of $\mathcal{M}$, we may express $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$, where $\langle a', b' \rangle$, $\langle a', c' \rangle$, $\langle b', c' \rangle$ are, respectively, the stabilizers of the face, edge, and vertex containing $\beta$. (More precisely, since $\mathcal{M}$ is regular, its blade set can be identified with the elements of $G$. The monodromy action is given by left multiplication by elements of $G$, and the elements $a', b', c'$ are simply the right actions by $a, b, c$, respectively.)

If $u \in \mathcal{U}(\mathbb{Z}_d)$ is an involutory unit, and if the assignment $\text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d)$ determined by $a' \mapsto -1, b' \mapsto -1, c' \mapsto -u,$

is a homomorphism, denote the corresponding homomorphism by $\alpha_{(u)} : \text{Aut}(\mathcal{M}) \to \mathcal{U}$. Note that since $\mathcal{M}$ is orientable, $\alpha_{(1)} : \text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d)$ is always a homomorphism, which we call the Steinberg homomorphism. Note that if $\mathcal{M}$ is regular with odd vertex valency then $\alpha_{(1)}$ is the only possible homomorphism $\text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)$ mapping both $a', b' \mapsto -1$. At the other extreme is the Accola homomorphism $\alpha_{(-1)} : \text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d)$.

In what follows, we shall denote the Steinberg and Accola homomorphisms by $\alpha_{\text{St}}$ and $\alpha_{\text{Acc}}$, respectively.

The following lemma indicates the ubiquity of the homomorphisms $\alpha_{(u)}$. We continue to assume that $\mathcal{M}$ is an arbitrary finite orientable map.
Lemma 2.2. Let $\mathcal{M}$ be a finite regular orientable map. Assume that $C \subseteq C_2(\mathcal{M})$ is an $\text{Aut}(\mathcal{M})$-submodule of $C_2(\mathcal{M})$ with $C_2(\mathcal{M})/C \cong \mathbb{Z}_d$. Then the induced $\text{Aut}(\mathcal{M})$-module structure on $\mathbb{Z}_d$ is induced by one of the homomorphisms $\alpha_{(u)}$, where $u$ is an involutory unit of $\mathbb{Z}_d$.

Proof. Note first of all that if $\theta, \theta' : C_2(\mathcal{M}) \rightarrow \mathbb{Z}_d$ are surjective homomorphisms, both having kernel $C$, then $\theta' = \theta \mu$ for some $\mu \in \mathcal{U}(\mathbb{Z}_d)$. If $\mathcal{M}$ has $r$ faces, then there exist $r$ blades $\beta = \beta_1, \beta_2, \ldots, \beta_r$, all in the same $G^+$-orbit and such that $\{\chi_{[\beta_1]} + W_{(a,b)}(\beta), \chi_{[\beta_2]} + W_{(a,b)}(\beta), \ldots, \chi_{[\beta_r]} + W_{(a,b)}(\beta)\}$ is a $\mathbb{Z}$-basis of $C_2(\mathcal{M})$. We write $m_i = \chi_{[\beta_i]} + W_{(a,b)}(\beta), i = 1, 2, \ldots, r$. As above, we set $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$. If $\text{Aut}(\mathcal{M})^+ = \langle a'b', a'c' \rangle$, then since $\text{Aut}(\mathcal{M})^+$ acts transitively on $m_1, m_2, \ldots, m_r$, we see that the $\text{Aut}(\mathcal{M})$-module homomorphism is uniquely determined by $\theta(m_1)$; furthermore, since $\theta$ is surjective, we may assume (by the first remark above) that $\theta(m_1) = 1$. Next, if $\alpha : \text{Aut}(\mathcal{M}) \rightarrow \mathcal{U}(\mathbb{Z}_d)$ is the corresponding homomorphism, note that since $m_1a' = -m_1 = m_1b'$, it follows that

$$-1 = -\theta(m_1) = \theta(-m_1) = \theta(m_1a') = \theta(m_1) \alpha(a') = \alpha(a'),$$

and so $\alpha(a') = -1$. Similarly, $\alpha(b') = -1$. Since clearly $\alpha(c') = -u$, for some involutory unit in $\mathbb{Z}_d$, we conclude that $\alpha = \alpha_{(u)}$.

Note that if $\alpha_{St} = \alpha_{(1)}$ is the Steinberg homomorphism, then $\text{Aut}(\mathcal{M})^+ \subseteq \ker \alpha$ and it follows that $\theta(m_i) = \theta(m_j)$ for all $i, j$; up to unit multiple this means that we may assume that $\theta(m_i) = 1, i = 1, 2, \ldots, r$. If $\alpha = \alpha_{(u)}$ where $u \neq 1$, then $K^+ = \ker (\alpha : \text{Aut}(\mathcal{M})^+ \rightarrow \mathcal{U}(\mathbb{Z}_d))$ has index 2 in $\text{Aut}(\mathcal{M})^+$, from which it follows easily that $r$ is even. Therefore, we may re-index the basis elements of $C_2(\mathcal{M})$ if necessary so that $m_2, m_4, \ldots, m_r$ and $m_1, m_3, \ldots, m_{r-1}$ are the two $K^+$-orbits. From this it follows that $\theta(m_i) = u^i, i = 1, 2, \ldots, r$.

Conversely, for the homomorphisms $\alpha_{(u)} : \text{Aut}(\mathcal{M}) \rightarrow \mathcal{U}(\mathbb{Z}_d)$, we see that the corresponding recipies for $\theta = \theta_{(u)} : C_2(\mathcal{M}) \rightarrow \mathbb{Z}_d$ do, in fact, give $\text{Aut}(\mathcal{M})$-module surjections, proving the lemma. $lacksquare$

We shall write $C_{(u)} = \ker(\theta_{(u)} : C_2(\mathcal{M}) \rightarrow \mathbb{Z}_d)$. Note that $H_2(\mathcal{M}) \subseteq C_{(u)}$ if and only if

$$\theta_{(u)}(m_1 + m_2 + \cdots + m_r) = 0 \in \mathbb{Z}_d,$$

i.e., if and only if

$$u + u^2 + \cdots + u^r = 0 \in \mathbb{Z}_d.$$
Note that if \( \alpha = \alpha_{St} \), i.e., if \( u = 1 \), then the above happens precisely when \( d | r \). On the other hand, if \( \alpha = \alpha_{Acc} \), and so \( u = -1 \), then \( r \) is even and \( H_2(\mathcal{M}) \) is always contained in the kernel of \( \theta_{Acc} = \theta(-1) \).

We shall write \( C_{St} = C(1) \) and \( C_{Acc} = C(-1) \). If \( \zeta = [z] \) and if \( \delta^1(\zeta) = \theta(u) : C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d \), we call \( \mathcal{M}_z \to \mathcal{M} \) a Steinberg covering if \( u = 1 \), and an Accola covering if \( u = -1 \). By [5, Lemma 6.4], the Steinberg and Accola coverings are connected and totally ramified over \( \mathcal{M} \).

As in [3, 4], Theorem 2.1 (1) can now be applied to obtain the following classification of regular cyclic coverings of Platonic maps. Thus, assume that \( \mathcal{M} \) is a tetrahedron, octahedron, cube, icosahedron or a dodecahedron. (We exclude the so-called hosohedron, which is a somewhat degenerate map of genus 0. This is the regular map on the sphere with two antipodal vertices, \( l \) edges and \( l \) faces (“lunes”).) Note that Accola homomorphisms \( \text{Aut}(\mathcal{M}) \to \mathcal{U}(\mathbb{Z}_d) \) exist only for the octahedron.

**Theorem 2.3.** Let \( \mathcal{M} \) be a Platonic map, let \( d \) be a fixed nonnegative integer, and let \( r \) be the number of faces in \( \mathcal{M} \). If \( d | r \), define \( \zeta = [z] \in \text{D}(\mathcal{M}; \mathbb{Z}_d) \) by setting \( \delta^1(\zeta) = \theta(1) \in \text{Hom}(C_2(\mathcal{M})/H_2(\mathcal{M}), \mathbb{Z}_d) \), where \( \theta(1)(m_i) = 1, i = 1, 2, \ldots, r \), and where \( \{m_1, m_2, \ldots, m_r\} \) is the basis of \( C_2(\mathcal{M}) \) given above. Then \( \mathcal{M}_z \to \mathcal{M} \) satisfies

(i) \( \mathcal{M}_z \) is connected and regular;
(ii) \( \mathcal{M}_z \to \mathcal{M} \) is unramified over vertices; and
(iii) \( \mathcal{M}_z \to \mathcal{M} \) has group of covering transformations isomorphic with \( \mathbb{Z}_d \).

Assuming that \( \mathcal{M} \) is not the octahedron, then up to \( \cong_{\mathcal{M}} \)-isomorphism, there are no other coverings \( \mathcal{M}' \to \mathcal{M} \) satisfying (i), (ii), and (iii), and if \( d \nmid r \) there are no coverings \( \mathcal{M}' \to \mathcal{M} \) satisfying (i), (ii), and (iii). \( \blacksquare \)

When \( \mathcal{M} \) is the octahedron, we have additional coverings corresponding to involutory units \( u \neq 1 \) in \( \mathcal{U}(\mathbb{Z}_d) \), such that \( \sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d \) as follows. In this case, \( \text{Aut}(\mathcal{M})^+ \cong S_4 \) (the symmetric group on 4 symbols), and has a unique subgroup \( K^+ \) of index 2. Assume that the basis elements \( m_1, \ldots, m_8 \in C_2(\mathcal{M}) \) have been indexed so that \( m_{2i}, i = 1, 2, 3, 4 \) and \( m_{2i-1}, i = 1, 2, 3, 4 \) are the two \( K^+ \)-orbits. Then the
mapping \( \theta(u) : C_2(\mathcal{M}) \rightarrow \mathbb{Z}_d \) given by \( \theta(u)(m_i) = (-1)^{i-1}, \ i = 1, 2, \ldots, 8 \) is surjective and its kernel is an \( \text{Aut}(\mathcal{M}) \)-submodule of \( C_2(\mathcal{M}) \) containing \( H_2(\mathcal{M}) \) precisely when \( \sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d \). Note that the Accola homomorphism \( \alpha_{\text{Acc}} : \text{Aut}(\mathcal{M}) \rightarrow \text{Aut}(\mathbb{Z}_d) \) exists for the automorphism group of the octahedron \( \mathcal{M} \), and so \( \mathcal{M} \) admits Accola coverings. Thus, we have the following; again, proofs can be found in [3, 4]:

**Theorem 2.4.** For the octahedron \( \mathcal{M} \), the \( \cong_\mathcal{M} \) classes of coverings \( \mathcal{M}' \rightarrow \mathcal{M} \) satisfying (i), (ii), and (iii) in Theorem 2.3 are in bijective correspondence with the involutory units \( u \in \mathbb{Z}_d \) such that \( \sum_{i=1}^{8} u^i = 0 \in \mathbb{Z}_d \). For each such involutory unit \( u \), the corresponding covering map has the form \( \mathcal{M}' = \mathcal{M}_{\zeta} \), where if \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d) \) and if \( \theta(u) : C_2(\mathcal{M}) \rightarrow \mathbb{Z}_d \) is as above, then \( \delta^1(\zeta) = \theta(u) \).

\[ \dot{\delta}^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \rightarrow \mathbb{Z}_d \]

is a surjective homomorphism.

In the preceding section, the cyclic coverings of the Platonic maps by regular maps were classified. Because the Platonic maps are simply connected, these coverings are totally ramified. In general, if \( \mathcal{M} \) is a regular orientable map, and if \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)_{\alpha} \), where \( \alpha : \text{Aut}(\mathcal{M}) \rightarrow U(\mathbb{Z}_d) \), then by [5, Corollary 3.7.2] together with [5, Lemma 6.4], the principal covering \( \mathcal{M}_\zeta \rightarrow \mathcal{M} \) is regular, connected, and totally ramified precisely when

\[ \delta^1(\zeta) : C_2(\mathcal{M})/H_2(\mathcal{M}) \rightarrow \mathbb{Z}_d \]

is a surjective homomorphism.

As in the preceding discussion, we write \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \), where \( \langle a', b' \rangle, \langle a', c' \rangle \), and \( \langle b', c' \rangle \) are, respectively, the stabilizers of the face, edge and vertex of a fixed blade \( \beta \). From Lemma 2.2, the actions of \( \text{Aut}(\mathcal{M}) \) on \( \mathbb{Z}_d \) must be of the form \( \alpha = \alpha(u) : \text{Aut}(\mathcal{M}) \rightarrow \text{Aut}(\mathbb{Z}_d) \), where \( u \) is an involutory unit in \( \mathbb{Z}_d \). Recall that if \( \zeta \in D(\mathcal{M}; \mathbb{Z}_d)_{\alpha_{\text{St}}} \), then \( \delta^1(\zeta) \) kills \( H_2(\mathcal{M}) \) if and only if \( d|r \), where \( r \) is the number of faces of \( \mathcal{M} \).

We shall investigate in detail the unramified, Steinberg, and Accola coverings of the regular affine maps. These are of the form

\[ \mathcal{M} = (B = \Delta(k, l)/K, s_1, s_2, s_3) \],
where
\[ K \leq \Delta(k, l) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_3)^2 = (s_1s_2)^k = (s_2s_3)^l = 1 \rangle, \]
and where \((k, l) = (4, 4)\) or \((3, 6)\).

### 3.1 The Regular Affine Maps of Type \((4, 4)\)

The focus of this subsection will be on the regular affine maps of type \((4, 4)\).

The proof of the following can be extracted from [1, Proposition 4.1] or from [2, Section 7]:

**Proposition 3.1.** There is an isomorphism
\[ \Delta(4, 4) \cong \langle i, \tau \rangle \rtimes \mathbb{Z}[i], \]
(where \(i\) acts by right multiplication on the Gaussian integers \(\mathbb{Z}[i]\) and \(\tau\) acts as complex conjugation) given by
\[ s_1 \mapsto (\tau, 0), \ s_2 \mapsto (-i\tau, 0), \ s_3 \mapsto (-\tau, 1). \]
(Note that \(\langle i, \tau \rangle \cong D_8\).)

