IB Further Mathematics
Abstract Algebra—Quiz 1

Name: ___________________________  Date: ___________

1. Consider the function \( f : \mathbb{R} \rightarrow \mathbb{R} \) by setting \( f(x) = x^2 \tan^{-1} x \). Prove that \( f \) is bijective. (Hint: use calculus!)

2. Define the binary operation \( \# \) on the set of real numbers by setting

\[
    a \# b = a + b + 1, \quad a, b \in \mathbb{R}.
\]

(a) Show that \( \# \) is both associative and commutative.

(b) Show that \( \mathbb{R} \) has an identity relative to \( \# \).

(c) Show that \( (\mathbb{R}, \#) \) is a group. (What’s left?)
3. Let $G$ be the group of permutations of the set $\{1, 2, 3\}$ with composition as its binary operation.

(a) Let
\[
\sigma = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}.
\]

Show that
\[
\sigma^3 = \tau^2 = (\sigma\tau)^2 = e \text{ (identity permutation)}.
\]

(b) Complete the multiplication table below:

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
<th>$\sigma$</th>
<th>$\sigma^2$</th>
<th>$\tau$</th>
<th>$\sigma\tau$</th>
<th>$\sigma^2\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$e$</td>
<td>$\sigma$</td>
<td>$\sigma^2$</td>
<td>$\tau$</td>
<td>$\sigma\tau$</td>
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</tr>
</tbody>
</table>
4. Let \( G \) be a group.
   
   (a) Prove that if \( x \in G \), then the inverse of \( x \) is **unique**.

   (b) Assume that for all elements \( g \in G \), we have that \( g^2 = e \), where \( e \in G \) is the identity. Prove that for all elements \( x, y \in G \) we have \( xy = yx \).

5. Let \( R \) be the relation on \( \mathbb{R} \times \mathbb{R} \) defined by setting \((x_1, y_1) R (x_2, y_2)\) if and only if \( y_1 + x_1^2 = y_2 + x_2^2 \).

   (a) Prove that \( R \) is an equivalence relation on \( \mathbb{R} \times \mathbb{R} \).

   (b) Describe the equivalence classes in \( \mathbb{R} \times \mathbb{R} \) relative to \( R \).
6. Let $Y$ be the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the relation $R$ on $Y$ by setting
\[ aRb \iff a^2 - b^2 \equiv 0 \pmod{5}, \]
where $a, b \in Y$.

(a) Show that $R$ is an equivalence relation.

(b) Define what is meant by the equivalence class containing $a$.

(c) Write down the equivalence classes in $Y$ relative to $R$. 
7. (a) The tables below are the Cayley tables for two groups of order 4. Are the groups isomorphic? Give a reason.

\[
\begin{array}{c|cccc}
  & e & x & y & xy \\
\hline
  e & e & x & y & xy \\
x & x & e & xy & y \\
y & y & xy & e & x \\
xy & xy & y & x & e \\
\end{array}
\quad
\begin{array}{c|cccc}
  & e & z & z^2 & z^3 \\
\hline
  e & e & z & z^2 & z^3 \\
z & z & z^2 & z^3 & e \\
z^2 & z^2 & z^3 & e & z \\
z^3 & z^3 & e & z & z^2 \\
\end{array}
\]

(b) Now define the real-valued functions \( f, g, h, j : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\} \) by setting

\[
f(x) = x, \quad g(x) = -x, \quad h(x) = \frac{1}{x}, \quad j(x) = -\frac{1}{x}.
\]

Relative to function composition, these four functions form a group; complete the Cayley table to the right:

\[
\begin{array}{c|cccc}
  \circ & f & g & h & j \\
\hline
  f & \cdot & g & h & j \\
g & & & & \\
h & & & & \\
j & & & & \\
\end{array}
\]

This group is isomorphic to one of the groups in part (a) above. Which one?

(c) The tables below are the Cayley tables for the integers modulo 4, relative to addition, and the nonzero integers modulo 5, relative to multiplication.

\[
\begin{array}{c|cccc}
+ & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 2 & 3 \\
1 & 1 & 2 & 3 & 0 \\
2 & 2 & 3 & 0 & 1 \\
3 & 3 & 0 & 1 & 2 \\
\end{array}
\quad
\begin{array}{c|cccc}
\times & 1 & 2 & 3 & 4 \\
\hline
1 & 1 & 2 & 3 & 4 \\
2 & 2 & 4 & 1 & 3 \\
3 & 3 & 1 & 4 & 2 \\
4 & 4 & 3 & 2 & 1 \\
\end{array}
\]

Are these groups isomorphic? Give a reason.