Part I. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n e^n}{n^2} \) diverges because

   I. The terms do not tend to 0 as \( n \) tends to \( \infty \).
   II. The terms are not all positive.
   III. \( \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} > 1 \).

   (A) I only
   (B) II only
   (C) III only
   (D) I and II only
   (E) I and III only

2. The interval of convergence for the series \( \sum_{n=1}^{\infty} \frac{(3x + 2)^{n+1}}{n^{5/2}} \) is

   (A) \(-1 \leq x < -\frac{1}{3}\)
   (B) \(-1 < x \leq -\frac{1}{3}\)
   (C) \(-1 \leq x \leq -\frac{1}{3}\)
   (D) \(\frac{1}{3} \leq x \leq 1\)
   (E) \(-1 < x < \frac{1}{3}\)

3. Given that \( f(x) = \sum_{n=0}^{\infty} \frac{n(x - a)^n}{2^n} \) on the interval of convergence of the Taylor series, \( f^{(4)}(a) = \)

   (A) 0
   (B) 6
   (C) 9
   (D) \(\frac{1}{4}\)
   (E) \(\frac{1}{4!}\)
4. Which of the following series converge?

I. \[ \sum_{n=1}^{\infty} \left( \frac{n^2 - n + 5}{n^{7/2} + 1} \right). \]

II. \[ \sum_{n=1}^{\infty} \frac{(-1)^n 3}{n}. \]

III. \[ \sum_{n=1}^{\infty} \left( \frac{\cos 2n\pi}{n^2} \right). \]

(A) I and II only
(B) I and III only
(C) II and III only
(D) They all do!
(E) None of them do!

5. \[ 1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} + \cdots + (-1)^n \frac{\pi^{2n}}{(2n)!} + \cdots = \]

(A) 0
(B) -1
(C) \( \pi \)
(D) 1
(E) \(-\pi\)
Part II. Free-Response Questions

1. A function \( f \) is defined by

\[
f(x) = \frac{1}{4} + \frac{2}{4^2} x + \frac{3}{4^3} x^2 + \cdots + \frac{n+1}{4^n+1} x^n + \cdots
\]

for all \( x \) in the interval of convergence of the given power series.

(a) (4 points) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) (3 points) Find \( \lim_{x \to 0} \frac{f(x) - \frac{1}{4}}{x} \).
#1, continued; \( f(x) = \frac{1}{4} + \frac{2}{4^2} x + \frac{3}{4^3} x^2 + \cdots + \frac{n+1}{4^{n+1}} x^n + \cdots \)

(c) **(3 points)** Write the first three nonzero terms and the general term for an infinite series that represents \( \int_0^2 f(x) \, dx \).

(d) **(4 points)** Find the sum of the series determined in part (c).
2. Let \( f \) be a function with derivatives of all orders and for which \( f(2) = 7 \). When \( n \) is odd, the \( n \)th derivative of \( f \) at \( x = 2 \) is 0. When \( n \) is even and \( n \geq 2 \), the \( n \)th derivative of \( f \) at \( x = 2 \) is given by \( f^{(n)}(2) = \frac{(n - 1)!}{3^n} \).

(a) **(4 points)** Write the sixth-degree Taylor polynomial for \( f \) about \( x = 2 \).

(b) **(3 points)** In the Taylor series for \( f \) about \( x = 2 \), what is the coefficient of \((x - 2)^2n\) for \( n \geq 1 \)?

(c) **(4 points)** Find the interval of convergence of the Taylor series for \( f \) about \( x = 2 \). Show the work that leads to your answer.
Part III. Multiple-Choice Questions (5 points each; please circle the correct answer.)