Next, from [1, Lemma 4.2], the normal subgroups of \(\Delta(4, 4)\) by which the quotients are nondegenerate maps can be identified with ideals \(I \subseteq \mathbb{Z}[i]\) which are invariant under complex conjugation. Since \(\mathbb{Z}[i]\) is a principal ideal domain, we may write any ideal as \(I = (a + bi)\mathbb{Z}[i]\); however, for \(I\) to be invariant under complex conjugation, there are only two possibilities:

1. \(I = m\mathbb{Z}[i]\), for some integer \(m\);
2. \(I = m(1 + i)\mathbb{Z}[i]\), for some integer \(m\).

Furthermore, note that under the above isomorphism
\[ s_2s_1s_2s_3 \mapsto (1, 1), \ s_1s_2s_3s_2 \mapsto (1, i); \]
also
\[ s_4s_2s_1s_3s_2s_3 \mapsto (1, 1 + i), \ s_2s_1s_3s_2s_3s_1 \mapsto (1, 1 - i). \]
Therefore, we conclude that in terms of the generators $s_1, s_2, s_3 \in \Delta(4, 4)$, these ideals of $\mathbb{Z}[i]$ correspond to normal subgroups of $\Delta(4, 4)$ as follows:

$$m \mathbb{Z}[i] \twoheadrightarrow \langle (s_2s_1s_2s_3)^m, (s_1s_2s_3s_2)^m \rangle, \quad m(1+i)\mathbb{Z}[i] \twoheadrightarrow \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m \rangle.$$

In turn, corresponding to these two possibilities, we have the following possible presentations for the monodromy group of the regular affine map $\mathcal{M}$ of type $(4, 4)$:

**Corollary 3.1.1.** The monodromy group of a finite regular (nondegenerate) affine map of type $(4, 4)$ has one of the following presentations:

1. $G = \langle a_1, b_1, c_1 | a_1^2 = b_1^2 = c_1^2 = (a_1c_1)^2 = (a_1b_1)^4 = (b_1c_1)^4 = (b_1a_1b_1c_1)^m = 1 \rangle$, for some positive integer $m$;

2. $G = \langle a_2, b_2, c_2 | a_2^2 = b_2^2 = c_2^2 = (a_2c_2)^2 = (a_2b_2)^4 = (b_2c_2)^4 = (a_2b_2a_2c_2b_2c_2)^m = 1 \rangle$, for some positive integer $m$. $\blacksquare$

Note first of all that the automorphism group of an affine map of type (1) above does not admit an Accola homomorphism unless $m$ is even. The automorphism group of type (2) always admits an Accola homomorphism. Next, notice that the affine map of type (1) has a “fundamental domain” in the shape of a square in the complex plane; that corresponding to type (2) has the shape of a diamond. Furthermore, if we denote the groups in family (1) above by $G(1, m)$ and those in family (2) by $G(2, m)$, where $m$ is a positive integer, then there is a two-fold covering $G(2, m) \to G(1, m)$, given by $a_2 \mapsto a_1$, $b_2 \mapsto b_1$, $c_2 \mapsto c_1$. Note that the kernel of the above mapping is the normal closure in $G(2, m)$ of the element $(b_2c_2b_2a_2)^m$. Similarly, there is a two-fold covering $G(1, 2m) \to G(2, m)$ given by $a_1 \mapsto a_2$, $b_1 \mapsto b_2$, $c_1 \mapsto c_2$, whose kernel is the normal closure in $G(1, 2m)$ of the element $(c_1b_1c_1a_1b_1a_1)^m$. Therefore, if $\mathcal{M}(1, m)$, $\mathcal{M}(2, m)$ are the corresponding regular affine maps, then there are unramified two-fold coverings

$$\mathcal{M}(2, m) \to \mathcal{M}(1, m), \quad \text{and} \quad \mathcal{M}(1, 2m) \to \mathcal{M}(2, m).$$

Let $\mathcal{M}$ be one of the affine maps $\mathcal{M}(1, m)$ or $\mathcal{M}(2, m)$. We may identify the blades of $\mathcal{M}$ with the group $G = \langle i, \tau \rangle \ltimes \mathbb{Z}[i]/I$. In turn the monodromy involutions of the monodromy group $G = \langle a, b, c \rangle$ are given by left multiplications by $a = (\tau, 0)$, $b = (-i\tau, 0)$, $c = (-\tau, 1)$, respectively. Similarly, the automorphism group $\text{Aut}(\mathcal{M})$ is generated by involutions $a', b', c'$ given by right multiplications.
by \((\tau, 0), (-i\tau, 0), (-\tau, 1)\), respectively. If we set \(D = \langle i, \tau \rangle \cong D_8\) (dihedral of order 8), we see that the faces of \(\mathcal{M}\) are of the form \(\{(x, w) \mid x \in D\}\), where \(w\) ranges over the cosets of \(I\) in \(\mathbb{Z}[i]\). As a result, we see that the normal subgroup \(N = \{(1, w) \mid w \in \mathbb{Z}[i]/I\}\) acts regularly by right multiplication on the faces of \(\mathcal{M}\). In other words, given the face \(f_w = \{(x, w) \mid x \in D\}\) and the element \(w' \in \mathbb{Z}[i]/I\), we have that

\[
f_w \cdot (1, w') = \{(x, w + w') \mid x \in D\}.
\]

Finally, from our observations above, we see that the corresponding fundamental groups are given by

\[
\pi_1(\mathcal{M}(1, m), (1, 0)) = \langle (s_2s_1s_2s_3)^m, (s_1s_2s_3)^m \rangle,
\]

\[
\pi_1(\mathcal{M}(2, m), (1, 0)) = \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m \rangle.
\]

When \(m = 2\) we can picture the blades of \(\mathcal{M} = \mathcal{M}(1, 2)\) as below. The darkened points represent the lattice points in the complex plane (and therefore parametrize the Gaussian integers \(\mathbb{Z}[i]\)). The blade \(\beta_0 = (1, 0)\) is indicated, as are the monodromy images \(a\beta_0, b\beta_0,\) and \(c\beta_0\):

![Figure 1. Monodromy Images of the Blade \(\beta_0\)](image)

### 3.1.1 The Regular Maps \(\mathcal{M}(1, m)\)—Unramified Coverings

In this subsection, we assume that \(\mathcal{M} = \mathcal{M}(1, m)\) and shall classify the unramified coverings \(\mathcal{M}' \to \mathcal{M}\) by regular connected maps and having a cyclic group \(\mathbb{Z}_d\) of covering transformations.
We have $\pi_1(\mathcal{M}(1,m),(1,0)) = \langle (s_2s_1s_2s_3)^m, (s_1s_2s_3s_2)^m \rangle$, which is a free abelian group of rank 2. Next, from [5, Section 5.3] we recall the homomorphism $h : \pi_1(\mathcal{M}, \beta_0) \to H_1(\mathcal{M}), \beta_0 = (1,0)$, inducing $\pi_1(\mathcal{M}, \beta_0) / \pi_1(\mathcal{M}, \beta_0) \cong H_1(\mathcal{M})$ ([5, Corollary 5.15.1]). By regularity, $\pi_1(\mathcal{M}(1,m),(1,0)) \leq G$, and so the homomorphism $h : \pi_1(\mathcal{M}(1,m),(1,0)) \to H_1(\mathcal{M}(1,m))$

is an $\text{Aut}(\mathcal{M})$-equivariant isomorphism, where the action of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ on $\pi_1(\mathcal{M}(1,m),(1,0))$ is simply conjugation by the elements $s_1, s_2, s_3$. We let $\epsilon : \mathbb{Z} \to \mathbb{Z}_d$ be the canonical projection, and set $\epsilon_* : H_1(\mathcal{M}) \to H_1(\mathcal{M}; \mathbb{Z}_d)$, the induced homomorphism in homology. Therefore, if we set $\gamma_x = s_2s_1s_2s_3, \gamma_y = s_1s_2s_3s_2$, and set

$$\partial / \partial x = \epsilon_* h(\gamma_x^m), \partial / \partial y = \epsilon_* h(\gamma_y^m) \in H_1(\mathcal{M}; \mathbb{Z}_d),$$

then we know that \{\partial / \partial x, \partial / \partial y\} is a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$. We denote by $dx, dy$ the corresponding dual $\mathbb{Z}_d$-basis elements of $H^1(M; \mathbb{Z}_d)$. Furthermore, by equivariance, we infer that

$$\partial / \partial x(a')_* = \epsilon_* h(s_1\gamma_x^m s_1), \partial / \partial y(a')_* = \epsilon_* h(s_1\gamma_y^m s_1)$$

with similar statements for the generators $b', c' \in \text{Aut}(\mathcal{M})$. Since $s_1\gamma_x s_1 = \gamma_x$, and since $s_1\gamma_y s_1 = \gamma_y^{-1}$ we conclude immediately that

$$\partial / \partial x(a')_* = \partial / \partial x, \partial / \partial y(a')_* = -\partial / \partial y.$$

Again, one likewise computes the effects of $b'$ and $c'$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$.

One has the following result.

**Proposition 3.2.** Assume that $\mathcal{M} = \mathcal{M}(1,m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(\mathcal{M})$ on $H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

$$(a')^* dx = dx, \quad (a')^* dy = -dy,$$

$$(b')^* dx = dy, \quad (b')^* dy = dx,$$

$$(c')^* dx = -dx, \quad (c')^* dy = dy$$

**Proof.** Very easy calculations show that that

$$(\partial / \partial x(a')^* = \partial / \partial x, \quad \partial / \partial y(a')^* = -\partial / \partial y,$$
\[ \frac{\partial}{\partial x}(b')_* = \frac{\partial}{\partial y}, \frac{\partial}{\partial y}(b')_* = \frac{\partial}{\partial x} \]
\[ \frac{\partial}{\partial x}(c')_* = -\frac{\partial}{\partial x}, \frac{\partial}{\partial y}(c')_* = \frac{\partial}{\partial y} \]

The corresponding representation of \( \text{Aut}(\mathcal{M}) \) on \( H^1(\mathcal{M}; \mathbb{Z}_d) \) is adjoint to the above (in terms of matrices over \( \mathbb{Z}_d \), the matrix representation is given by the transpose of the matrix representation on homology). The result follows. ■

In terms of the ordered \( \mathbb{Z}_d \)-basis \( (dx, dy) \) of \( H_1(\mathcal{M}; \mathbb{Z}_d) \), we have the matrix representation of \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \) below:

\[
\begin{align*}
 a' & \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\
b' & \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\
c' & \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{align*}
\]

Furthermore, an \( \alpha \)-isotypical class in \( H^1(\mathcal{M}; \mathbb{Z}_d) \) is realized by a “simultaneous eigenvector” for the above matrices. Thus, if \( \zeta = xdx + ydy \) is an \( \alpha \)-isotypical class, then the homomorphism \( \theta = \delta(\zeta) : H_1(\mathcal{M}) \to \mathbb{Z}_d \) is determined by \( \theta(\partial/\partial x) = x, \theta(\partial/\partial y) = y \). Therefore, we see that the principal derived map \( \mathcal{M}_\zeta \) is connected precisely when \( x, y \) together generate \( \mathbb{Z}_d \). However, note that \( \alpha(b')\zeta = (b')^*(\zeta) = ydx + xdy \), from which we conclude that \( \alpha(b')x = y, \alpha(b')y = x \).

Therefore, we conclude that
\[
\text{im}(\theta : H_1(\mathcal{M}) \to \mathbb{Z}_d) = \langle x \rangle = \langle y \rangle \subseteq \mathbb{Z}_d,
\]
from which it follows that \( x, y \) are units in \( \mathbb{Z}_d \).

Next applying the condition \( \alpha(a')\zeta = (a')^*\zeta = xdx - ydy \), we obtain the conditions
\[
\alpha(a')x = x, \quad \alpha(a')y = -y.
\]

Since \( x, y \in \mathbb{Z}_d \) are units, we infer that \( 1 = \alpha(a') = -1 \) in \( \mathbb{Z}_d \), which forces \( d = 2 \). It is easy to check that in this case, there is a unique \( \alpha \)-isotypical class, viz.,
\[
\zeta = dx + dy \in H^1(\mathcal{M}; \mathbb{Z}_2).
\]

Note that we have already observed above that \( \mathcal{M}(2, m) \to \mathcal{M}(1, m) \) is unramified and \( \mathcal{M}(2, m) \) is regular. We have, therefore, the following classification theorem.

**Theorem 3.3.** Up to \( \cong_{\mathcal{M}(1,m)} \), there is a unique nontrivial unramified covering \( \mathcal{M}' \to \mathcal{M}(1, m) \) by a regular connected map \( \mathcal{M}' \) and having cyclic group of covering transformations, namely, the 2-fold covering \( \mathcal{M}(2, m) \to \mathcal{M}(1, m) \). ■
3.1.2 The Regular Maps $\mathcal{M}(1,m)$—The Steinberg Coverings

In this subsection we determine the Steinberg coverings of the affine map $\mathcal{M} = \mathcal{M}(1,m)$, and having cyclic group $\mathbb{Z}_d$ of covering transformations. Recall that since $\mathcal{M}$ has $m^2$ faces, then $d$ must satisfy the condition $d \mid m^2$.

Let $\beta_1, \beta_2, \ldots, \beta_{m^2}$ be blades in the same $G^+$-orbit generating the $m^2$ faces of $\mathcal{M}$, and set $m_i = \chi(\beta_i)_{p+} + W^{(a,b)} \in C_2(\mathcal{M}), i = 1, 2, \ldots, m^2$. Then, up to a unit multiple in $\mathbb{Z}_d$, the Aut($\mathcal{M}$)-module homomorphism $\theta = \theta_{St} : C_2(\mathcal{M}) \to \mathbb{Z}_d$ satisfies $\theta(m_i) = 1$, $i = 1, 2, \ldots, m^2$.

