SURGERY NUMBERS OF 3-MANIFOLDS: A HYPERBOLIC EXAMPLE

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ABSTRACT. This paper defines the surgery number and the Dehn surgery number of any 3-manifold. Many examples which illustrate the difficulties involved in computing surgery numbers are given, and it is shown that the Dehn surgery number is non-trivial for some hyperbolic manifolds.

In this article, we will define the surgery number and the Dehn surgery number of any 3-manifold. These numbers are topological invariants by their very definition, but they are nearly impossible to compute in general. We will give a series of examples which suggest questions and demonstrate the difficulties involved in computing surgery numbers. The last example in this paper is a hyperbolic manifold with non-trivial Dehn surgery number. In particular, we answer question 3.6(c) from the 1978 Kirby problem list by constructing an irreducible and in fact hyperbolic homology sphere which is not Dehn surgery on any knot [6].

Surgery and Dehn surgery are generic ways of constructing 3-manifolds. A manifold, \( N \), is obtained by surgery on a link in a manifold \( M \), if

\[
N - M = \partial [I \times M \bigcup_{n(S^1 \times D^2)} n(D^2 \times D^2)]
\]

where the isotopy class of the identification of \( n(S^1 \times D^2) \) with \( \{1\} \times N(L) \) is called a framing. Likewise, a manifold is Dehn surgery on a link in \( M \) if

\[
N = (M - \tilde{N}(L)) \bigcup_{nT^2} n(D^2 \times S^1)
\]

Any surgery description of \( N \) from \( M \) has a corresponding Dehn surgery description.

A Dehn surgery construction corresponds to a surgery construction if and only if the new meridian intersects the old meridian in exactly

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one point where the meridian of a knot is the boundary of a fiber in a tubular neighborhood. Every closed orientable 3-manifold may be obtained from surgery on $S^3$.

**Definition.** The *surgery metric* is:

$$S(M, N) = \text{The smallest number of components in a link in } M \text{ in a surgery description of } N.$$  

The *Dehn surgery metric* is:

$$D(M, N) = \text{The smallest number of components in a link in } M \text{ in a Dehn surgery description of } N.$$  

Let the (Dehn) surgery number of $M$ be $(DS(M) = DS(S^3, M))$, $S(M) = S(S^3, M)$. Let $h(M)$ be the size of a minimal generating set for $H_1(M)$. The weight of a group is the size of the smallest set which normally generates the group. Put another way, it is the smallest number of new relations that can be added to the group in order to kill it.

**Fact.** $h(M) \leq Wt(\pi_1(M)) \leq DS(M) \leq S(M).$

The first inequality follows because $H_1(M)$ is the abelianization of $\pi_1(M)$. The last is a trivial consequence of the observation that surgery is a special case of Dehn surgery.

If $M$ is Dehn surgery on a link, $L$, then

$$M = (S^3 - \overset{o}{N}(L)) \bigcup_{nT^2} n(D^2 \times S^1).$$

Consider,

$$M \bigcup_{nT^2} N(L) = S^3 \bigcup_{nT^2} n(D^2 \times S^1).$$

By Van Kampen's Theorem, $M \bigcup_{nT^2} N(L)$ is simply connected and $\pi_1(M \bigcup_{nT^2} N(L))$ is obtained from $\pi_1(M)$ by adding $n$ relations, thus $Wt(\pi_1(M)) \leq n$.

The weight is not as useful as it first appears. It is difficult to compute the weight of a group, and the weight is often not equal to the Dehn surgery number. We will prove that the manifold in Figure 1 has Dehn surgery number 2, but the weight of its fundamental group is 1.

It is a straightforward exercise to compute

$$\pi_1(\Sigma\# - \Sigma) = \pi_1(\Sigma)\#\pi_1(\Sigma) \text{ where } \pi_1(\Sigma) = \langle x, y \mid (xy)^2 = x^3 = y^5 \rangle.$$  

The fundamental group of $\Sigma$ is normally generated by an element of order 5 and it is also normally generated by an element of order 3, so $\pi_1(\Sigma\# - \Sigma)$ is normally generated by the word which sets the element of
order 5 in the first factor equal to the element of order 3 in the second factor.

The Dehn surgery number is also related to the unknotting number. To see how, assume that \( K \) has unknotting number \( n \). Let \( M_K \) be the 2-fold branched cover of \( S^3 \) branched over \( K \), and take one small ball around each of the \( n \) crossings used to unknott \( K \). The inverse image of one of the small balls is a \( D^2 \times S^1 \) (see Figure 2).

Cutting out a \( S^1 \times D^2 \) in the cover and gluing it in differently corresponds to cutting out a ball with two arcs in the base of the covering and gluing the ball in with the opposite crossing. The 2-fold cover of \( S^3 \) branched over the unknot is \( S^3 \), so \( M_K \) is Dehn surgery on a link with \( n \) components. We, therefore, have:

**Fact.** \( DS(M_K) \leq u(K) \).

It seems plausible that the number of non-trivial, non-parallel 2-spheres in a manifold should give a lower bound on the Dehn surgery number of the manifold.

