STABILITY OF $C^*$-ALGEBRAS ASSOCIATED TO $k$-GRAPHS

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Abstract. We give an emended proof of a result in the literature characterizing which graphs yield stable $C^*$-algebras. We strengthen this result by adding another necessary condition. We characterize stability of $C^*$-algebras associated to certain higher-rank graph $C^*$-algebras.

1. Introduction

A $C^*$-algebra $A$ is said to be stable if $A \cong A \otimes K$, where $K$ denotes the $C^*$-algebra of compact operators on an infinite-dimensional separable Hilbert space. Blackadar in [2, Theorem 4.10] showed that an AF algebra is stable if and only if it admits no tracial states. Hjelmborg and Rørdam in [5, Theorem 3.3] characterized stability for $C^*$-algebras which admit approximate identities consisting of projections. Hjelmborg in [4, Theorem 2.14] used this characterization to characterize stability for Cuntz-Krieger algebras affiliated to locally finite graphs. His description introduced two new graph-theoretic concepts. A vertex is said to be left-finite if it can reach only finitely many vertices by following directed paths. Certain left-finite vertices in a graph yield nonzero unital quotients of the graph algebra; hence these vertices are obstructions to stability of the graph algebra. A graph trace is a $\mathbb{R}^+$-valued function on the vertices which satisfies certain Cuntz-Krieger-type relations. A graph trace $g$ with $\|g\|_1 = 1$ yields a tracial state on $C^*(E)$; thus these graph traces are also obstructions to stability. Tomforde in [11, Theorem 3.2] characterized stability of $C^*$-algebras associated to arbitrary graphs, showing that a graph $E$ yields a stable graph algebra if and only if $E$ has no left-finite cycles and no bounded graph traces. However, the proof in [11] is flawed. In this paper we give an emended proof which should generalize to more contexts, as it does not involve AF graph algebras. (It is hard to generalize proofs involving AF graph algebras to the context of $k$-graphs as it is difficult to say when a $k$-graph yields an AF $C^*$-algebra, see [3].)

A $k$-graph is a higher-dimensional generalization of a directed graph. Roughly speaking, a $k$-graph is formed from equivalence classes of paths within a colored directed graph. To any well-behaved $k$-graph one can affiliate a universal $C^*$-algebra generated by partial isometries satisfying Cuntz-Krieger relations. Owing to their combinatorial richness we impose restrictions on $k$-graphs when describing stability for stable $k$-graph algebras. A $k$-graph is balanced if given any vertices $v$ and $w$, and any degrees $m, n \in \mathbb{N}^k$, we have $|v\Lambda^m w| = |v\Lambda^m w|$ if $|m| = |n|$ (where $|n| = \sum_{i=1}^k n_i$). In Theorem 6.10 we show that a balanced row-finite $k$-graph with no sources yields a stable $k$-graph algebra if and only if it has no left-finite cycles and no bounded $k$-graph traces.
In this section we review background material on graph algebras. A \((\text{directed})\) graph consists of a quadruple \(E = (E^0, E^1, r, s)\) where \(E^0\) and \(E^1\) are sets called, respectively, the vertices and edges of \(E\), and \(r\) and \(s\) are maps from \(E^1\) to \(E^0\) called the range and source maps. A vertex \(v \in E^0\) is called \emph{regular} if it receives a finite and positive number of edges; that is, if \(0 < |r^{-1}(v)| < \infty\). If a vertex is not regular then it is said to be \emph{singular}, either a source \((|r^{-1}(v)| = 0)\) or an infinite receiver \((|r^{-1}(v)| = \infty)\). A graph \(E\) is said to be \emph{row-finite} if no vertex receives infinitely many edges, and \(E\) is said to have no \emph{sources} if every vertex receives at least one edge. Thus a graph is row-finite with no sources if and only if each of its vertices is regular. The (finite) \emph{path space} of \(E\), denoted \(E^*\), is the set of all finite sequences \(e_1 e_2 \ldots e_n\) with \(s(e_i) = r(e_{i+1})\) for \(i = 1, \ldots, n - 1\). The range of a path \(\lambda = e_1 \ldots e_n\) is defined to be \(r(\lambda) := r(e_1)\) and the source is \(s(\lambda) := s(e_n)\). The \emph{length} of \(\lambda = e_1 \ldots e_n\) is defined to be \(|\lambda| = n\). We include the vertices \(E^0\) in \(E^*\) as paths of length zero with \(r(v) = s(v)\). A \emph{cycle} is a directed path \(\lambda \in E^*\) with \(|\lambda| > 0\) and \(s(\lambda) = r(\lambda)\). (Note that we are using the right to left orientation for arrows as in [7], which is different from that of [11].)