We shall begin by explicitly constructing an element $z \in (\delta^1)^{-1}(\theta) \subseteq D(\mathcal{M}; \mathbb{Z}_d)$ as follows. Start with the blade $\beta_0 = (1,0)$, and define the blade $\beta_1 = b(1,0) = (-i\tau,0)(1,0) = (-i\tau,0)$. Define the voltages $z_1, z_2$ via the assignments

\[
\begin{align*}
z_1 : & \beta_1 \mapsto -1, \quad c\beta_1 \mapsto -1, \quad a\beta_1 \mapsto 1, \quad ac\beta_1 \mapsto 1, \\
z_2 : & \beta_0 \mapsto 1, \quad c\beta_0 \mapsto 1, \quad a\beta_0 \mapsto -1, \quad ac\beta_0 \mapsto -1,
\end{align*}
\]

and where all remaining blades are mapped to 0. Next, define the automorphisms $\sigma_1, \sigma_2 \in \text{Aut}(\mathcal{M})$ to be translations one unit in the horizontal and vertical directions, respectively. Therefore,

$$\beta = (x,w) \mapsto (x,w+1) = \beta \sigma_1, \quad \text{and} \quad \beta = (x,w) \mapsto (x,w+i) = \beta \sigma_2.$$ 

Therefore, $\sigma_1$ and $\sigma_2$ represent right multiplications in $\langle i, \tau \rangle \ltimes \mathbb{Z}[i]$ by $(1,1)$ and $(1,i)$, respectively.

In terms of the translations $\sigma_1, \sigma_2 \in \text{Aut}(\mathcal{M})$ defined above, we define

$$z'_1 = \sum_{k=0}^{m-2} \sum_{j=0}^{m-1} (k+1)(\sigma_2^{-k})^*(\sigma_1^{-j})^* z_1.$$ 

We indicate the values of $z'_1$ for $m = 4$ below:
To the voltage $z'_1$ we add the voltage $z'_2$ given by

$$z'_2 = \sum_{k=0}^{m-2} (k + 1)m(\sigma_1^{-k})^*\sigma_2^*z_2,$$

obtaining the voltage $z_{St} = z'_1 + z'_2$. In what follows, we shall identify the voltage $z_{St}$ with the voltage class it determines in $D(M; \mathbb{Z}_d)$.

If $m = 4$, $z_{St}$ is as follows:
One checks that if $d|m^2$, then $\delta^1(z_{St}) = \theta$. Note, however, that even though $\theta$ is a surjective $\text{Aut}(M)$-module homomorphism $C_2(M)/H_2(M) \to \mathbb{Z}_d$, one cannot conclude that $z_{St} \in D(M; \mathbb{Z}_d)$. This will be addressed momentarily.

We denote by $\mathbb{Z}_d(\theta)$ the $\mathbb{Z}_d$-submodule of $\text{Hom}(C_2(M)/H_2(M), \mathbb{Z}_d)$ generated by $\theta$ and set $D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d([z_{St}]) + H^1(M; \mathbb{Z}_d)$. Since $\theta : C_2(M)/H_2(M) \to \mathbb{Z}_d$ is surjective, it has order $d$ in the abelian group $\text{Hom}(C_2(M)/H_2(M), \mathbb{Z}_d)$. Therefore, $z_{St}$ likewise must have order $d$ in the abelian group $D(M; \mathbb{Z}_d)$, from which it follows that the above sum is direct: $D_\theta = \mathbb{Z}_d([z_{St}]) \oplus H^1(M; \mathbb{Z}_d)$. In turn, one easily sees that $\{z_{St}, dx, dy\}$ is a $\mathbb{Z}_d$-basis of the free $\mathbb{Z}_d$-module $D_\theta$.

In terms of this basis, the action of $\text{Aut}(M)$ on $D_\theta$ is given below:
Proposition 3.4. Assume that \( \mathcal{M} = \mathcal{M}(1, m) \), and that \( d \) is a positive integer dividing \( m^2 \). Then the action of \( \text{Aut}(\mathcal{M}) \) on \( D_0 = \mathbb{Z}_d([z_{St}]) \oplus H^1(\mathcal{M}; \mathbb{Z}_d) \) is given by

\[
\begin{align*}
(a')^* dx &= dx, & (a')^* dy &= -dy, \\
(b')^* dx &= dy, & (b')^* dy &= dx, \\
(c')^* dx &= -dx, & (c')^* dy &= dy, \\
(a')^* z_{St} &= -z_{St} - m dx, & (b')^* z_{St} &= -z_{St}, & (c')^* z_{St} &= -z_{St} + 2m dy.
\end{align*}
\]

Proof. Note first that if \( g' \in \{a', b', c'\} \), then since \( \alpha_{St}(g') = -1 \) we infer that \( g'z_{St} \equiv -z_{St} \pmod{H_1(\mathcal{M}; \mathbb{Z}_d)} \). Therefore, if \((g')^* z_{St} = -z_{St} + x dx + y dx\), then the coefficients \( x, y \) can be computed via

\[
\begin{align*}
x &= \partial / \partial x((g')^* z_{St} + z_{St}), \\
y &= \partial / \partial y((g')^* z_{St} + z_{St}).
\end{align*}
\]

For example, if \( m = 4 \), we illustrate \((c')^* z_{St} + z_{St}\), together with \( \partial / \partial x, \partial / \partial y \) below:

\[
\begin{array}{cccccccc}
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
-8 & -8 & -8 & -8 & -8 & -8 & -8 & -8 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\partial / \partial x \\
\beta_0 \\
\end{array}
\]

Figure 4. The Voltage \((c')^* z_{St} + z_{St}\), and the Homology Basis Elements \( \partial / \partial x, \partial / \partial y \)
where in the above, a subdiagram of the form

\[
\begin{array}{c}
\beta \\
\downarrow \\
c\beta \\
\end{array}
\]

represents \( \chi_{\beta} + \chi_{c\beta} + W^{(a,c)} \in C_1(\mathcal{M}; \mathbb{Z}_d) \).

Thus in the above we see that \((c')^*z_{\text{St}} + z_{\text{St}} = 8dy\); in general it is easy to see that \((c')^*z_{\text{St}} + z_{\text{St}} = 2mdy\). The remaining calculations are entirely similar.

In terms of matrices over \( \mathbb{Z}_d \), and relative to the ordered basis \((z_{\text{St}}, dx, dy)\) of \( \mathbb{Z}_d(z_{\text{St}}) \oplus H^1(\mathcal{M}; \mathbb{Z}_d) \), we have the representation of \( \text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle \) given by

\[
\begin{align*}
a' &\mapsto \begin{pmatrix} -1 & 0 & 0 \\ -m & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\
b' &\mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
c' &\mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2m & 0 & 1 \end{pmatrix}.
\end{align*}
\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map \( \mathcal{M}(1, m) \). Note that if \( \zeta \in (\delta^1)^{-1}(\theta) \subseteq D_\theta \) is \( \alpha \)-isotypical for some \( \alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), then \( \zeta \) is of the form \( \zeta = z_{\text{St}} + xd + yd \) for suitable \( x, y \in \mathbb{Z}_d \). We have the following:

**Theorem 3.5.** Let \( \mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m) \) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \( \mathbb{Z}_d \) and inducing the action \( \alpha = \alpha_{\text{St}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), by a regular connected map. Then \( d|m \); furthermore, up to \( \cong_{\mathcal{M}(1, m)}, \mathcal{M}' = \mathcal{M}_\zeta \), where \( \zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d) \) as below:

1. If \( d \) is odd, then up to unit multiples in \( \mathbb{Z}_d \), \( \zeta = z_{\text{St}} \in D(\mathcal{M}; \mathbb{Z}_d) \) is the unique such class;
2. If \( d = 2d_0 \), then up to unit multiples in \( \mathbb{Z}_d \), there are two possibilities for \( \zeta \), namely, \( \zeta = z_{\text{St}} \) and \( \zeta = z_{\text{St}} + d_0dx + d_0dy \).
Proof. Note first that since \( \alpha_{St}(a') = \alpha_{St}(b') = \alpha_{St}(c') = -1 \), then an \( \alpha_{St} \)-isotypical voltage \( \zeta = z_{St} + x dx + y dy \), must satisfy 
\[-(z_{St} + x dx + y dy) = -\zeta = \alpha_{St}(a') \zeta = \alpha_{St}(b') \zeta = \alpha_{St}(c') \zeta = \alpha_{St} - (m - x) dx - y dy, \]
from which we conclude that \( 2x = m \). Next, 
\[-\zeta = \alpha_{St}(b') \zeta = \alpha_{St}(c') \zeta = \alpha_{St}(a') \zeta = -(z_{St} + x dx + y dy) = -z_{St} + y dx + x dy, \]
and so \( y = -x \). Finally, we have 
\[-(z_{St} + x dx + y dy) = -\zeta = \alpha_{St}(a') \zeta = \alpha_{St}(b') \zeta = \alpha_{St}(c') \zeta = -(z_{St} + x dx + (2m + y) dy), \]
and so \( -2y = 2m \). Therefore, \( m = 2x = 2y = 2m \) and so \( m = 0 \), i.e., \( d|m \), as claimed. From this the remaining assertions are entirely routine. 

3.1.3 The Regular Maps \( M(1, m) \)—The Accola Coverings

We turn now to the Accola coverings of \( M(1, m) \). Therefore, we must assume that \( m \) is even. Note first that in terms of the identification of \( \Delta(4, 4) \) with \( \langle i, \tau \rangle \times Z[2i] \), then \( \Delta^+ = \langle i \rangle \times Z[i] \). If \( \text{Aut}(M) \) is the automorphism group of \( M(1, m) \), with subgroup \( M(1, m)^+ \), then it is routine to check that 
\[ K^+ := \ker(\alpha_{Acc} : \text{Aut}(M)^+ \to \{\pm 1\}) = \langle i \rangle \times (1 + i)Z[i]. \]
Therefore, we see that the faces in a \( K^+ \)-orbit form a “checkerboard” pattern inasmuch as no two faces sharing an edge can be in the same \( K^+ \)-orbit. From this, it is easy to determine an element \( z_{Acc} \) such that \( \delta^1(z_{Acc}) = \theta_{Acc} : C_2(M)/H_2(M) \to Z_d \).

To do this, let \( \beta_1 = b\beta_0 \) and recall the voltage \( z_1 \) given in the previous subsection:
\[ z_1 : \beta_1 \mapsto -1, \ c\beta_1 \mapsto -1, \ a\beta_1 \mapsto 1, \ ac\beta_1 \mapsto 1. \]
Let \( m = 2n \) and set
\[ z_{Acc} = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} (-1)^j (\sigma^{-2k}_2)^* (\sigma^{-j}_1)^* z_1. \]
As above, we shall not draw a distinction between the voltage \( z_{Acc} \) and the class it determines in \( D(M; Z_d) \).

If \( m = 4 \), then \( z_{Acc} \) is depicted as below:
We set $\theta = \theta_{\text{Acc}}$ and set $D_\theta = (\delta^1)^{-1}(Z_d(\theta)) = Z_d([z_{\text{Acc}}]) \oplus H^1(\mathcal{M}; Z_d)$. Note that

$$(a')^*z_{\text{Acc}} + z_{\text{Acc}}, (b')^*z_{\text{Acc}} + z_{\text{Acc}}, (c')^*z_{\text{Acc}} - z_{\text{Acc}} \in H^1(\mathcal{M}; Z_d).$$

Arguing as with the Steinberg voltage (the present case is even easier), one obtains

\begin{align*}
(a')^*dx &= dx, \quad (a')^*dy = -dy, \\
(b')^*dx &= dy, \quad (b')^*dy = dx, \\
(c')^*dx &= -dx, \quad (c')^*dy = dy,
\end{align*}

Figure 5. The Voltage $z_{\text{Acc}}$
\[(a')^* z_{\text{Acc}} = -z_{\text{Acc}},\]
\[(b')^* z_{\text{Acc}} = -z_{\text{Acc}},\]
\[(c')^* z_{\text{Acc}} = z_{\text{Acc}},\]

In terms of matrices over \(\mathbb{Z}_d\), and relative to the ordered basis \((z_{\text{Acc}}, dx, dy)\) of \(\mathbb{Z}_d \langle z_{\text{Acc}} \rangle \oplus H^1(M;\mathbb{Z}_d)\) the representation of \(\text{Aut}(M) = \langle a', b', c' \rangle\) given by

\[
a' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix},
\quad b' \mapsto \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix},
\quad c' \mapsto \begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

The above representation easily implies the following classification of the Accola coverings of \(M(1, m)\).

**Theorem 3.7.** Let \(M' \to M = M(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{Acc}} : \text{Aut}(M) \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then up to \(\cong_{M(1,m)}\), \(M' = M_z\), where \(\zeta = [z] \in \text{D}(M;\mathbb{Z}_d)\) as below:

1. If \(d\) is odd, then \(\zeta = z_{\text{Acc}} \in \text{D}(M;\mathbb{Z}_d)\) is the unique such class;
2. If \(d = 2d_0\) is even, there are two possibilities for \(\zeta\), namely, \(\zeta = z_{\text{Acc}}\) and \(\zeta = z_{\text{Acc}} + d_0 dx + d_0 dy\). ■

**Remark:** Note that if \(d = 2\) then the Steinberg and Accola coverings of \(M(1, m)\) coincide; up to \(\cong_{M(1,m)}\); there are two such.

### 3.1.4 The Regular Maps \(M(2, m)\)—Unramified Coverings

We turn now to the maps \(M = M(2, m)\) and start by determining the unramified coverings by connected regular maps. We shall first determine \(\mathbb{Z}_d\)-bases for \(H_1(M;\mathbb{Z}_d)\) and \(H^1(M;\mathbb{Z}_d)\). We recall that the fundamental group of \(M\) is given by \(\pi_1(M;\beta_0) = \langle (s_1s_2s_1s_3s_2s_3)^m, (s_2s_1s_3s_2s_3s_1)^m, \rangle\), where \(\beta_0 = (1, 0)\). Set \(\gamma_1 = (s_1s_2s_1s_3s_2s_3), \gamma_2 = (s_2s_1s_3s_2s_3s_1)\) and define the cycles \(\partial/\partial \alpha_1 = \epsilon, h(\gamma_1^m), \partial/\partial \alpha_2 = \epsilon, h(\gamma_2^m) \in H_1(M;\mathbb{Z}_d), \) where, as usual, \(\epsilon : \mathbb{Z} \to \mathbb{Z}_d\) is the projection map. As
above, we know that \( \{ \partial/\partial \alpha_1, \partial/\partial \alpha_2 \} \) is a \( \mathbb{Z}_d \)-basis of \( H_1(\mathcal{M}; \mathbb{Z}_d) \). We denote by \( \{ d\alpha_1, d\alpha_2 \} \) the corresponding \( \mathbb{Z}_d \)-dual basis of \( H^1(\mathcal{M}; \mathbb{Z}_d) \).

