Part I. Multiple-Choice Questions

1. The graph in the $xy$-plane represented by $x = 3 \sin t$ and $y = 2 \cos t$ is
   (A) a circle (B) an ellipse (C) a hyperbola (D) a parabola (E) a line

2. If a particle moves in the $xy$-plane so that at time $t > 0$ its position vector is $(e^{t^2}, e^{-t^3})$, then its velocity vector at time $t = 3$ is
   (A) $(\ln 6, \ln(-27))$
   (B) $(\ln 9, \ln(-27))$
   (C) $(e^9, e^{-27})$
   (D) $(6e^9, -27e^{-27})$
   (E) $(9e^9, -27e^{-27})$

3. A particle moves along a path described by $x = \cos^3 t$ and $y = \sin^3 t$. The distance that the particle travels along the path from $t = 0$ to $t = \frac{\pi}{2}$ is
   (A) 0.75 (B) 1.50 (C) 0 (D) -3.50 (E) -0.75

4. The area enclosed by the polar equation $r = 4 + \cos \theta$, for $0 \leq \theta \leq 2\pi$, is
   (A) 0 (B) $\frac{9\pi}{2}$ (C) $18\pi$ (D) $\frac{33\pi}{2}$ (E) $\frac{33\pi}{4}$

5. If, for $t > 0$, $x = t^2$ and $y = \cos t^2$, then $\frac{dy}{dx} =$
   (A) $\cos t^2$ (B) $-\sin t^2$ (C) $-\sin 2t$ (D) $\sin t^2$ (E) $\cos 2t$
6. Find the area inside one loop of the curve $r = \sin 2\theta$.

(A) $\frac{\pi}{16}$  (B) $\frac{\pi}{8}$  (C) $\frac{\pi}{4}$  (D) $\frac{\pi}{2}$  (E) $\pi$

7. Find the length of the arc of the curve defined by $x = \frac{1}{2}t^2$ and $y = \frac{1}{9}(6t + 9)^{\frac{3}{2}}$, from $t = 0$ to $t = 2$.

(A) 8  (B) 10  (C) 12  (D) 14  (E) 16

8. If $f$ is a vector-valued function defined by $f(t) = (\sin 2t, \sin^2 t)$, then $f''(t) =$

(A) $(-4 \sin 2t, 2 \cos 2t)$
(B) $(-\sin 2t, -\cos^2 t)$
(C) $(4 \sin 2t, \cos^2 t)$
(D) $(4 \sin 2t, -2 \cos 2t)$
(E) $(2 \cos 2t, -4 \sin 2t)$

9. A solid is formed by revolving the region bounded by the $x$-axis, the lines $x = 0$ and $x = 1$ and the parametric curve $x = \sin t$, $y = 1 + \cos^2(2t)$ about the $x$-axis. Write down the integral which will compute the volume of the solid.

(A) $\pi \int_{0}^{\frac{\pi}{2}} (1 + \cos^4(2t)) \, dt$  (B) $\pi \int_{0}^{\frac{\pi}{2}} (1 + \cos^2(2t))^2 \, dt$

(D) $2\pi \int_{0}^{\frac{\pi}{2}} (1 + \cos^2(2t))^2 \, dt$  (E) $\pi \int_{0}^{\frac{\pi}{2}} \cos t(1 + \cos^2(2t))^2 \, dt$

(C) $\pi \int_{0}^{\frac{\pi}{2}} (\sin^2 t + (1 + \cos^2(2t))^2) \, dt$
10. The length of the curve determined by $x = 3t$ and $y = 2t^2$ from $t = 0$ to $t = 9$ is

(A) $\int_0^9 \sqrt{9t^2 + 4t^4} \, dt$

(B) $\int_0^{162} \sqrt{9 - 16t^2} \, dt$

(C) $\int_0^{162} \sqrt{9 + 16t^2} \, dt$

(D) $\int_0^3 \sqrt{9 - 16t^2} \, dt$

(E) $\int_0^9 \sqrt{9 + 16t^2} \, dt$

11. The length of the curve determined by $x = 2t^3$ and $y = t^3$ from $t = 0$ to $t = 1$ is

(A) $\frac{5}{7}$

(B) $\frac{\sqrt{5}}{2}$

(C) $\frac{3}{2}$

(D) $\sqrt{5}$

(E) 3

12. The area of the region inside the polar curve $r = 4 \sin \theta$ but outside the polar curve $r = 2\sqrt{2}$ is given by