The correct conjecture is hard to make because of examples like Figure 4. See [8]. It would also be hard to come up with a correct conjecture as to when incompressible tori imply large surgery number, because of examples like Figure 5.
Figure 3. This probably has $DS(M) = 4$.

Figure 4. The trouble with 2-sphere bounds.
Figure 5. An interesting $T^2$.

Figure 5 shows a $T^2 - 2B^2$ embedded in $S^3$ with boundary on the boundary of a regular neighborhood of the figure eight knot. This completes to an incompressible torus in $-4$-framed surgery on the figure eight knot.

It is intuitively clear that it should be possible to construct a homology sphere which is not surgery on any knot by gluing two knot exteriors together. This does not always work. The manifold in Figure 6 is the union of two non-trivial knot exteriors, but it is surgery on a knot.

One indication of how difficult it is to prove anything about surgery numbers is given by the Gordon-Luecke theorem [3].

**Theorem.** (Gordon-Luecke) Non-trivial Dehn surgery on a non-trivial not has Dehn surgery number 1.

If a manifold is non-trivial Dehn surgery on a non-trivial knot, then its Dehn surgery number is less than or equal to 1. The point is that the manifold could be $S^3$ in which case the Dehn surgery number would be 0. The theorem is that this is not the case. The Gordon-Luecke theorem was around as the $P'$ conjecture for about eighty years.

In light of the above facts, it is natural to ask if there is an irreducible (i.e., no non-trivial 2 spheres) homology 3-sphere ($h(M) = 0$) with Dehn surgery number bigger than one. What about a hyperbolic example? It would also be interesting to have examples of 3-manifolds so that $h(M) \ll \text{Wt}(\pi_1(M))$, $\text{Wt}(\pi_1(M)) \ll DS(M)$ and $DS(M) \ll S(M)$. These conditions are most-likely generic.

Consider lens spaces as examples for the last inequality. John Berge has the following theorem.
Figure 6. Union of a right and a left trefoil.
Theorem. (Berge) If $K$ is a simple closed curve on the boundary of a standardly embedded genus two handlebody in $S^3$ which represents a free generator of both the fundamental group of the handlebody and the fundamental group of its complement, then there is an integral surgery on $K$ which yields a lens space.

He conjectures that these are the only knots with surgeries which yield lens spaces. From this conjectured form, it is possible to write down a conjectural list of all lens spaces with surgery number one. Since every lens space has Dehn surgery number one, all of the remaining lens spaces would have $SL < DS(L)$.

The linking pairing of a rational homology sphere is a slightly obscure tool which can be used to show that the surgery number is bigger than the Dehn surgery number [4]. If $M$ is a rational homology sphere, then $H_1(M)$ is finite, so we may define a pairing on $H_1(M)$ as follows. Let $\alpha$ and $\beta$ be homology cycles. Some multiple of $\alpha$, say $n\alpha$, is the boundary of a surface, $F$. Define the linking pairing of $\alpha$ and $\beta$, $\lambda(\alpha, \beta)$, to be the intersection number of $F$ and $\beta$ divided by $n$. This is well defined mod $\mathbb{Z}$.

As a concrete example consider $-L(5, 2)$. This lens space is $5/2$ Dehn surgery on the unknot in $S^3$. $H_1$ is $\mathbb{Z}_5$ generated by the meridian, $\mu$. Five times the meridian is homologous to twice the longitude which bounds a singular disk intersecting the meridian twice, thus $\lambda(\mu, \mu) = 2/5$. See figure 7.

If $M$ is the boundary of a simply connected 4-manifold, $W$, then the linking pairing of $M$ may be computed from the intersection form of $W$. 
Let
\[ Q_W : H^2(W; \partial W) \otimes H^2(W; \partial W) \xrightarrow{\cup} H^4(W; \partial W) \cong \mathbb{Z} \]
be the intersection pairing. This induces a pairing
\[ Q'_W : U \times U \to \mathbb{Q}/\mathbb{Z} \]
where \( U = H^2(W; \partial W) \otimes \mathbb{Q}/H^2(W; \partial W) \).

A straightforward argument with Poincaré duality and the homology exact sequence of the pair \((W, M)\) will show that the above pairing is isomorphic to the linking pairing on \(M\). Looking back at our example, we can see that if \(-L(5, 2)\) were surgery on a knot then the framing would have to be \(\pm 5\). This would imply that \(-L(5, 2)\) was the boundary of a simply connected 4-manifold with intersection form
\[ Q_W : \langle t \rangle \otimes \langle t \rangle \to \mathbb{Z} , \quad Q_W(t, t) = \pm 5 \]
so that the linking pairing would be isomorphic to
\[ Q'_W : \langle u \mid 5u = 0 \rangle \times \langle u \mid 5u = 0 \rangle \to \mathbb{Q}/\mathbb{Z} , \text{ where } u = 1/5t , \]
so
\[ Q'_W(Ku, Ku) = \pm K^2/5 . \]

This contradicts our computation of the linking pairing since 2 is not a square mod 5. We must, therefore, conclude that \(-L(5, 2)\) is not surgery on any knot. It is, however, surgery on the Hopf link so \(DS(-L(5, 2)) = 1\) and \(S(-L(5, 2)) = 2\).