The action of \( \text{Aut}(\mathcal{M}) \) on \( H^1(\mathcal{M}; \mathbb{Z}_d) \), for any non-negative integer \( d \), is determined below.

**Proposition 3.8.** Assume that \( \mathcal{M} = \mathcal{M}(2, m) \), and that \( d \) is a non-negative integer. Then the action of \( \text{Aut}(\mathcal{M}) \) on \( H^1(\mathcal{M}; \mathbb{Z}_d) \) is given by

\[
\begin{align*}
(a')^*(d\alpha_1) &= d\alpha_2, \\
(b')^*(d\alpha_1) &= d\alpha_1, \quad (b')^*(d\alpha_2) = -d\alpha_2, \\
(c')^*(d\alpha_1) &= -d\alpha_2.
\end{align*}
\]

**Proof.** The above result is obtained by first computing the effects of the generators \( a', b', c' \in \text{Aut}(\mathcal{M}) \) on \( H_1(\mathcal{M}; \mathbb{Z}_d) \), which are computed by conjugating the elements \( \gamma_1^m \) and \( \gamma_2^m \) by the elements \( s_1, s_2, s_3 \), respectively. The action on the \( \mathbb{Z}_d \)-dual \( H^1(\mathcal{M}; \mathbb{Z}_d) \) is obtained by dualizing (take matrix transposes). \( \square \)

In analogy with the case \( \mathcal{M} = \mathcal{M}(1, m) \), one easily concludes that the only nontrivial isotypical class in \( H^1(\mathcal{M}; \mathbb{Z}_d) \) giving a connected principal derived map is \( \zeta = d\alpha_1 + d\alpha_2 \in H^1(\mathcal{M}; \mathbb{Z}_2) \). The following is now immediate:

**Theorem 3.9.** Up to \( \cong_{\mathcal{M}(2,m)} \), there is a unique nontrivial unramified covering \( \mathcal{M}' \rightarrow \mathcal{M}(2, m) \) by a regular connected map \( \mathcal{M}' \) and having cyclic group of covering transformations, namely, the 2-fold covering \( \mathcal{M}(1, 2m) \rightarrow \mathcal{M}(2, m) \). \( \square \)

We turn now to the classification of the Steinberg and Accola coverings of \( \mathcal{M}(2, m) \).

**3.1.5 The Regular Maps \( \mathcal{M}(2, m) \)—The Steinberg Coverings**

In order to compute the totally ramified coverings of \( \mathcal{M}(2, m) \), it shall be convenient to capitalize on the 2-fold covering \( p: \mathcal{M}(2, m) \rightarrow \mathcal{M}(1, m) \). To this end, note first that the calculations

\[
s_1s_2s_1s_3s_2s_3 = s_2s_1s_2s_3 \cdot s_1s_2s_3s_2 \quad \text{and} \quad s_2s_1s_3s_2s_3s_1 = s_2s_1s_2s_3 \cdot (s_1s_2s_3s_2)^{-1}
\]
imply that under the induced mapping in homology \( p_* : H_1(M(2, m); \mathbb{Z}_d) \to H_1(M(1, m); \mathbb{Z}_d) \) we have

\[ p_* : \partial/\partial \alpha_1 \mapsto \partial/\partial x + \partial/\partial y, \quad \partial/\partial \alpha_2 \mapsto \partial/\partial x - \partial/\partial y. \]

Therefore, if \( p^*(dx) = a_1 d\alpha_1 + a_2 d\alpha_2 \), then

\[ a_1 = \partial/\partial \alpha_1 (a_1 d\alpha_1 + a_2 d\alpha_2) = \partial/\partial \alpha_1 (p^*(dx)) = (\partial/\partial x + \partial/\partial y)(dx) = 1. \]

Likewise, one obtains \( a_2 = 1 \), forcing \( p^*(dx) = d\alpha_1 + d\alpha_2 \). Similarly, one obtains

\[ p^*(dy) = d\alpha_1 - d\alpha_2. \]

We summarize:

**Lemma 3.10.** Relative to the 2-fold unramified covering \( p : M(2, m) \to M(1, m) \), we have

\[ p^*(dx) = d\alpha_1 + d\alpha_2, \quad p^*(dy) = d\alpha_1 - d\alpha_2. \]

We have \( \text{Aut}(M(1, m)) = \langle a', b', c' \rangle \); since \( M(2, m) \) is regular, then \( a', b', c' \) lift to generators of \( \text{Aut}(M(2, m)) \); we shall, for convenience (and simplicity), continue to denote these lifted automorphisms of \( M(2, m) \) also as \( a', b', \) and \( c' \). As a result, we see that if \( \zeta \in D(M(1, m); \mathbb{Z}_d) \), then

\[ (a')^* p^*(\zeta) = p^*((a')^*\zeta) \]

with similar results for \( b', c' \).

In analyzing the Steinberg coverings of \( M(2, m) \) we start by recalling the voltage \( z^1_{St} := z_{St} \in D(M(1, m); \mathbb{Z}_d) \) defined in Subsection 3.1.2; in terms of this, define

\[ z^2_{St} = p^*(z^1_{St}) \in D(M(2, m); \mathbb{Z}_d). \]

It is clear that \( \delta^1(z^2_{St}) = \theta_{St} : C_2(M(2, m))/H_2(M(2, m)) \to \mathbb{Z}_d \). Therefore, if \( \theta = \theta_{St} \) then \( (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(z^2_{St}) \oplus H^1(M(2, m); \mathbb{Z}_d) \). We already have enough information to determine the action of \( \text{Aut}(M(2, m)) \) on \( D_{\theta} = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) \). Using the results of **Proposition 3.4**, we compute:
\[(a')^*(z_{St}^2) = (a')^*p^*(z_{St}^1) = p^*((a')^*(z_{St}^1)) = p^*(-z_{St}^1 - mdx) = -z_{St}^2 - md\alpha_1 - md\alpha_2.\]

Next,
\[(b')^*(z_{St}^2) = (b')^*p^*(z_{St}^1) = p^*((b')^*(z_{St}^1)) = p^*(-z_{St}^1) = -z_{St}^2.\]

\[(c')^*(z_{St}^2) = (c')^*p^*(z_{St}^1) = p^*((c')^*(z_{St}^1)) = p^*(-z_{St}^1 + 2mdy) = -z_{St}^2 + 2md\alpha_1 - 2md\alpha_2.\]

We summarize below the action of \(\text{Aut}(\mathcal{M}(2, m))\) on \(D_\theta\):

**Proposition 3.11.** Assume that \(\mathcal{M} = \mathcal{M}(2, m)\) and that \(d\) is a positive integer dividing \(2m^2\). Then the action of \(\text{Aut}(\mathcal{M})\) on \(D_\theta = \mathbb{Z}_d\langle z_{St}^2 \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)\) is determined by

\[
\begin{align*}
(a')^*(d\alpha_1) &= d\alpha_2, \\
(b')^*(d\alpha_1) &= d\alpha_1, \quad (b')^*(d\alpha_2) = -d\alpha_2, \\
(c')^*(d\alpha_1) &= -d\alpha_2, \\
(a')^*(z_{St}^2) &= -z_{St}^2 - md\alpha_1 - md\alpha_2, \\
(b')^*(z_{St}^2) &= -z_{St}^2, \\
(c')^*(z_{St}^2) &= -z_{St}^2 + 2md\alpha_1 - 2md\alpha_2.
\end{align*}
\]
In terms of the ordered $\mathbb{Z}_d$-basis $(z_2^{\text{St}}, d\alpha_1, d\alpha_2)$ of $D_\theta$, we have the matrix representation of $\text{Aut}(M) = \langle a', b', c' \rangle$:

$$
ad' \mapsto \begin{bmatrix} -1 & 0 & 0 \\
-m & 0 & 1 \\
-m & 1 & 0 
\end{bmatrix},

b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 
\end{bmatrix},

c' \mapsto \begin{bmatrix} -1 & 0 & 0 \\
2m & 0 & -1 \\
-2m & -1 & 0 
\end{bmatrix}.$$

The above provides the following classification of the Steinberg coverings of the regular affine map $M(2,m)$.

**Theorem 3.12.** Let $M' \rightarrow M = M(2,m)$ be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with $\mathbb{Z}_d$ and inducing the action $\alpha = \alpha_\text{St} : \text{Aut}(M) \rightarrow \text{Aut}(\mathbb{Z}_d)$, by a regular connected map. Then $d|m$; furthermore, up to $\cong_{M(2,m)}$, $M' = M_\zeta$, where $\zeta = [z] \in D(M; \mathbb{Z}_d)$ as below:

1. If $d$ is odd, then up to unit multiples in $\mathbb{Z}_d$, $\zeta = z_2^{\text{St}} \in D(M; \mathbb{Z}_d)$ is the unique such class;

2. If $d = 2d_0$, then up to unit multiples in $\mathbb{Z}_d$, there are two possibilities for $\zeta$, namely, $\zeta = z_2^{\text{St}}$ and $\zeta = z_2^{\text{St}} + d_0 d\alpha_1 + d_0 d\alpha_2$.

**Proof.** We let $\zeta = z_2^{\text{St}} + a_1 d\alpha_1 + a_2 d\alpha_2 \in D(M; \mathbb{Z}_d)$ be an $\alpha_\text{St}$-isotypical vector. From $(a')^* \zeta = -\zeta$ we obtain the condition $a_1 + a_2 = m$. From $(b')^* \zeta = -\zeta$ we learn that $2a_1 = 0 \in \mathbb{Z}_d$. Finally from $(c')^* \zeta = -\zeta$ we have $a_2 - a_1 = 2m$. These conditions jointly imply that $m = 0 \in \mathbb{Z}_d$ (and so $d|m$) and so $a_2 = -a_1 = a_1$. Everything follows. $

3.1.6 The Regular Maps $M(2,m)$—The Accola Coverings

Finally, we turn to the classification of the Accola coverings of the maps $M(2,m)$. As in the case $M = M(1,m)$ we construct an element $z_2^{\text{Acc}} \in (\delta^1)^{-1}(\theta)$, where $\theta = \theta_\text{Acc}$, as follows. We set

$$z_2^{\text{Acc}} = p^*(z_2^{\text{Acc}}) \in D(M(2,m); \mathbb{Z}_d),$$

where $z_2^{\text{Acc}} := z_2^{\text{Acc}}$ is as in Subsection 3.1.3; note that $\delta^1(z_2^{\text{Acc}}) = \theta_\text{Acc}$.

Using the results of Proposition 3.6 together with Lemma 3.10 we can compute the effect of $a', b'$, and $c'$ on the $\mathbb{Z}_d$-basis elements $z_2^{\text{Acc}}, d\alpha_1, d\alpha_2$ of $(\delta^1)^{-1}(\theta) = \mathbb{Z}_d(z_2^{\text{Acc}}) \oplus H^1(M(2,m); \mathbb{Z}_d)$. First, we have
\[(a')^*(z_{\text{Acc}}^2) = \begin{array}{c}
(a')^*p^*(z_{\text{Acc}}^1) \\
p^*((a')^*(z_{\text{Acc}}^1)) \\
p^*(-z_{\text{Acc}}^1) \\
-z_{\text{Acc}}^2.
\end{array}\]

Next,
\[(b')^*(z_{\text{Acc}}^2) = \begin{array}{c}
(b')^*p^*(z_{\text{Acc}}^1) \\
p^*((b')^*(z_{\text{Acc}}^1)) \\
p^*(-z_{\text{Acc}}^1) \\
-z_{\text{Acc}}^2.
\end{array}\]

\[(c')^*(z_{\text{Acc}}^2) = \begin{array}{c}
(c')^*p^*(z_{\text{Acc}}^1) \\
p^*((c')^*(z_{\text{Acc}}^1)) \\
p^*(z_{\text{Acc}}^1) \\
z_{\text{Acc}}^2.
\end{array}\]

One therefore has the following matrix representation of \(\text{Aut}(\mathcal{M}(2, m)) = \langle a', b', c' \rangle\) on \(\mathbb{Z}_d(z_{\text{Acc}}^2) \oplus H^1(\mathcal{M}(2, m); \mathbb{Z}_d)\) relative to the \(\mathbb{Z}_d\)-basis \((z_{\text{Acc}}^2, d\alpha_1, d\alpha_2)\):

\[
a' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}, \quad b' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad c' \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]

From the above, we may extract the classification of the Accola coverings of the map \(\mathcal{M}(2, m)\):

**Theorem 3.13.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(2, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{Acc}} : \mathcal{M} \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then up to \(\cong_{\mathcal{M}(2, m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(z = [z] \in D(\mathcal{M}; \mathbb{Z}_d)\) as below:

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(1) If \( d \) is odd, then \( \zeta = z_{\text{Acc}}^2 \in D(M; \mathbb{Z}_d) \) is the unique such class;

(2) If \( d = 2d_0 \) is even, there are two possibilities for \( \zeta \), namely, \( \zeta = z_{\text{Acc}}^2 \) and 
\( \zeta = z_{\text{Acc}}^2 + d_0d\alpha_1 + d_0d\alpha_2 \).

**Remark:** Note that if \( d = 2 \) then the Steinberg and Accola coverings of \( M(2,m) \) coincide (up to \( \cong_{M(1,m)} \), there are two such).

