I’ve gotten quite a few questions about the explicit calculation of the conjugacy classes of $S_3$. Just for the record, here’s the result:

$$C_{S_3}(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}) = \{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \}.$$  
$$C_{S_3}(\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}) = \{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \}.$$  
$$C_{S_3}(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}) = \{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \}.$$  

In the above, note that the the second conjugacy class consists precisely of the elements of order 2 in $S_3$, and the third conjugacy class consists precisely of the elements of order 3. While this doesn’t always happen quite so nicely, it is true that conjugate elements do have the same order:

**Lemma.** Let $G$ be a group and let $x, y \in G$. If $x$ and $y$ are conjugate, then $x$ and $y$ have the same order.

**Proof:** By assumption, there exists an element $g \in G$ with $y = gxg^{-1}$. Assume that $o(x) = n$, $o(y) = m$. Then

$$e = y^m = (gxg^{-1})(gxg^{-1}) \cdots (gxg^{-1}) = gx^m g^{-1}.$$  
$m$ factors

Therefore, it follows that $x^m = g^{-1}eg = e$ and so $n|m$. In an entirely similar way, $m|n$, and the result follows.

Use the above helpful hint to compute the conjugacy classes of the symmetric group $S_4$. (It turns out that there are five conjugacy classes in all. Warning: the elements of order 2 actually form two conjugacy classes, not one!)

Consider the group with “generators” $x, y$ and having rules of calculation $x^5 = y^2 = e$, $yx = x^4$ ($= x^{-1}$). If we call this group $D_{10}$, then the elements of $D_{10}$ can be written out at

$$D_{10} = \{ e, x, x^2, x^3, x^4, y, yx, yx^2, yx^3, yx^4 \}.$$  

Compute the conjugacy classes of $D_{10}$. (You might want to work out the multiplication table first. It’s especially helpful if you know first all of the orders of the elements.)
Consider the group with “generators” $x, y$ and having rules of calculation $x^6 = y^2 = e$, $yxy = x^5 (= x^{-1})$. If we call this group $D_{12}$, then the elements of $D_{12}$ can be written out at

$$D_{10} = \{e, x, x^2, x^3, x^4, x^5, yx, yx^2, yx^3, yx^4, yx^5\}.$$ 

Compute the conjugacy classes of $D_{12}$. (The answer here is slightly different from the calculation in $D_{10}$.)