1. The series \( \sum_{n=1}^{\infty} \frac{\sqrt{n^p - 1}}{n^{p+2} + 1} \) will converge, provided that
   (A) \( p > 1 \)
   (B) \( p > 2 \)
   (C) \( p > -1 \)
   (D) \( p > -2 \)
   (E) \( p > 0 \)

2. The graph of the function represented by the Taylor series \( \sum_{n=0}^{\infty} n(x + 1)^{n-1} \) intersects the graph of \( y = \ln x \) at \( x \approx \)
   (A) 1.763
   (B) 0.703
   (C) 1.532
   (D) 0.567
   (E) 1.493

3. Using the fourth-degree Maclaurin polynomial of the function \( f(x) = e^x \) to estimate \( e^{-2} \), this estimate is
   (A) 7.000
   (B) 0.333
   (C) 0.135
   (D) 0.067
   (E) 0.375
4. What is the approximation of the value of \( \cos(2^\circ) \) obtained by using the sixth-degree Taylor polynomial about \( x = 0 \) for \( \cos x \)?

(A) \( 1 - 2 + \frac{2}{3} - \frac{4}{45} \)

(B) \( 1 + 2 + \frac{16}{24} + \frac{64}{720} \cdot \)

(C) \( 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} \)

(D) \( 1 - \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} - \frac{\pi^6}{6! \cdot 90^6} \)

(E) \( 1 + \frac{\pi^2}{2 \cdot 90^2} + \frac{\pi^4}{4! \cdot 90^4} + \frac{\pi^6}{6! \cdot 90^6} \)

5. Which of the following gives a Taylor polynomial approximation about \( x = 0 \) for \( \sin 0.5 \), correct to four decimal places?

(A) \( 0.5 + \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} \)

(B) \( 0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} \)

(C) \( 0.5 - \frac{(0.5)^3}{3} + \frac{(0.5)^5}{5} \)

(D) \( 0.5 + \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} + \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!} \)

(E) \( 0.5 - \frac{(0.5)^2}{2!} + \frac{(0.5)^3}{3!} - \frac{(0.5)^4}{4!} + \frac{(0.5)^5}{5!} \)
1. The function $f$ has derivatives of all orders for all real numbers $x$. Assume $f(2) = -3$, $f'(2) = 5$, $f''(2) = 3$, and $f'''(2) = -8$.

   (a) **(3 points)** Write the third-degree Taylor polynomial for $f$ about $x = 2$ and use it to approximate $f(1)$.

   (b) **(4 points)** The fourth derivative of $f$ satisfies the inequality $|f^{(4)}(x)| \leq 3$ for all $x$ in the closed interval $[1, 2]$. Use the Lagrange error bound on the approximation to $f(1)$ found in part (a) to explain why $f(1) \neq -5$.

   (c) **(4 points)** Write the fourth-degree Taylor polynomial, $P(x)$, for $g(x) = f(x^2 + 2)$ about $x = 0$. Use $P$ to explain why $g$ must have a relative minimum at $x = 0$. 
2. Let $f$ be a function having derivatives of all orders for all real numbers. The third-degree Taylor polynomial for $f$ about $x = 2$ is given by

$$T(x) = -5(x - 2)^2 - 3(x - 2)^3.$$

(a) *(2 points)* Find $f(2)$ and $f''(2)$.

(b) *(4 points)* Is there enough information given to determine whether $f$ has a critical point at $x = 2$? If not, explain why not. If so, determine whether $f(2)$ is a relative maximum, a relative minimum, or neither, and justify your answer.
#2, continued; \[ T(x) = -5(x - 2)^2 - 3(x - 2)^3. \]

(c) \textbf{(4 points)} Use \( T(x) \) to find an approximation for \( f(0) \). Is there enough information given to determine whether \( f \) has a critical point at \( x = 0 \)? If not, explain why not. If so, determine whether \( f(0) \) is a relative maximum, a relative minimum, or neither, and justify your answer.

(d) \textbf{(4 points)} The fourth derivative of \( f \) satisfies the inequality \( |f^{(4)}(x)| \leq 5 \) for all \( x \) in the closed interval [0, 2]. Use the Lagrange error bound on the approximation to \( f(0) \) found in part (c) to explain why \( f(0) \) is positive.