### 3.2 The Regular Affine Maps of Type \((3,6)\)

In this subsection we consider the regular affine maps of type \((3,6)\). These maps are represented in the form

\[ M = M(\Delta/K, s_1, s_2, s_3), \]

where \( \Delta = \Delta(3,6) \),

\[ \Delta(3,6) = \langle s_1, s_2, s_3 \mid s_i^2 = (s_1s_3)^2 = (s_1s_2)^3 = (s_2s_3)^6 = 1 \rangle, \]

and where \( K \triangleleft \Delta \). Again, applying [1, Proposition 4.1] or [2, Section 7], we have:

**Proposition 3.14.** There is an isomorphism

\[ \Delta(3,6) \cong \langle \omega, \tau \rangle \ltimes \mathbb{Z}[\omega], \]

where \( \omega \) acts by right multiplication on \( \mathbb{Z}[\omega] \) and \( \tau \) acts as complex conjugation. An isomorphism is given by

\[ s_1 \mapsto (-\tau, 1), \quad s_2 \mapsto (\tau\omega, 0), \quad s_3 \mapsto (\tau, 0). \]

(Note that \( \langle \omega, \tau \rangle \cong D_{12} \).)

With the above isomorphism, we may view the blades of an affine map of type \((3,6)\) as below. The *bullets* (●) indicate the vertices of an affine map having vertex valency 6 and face valency 3; in viewing the affine maps of vertex valency 3 and face valency 6, the bullets identify the faces. Under the above isomorphism \( \Delta \cong \langle \omega, \tau \rangle \ltimes \mathbb{Z}[\omega] \), if \( \beta_0 = (1, 0) \), then \( \beta_0, a\beta_0, b\beta_0, \) and \( c\beta_0 \) are marked below.
The following is routine. Note first that it is well known that $\mathbb{Z}[\omega]$ is a principal ideal domain.

**Lemma 3.15.** The only ideals of $\mathbb{Z}[\omega]$ invariant under complex conjugation are of the form $a\mathbb{Z}[\omega]$ and $(a + a\omega)\mathbb{Z}[\omega]$, where $a \in \mathbb{Z}$.

We shall denote by $\mathcal{M}(1, m)$ the regular maps corresponding to the ideal $a\mathbb{Z}[\omega]$ and by $\mathcal{M}(3, m)$ the regular maps corresponding to the ideal $(a + a\omega)\mathbb{Z}[\omega]$. Note that $[a\mathbb{Z}[\omega] : (a + a\omega)\mathbb{Z}[\omega]] = 3 = [(a + a\omega)\mathbb{Z}[\omega] : 3a\mathbb{Z}[\omega]]$. Therefore, we have three-fold coverings $\mathcal{M}(3, m) \to \mathcal{M}(1, m)$ and $\mathcal{M}(1, 3m) \to \mathcal{M}(3, m)$.
Note that under the above isomorphism $\Delta \cong \langle \omega, \tau \rangle \times \mathbb{Z}[\omega]$, we have that $s_2s_3s_2s_3s_2s_1 \mapsto (1, 1)$ and $s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1 \mapsto (1, 1 + \omega)$. Therefore, the monodromy groups and fundamental groups of the maps $\mathcal{M}(1, m)$ and $\mathcal{M}(3, m)$ are as given below:

Type $\mathcal{M}(1, m)$:

$$G = \langle a_1, b_1, c_1 \mid a_1^2 = b_1^2 = c_1^2 = (a_1c_1)^2 = (a_1b_1)^3 = (b_1c_1)^6 = (b_1c_1b_1c_1b_1a_1)^m = 1 \rangle,$$

for some positive integer $m$, with fundamental group

$$\pi_1(\mathcal{M}(1, m), (1, 0)) = \langle (s_2s_3s_2s_3s_2s_1)^m, (s_3s_2s_3s_2s_3s_1s_2)^m \rangle.$$

Type $\mathcal{M}(3, m)$:

$$G = \langle a_3, b_3, c_3 \mid a_3^2 = b_3^2 = c_3^2 = (a_3c_3)^2 = (a_3b_3)^3 = (b_3c_3)^6 = (c_3b_3c_3b_3c_3a_3b_3c_3b_3a_3)^m = 1 \rangle,$$

for some positive integer $m$ with fundamental group

$$\pi_1(\mathcal{M}(1, 3m), (1, 0)) = \langle (s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1)^m, (s_3s_1s_2s_3s_2s_3s_1s_3s_2s_3s_2)^m \rangle;$$

(note that $s_3s_1s_2s_3s_2s_3s_1s_3s_2s_3 \mapsto (1, -1 + \omega^2)$.

Thus, denoting the monodromy groups of the maps $\mathcal{M}(1, m)$ by $G(1, m)$ and denoting the monodromy groups of the maps $\mathcal{M}(3, m)$ by $G(3, m)$ we see that $a_3 \mapsto a_1$, $b_3 \mapsto b_1$ and $c_3 \mapsto c_1$ gives a three-fold covering $G(3, m) \rightarrow G(1, m)$ whose kernel is the normal closure in $G(3, m)$ of the element $(b_3c_3b_3c_3b_3a_3)^m$. Likewise, there is a three-fold covering $G(1, 3m) \rightarrow G(3, m)$ given by $a_1 \mapsto a_3$, $b_1 \mapsto b_3$ and $c_1 \mapsto c_3$ and whose kernel is the normal closure in $G(1, 3m)$ of the element $(c_1b_1c_1b_1c_1b_1c_1b_1a_1)^m$.

We turn now to the cyclic coverings of the regular maps of type $(3, 6)$.

### 3.2.1 The Regular Maps $\mathcal{M}(1, m)$—Unramified Coverings

As in the previous sections, if we denote $\gamma_x = s_2s_3s_2s_3s_2s_1$, $\gamma_\omega = s_3s_2s_3s_2s_1s_2$, then the fundamental group of the affine map $\mathcal{M} = \mathcal{M}(1, m)$ is generated by $\gamma_x^m$, $\gamma_\omega^m$. Therefore, the elements

$$\partial / \partial x = \epsilon_* h(\gamma_x^m), \quad \partial / \partial \omega = \epsilon_* h(\gamma_\omega^m) \in H_1(\mathcal{M}; \mathbb{Z}_d)$$

together form a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$. We denote by $\{dx, d\omega\}$ the corresponding dual basis of $H^1(\mathcal{M}; \mathbb{Z}_d)$. The action of $\mathcal{M} = \langle a', b', c' \rangle$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$ is given by
conjugation by $s_1, s_2, s_3$ on the elements $\gamma_x, \gamma_\omega$. Since $s_1 \gamma_x s_1 = \gamma_x^{-1}, s_1 \gamma_\omega s_1 = \gamma_x^{-1} \gamma_\omega$ we infer that

$$\partial / \partial x(a')_* = -\partial / \partial x, \quad \partial / \partial \omega(a')_* = -\partial / \partial x + \partial / \partial \omega.$$  

Likewise, one computes that

$$\partial / \partial x(b')_* = \partial / \partial \omega, \quad \partial / \partial x(c')_* = \partial / \partial \omega, \quad \partial / \partial x(c')_* = \partial / \partial x - \partial / \partial \omega.$$  

Taking adjoints, one immediately obtains the following result:

**Proposition 3.16.** Assume that $M = M(1, m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(M)$ on $H^1(M; Z_d)$ is given by

$$(a')^*(dx) = -dx - d\omega, \quad (a')^*(d\omega) = d\omega,$$

$$(b')^*(dx) = d\omega, \quad (c')^*(dx) = dx + d\omega, \quad (c')^*(d\omega) = -d\omega.$$  

In turn, from the above one concludes that if $\zeta \in H^1(M; Z_d)$ is an $\alpha$-isotypical class yielding a regular, connected, nontrivial unramified covering of $M$ having cyclic group $Z_d$ of covering transformations, then $d = 3$ and (up to unit multiple), $\zeta = dx - d\omega$.

Note that we have already observed above that $M(3, m) \to M(1, m)$ is unramified and $M(2, m)$ is regular. We have, therefore, the following classification theorem.

**Theorem 3.17.** Up to $\cong_{M(1,m)}$, there is a unique nontrivial unramified covering $M' \to M(1, m)$ by a regular connected map $M'$ and having cyclic group of covering transformations, namely, the 3-fold covering $M(3, m) \to M(1, m)$.

We turn next to the Steinberg and Accola coverings of $M(1, m)$.

### 3.2.2 The Regular Maps $M(1, m)$—The Steinberg Coverings

In this subsection we determine the Steinberg coverings of the affine map $M = M(1, m)$, and having cyclic group $Z_d$ of covering transformations. Since it is easy to verify that $M$ has $2m^2$ faces, it follows that $d$ must satisfy the condition $d|2m^2$. 


Let $\beta_1, \beta_2, \ldots, \beta_{2m^2}$ be blades in the same $G^+$-orbit generating the $2m^2$ faces of $\mathcal{M}$, and set $m_i = \chi[\beta_i]_{p+} + W^{(a,b)} \in C_2(\mathcal{M}), i = 1, 2, \ldots, 2m^2$. Then, up to a unit multiple in $\mathbb{Z}_d$, the $\text{Aut}(\mathcal{M})$-module homomorphism $\theta = \theta_{\text{st}} : C_2(\mathcal{M}) \to \mathbb{Z}_d$ satisfies $\theta(m_i) = 1, i = 1, 2, \ldots, 2m^2$.

We shall begin by explicitly constructing an element $z \in (\delta^1)^{-1}(\theta)$ as follows. Start with the blade $\beta_0 = (1, 0)$, and define the blade $\beta_1 = b(1, 0) = (0, i\tau)(0, 1) = (0, i\tau)$. Define the voltages $z_0, z_1$ via the assignments

\[ z_0 : a\beta_0, ac\beta_0 \mapsto 1, \beta_0, c\beta_0 \mapsto -1, \]
\[ z_1 : ba\beta_0, cba\beta_0 \mapsto 1, aba\beta_0, acba\beta_0 \mapsto -1, \text{ and} \]
\[ z_2 : b\beta_0, cb\beta_0 \mapsto 1, ab\beta_0, acb\beta_0 \mapsto -1, \]

with all remaining blades mapped to zero.

Next, define the automorphisms $\sigma_1, \sigma_\omega \in \text{Aut}(\mathcal{M})$, $\mathcal{M} = \mathcal{M}(1, m)$ to be the translations one unit in the directions of the real unit $1$ and the complex unit $\omega$, respectively. Therefore,

\[ \sigma_1 : (g, \zeta) \mapsto (g, \zeta + 1) = (g, \zeta)(1, 1), \text{ and} \]
\[ \sigma_\omega : (g, \zeta) \mapsto (g, \zeta + \omega) = (g, \zeta)(1, \omega). \]

In terms of the translations $\sigma_1, \sigma_\omega$ defined above, we now define the voltages

\[ z'_0 = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} 2k(\sigma^{-k}_\omega)^*(\sigma^{-j}_1)^* z_0, \]
\[ z'_1 = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (2k + 1)(\sigma^{-k}_\omega)^*(\sigma^{-j}_1)^* z_1, \]

and set $z' = z'_0 + z'_1$.

The voltage $z'$, together with $\partial/\partial x$ and $\partial/\partial \omega$ are depicted below, again for $m = 3$.†
As usual, the convention is that for each blade $\beta$ labeled with value $z_\beta'$, then the blade $a\beta$ carries the value $-z_\beta'$.

Next, define voltages
\[ z_1'' = \sum_{k=0}^{m-1} 2mk(\sigma_1^{-k})^* \sigma_2^* z_1, \quad z_2'' = \sum_{k=0}^{m-1} 2mk(\sigma_1^{-k})^* \sigma_2^* z_2. \]

Finally, set $z_{St} = z' + z_1'' + z_2''$. This voltage is illustrated below for $m = 3$ (with the usual conventions):
One checks that if $d|2m^2$, then $\delta^1(\mathbf{z}_{St}) = \theta$. As before, even though $\theta$ is a surjective \text{Aut}(\mathcal{M})\text{-module homomorphism} $C_2(\mathcal{M})/H_2(\mathcal{M}) \to \mathbb{Z}_d$, one cannot conclude that $\mathbf{z}_{St} \in D(\mathcal{M}; \mathbb{Z}_d)_\alpha$ for any homomorphism $\alpha : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)$. This is remedied through the actions of the automorphisms $\alpha'$, $\beta'$, and $\gamma'$ on $D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d\langle \theta \rangle) = \mathbb{Z}_d\langle \mathbf{z}_{St} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$. Routine calculations reveal that this action is as below:

**Proposition 3.18.** Assume that $\mathcal{M} = \mathcal{M}(1,m)$, and that $d$ is a positive integer dividing $2m^2$. Then the action of \text{Aut}(\mathcal{M}) on $D_\theta = \mathbb{Z}_d\langle \mathbf{z}_{St} \rangle \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$ is given by

$$(a')^*d\mathbf{x} = -d\mathbf{x} - d\omega, \quad (a')^*d\omega = d\omega,$$

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\[(b')^*\text{dx} = \text{d}\omega,\]
\[(c')^*\text{dx} = \text{dx} + \text{d}\omega, \quad (c')^*\text{d}\omega = -\text{d}\omega,\]
\[(a')^*\text{z}_{\text{St}} = -\text{z}_{\text{St}} - m(m + 2)\text{d}\omega,\]
\[(b')^*\text{z}_{\text{St}} = -\text{z}_{\text{St}},\]
\[(c')^*\text{z}_{\text{St}} = -\text{z}_{\text{St}} + m^2\text{d}\omega.\]

In terms of matrices over \(\mathbb{Z}_d\), we have the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\) given by

\[a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -m(m + 2) & -1 & 1 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ m^2 & 1 & -1 \end{bmatrix}.\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map of type \((3, 6)\). Note that if \(\zeta \in (\delta^1)^{-1}(\theta) \subseteq D_\theta\) is \(\alpha\)-isotypical for \(\alpha = \alpha_{\text{St}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), then \(\zeta\) is of the form \(\zeta = \text{z}_{\text{St}} + x\text{dx} + y\text{d}\omega\) for suitable \(x, y \in \mathbb{Z}_d\). We have the following:

**Theorem 3.19.** Let \(\mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{St}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then \(\text{d}|2m\); furthermore, up to \(\cong_{\mathcal{M}(1,m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = \text{z}_{\text{St}} - m^2\text{dx} + m^2\text{d}\omega \in D(\mathcal{M}; \mathbb{Z}_d)\).

