I. Definition and elementary examples

A sequence of real numbers is intuitively just a list of real numbers, often expressed simply as \(a_1, a_2, a_3, \ldots\), where each \(a_n \in \mathbb{R}, \ n = 1, 2, 3 \ldots\) is a real number. Of course, there is no compelling reason for the sequence to start at \(n = 1\); we might wish to consider a sequence \(b_0, b_1, b_2, \ldots\), where each \(b_n \in \mathbb{R}, \ n = 0, 1, 2 \ldots\), or even a sequence \(c_4, a_5, a_6, \ldots\), where each \(c_n \in \mathbb{R}, \ n = 4, 5, 6 \ldots\).

Formal Definition of Sequence. There are many definitions of sequences (not all entirely equivalent), but the one I prefer is as follows. First of let \(\mathbb{N}_{\geq k} = \{k, k + 1, k + 2, \ldots\}\). A (real) sequence is a function

\[ f : \mathbb{N}_{\geq k} \rightarrow \mathbb{R}, \]

for some integer \(k\). We often denote the values \(f(n), \ n \in \mathbb{N}_{\geq k}\) as subscripted quantities, such as \(f(n) = a_n\).

Simple examples of sequences defined by formula.

1. Set \(a_n = 2 + 3n, \ n = 1, 2, \ldots\). Expanded, this looks like 5, 8, 11, 14, \ldots. This is, of course, a very simple arithmetic sequence. Presumably, you studied these either in your Algebra II or in Precalculus class (or both!).

2. Set \(a_n = 5 - 2n, \ n = 0, 1, 2, \ldots\). This is another arithmetic sequence which when expanded looks like 5, 3, 1, -1, -3, -5.

3. Set \(b_n = \frac{2(-1)^n}{3^n}, \ n = 0, 1, 2, \ldots\). This is a familiar geometric sequence, which when expanded looks like \(2, -\frac{2}{3}, \frac{2}{9}, -\frac{2}{27}, \ldots\).

4. Set \(c_n = \frac{n}{n^2 + 1}, \ n = 0, 1, 2, \ldots\). When expanded this looks like \(0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots\) and is easily seen to be neither arithmetic nor geometric.

5. Set \(a_n = \frac{n}{n^2 - 7n + 10}, \ n \geq 0\). This example is mildly problematical because the denominator (which equals \((n - 2)(n - 5)\)) vanishes at \(n = 2\) and at \(n = 5\).
and so the domain contains $\mathbb{N}_{\geq k}$ only where $k = 6$. This is not such a huge problem; starting with $n = 6$ we can still list the elements of this sequence:

$$\frac{6}{4} = \frac{3}{2}, \frac{7}{10}, \frac{8}{18} = \frac{4}{9}, \ldots$$

6. Recall the definition of $n$ factorial, given by $n! = 1 \cdot 2 \cdot 3 \cdots n$, $n \in \{1, 2, 3, \cdots\}$. Also, one defines $0! = 1$. (Do you know why?) At any rate, the above sequence looks like

$$1, 1, 2, 6, 24, \ldots,$$

a very rapidly growing sequence.

II. Continuous analogs of sequences and l’Hôpital’s Rule

Most of the above examples can be extended to functions defined on (most of) the real line. For example the simple arithmetic sequence defined by $a_n = 3 + 5n$, $n = 0, 1, 2, \ldots$ can be extended to the simple linear function $f(x) = 3 + 5x$, $x \in \mathbb{R}$. This is a convenient observation because we often will have occasion to compute limits, especially those of the form

$$\lim_{n \to \infty} a_n,$$

whose computation will require an appeal to l’Hôpital’s Rule. For example, in computing

$$\lim_{n \to \infty} \frac{n \ln n}{n^2 - n + 4},$$

we can’t just use l’Hôpital’s Rule directly, since the derivative of a sequence doesn’t make sense. However, if there is a analogous function defined for real numbers which restricts to the given sequence, then we can apply l’Hôpital’s Rule to this. For example:

$$\lim_{n \to \infty} \frac{n \ln n}{n^2 - n + 4} = \lim_{x \to \infty} \frac{x \ln x}{x^2 - x + 4} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{\ln x + 1}{2x - 1} \overset{\text{H}}{=} \lim_{x \to \infty} \frac{1}{2x} = 0$$

From the above, we can now safely say that $\lim_{n \to \infty} \frac{n \ln n}{n^2 - n + 4} = 0$.