\textbf{Definition 2.1.} Let \(E\) be a directed graph and let \(B\) be a \(C^*\)-algebra. A \emph{Cuntz-Krieger} \(E\)-family in \(B\) is a collection \(\{S_e, P_v\}_{e \in E^1, v \in E^0} \subset B\), where the \(S_e\) are partial isometries with mutually orthogonal range projections and the \(P_v\) are mutually orthogonal projections, satisfying the following \emph{Cuntz-Krieger relations}:

\begin{enumerate}
\item \(S_e S_e^* = P_{s(e)}\)
\item \(S_{e} S_{e'} \leq P_{r(e)}\);
\item \(v \in E^0\) is regular, then \(\sum_{r(e) = v} S_e S_e^* = P_v\).
\end{enumerate}

Typically we denote \(\{S_e, P_v\}_{e \in E^1, v \in E^0}\) simply by \(\{S, P\}\). The \(C^*\)-algebra generated by a Cuntz-Krieger family \(\{S, P\}\) \(\subset B\) is denoted by \(C^*(S, P) \subset B\). The \emph{graph algebra} of \(E\), denoted \(C^*(E)\), is the universal \(C^*\)-algebra generated by a Cuntz-Krieger \(E\)-family \(\{s, p\}\).

If \(\mu = e_1 \ldots e_n\) is a directed path in \(E\), then by \(s_\mu\) we denote the partial isometry \(s_\mu = s_{e_1} \ldots s_{e_n}\) in \(C^*(E)\). The following properties are well-known consequences of the Cuntz-Krieger relations and describe the \(*\)-algebraic structure of \(C^*(E)\). We will use them without referring to this corollary. (Recall that for projections \(p, q\) the notation \(p \perp q\) means \(pq = 0\).)

\textbf{Corollary 2.2 ([7 Corollary 1.14])}. Suppose that \(E\) is a graph. Let \(\mu, \nu \in E^*\). Then the following hold:

\begin{enumerate}
\item \(\text{if } |\mu| = |\nu| \text{ and } \mu \neq \nu\), then \(s_\mu s_\nu^* \perp s_\nu s_\mu^*\);
\item \(s_\mu^* s_\nu = \begin{cases} s_\nu & \text{if } \mu = \nu \mu' \\
 s_\nu' & \text{if } \nu = \mu \nu' \\
 0 & \text{otherwise} \end{cases}\)
\item \(\text{if } s_\mu s_\nu \neq 0\), then \(\mu \nu\) is a path \((s(\mu) = r(\nu))\) and \(s_\mu s_\nu = s_{\mu \nu}\);
\item \(\text{if } s_\mu^* s_\nu \neq 0\), then \(s(\mu) = s(\nu)\).
\end{enumerate}

Recall that if \(p\) and \(q\) are projections in a \(C^*\)-algebra \(A\), then we say that \(p\) is \emph{subequivalent} to \(q\), denoted \(p \lesssim q\), if there is an element \(x \in A\) such that \(x^* x = p\) and \(x x^* \leq q\). By a \emph{comparison} between two projections we mean such a partial isometry. Much of this paper is devoted to the construction or non-existence of
comparisons between projections. The following lemma describes two different ways to compare vertex projections in a graph algebra.

**Lemma 2.3.** Let $E$ be a directed graph.

1. If $v$ and $w$ are vertices of $E$ and there is a path from $w$ to $v$, then $p_w \lesssim p_v$.

2. If $v$ is a regular vertex, and if for every $e \in r^{-1}(v)$, there is a path $\lambda_e$ with $s(\lambda_e) = s(e)$, such that none of the paths $\lambda_e$ contains the other as initial prefix, then $p_v \lesssim \sum_{w = r(\lambda_e)} p_w$ (where we don’t repeat vertex projections in the sum).