Up to now, we have just used tools from Algebraic topology to address surgery number questions. Gauge theory may also be applied to these questions. The relevant theorem is in [1]. It is really a corollary of a theorem of Taubes [11].

**Theorem.** Let \(K\) be a compact simply connected 4-manifold with boundary and a non-standard unimodular definite intersection form. If \(\Sigma = \partial K\) is a homology sphere and \(T\) is a compact oriented 4-manifold such that
\[ \partial T = \Sigma \perp - \Sigma \perp M \]
\[ H_2(T) = 0 \]
and \( \pi_1(T) = \text{Normal Closure}(\pi_1(M)) \).

Then, \(M\) is not the boundary of any simply connected definite 4-manifold.

This is related to the surgery number question via the Lemma below.

**Lemma.** If \(M\) is a homology sphere which is Dehn surgery on a knot, then \(M\) is the boundary of a simply connected, definite 4-manifold.
See Figure 8 for a pictorial proof of this.

The Poincaré homology sphere, $\Sigma$, is the boundary of the non-standard $E_8$ manifold. We can construct a $T$ with boundary $\Sigma \sqcup \Sigma$ by adding a 3-handle to $I \times (\Sigma\# - \Sigma)$. See Figure 9.

It is easy to see that this $T$ satisfies the requirements of the theorem so we conclude that $\Sigma\# - \Sigma$ is not Dehn surgery on any knot. To construct an irreducible or a hyperbolic example, we will add a cobordism onto the bottom of the $T$.

In our preprint we constructed cobordism to an irreducible example by adding a 1-handle – 2-handle pair to $I \times (\Sigma\# - \Sigma)$ [1]. In [7] Chuck Livingston used link concordance to prove that every homology 3-sphere is homology cobordant to an irreducible homology 3-sphere. Robert Myers showed that Livingston’s construction actually produces a homology cobordism to a simple Haken and therefore hyperbolic 3-manifold [9]. We only need to check that the fundamental group of this cobordism is normally generated by the hyperbolic component, and it will follow that the longneck $T$ satisfies the hypothesis of the theorem. Before we describe the Livingston-Myers cobordism, we will describe the example in our preprint. This is interesting because our original construction gives an explicit surgery description of the irreducible example.
Figure 9. How to add a 3-handle.

When a 1-handle – 2-handle pair is added to $I \times (\Sigma \# - \Sigma)$ we get the manifold in figure 11. Performing a $K$-move on the unknotted 1-framed component will untie the $-1$-framed component. Then a $K$-move on the $-1$-framed component will generate the following picture of the same manifold. Figure 12 is an explicit surgery description of an irreducible homology sphere with Dehn surgery number 2.

We will now go back to the Livingston-Myers cobordism. To construct the cobordism, Myers began with $4D^2$'s embedded in a $D^4$. We will describe this embedding by pictures at $t = 5$, 4, and 3 in $D^4 = [0,5] \times D^3$, where $t$ is the coordinate of the first factor. See Figure 13.

Disks embedded in $D^4$ in this manner form an object called a tangle concordance. Let $X^4 = D^4 - \tilde{N}(4D^2)$ and let $S_t = (\{t\} \times D^3) \cap X$. Then

$$\partial X = S_0 \cup \left( \bigcup_{F_4 \times \{0\}} (F_4 \times I) \right) \cup \left( \bigcup_{F_4 \times \{1\}} S_8 \right)$$

where $F_4$ is the surface of genus four. Now $\Sigma \# - \Sigma$ has a genus 4 Heegaard splitting. This means that

$$\Sigma \# - \Sigma = (D^2 - 4D^2) \times I \bigcup_f (D^2 - 4D^2) \times I$$

where $f : F_4 \to F_4$ is an orientation reversing homeomorphism [10].
Figure 10. Longneck $T$.

Figure 11. The Thing.
Figure 12. The same thing.
FIGURE 13. The concordance.
This homeomorphism can be extended to a homeomorphism,

\[ \bar{f} : F_4 \times I \to F_4 \times I \]

We put all of this together to construct the cobordism, \( W \). Let \( W = X \cup_{f} (-X) \). Myers proved that \( W \) is a homology cobordism between \( \Sigma\# - \Sigma \) and a hyperbolic homology sphere. It only remains to check that \( W \) satisfies the fundamental group condition. The condition on the fundamental group follows from Van Kampen's theorem by the same argument that was used to prove that the weight of the fundamental group is a bound on the Dehn surgery number.

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