**Proof.** We set \(\zeta = \text{z}_{\text{St}} + x\text{dx} + y\text{d}\omega\); since \(\alpha_{\text{St}}(a') = \alpha_{\text{St}}(b') = \alpha_{\text{St}}(c') = -1\), then the condition
\[-\text{z}_{\text{St}} - x\text{dx} - y\text{d}\omega = -\zeta = \alpha_{\text{St}}(a')\zeta = (a')^*\zeta = -\text{z}_{\text{St}} - x\text{dx} + (-m(m + 2) - x + y)\text{d}\omega\]
yields \(2y - x = m(m + 2)\). Next, the condition \(-\zeta = \alpha_{\text{St}}(b')\zeta = (b')^*\zeta\) clearly yields \(x = -y\). Therefore, it follows that \(3y = m(m + 2)\). From
\[-\text{z}_{\text{St}} - x\text{dx} - y\text{d}\omega = -\zeta = \alpha_{\text{St}}(c')\zeta = (c')^*\zeta = -\text{z}_{\text{St}} + x\text{dx} + (m^2 + x - y)\text{d}\omega,\]
we obtain \(2x = 0\) and \(x = -m^2 = -y\). Finally,

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\[ m(m + 2) = 3y = -3x = 3m^2; \]
since we know a priori that \(2m^2 = 0 \in \mathbb{Z}_d\), we conclude that \(2m = 0 \in \mathbb{Z}_d\), proving that \(d|2m\).

3.2.3 The Regular Maps \(\mathcal{M}(1, m)\)—The Accola Coverings

Note that the Accola homomorphism exists for both the maps \(\mathcal{M}(1, m)\) and \(\mathcal{M}(3, m)\), for all positive integers \(m\). Furthermore a construction of \(z_{\text{Acc}} \in D(\mathcal{M}; \mathbb{Z}_d)\), \(\mathcal{M} = \mathcal{M}(1, m)\) is obtained very easily, as follows. If we set \(\beta = ba\beta_0\), then we define the voltage \(z\) by setting

\[
z : \beta, c\beta \mapsto 1, \ a\beta, ac\beta \mapsto -1.
\]

Set

\[
z_{\text{Acc}} = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (\sigma_1^k)^*(\sigma_2^j)^*z.
\]

For \(m = 3\), this voltage is illustrated below (with the usual conventions):
It is clear that \( z_{\text{Acc}} \) induces the Accola homomorphism, i.e., that \( \delta^1(z_{\text{Acc}}) = \theta_{\text{Acc}} : C_2(M)/H_2(M) \rightarrow \mathbb{Z}_d \). It is obvious that \( (b')^*z_{\text{Acc}} = -z_{\text{Acc}} \); routine calculations reveal that \( (a')^*z_{\text{Acc}} + z_{\text{Acc}} + md\omega = (c')^*z_{\text{Acc}} - z_{\text{Acc}} + md\omega = 0 \), from which we conclude that \( (a')^*z_{\text{Acc}} = -z_{\text{Acc}} - md\omega \) and that \( (c')^*z_{\text{Acc}} = z_{\text{Acc}} - md\omega \).

We have, therefore, the following:

**Proposition 3.20.** If \( \theta = \theta_{\text{Acc}} : C_2(M)/H_2(M) \rightarrow \mathbb{Z}_d \), then the action of \( \text{Aut}(M) \) on \( D_\theta = \mathbb{Z}_d([z_{\text{Acc}}]) \oplus H^1(M; \mathbb{Z}_d) \) \((M = M(1, m))\) is given by

\[
(a')^*dx = -dx - d\omega, \quad (a')^*d\omega = d\omega,
\]
\[(b')^*dx = d\omega,\]
\[(c')^*dx = dx + d\omega, \quad (c')^*d\omega = -d\omega,\]
\[(a')^*z_{\text{Acc}} = -z_{\text{Acc}} - md\omega,\]
\[(b')^*z_{\text{Acc}} = -z_{\text{Acc}},\]
\[(c')^*z_{\text{Acc}} = z_{\text{Acc}} + md\omega.\]

In terms of matrices over \(\mathbb{Z}_d\), and relative to the ordered basis \((z_{\text{Acc}}, dx, d\omega)\) of \(\mathbb{Z}_d\langle z_{\text{Acc}} \rangle \oplus H^1(M; \mathbb{Z}_d)\), the representation of \(\text{Aut}(M) = \langle a', b', c' \rangle\) is given by

\[
a' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
-m & -1 & 1
\end{bmatrix},
\ b' \mapsto \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\ c' \mapsto \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
m & 1 & -1
\end{bmatrix}.
\]

From the above calculations, we may infer the following classification of the Accola coverings of the affine \((3,6)\)-map \(M(1,m)\):

**Theorem 3.21.** Let \(M' \rightarrow M = M(1,m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{\text{Acc}} : \text{Aut}(M) \rightarrow \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then we must have \(d \mid 2m\). Furthermore, up to \(\simeq_{M(1,m)}\), \(M' = M_z\), where \(\zeta = [z] = z_{\text{Acc}} - xdx + x d\omega\) and where \(x \in \mathbb{Z}_d\) is any element satisfying \(3x = m\). □

### 3.2.4 The Regular Maps \(M(3,m)\)—Unramified Coverings

We turn now to the maps \(M = M(3,m)\) and start by determining the unramified coverings by connected regular maps. To do this, we shall first determine \(\mathbb{Z}_d\)-bases for \(H_1(M; \mathbb{Z}_d)\) and \(H^1(M; \mathbb{Z}_d)\). From page 27, the fundamental group of \(M\) is given by

\[
\pi_1(M; \beta_0) = \langle (s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1)^m, (s_3s_1s_2s_3s_2s_1s_3s_2s_3s_2s_1)^m \rangle,
\]
where $\beta_0 = (1, 0)$. Set
\[ \gamma_1 = (s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1), \quad \gamma_2 = (s_3s_1s_2s_3s_2s_1s_3s_2s_3), \]
and define the cycles $\partial/\partial \tau_1 = \epsilon_s h(\gamma_1^m)$, $\partial/\partial \tau_2 = \epsilon_s h(\gamma_2^m) \in H_1(\mathcal{M}; \mathbb{Z}_d)$, where $\epsilon : \mathbb{Z} \to \mathbb{Z}_d$ is the projection map. As above, we know that $\{\partial/\partial \tau_1, \partial/\partial \tau_2\}$ is a $\mathbb{Z}_d$-basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$; denote by $\{d\tau_1, d\tau_2\}$ the corresponding $\mathbb{Z}_d$-dual basis of $H_1(\mathcal{M}; \mathbb{Z}_d)$. By conjugating $\gamma_1^m$, $\gamma_2^m$ by the elements $s_1, s_2, s_3$ we may compute the effect of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$. In turn, the representation of $\text{Aut}(\mathcal{M})$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$ is obtained by dualizing (i.e., by taking adjoints). The results are summarized below:

**Proposition 3.22.** Assume that $\mathcal{M} = \mathcal{M}(3, m)$, and that $d$ is a non-negative integer. Then the action of $\text{Aut}(\mathcal{M})$ on $H_1(\mathcal{M}; \mathbb{Z}_d)$ is given by
\[
(a')^*(d\tau_1) = d\tau_2, \\
(b')^*(d\tau_1) = d\tau_1 - d\tau_2, \\
(c')^*(d\tau_1) = -d\tau_2.
\]

In turn, from the above, one easily determines that if $\zeta \in H_1(\mathcal{M}; \mathbb{Z}_d)$ is an $\alpha$-isotypical class yielding a regular, connected, unramified covering of $\mathcal{M}$ having cyclic group $\mathbb{Z}_d$ of covering transformations, then $d = 3$ and (up to unit multiple), $\zeta = d\tau_1 + d\tau_2$.

We have already observed above that $\mathcal{M}(1, 3m) \to \mathcal{M}(3, m)$ is unramified and that $\mathcal{M}(1, 3m)$ is regular. We have, therefore, the following classification theorem.

**Theorem 3.23.** Up to $\cong_{\mathcal{M}(3, m)}$, there is a unique nontrivial unramified covering $\mathcal{M}' \to \mathcal{M}(3, m)$ by a regular connected map $\mathcal{M}'$ and having cyclic group of covering transformations, namely, the 3-fold covering $\mathcal{M}(1, 3m) \to \mathcal{M}(3, m)$.

We turn now to the classification of the Steinberg and Accola coverings of $\mathcal{M}(3, m)$.

### 3.2.5 The Regular Maps $\mathcal{M}(3, m)$—The Steinberg Coverings

In order to compute the totally ramified coverings of $\mathcal{M}(3, m)$, we capitalize on the 3-fold covering $p : \mathcal{M}(3, m) \to \mathcal{M}(1, m)$. In this case, we have that
\[
s_3s_2s_3s_2s_3s_1s_2s_3s_2s_1 = s_2s_3s_2s_3s_2s_1 \cdot s_3s_2s_3s_2s_1s_2,
\]

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and so it follows that under the induced mapping \( p_* : H_1(M(3, m); \mathbb{Z}_d) \to H_1(M(1, m); \mathbb{Z}_d) \), we have \( \partial/\partial \tau_1 p_* = \partial/\partial x + \partial/\partial \omega \). Also

\[
s_3s_1s_2s_3s_2s_1s_3s_2s_3s_2 = (s_2s_3s_2s_3s_1)^{-2} \cdot s_3s_2s_3s_2s_1s_2,
\]

from which we conclude that \( \partial/\partial \tau_2 p_* = -2\partial/\partial x + \partial/\partial \omega \).

The lemma below gives the pull-backs of the basis elements \( dx \) and \( d\omega \) of \( H^1(M(1, m); \mathbb{Z}_d) \) in terms of the basis elements \( d\tau_1 \) and \( d\tau_2 \) of \( H^1(M(3, m); \mathbb{Z}_d) \).

**Lemma 3.24.** Relative to the covering \( p : M(3, m) \to M(1, m) \), we have

\[
p^*(dx) = d\tau_1 - 2d\tau_2; \quad p^*(d\omega) = d\tau_1 + d\tau_2.
\]

**Proof.** As computed above,

\[
\partial/\partial \tau_1 p_* = \partial/\partial x + \partial/\partial \omega, \quad \partial/\partial \tau_2 p_* = -2\partial/\partial x + \partial/\partial \omega.
\]

Therefore, it follows that if \( p^*(dx) = \tau_1 d\tau_1 + \tau_2 d\tau_2 \), then

\[
\tau_1 = \partial/\partial \tau_1 (p^* dx) = (\partial/\partial \tau_1 p_*) dx = (\partial/\partial x + \partial/\partial \omega)(dx) = 1.
\]

Also,

\[
\tau_2 = \partial/\partial \tau_2 (p^* dx) = (\partial/\partial \tau_2 p_*) dx = (-2\partial/\partial x + \partial/\partial \omega)(dx) = -2;
\]

One similarly verifies the corresponding result for \( p^*(d\omega) \).

We have \( \text{Aut}(M(1, m)) = \langle a', b', c' \rangle \); since \( M(3, m) \) is regular, then \( a', b', c' \) lift to generators of \( \text{Aut}(M(3, m)) \); we shall, for convenience (and simplicity) continue to denote these lifted automorphisms of \( M(3, m) \) as \( a', b', c' \). As a result, we see that if \( \zeta \in D(M(1, m); \mathbb{Z}_d) \), then

\[
(a')^* p^* (\zeta) = p^* (a')^* (\zeta),
\]

with similar results for \( b', c' \).

In analyzing the Steinberg covering of \( M(3, m) \) we proceed as in the case of the type \((4,4)\) affine maps, recalling the voltage \( z^1_{\text{St}} := z_{\text{St}}^1 \in D(M(1, m); \mathbb{Z}_d) \) given in Subsection 3.2.2. We now define

\[
Z^2_{\text{St}} = p^*(z^1_{\text{St}}) \in D(M(3, m); \mathbb{Z}_d).
\]
It is clear that $\delta^1(\mathbb{Z}_{St}^2) = \theta_{St} : C_2(\mathcal{M}(3, m))/H_2(\mathcal{M}(3, m)) \rightarrow \mathbb{Z}_d$. Therefore, if $\theta = \theta_{St}$ then $(\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(\mathbb{Z}_{St}^2) \oplus H^1(\mathcal{M}(3, m); \mathbb{Z}_d)$. We already have enough information to determine the action of $\text{Aut}(\mathcal{M}(3, m))$ on $D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta))$. Using the results of Proposition 3.18, we compute:

\[
(a')^*(\mathbb{Z}_{St}^2) = (a')^*p^*(\mathbb{Z}_{St}^1)
= p^*((a')^*(\mathbb{Z}_{St}^1))
= p^*(-z_{St}^1 - m(m + 2)d\omega)
= -z_{St}^2 - m(m + 2)d\tau_1 - m(m + 2)d\tau_2.
\]

Next,

\[
(b')^*(\mathbb{Z}_{St}^2) = (b')^*p^*(\mathbb{Z}_{St}^1)
= p^*((b')^*(\mathbb{Z}_{St}^1))
= p^*(-z_{St}^1))
= -z_{St}^2.
\]

\[
(c')^*(\mathbb{Z}_{St}^2) = (c')^*p^*(\mathbb{Z}_{St}^1)
= p^*((c')^*(\mathbb{Z}_{St}^1))
= p^*(-z_{St}^1) + m^2d\omega)
= -z_{St}^2 + m^2d\tau_1 + m^2d\tau_2.
\]

We summarize below the action of $\text{Aut}(\mathcal{M}(3, m))$ on $D_\theta$:

**Proposition 3.25.** Assume that $\mathcal{M} = \mathcal{M}(3, m)$ and that $d$ is a positive integer dividing $6m^2$. Then the action of $\text{Aut}(\mathcal{M})$ on $D_\theta = \mathbb{Z}_d(\mathbb{Z}_{St}^2) \oplus H^1(\mathcal{M}; \mathbb{Z}_d)$ is determined by

- $(a')^*(d\tau_1) = d\tau_2$,
- $(b')^*(d\tau_1) = d\tau_1 - d\tau_2$, $(b')^*(d\tau_2) = -d\tau_2$,
- $(c')^*(d\tau_1) = -d\tau_2$,
- $(a')^*(\mathbb{Z}_{St}^2) = -z_{St}^2 - m(m + 2)d\tau_1 - m(m + 2)d\tau_2$,
- $(b')^*(\mathbb{Z}_{St}^2) = -z_{St}^2$.  


