1. Let $G_1$ and $G_2$ be groups and form the cartesian product

$$G_1 \times G_2 = \{ (g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2 \}.$$ 

Define the product simply by setting $(g_1, g_2)(g'_1, g'_2) = (g_1g'_1, g_2g'_2)$, where $g_1, g'_1 \in G_1, g_2, g'_2 \in G_2$. Show that relative to this binary operation, $G_1 \times G_2$ is a group. Under what circumstances is the group $G_1 \times G_2$ an abelian group?
2. Consider the multiplicative group $G = \mathbb{Z}_{17}^\times = \mathbb{Z}_{17} - \{0\}$.

(a) Find the order of the element 7.

(b) For each divisor of 16, find an element of that order.
3. Let $S$ be the group of permutations of the set $\{1, 2, 3, 4\}$, relative to composition as the binary operation.

(a) Is $S$ abelian? Explain.

(b) Find a noncyclic subgroup of order 4.

(c) Find a subgroup of order 6.
4. Let $G$ be a group.

(a) Define the center $Z$ of the group $G$ to be

$$Z = \{ z \in G \mid zg = zg \text{ for all elements } g \in G \}.$$  

In other words, the center $Z$ consists of all elements of $G$ that commute with all other elements of $G$.

Show that $Z$ is an abelian subgroup of $G$.

(b) Now let $x \in G$ be a fixed element of $G$. Define the centralizer of $x$ by setting

$$C(x) = \{ z \in G \mid zx = xz \}.$$  

That is, the centralizer of $x$ consists of all elements of $G$ that commute with $x$.

Show that $C(x)$ is a subgroup of $G$. 
5. Let $G$ be the group of permutations of the set $\{1, 2, 3\}$ and let $H$ be the cyclic group generated by $\tau$, where

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$ 

Define the relation “$\equiv \pmod{H}$” by setting $g_1 \equiv g_2 \pmod{H} \iff g_2^{-1}g_1 \in H$, $g_1, g_2 \in G$.

(a) Write down the equivalence class containing the element $x$, where

$$x = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$ 

(b) How many equivalence classes in $G$ relative to $\equiv \pmod{H}$ are there?
6. Let $G$ be an abelian group and define the mapping $f : G \to G$ by setting $f(g) = g^2$, for all $g \in G$.

(a) Show that $f$ is a homomorphism of $G$ into itself.

(b) Now suppose that $G$ is finite of odd order. Show that $f$ is actually an isomorphism of $G$ onto itself.
7. Let $G_1$ and $G_2$ be groups and let $f : G_1 \to G_2$ be a **surjective** homomorphism. Show that if $G_1$ is abelian, then $G_2$ is also abelian.