IB Further Mathematics
Discrete Mathematics
Mid-Unit Test (65 Points Total)

Name: ___________________________ Date: __________________

1. Consider the Diophantine equation $15x + 24y = 123$.
   
   (a) (3 points) Explain briefly why there can only be finitely many integer solutions $(x, y)$ such that $x, y \geq 0$.

   (b) (7 points) Solve the above equation, subject to $x, y \geq 0$.

2. (5 points) You are given that
   
   $2417$ is a prime, and that $2 \cdot 2417 - 3 \cdot 1611 = 1$.

   Find the least positive integer solution of the congruence $1611x^{4833} \equiv 5 \pmod{2417}$. 
3. (a) **(5 points)** Write the base 6 decimal number 245.32 as a base 10 decimal (possibly repeating).

(b) **(5 points)** Let $x$ be a positive integer, and assume that $x_6 = a_n a_{n-1} \cdots a_1 a_0$ is the base 6 representation of $x$.

   i. Prove that $x \equiv a_0 + a_1 + \cdots + a_{n-1} + a_n \pmod{5}$.

   ii. **(4 points)** Hence, or otherwise, show that $x$ is divisible by 5 if and only if $a_0 + a_1 + \cdots + a_n$ is divisible by 5.

   iii. **(4 points)** More generally, let $x$ be a positive integer and let $x_m = b_0 b_1 \cdots b_n$ be the base $m$ representation of $x$. What is the natural generalization of the statement given in ii., above. (You don’t need to prove this generalization.)
4. **(6 points)** For any positive integers $a$ and $b$, let $\gcd(a, b)$ and $\text{lcm}(a, b)$ denote the greatest common divisor and the least common multiple of $a$ and $b$, respectively. Prove that $ab = \gcd(a, b)\text{lcm}(a, b)$. (Hint: if $d = \gcd(a, b)$, then $\gcd \left( \frac{a}{d}, \frac{b}{d} \right) = 1$.)

5. **(a) (3 points)** What is the smallest value of $\lambda \geq 8$ such that the Diophantine equation $\lambda x - 6y = 1$ has a solution?

   **(b) (5 points)** For the value of $\lambda$ determined in (a), find the general solution of the above equation.
6. **(10 points)** Find the least positive integer solution of the congruences

\[
\begin{align*}
m &\equiv 7 \pmod{10} \\
m &\equiv 5 \pmod{26} \\
m &\equiv 1 \pmod{12}.
\end{align*}
\]
7. **(4 points)** Consider the linear homogeneous recurrence equation
\[ u_n = u_{n-1} + 2u_{n-2}, \quad n \geq 2. \]

(a) Find the general solution of the above equation.

(b) **(4 points)** Find the particular solution of the above recurrence equation, given the initial conditions \( u_0 = 0, \quad u_1 = 1. \)