\((c')^*(z^2_{St}) = -z^2_{St} + m^2d\tau_1 + m^2d\tau_2.\)  

In terms of the ordered \(\mathbb{Z}_d\)-basis \((z^2_{St}, d\alpha_1, d\alpha_2)\) of \(D_\theta\), we have the matrix representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c'\rangle\):

\[
\begin{align*}
a' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ -m(m+2) & 0 & 1 \\ -m(m+2) & 1 & 0 \end{bmatrix}, \\
b' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \\
c' &\mapsto \begin{bmatrix} -1 & 0 & 0 \\ m^2 & 0 & -1 \\ m^2 & -1 & 0 \end{bmatrix}.
\end{align*}
\]

The above provides the following classification of the Steinberg coverings of the regular affine map \(\mathcal{M}(3, m)\).

**Theorem 3.26.** Let \(\mathcal{M}' \rightarrow \mathcal{M} = \mathcal{M}(3, m)\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}) \rightarrow \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then \(d\mid 2m\); furthermore, up to \(\cong_{\mathcal{M}(3, m)}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = [z] \in D(\mathcal{M}; \mathbb{Z}_d)\) is given uniquely up to unit multiples in \(\mathbb{Z}_d\) by \(\zeta = z^2_{St} + m^2d\tau_2 \in D(\mathcal{M}; \mathbb{Z}_d)\).

**Proof.** If one writes the \(\alpha_{St}\)-isotypical class as \(\zeta = z^2_{St} + \tau_1d\tau_1 + \tau_2d\tau_2\), then

\[-\zeta = \alpha_{St}(a')\zeta = (a')^*(\zeta)\]

implies that

\[-z^2_{St} - \tau_1d\tau_1 - \tau_2d\tau_2 = -z^2_{St} + (\tau_2 - m(m+2))d\tau_1 + (\tau_1 - m(m+2))d\tau_2.\]

Therefore, it follows that \(\tau_1 + \tau_2 = m(m+2)\). Next, the equation \(-\zeta = \alpha_{St}(b')\zeta = (b')^*(\zeta)\) clearly implies that \(\tau_2 = \tau_1 + \tau_2\), and so \(\tau_1 = 0\). Finally, \(-\zeta = \alpha_{St}(c')\zeta = (c')^*(\zeta)\) implies that \(\tau_2 = m^2\); in turn this implies that \(m^2 = \tau_1 + \tau_2 = m(m+2)\), and so \(2m = 0 \in \mathbb{Z}_d\), as claimed.

**3.2.6 The Regular Maps \(\mathcal{M}(3, m)\)—The Accola Coverings**

Finally, we turn to the classification of the Accola coverings of the maps \(\mathcal{M}(3, m)\). As in the case \(\mathcal{M} = \mathcal{M}(1, m)\), we construct an element \(z^2_{Acc} \in (\delta^1)^{-1}(\theta)\), where \(\theta = \theta_{Acc}\), as follows. We set

\[z^2_{Acc} = p^*(z^1_{Acc}) \in D(\mathcal{M}(3, m); \mathbb{Z}_d),\]

where \(z^1_{Acc} = z_{Acc}\) is as in Subsection 3.2.3; note that \(\delta^1(z^2_{Acc}) = \theta_{Acc}\).
Using the results of Proposition 3.20 together with pull backs under \( p : M(3, m) \rightarrow M(1, m) \) of \( dx, d\omega \) to \( \mathbb{Z}_d \)-linear combinations of \( d\tau_1, d\tau_2 \), we can compute the effect of \( a', b', \) and \( c' \) on the \( \mathbb{Z}_d \)-basis elements \( \mathcal{Z}^2_{\text{Acc}}, d\tau_1, d\tau_2 \) of \( (\delta^1)^{-1}(\theta) = \mathbb{Z}_d(\mathcal{Z}^2_{\text{Acc}}) \oplus H^1(M(3, m); \mathbb{Z}_d) \). First, we have

\[
(a')^* (\mathcal{Z}^2_{\text{Acc}}) = (a')^* p^* (\mathcal{Z}^1_{\text{Acc}}) \\
= p^* ((a')^* (\mathcal{Z}^1_{\text{Acc}})) \\
= p^* (-\mathcal{Z}^1_{\text{Acc}} - m d \omega) \\
= -\mathcal{Z}^2_{\text{Acc}} - m d \tau_1 - m d \tau_2.
\]

Next,

\[
(b')^* (\mathcal{Z}^2_{\text{Acc}}) = (b')^* p^* (\mathcal{Z}^1_{\text{Acc}}) \\
= p^* ((b')^* (\mathcal{Z}^1_{\text{Acc}})) \\
= p^* (-\mathcal{Z}^1_{\text{Acc}}) \\
= -\mathcal{Z}^2_{\text{Acc}}.
\]

\[
(c')^* (\mathcal{Z}^2_{\text{Acc}}) = (c')^* p^* (\mathcal{Z}^1_{\text{Acc}}) \\
= p^* ((c')^* (\mathcal{Z}^1_{\text{Acc}})) \\
= p^* (\mathcal{Z}^1_{\text{Acc}} - m d \omega) \\
= \mathcal{Z}^2_{\text{Acc}} - m d \tau_1 - m d \tau_2.
\]

One therefore has the following matrix representation of \( \text{Aut}(M(3, m)) = \langle a', b', c' \rangle \) on \( \mathbb{Z}_d(\mathcal{Z}^2_{\text{Acc}}) \oplus H^1(M(3, m); \mathbb{Z}_d) \) relative to the \( \mathbb{Z}_d \)-basis \( \langle \mathcal{Z}^2_{\text{Acc}}, d\tau_1, d\tau_2 \rangle \):

\[
a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ -m & 0 & 1 \\ -m & 1 & 0 \end{bmatrix}, \quad b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad c' \mapsto \begin{bmatrix} 1 & 0 & 0 \\ -m & 0 & -1 \\ -m & -1 & 0 \end{bmatrix}.
\]

From the above calculations, we may infer the following classification of the Accola coverings of the affine \( (3, 6) \)-map \( M(3, m) \):
Theorem 3.27. Let \( \mathcal{M}' \to \mathcal{M} = \mathcal{M}(1, m) \) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \( \mathbb{Z}_d \) and inducing the action \( \alpha = \alpha_{\text{Acc}} : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathbb{Z}_d) \), by a regular connected map. Then we must have \( d \mid 2m \). Furthermore, up to \( \cong_{\mathcal{M}(3,m)} \), \( \mathcal{M}' = \mathcal{M}_z \), where \( z = [z] = z_{\text{Acc}} + m\mathbf{d}\tau_2 \).

Remark: Note that if \( d = 2 \) then the Steinberg and Accola coverings of \( \mathcal{M}(3, m) \) coincide (up to \( \cong_{\mathcal{M}(3,m)} \)).

3.3 The Regular Affine Maps of Type \((6,3)\)

Since the affine maps of type \((6,3)\) are the duals of the corresponding maps of type \((3,6)\), much of the work in the preceding subsection can be applied. Note, in particular, that an unramified covering of an affine map of type \((3,6)\) gives an unramified covering of the corresponding dual map of type \((6,3)\). We retain the notation of Subsection 3.2 and denote by \( \mathcal{M}(1, m) \) and \( \mathcal{M}(3, m) \) the regular affine maps of type \((3,6)\) containing \(2m^2\) and \(6m^2\) faces respectively, and by \( \mathcal{M}(1, m)^* \) and \( \mathcal{M}(3, m)^* \) their duals, thereby interchanging the vertices and faces. Therefore, there are the 3-fold unramified coverings \( \mathcal{M}(1,3m)^* \to \mathcal{M}(3, m)^* \to \mathcal{M}(1, m)^* \). Furthermore, the above remarks guarantee the following analogs of Theorems 3.17 and 3.23:

Theorem 3.28. The unramified coverings of the regular affine maps of type \((6,3)\) are as follows.

(i) Up to \( \cong_{\mathcal{M}(1,m)^*} \), there is a unique nontrivial unramified covering \( \mathcal{M}' \to \mathcal{M}(1, m)^* \) by a regular connected map \( \mathcal{M}' \) and having cyclic group of covering transformations, namely, the 3-fold covering \( \mathcal{M}(3, m)^* \to \mathcal{M}(1, m)^* \).

(ii) Up to \( \cong_{\mathcal{M}(3,m)^*} \), there is a unique nontrivial unramified covering \( \mathcal{M}' \to \mathcal{M}(3, m)^* \) by a regular connected map \( \mathcal{M}' \) and having cyclic group of covering transformations, namely, the 3-fold covering \( \mathcal{M}(1,3m)^* \to \mathcal{M}(3, m)^* \).

Since the face valency of the regular maps of type \((6,3)\) is odd (\(= 3\)), there are no Accola coverings of the maps \( \mathcal{M}(1, m)^* \), \( \mathcal{M}(3, m)^* \). Therefore we need only classify the Steinberg coverings. Note that in the present case, if \( \mathcal{M} \) is a map of type \((3,6)\), and if \( \mathcal{M}^* \) is the corresponding dual map, voltages \( z : B \to \mathbb{Z}_d \), where \( B \) is the blade.
set of $\mathcal{M}^*$ (= blade set of $\mathcal{M}$) will satisfy

$$z(a\beta) = z(\beta) = -z(c\beta),$$

for all $\beta \in B$. This is obvious, since the duality operation $\mathcal{M} \rightarrow \mathcal{M}^*$ simply interchanges the roles of the monodromy involutions $a$ and $c$.

### 3.3.1 The Regular Maps $\mathcal{M}(1, m)^*$ and $\mathcal{M}(3, m)^*$—The Steinberg Coverings

We note first of all that the fundamental groups of both $\mathcal{M}(1, m)^*$ and $\mathcal{M}(3, m)^*$ are the same as that of $\mathcal{M}(1, m)$ and $\mathcal{M}(3, m)$, respectively. However, the mapping $h : \pi_1(\mathcal{M}^*, \beta_0) \rightarrow H_1(\mathcal{M}^*; \mathbb{Z})$ ($\mathcal{M} = \mathcal{M}(1, m)$ or $\mathcal{M}(3, m)$) is slightly different in that the roles of $s_1$ and $s_3$ need to be reversed. The action of the automorphism group on $H_1(\mathcal{M}^*; \mathbb{Z})$ can, exactly as before, be computed via conjugation on the fundamental group. Therefore, the results will be exactly as in the previous subsection. In particular, if $d$ is a positive integer, and if $\epsilon : \mathbb{Z} \rightarrow \mathbb{Z}_d$ is the projection map, we obtain $\mathbb{Z}_d$-bases $\{\partial/\partial x^*, \partial/\partial \omega^*\}$ and $\{\partial/\partial \tau_1^*, \partial/\partial \tau_2^*\}$ of $H_1(\mathcal{M}(1, m)^*; \mathbb{Z}_d)$ and $H_1(\mathcal{M}(3, m)^*; \mathbb{Z}_d)$, respectively, via

$$\mathcal{M}(1, m)^*: \partial/\partial x^* = \epsilon_* h((s_2s_1s_2s_1s_3s_2s_3)^m), \partial/\partial \omega^* = \epsilon_* h((s_1s_2s_1s_3s_2s_3s_2s_1)^m);$$

$$\mathcal{M}(3, m)^*: \partial/\partial \tau_1^* = \epsilon_* h((s_1s_2s_1s_3s_2s_3s_2s_1s_3s_2s_1s_3s_2s_1)^m), \partial/\partial \tau_2^* = \epsilon_* h((s_1s_3s_2s_1s_3s_2s_3s_2s_1s_3s_2s_1s_3s_2s_1)^m).$$

We denote by $\{dx^*, d\omega^*\}$ and $\{d\tau_1^*, d\tau_2^*\}$ the corresponding dual $\mathbb{Z}_d$-bases of $H^1(\mathcal{M}(1, m)^*; \mathbb{Z}_d)$ and $H^1(\mathcal{M}(3, m)^*; \mathbb{Z}_d)$, respectively. We have the following analogues of Proposition 3.16 and Proposition 3.22 below:

**Proposition 3.29.** The action of $\text{Aut}(\mathcal{M})^*$ on $H^1(\mathcal{M}^*; \mathbb{Z}_d)$ is given below:

**\(\mathcal{M}(1, m)^*\):** $\text{Aut}(\mathcal{M}(1, m)^*) = \langle a', b', c' \rangle$ and

$$(a')^*(dx^*) = dx^* + d\omega^*,\quad (a')(d\omega^*) = -d\omega^*,$$

$$(b')^*(dx^*) = d\omega^*,$$

$$(c')^*(dx^*) = -dx^* - d\omega^*,\quad (c')(d\omega^*) = d\omega^*.$$

**\(\mathcal{M}(3, m)^*\):** $\text{Aut}(\mathcal{M}(3, m)^*) = \langle a', b', c' \rangle$ and

$$(a')^*(d\tau_1^*) = -d\tau_2^*.$$
\[(b')^*(d\tau_1^*) = d\tau_1^* - d\tau_2^*, \quad (b')^*(d\tau_2^*) = -d\tau_2^*,\]
\[(c')^*(d\tau_1^*) = d\tau_2^*.\]

Furthermore, if \(p : M(3, m)^* \to M(1, m)^*\) is the 3-fold covering, then the pull-back \(p^* : H^1(M(1, m)^*; \mathbb{Z}_d) \to H^1(M(3, m)^*; \mathbb{Z}_d)\) is exactly as in Lemma 3.24:

**Lemma 3.30.** With respect to the 3-fold covering mapping \(p : M(3, m)^* \to M(1, m)^*\), the pull-back \(p^* : H^1(M(1, m)^*; \mathbb{Z}_d) \to H^1(M(3, m)^*; \mathbb{Z}_d)\) is given by

\[p^*(dx^*) = d\tau_1^* - 2d\tau_2^*, \quad p^*(d\omega^*) = d\tau_1^* + d\tau_2^*.\]

Next, in determining the Steinberg coverings of the affine map \(M^* = M(1, m)^*\), and having cyclic group \(\mathbb{Z}_d\) of covering transformations, we recall that since \(M^*\) has \(m^2\) faces, then \(d|m^2\).