Note that Example 3 above is a bit of a problem, for how does one define $\frac{2(-1)^x}{3^x}$ for $x \in \mathbb{R}$. For example, if $x = 1/2$, then $(-1)^x = i$, the familiar imaginary number. Therefore, the “extension” of this sequence therefore becomes non-real, and is therefore outside our purview.
Perhaps a more compelling example comes from Example 6, above. That is to say, how might one define $x!$, where $x$ is an arbitrary real number. What’s interesting is that one can almost do this. Below is some interesting, but optional reading.

Optional Reading—the Gamma Function

The so-called **Gamma function** is defined by an integral, as follows. Let $x \in \mathbb{R}$ and set

$$
\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.
$$

Let’s compute a few values:

1. $\Gamma(1) = \int_0^\infty e^{-t} \, dt = -e^{-t}\bigg|_0^\infty = 1$.

2. $\Gamma(2) = \int_0^\infty te^{-t} \, dt = -te^{-t} - e^{-t}\bigg|_0^\infty = 1 \text{ (Integration by parts was used.)}$

3. In general, for any positive integer $n \geq 2$,

$$
\Gamma(n) = \int_0^\infty t^{n-1}e^{-t} \, dt \\
= -t^{n-1}e^{-t}\bigg|_0^\infty + (n-1)\int_0^\infty t^{n-2}e^{-t} \, dt \\
= 0 + (n-1)\int_0^\infty t^{n-2}e^{-t} \, dt \\
= (n-1)\Gamma(n-1).
$$

This says that $\Gamma(3) = 2\Gamma(2) = 2$, $\Gamma(4) = 3\Gamma(3) = 6 = 3!$. From this, it follows that when $n \geq 1$ is an integer, $\Gamma(n) = (n-1)!$. Therefore the Gamma function gives us an extension of ordinary factorial to a function defined on (most of) the real line.

**Exercises.**

1. Show that for any real number $x > 1$, $\Gamma(x + 1) = x\Gamma(x)$.

2. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

3. Compute $\Gamma\left(\frac{3}{2}\right)$ (Can you use Exercise #1 to do this?)

---

1The Gamma function is not defined at $x = -n$, where $n = 0, 1, 2, \ldots$
4. For any integer \( n > 1 \), compute \( \Gamma \left( n + \frac{1}{2} \right) \).

Let’s return to a few more examples of examples of sequences. These examples, however, will be of a rather different nature.

III. Recursive examples of sequences

1. Consider the sequence \( a_1, a_2, \ldots \) defined by setting

\[
a_1 = 5, \ a_n = 3 + a_{n-1}, \ n > 1.
\]

This example doesn’t provide an explicit formula for the \( n \)-th term of the sequence; rather it gives a **recursive** definition of \( a_n \) in terms of the value of \( a_{n-1} \). In this particular example we infer immediately that \( a_n - a_{n-1} = 3 \) (a constant) and so the sequence is automatically arithmetic. Furthermore, a moment’s thought shows that this is the same sequence given in Example #1 above, namely that \( a_n = 2 + 3n, \ n = 1, 2, 3, \ldots \).

2. Define the sequence \( b_0, b_1, \ldots \) by stipulating that

\[
b_0 = 2, \ b_n = -\frac{1}{3}b_{n-1}, \ n > 1.
\]

One sees immediately that this is the geometric sequence given in Example #3 above with first term 2 and common ratio \(-\frac{1}{3}\). In particular, the explicit formula expressing \( b_n \) is \( b_n = \frac{2(-1)^n}{3^n} \).