(A) $2 \int_{\pi/4}^{3\pi/4} (4 \sin^2 \theta - 1) \, d\theta$

(B) $\frac{1}{2} \int_{\pi/4}^{3\pi/4} (4 \sin \theta - 2\sqrt{2})^2 \, d\theta$

(C) $\frac{1}{2} \int_{\pi/4}^{3\pi/4} (4 \sin \theta - 2\sqrt{2}) \, d\theta$

(D) $\frac{1}{2} \int_{\pi/4}^{3\pi/4} (16 \sin^2 \theta - 8) \, d\theta$

(E) $\frac{1}{2} \int_{\pi/4}^{3\pi/4} (4 \sin^2 \theta - 1) \, d\theta$. 
13. A particle moves on a plane curve so that at any time \( t > 0 \) its position is defined by the parametric equations \( x(t) = 3t^2 - 7 \) and \( y(t) = \frac{4t^2 + 1}{3t} \). The acceleration vector of the particle at \( t = 2 \) is

(A) \( (6, \frac{1}{12}) \)  (B) \( (17, \frac{17}{6}) \)  (C) \( (12, \frac{47}{12}) \)  (D) \( (12, \frac{33}{12}) \)  (E) \( (6, \frac{17}{6}) \)

14. The acceleration of a particle is described by the parametric equation \( x''(t) = \frac{t^2}{4} + t \) and \( y''(t) = \frac{1}{3t} \). If the velocity vector of the particle when \( t = 2 \) is \( (4, \ln 2) \), what is the velocity vector of the particle when \( t = 1 \)?

(A) \( \left( \frac{5}{4}, \frac{1}{3} \right) \)

(B) \( \left( \frac{23}{12}, \frac{\ln 4}{3} \right) \)

(C) \( \left( \frac{23}{12}, \frac{\ln 2}{3} \right) \)

(D) \( \left( \frac{5}{4}, \frac{2 \ln 2}{3} \right) \)

(E) \( \left( \frac{23}{12}, \frac{1}{3} \ln 2 \right) \)
Part II. Free-Response Questions

1. A moving particle has position \((x(t), y(t))\) at time \(t\). The position of the particle at time \(t = 1\) is \((2, 6)\), and the velocity vector at any time \(t > 0\) is given by \(\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)\).

(a) Find the acceleration vector at time \(t = 3\).

(b) Find the position of the particle at time \(t = 3\).

(c) For what time \(t > 0\) does the line tangent to the path of the particle at \((x(t), y(t))\) have a slope of 8?

(d) The particle approaches a line as \(t \to \infty\). Find the slope of this line. Show that work that leads to your conclusion.

2. The figure above shows the graphs of the line \(x = \frac{5}{3}y\) and the curve \(C\) is given by \(x = \sqrt{1 + y^2}\). Let \(S\) be the region bounded by the two graphs and the \(x\)-axis. The line and the curve intersect at point \(P\).

(a) Curve \(C\) is a part of the curve \(x^2 - y^2 = 1\). Show that \(x^2 - y^2 = 1\) can be written as the polar equation \(r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}\).

(b) Use the polar equation given above to set up an integral expression with respect to the polar angle \(\theta\) that represents the area of \(S\).
3. The figure above shows the path traveled by a roller coaster car over the time interval $0 \leq t \leq 18$ seconds. The position of the car at time $t$ can be modeled parametrically by

$$
\begin{align*}
  x(t) &= 10t + 4 \sin t \\
  y(t) &= (20 - t)(1 - \cos t)
\end{align*}
$$

where $x$ and $y$ are measured in meters. The derivatives of these functions are given by

$$
\begin{align*}
  x'(t) &= 10 + 4 \cos t \\
  y'(t) &= (20 - t) \sin t + \cos t - 1
\end{align*}
$$

(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car’s horizontal position is $x = 140$.

(c) Find the time $t$ at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.
(d) For $0 < t < 18$, there are two times at which the car is at ground level $(y = 0)$. Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times.
4. A particle moves in the $xy$-plane so that its position at any time $t$, for $-\pi \geq t \geq \pi$, is given by $x(t) = \sin 3t$ and $y(t) = 2t$.

(a) Sketch the path of the particle in the $xy$-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of $t$ for which the $x$-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $\pi$ greater than $5\pi$. Justify your result.