As usual, let \(\beta_1, \beta_2, \ldots, \beta_{m^2}\) be blades in the same \(G^+\)-orbit generating the \(m^2\) faces of \(M^*\), and set \(m_i = \chi(\beta_i)_{\mathcal{F}^+} + W^{(b,c)} \in C_2(M), i = 1, 2, \ldots, m^2\). Then, up to a unit multiple in \(\mathbb{Z}_d\), the \(\text{Aut}(M^*)\)-module homomorphism \(\theta = \theta_{\text{St}} : C_2(M^*) \to \mathbb{Z}_d\) satisfies \(\theta(m_i) = 1, i = 1, 2, \ldots, m^2\).

In close analogy with our earlier work, we begin by explicitly constructing an element \(z_{\text{St}} = z \in (\delta^1)^{-1}(\theta)\). Start with the blade \(\beta_0 = (1, 0)\), and define the blade \(\beta_1 = b(1, 0) = (\tau\omega, 0)(1, 0) = (\tau\omega, 0)\). Define the voltages \(z_1, z_2\) via the assignments

\[z_1 : \beta_1 \mapsto -1, \quad a\beta_1 \mapsto -1, \quad c\beta_1 \mapsto 1, \quad ac\beta_1 \mapsto 1,\]
\[z_2 : \beta_0 \mapsto 1, \quad a\beta_0 \mapsto 1, \quad c\beta_0 \mapsto -1, \quad ac\beta_0 \mapsto -1,\]

with all remaining blades mapped to zero.

In terms of the translations \(\sigma_1, \sigma_\omega\) we now define the voltage

\[z_1' = \sum_{k=0}^{m-1} \sum_{j=0}^{m-1} (k + 1)(\sigma_\omega^{-k})(\sigma_1^{-j})^*z_1.\]

The voltage \(z_1'\), is depicted below for \(m = 3\).
To the voltage $z'_1$ we add the voltage $z'_2$, defined by

$$z'_2 = \sum_{k=0}^{m-1} m(k + 1)(\sigma_1^{-k})^* z_2,$$

and set $z'_S = z'_1 + z'_2$, illustrated below for $m = 3$ (together with $\partial/\partial x^*, \partial/\partial \omega^* \in H_1(\mathcal{M}(1, m)^*; \mathbb{Z}_d)$)
One checks that if \( d\mid m^2 \), then \( \delta^1(z_{St}^*) = \theta \). As usual, even though \( \theta \) is a surjective \( \text{Aut}(\mathcal{M}^*) \)-module homomorphism \( C_2(\mathcal{M}^*)/H_2(\mathcal{M}^*) \to \mathbb{Z}_d \), one cannot conclude that \( z_{St}^* \in D(\mathcal{M}^*; \mathbb{Z}_d) \) for any homomorphism \( \alpha : \text{Aut}(\mathcal{M}^*) \to \text{Aut}(\mathbb{Z}_d) \). This is remedied through the action of the automorphisms \( a', b', c' \) on \( D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d[z_{St}^*] \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d) \), given below:

**Proposition 3.31.** Assume that \( \mathcal{M} = \mathcal{M}(1, m) \), and that \( d \) is a positive integer dividing \( m^2 \). Then the action of \( \text{Aut}(\mathcal{M}^*) \) on \( D_\theta = \mathbb{Z}_d[z_{St}^*] \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d) \) is given by

![Figure 11. The Voltage \( z_{St}^* \)](image)
\[(a') \ast dx^* = dx^* + d\omega^*, \ (a') \ast d\omega^* = -d\omega^*,\]

\[(b') \ast dx^* = d\omega^*,\]

\[(c') \ast dx^* = -dx^* - d\omega^*, \ (c') \ast d\omega^* = d\omega^*,\]

\[(a') \ast z_{St}^* = z_{St}^* - m(m+1)dx^* - \frac{1}{2}m(m+1)d\omega^* = -z_{St}^* - mdx^* - \frac{1}{2}m(m+1)d\omega^*\]

\[(\text{since } m^2 = 0 \in \mathbb{Z}_d),\]

\[(b') \ast z_{St}^* = -z_{St}^*,\]

\[(c') \ast z_{St}^* = -z_{St}^* + \frac{1}{2}m(m+5)d\omega^*.\]

\[\blacksquare\]

In terms of matrices over \(\mathbb{Z}_d\), we have the representation of \(\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle\) given by

\[a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ -m & 1 & 0 \\ -\frac{1}{2}m(m+1) & 1 & -1 \end{bmatrix}, \ b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ c' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \frac{1}{2}m(m+5) & -1 & 1 \end{bmatrix}.\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map \(\mathcal{M}^* = \mathcal{M}(1, m)^*\) of type \(6, 3\). Note that if \(\zeta \in D_9\) is \(\alpha\)-isotypical for \(\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)\), then we may assume that \(\zeta\) is of the form \(\zeta = z_{St}^* + xdx^* + yd\omega^*\) for suitable \(x, y \in \mathbb{Z}_d\). We have the following.

**Theorem 3.32.** Let \(\mathcal{M}' \rightarrow \mathcal{M}^* = \mathcal{M}(1, m)^*\) be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with \(\mathbb{Z}_d\) and inducing the action \(\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)\), by a regular connected map. Then \(d|m\); furthermore, up to \(\cong_{\mathcal{M}(1, m)^*}\), \(\mathcal{M}' = \mathcal{M}_z\), where \(\zeta = z_{St}^* + \frac{1}{2}m(m+1)dx^* + \frac{1}{2}m(m+1)d\omega^* \in D(\mathcal{M}^*; \mathbb{Z}_d)\).

**Proof.** Since \(\alpha_{St}(a') = \alpha_{St}(b') = \alpha_{St}(c') = -1\), and if \(\zeta = z_{St}^* + xdx^* + yd\omega^*\), then the condition

\[-z_{St}^* - xdx^* - yd\omega^* = -\zeta = \alpha_{St}(a')\zeta = (a')\ast \zeta = -z_{St}^* - (m-x)dx^* - (\frac{1}{2}m(m+1)-x+y)d\omega^*\]

yields \(x = \frac{1}{2}m(m+1)\). Next, the condition \(-\zeta = \alpha_{St}(b')\zeta = (b')\ast \zeta\) clearly yields \(x = -y\). From

\[-z_{St}^* - xdx^* - yd\omega^* = -\zeta = \alpha_{St}(c')\zeta = (c')\ast \zeta = -z_{St}^* - xdx^* - (\frac{1}{2}m(m+5)-x+y)d\omega^*,\]

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we obtain

\[ 2x = -2y = \frac{1}{2}m(m + 5) - x = \frac{1}{2}m(m + 5) - \frac{1}{2}m(m + 1) = 2m. \]

Therefore \( m(m + 1) = 2x = 2m \) and so it follows that \( m = 0 \in \mathbb{Z}_d \) (since \( m^2 = 0 \)), i.e., that \( d \mid m \). Therefore, \( 2x = 2y = 0 \) and so we infer that \( y = \frac{1}{2}m(m + 1) \).

Finally, we turn to the Steinberg coverings of \( M(3, m)^* \). We proceed as in the earlier subsections, starting with the Steinberg voltage \( z_{\text{St}}^1 = z_{\text{St}}^* \in D(M(1, m^*; \mathbb{Z}_d), \text{given above}) \). Relative to the unramified covering \( p : M(3, m)^* \to M(1, m)^* \), we now define

\[ z_{\text{St}}^{*2} = p^*(z_{\text{St}}^{*1}) \in D(M(3, m)^*; \mathbb{Z}_d). \]

It is clear that \( \delta^1(z_{\text{St}}^{*2}) = \theta_{\text{St}} : C_2(M(3, m)^*)/H_2(M(3, m)^*)) \to \mathbb{Z}_d. \) Therefore, if \( \theta = \theta_{\text{St}} \), then \( (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) = \mathbb{Z}_d(z_{\text{St}}^{*2}) \oplus H_1(M(3, m)^*; \mathbb{Z}_d). \) Using the results of Proposition 3.31, we have

\[
(a')^*(z_{\text{St}}^{*2}) = (a')^*p^*(z_{\text{St}}^{*1})
= p^*((a')^*(z_{\text{St}}^{*1}))
= p^*(-z_{\text{St}}^{*1} - \frac{1}{2}m(m + 1)d\omega^*)
= -z_{\text{St}}^{*2} - \frac{1}{2}m(m + 3)d\tau_1^* - \frac{1}{2}m(m - 3)d\tau_2^*.
\]

Next,

\[
(b')^*(z_{\text{St}}^{*2}) = (b')^*p^*(z_{\text{St}}^{*1})
= p^*((b')^*(z_{\text{St}}^{*1}))
= p^*(-z_{\text{St}}^{*1})
= -z_{\text{St}}^{*2},
\]

\[
(c')^*(z_{\text{St}}^{*2}) = (c')^*p^*(z_{\text{St}}^{*1})
= p^*((c')^*(z_{\text{St}}^{*1}))
= p^*(-z_{\text{St}}^{*1} + \frac{1}{2}m(m + 5)d\omega^*)
= -z_{\text{St}}^{*2} + \frac{1}{2}m(m + 5)d\tau_1^* + \frac{1}{2}m(m + 5)d\tau_2^*.
\]

Therefore, the action of \( \text{Aut}(M(3, m)^*) \) on \( D_\theta = (\delta^1)^{-1}(\mathbb{Z}_d(\theta)) \) is as given below:
Proposition 3.33. Assume that $\mathcal{M} = \mathcal{M}(1, m)$, and that $d$ is a positive integer dividing $3m^2$. Then the action of $\text{Aut}(\mathcal{M}^*)$ on $D_0 = \mathbb{Z}_d\langle [z_{St}] \rangle \oplus H^1(\mathcal{M}^*; \mathbb{Z}_d)$ is given by
\[
(a')^*d\tau_1^* = -d\tau_2^*, \\
(b')^*d\tau_1^* = d\tau_1^* - d\tau_2^*, (b')^*d\tau_2^* = -d\tau_2^*, \\
(c')^*d\tau_1^* = d\tau_2^*, \\
(a')^*z_{St}^2 = -z_{St}^1 - \frac{1}{2}m(m + 3)d\tau_1^* - \frac{1}{2}m(m - 3)d\tau_2^*, \\
(b')^*z_{St}^* = -z_{St}^*, \\
(c')^*z_{St}^* = -z_{St}^* + \frac{1}{2}m(m + 5)d\tau_1^* + \frac{1}{2}m(m + 5)d\tau_2^*. 
\]

In terms of matrices over $\mathbb{Z}_d$, we have the representation of $\text{Aut}(\mathcal{M}) = \langle a', b', c' \rangle$ given by
\[
\begin{align*}
&a' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ -\frac{1}{2}m(m + 3) & 0 & -1 \\ -\frac{1}{2}m(m - 3) & -1 & 0 \end{bmatrix}, \\
&b' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \\
&c' \mapsto \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{2}m(m + 5) & 0 & 1 \\ \frac{1}{2}m(m + 5) & 1 & 0 \end{bmatrix}. 
\end{align*}
\]

The above can be used to obtain a classification of the Steinberg coverings of the regular affine map $\mathcal{M}^* = \mathcal{M}(3, m)^*$ of type $(6, 3)$. Again, if $\zeta \in D_0$ is $\alpha$-isotypical for $\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)$, then we may assume that $\zeta$ is of the form $\zeta = z_{St}^2 + \tau_1 d\tau_1^* + \tau_2 d\tau_2^*$ for suitable $\tau_1, \tau_2 \in \mathbb{Z}_d$. We have the following; the proof is entirely routine.

Theorem 3.34. Let $\mathcal{M}' \rightarrow \mathcal{M}^* = \mathcal{M}(3, m)^*$ be a totally ramified covering (unramified over vertices) having cyclic group of covering transformations isomorphic with $\mathbb{Z}_d$ and inducing the action $\alpha = \alpha_{St} : \text{Aut}(\mathcal{M}^*) \rightarrow \text{Aut}(\mathbb{Z}_d)$, by a regular connected map. Then $d|m$; furthermore, up to $\cong_{\mathcal{M}(3, m)^*}$, $\mathcal{M}' = \mathcal{M}_z$, where $\zeta = z_{St}^2 + \frac{1}{2}m(m + 3)d\tau_2^* \in D(\mathcal{M}^*; \mathbb{Z}_d)$.

Remark: If $d$ is odd then it follows easily that $\frac{1}{2}m(m + 3) = 0 \in \mathbb{Z}_d$. Therefore, the unique isotypical class in this case is $\zeta = z_{St}^2 \in D(\mathcal{M}(3, m)^*; \mathbb{Z}_d)$. 

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4 Concluding Remarks.

Our objective has been to apply the homological groundwork laid in [5] to not only
the classification of the unramified coverings of regular affine maps by regular (affine)
maps (with cyclic group of covering automorphisms), but also to the classification of
two very wide classes of totally ramified coverings of regular affine maps by regular
maps (again having cyclic group of covering automorphisms). As already noted in
the Introduction, this actually classifies all coverings of regular affine maps by regular
maps having as totally ramified factor either a Steinberg or an Accola factor. Note,
however, that if the group of covering automorphisms is cyclic of order $d$, and if
$\text{Aut}(\mathbb{Z}_d)$ has more than one involution, then there is an a priori possibility of having
totally ramified coverings outside the perview of the present investigations. At the
other extreme, note that if $d$ is either prime or of the form $d = 2p$, where $p$ is an odd
prime, then $\text{Aut}(\mathbb{Z}_d)$ has exactly one involution, and so all totally ramified coverings
are either Steinberg or Accola.

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