3. Many of you will be familiar with this example. Define \( F_0, F_1, F_2, \ldots \) by defining

\[
F_0 = 1, \ F_1 = 1, \ F_n = F_{n-1} + F_{n-2}, \ n \geq 2.
\]

Of course, this is the famous **Fibonacci** sequence, whose first several terms are easily obtained as

\[
1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots
\]

In light of the above two examples one is led to ask whether there is an explicit formula which computes the general term of the sequence. That is to say, is there a **non-recursive** definition? In this example there is, but\(^2\) the formula

\(^2\) unless you’ve studied what are called “homogeneous linear recursion equations"
is a bit non-intuitive: it turns out that, in fact,

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}. \]

Incredible, perhaps, but true! Try the first few values. Also check for yourselves that the above formula for \( F_n \) does indeed satisfy \( F_n = F_{n-1} + F_{n-2} \). (To do this, check first that the numbers \( \left( \frac{1+\sqrt{5}}{2} \right) \) and \( \left( \frac{1-\sqrt{5}}{2} \right) \) are both solutions of the equation \( x^2 = x + 1 \).) At any rate, the above shows that the Fibonacci sequence has a very close relationship with the **Golden Ratio** \( \left( \frac{1+\sqrt{5}}{2} \right) \).

4. Now let’s get ugly!\(^3\) Define the recursive sequence by setting

\[ Q_1 = Q_2 = 1, \quad Q_n = Q_{n-1} + Q_{n-2}, \quad n \geq 3. \]

The first few terms can be computed by hand:

1, 1, 2, 3, 3, 4, 5, 5, 6, 6, 8, 8, 8, 10, 9, 10, ... 

While the sequence is defined by a relatively simple mathematical rule, the terms exhibit chaotic behavior, which gets more noticeable as you go further out into the sequence. It’s very simple to write a TI-84 code which will generate the first 999 terms of this sequence (the maximum allowable by your calculator). Below is the code, together with a graph of the terms of the sequence. While this “\( Q \)-sequence” resembles the Fibonacci sequence inasmuch as having each term being the sum of two earlier terms, there is no known non-recursive formula which gives the terms of the sequence.

---

\(^3\)This example comes from the book, Gödel, Escher, Bach: an Eternal Golden Braid, by Douglas R. Hofstadter, p137. See also http://local.wasp.uwa.edu.au/~pbourke/fractals/qseries/
IV. The Euler-Masceroni number $\gamma$

Let’s define a sequence $a_1, a_2, a_3, \ldots$ in the following way:

\[
a_1 = 1 - \ln 1 = 1,
\]

\[
a_2 = 1 + \frac{1}{2} - \ln 2 \approx 0.807,
\]

\[
a_3 = 1 + \frac{1}{2} + \frac{1}{3} - \ln 3 \approx 0.735,
\]

\[
\vdots
\]

\[
a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n = \sum_{k=1}^{n} \frac{1}{k} - \ln n.
\]

A simple TI-84 code is shown below which will generate the first 20 terms of this sequence. Also, the graph of the sequence is shown.

\[
\begin{align*}
\text{PROGRAM: EULER} \\
\text{ClrList L3} \\
\text{0} \rightarrow \text{S} \\
\text{For}(N, 1, 20) \\
\text{S} + N^{-1} \rightarrow S \\
\text{S} - \text{ln(N)} \rightarrow \text{L3(N)} \\
\text{End} \\
\text{Stop}
\end{align*}
\]

From the above, it’s reasonable to surmise that (i) the terms of the above sequence are all positive, and that (ii) the terms are decreasing. This will establish that the sequence has a limit. We itemize the proofs of (i) and (ii) below:

We’ll itemize a few easily-established facts about this sequence.

(i) Each term $a_n, \ n \geq 1$ is positive. To see this, consult the figure below, which shows the graph of $y = \frac{1}{x}$ lying underneath rectangles of heights $1/n, \ n \geq 1$. The picture shows clearly that for each $n \geq 2$,

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \geq \text{area under graph of } y = \frac{1}{x}, \ (1 \leq x \leq n) = \ln n
\]

A fortiori, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} \geq \ln n$, which implies that each $a_n > 0$. 

(ii) For each \(n \geq 1\), \(a_{n+1} < a_n\). To see this, note first that \(a_n - a_{n+1} = \ln \frac{n+1}{n} - \frac{1}{n+1}\).

Since \(\ln \frac{n+1}{n}\) is the area under the graph of \(y = \frac{1}{x}\), \(n \leq x \leq n + 1\), a quick glimpse at the figure shows that this quantity is clearly positive.

**Exercise.** Show that for each \(n \geq 1\), \(a_n > \frac{1}{2} + \frac{1}{2n}\). (Hint: show that

\[
1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} - \ln n > \frac{1}{2} - \frac{1}{2n}.
\]

This follows again by looking at the figure and summing a telescoping series.)