5. A moving particle has position $(x(t), y(t))$ at time $t$. The position of the particle at time $t = 1$ is $(7, 0)$, and the velocity vector at any time $t > 0$ is given by $\left(3 - \frac{3}{t^2}, 4 + \frac{2}{t^2}\right)$.

(a) Find the position of the particle at $t = 3$.

(b) Will the line tangent to the path of the particle at $(x(t)), y(t))$ ever have a slope of zero? If so, when? If not, why not?
6. A particle starts at point $A$ on the positive $x$-axis at time $t = 0$ and travels along the curve from $A$ to $B$ to $C$ to $D$, as shown above. The coordinates of the particle’s position $(x(t), y(t))$ are differentiable functions of $t$, where

$$x'(t) = \frac{dx}{dt} = -9 \cos \left( \frac{\pi t}{6} \right) \sin \left( \frac{\sqrt{t} + 1}{2} \right), \quad y'(t) = \frac{dy}{dt}$$

is not explicitly given. At time $t = 9$, the particle reaches its final position at point $D$ on the positive $x$-axis.

(a) At point $C$, is $\frac{dy}{dt}$ positive? At point $C$, is $\frac{dx}{dt}$ positive? Give a reason for each answer.

(b) The slope of the curve is undefined at point $B$. At what time $t$ is the particle at point $B$?

(c) The line tangent to the curve at the point $(x(8), y(8))$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.

(d) How far apart are points $A$ and $D$, the initial and final positions, respectively, of the particle?
7. The figure above shows the graphs of the circles
\[ x^2 + y^2 = 2 \quad \text{and} \quad (x - 1)^2 + y^2 = 1. \]
The graphs intersect at the points \((1, 1)\) and \((1, -1)\). Let \(R\) be the region in the first quadrant bounded by the two circles.

(a) Write the polar equations of the two circles.

(b) Write down an integral in terms of the polar angle \(\theta\) which computes the area of \(R\).

8. A particle moves in the \(xy\)-plane so that the position of the particle at any time \(t\) is given by
\[ x(t) = 2e^{3t} + e^{-7t} \quad \text{and} \quad y(t) = 3e^{3t} - e^{-2t}. \]

(a) Find the velocity vector of the particle in terms of \(t\), and find the speed of the particle at time \(t = 0\).

(b) Find \(\frac{dy}{dx}\) in terms of \(t\), and find \(\lim_{t \to \infty} \frac{dy}{dx}\).

(c) Find each value \(t\) at which the line tangent to the path of the particle is horizontal, or explain why none exists.

(d) Find each value \(t\) at which the line tangent to the path of the particle is vertical, or explain why none exists.
9. An object moving along a curve in the $xy$-plane has position $(x(t), y(t))$ at time $t \geq 0$ with $\frac{dx}{dt} = 3 + \cos t^2$. The derivative $\frac{dy}{dx}$ is not explicitly given. At time $t = 2$, the object is at position $(1, 8)$.

(a) Find the $x$-coordinate of the object at time $t = 4$.

(b) At time $t = 2$, the value of $\frac{dy}{dt}$ is $-7$. Write an equation for the line tangent to the curve at the point $(x(2), y(2))$.

(c) Find the speed of the object at time $t = 2$.

(d) For $t \geq 3$, the line tangent to the curve at $(x(t), y(t))$ has a slope of $2t + 1$. Find the acceleration vector of the object at time $t = 4$.

10. A particle moving along a curve in the plane has position $(x(t), y(t))$ at time $t$, where $\frac{dx}{dt} = \sqrt{t^4 + 9}$ and $\frac{dy}{dt} = 2e^t + 5e^{-t}$ for all real values of $t$. At time $t = 0$, the particle is at the point $(4, 1)$.

(a) Find the speed of the particle and its acceleration vector at time $t = 0$.

(b) Find an equation of the line tangent to the path of the particle at time $t = 0$.

(c) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

(d) Find the $x$-coordinate of the position of the particle at time $t = 3$. 
11. The curve above is drawn in the $xy$-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$, $0 \leq \theta \leq \pi$, where $r$ is measured in meters and $\theta$ is measured in radians. The derivative of $r$ with respect to $\theta$ is given by $\frac{dr}{d\theta} = 1 + 2 \cos(2\theta)$.

(a) Find the area bounded by the curve and the $x$-axis.

(b) Find the angle $\theta$ that corresponds to the point on the curve with $x$-coordinate $-2$.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about $r$? What does this fact say about the curve?

(d) Find the value of $\theta$ in the interval $0 \leq \theta \leq \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.