**Proof.**

1. Let $\lambda$ be a directed path with source $w$ and range $v$. Then $s_\lambda^* s_\lambda = p_{s(\lambda)}$ and $s_\lambda s_\lambda^* \leq p_{r(\lambda)}$, so that $p_w \lesssim p_v$.

2. Applying the Cuntz-Krieger relations we obtain $p_v = \sum_{r(e) = v} s_e s_e^*.$

Now for each $e$ we have $s_e s_e^* \sim s_e^* s_e = p_{s(e)} = p_{s(\lambda_e)} \sim s_{\lambda_e} s_{\lambda_e}^* \leq p_{r(\lambda_e)}.$

As none of the paths $\lambda_e$ overlap, the range projections $s_{\lambda_e} s_{\lambda_e}^*$ are pairwise orthogonal and we can add them to obtain $p_v = \sum_{r(e) = v} s_e s_e^* \lesssim \sum_{r(e) = v} s_{\lambda_e} s_{\lambda_e}^*.$

The right-hand projection is dominated by $\sum_{w; w = r(\lambda_e)} p_w$, as one can verify by using the Cuntz-Krieger relations.

\[ \Box \]

### 3. Stability

This section reviews the basic properties of stable $C^*$-algebras that we will use to characterize stable graph algebras.

**Definition 3.1.** A $C^*$-algebra $A$ is stable if $A \cong A \otimes K$, where $K$ denotes the $C^*$-algebra of compact operators on an infinite-dimensional separable Hilbert space. The tensor product $A \otimes K$ is called the stabilization of $A$.

**Remark.** The stabilization of any $C^*$-algebra is stable. As $K$ is nuclear, there is no need to specify a tensor norm for $A \otimes K$.

The following lemma describes stable $C^*$-algebras as inductive limits.

**Lemma 3.2.** Let $A$ be a stable $C^*$-algebra. Then $A \cong \lim_{\to} M_n(A)$, where $M_n(A) \hookrightarrow M_{n+1}(A)$ is the embedding into the upper-left corner.

We can see from the lemma that any stable $C^*$-algebra is non-unital. For more background on stability see [11] and [5].

**Proposition 3.3** ([8 Cor. 2.3(ii)]). Any ideal or quotient of a stable $C^*$-algebra is stable.

**Corollary 3.4.** A stable $C^*$-algebra has no nonzero unital quotients.
**Definition 3.5.** If \( v \) and \( w \) are vertices of \( E \) then we write \( v \leq w \) if there is a directed path \( \lambda \in E^* \) with \( r(\lambda) = v \) and \( s(\lambda) = w \). For any \( v \) we have \( v \leq v \) via a path of length 0. We say that a vertex \( w \) is left infinite (resp. left finite) if the set \( L(w) = \{ v \in E^0 : v \leq w \} \) is infinite (resp. finite). A directed cycle \( \lambda = e_1 \ldots e_n \) is left finite (resp. left infinite) if \( r(e_1) \) is left finite (resp. left infinite).

The ideal structure of a graph algebra is described by certain sets of vertices.

**Definition 3.6.** Let \( E \) be a directed graph. A subset \( H \subseteq E^0 \) is hereditary if for any \( e \in E^1 \), \( r(e) \in H \) implies \( s(e) \in H \). The subset \( H \) is saturated if for any regular vertex \( v \), the inclusion \( s(r^{-1}(v)) = \{ s(e) : r(e) = v \} \subseteq H \) implies \( v \in H \).

The basic example of a saturated and hereditary subset of \( E^0 \) is \( H_I = \{ v \in E^0 : p_v \in I \} \), where \( I \) is any ideal of \( C^*(E) \). The following result is used to determine which graph algebras have unital quotients. It is a generalization of part of the proof of [4, Theorem 2.14], which was also used in [11, Theorem 3.2].

**Lemma 3.7.** Let \( E \) be a graph and let \( v \in E^0 \). Then \( H = E^0 \setminus L(v) \), the set of all vertices which do not lie on paths with source \( v \), is a hereditary subset of \( E^0 \). The set \( H \) is saturated if and only if \( v \) lies on a cycle or \( v \) is singular.

**Proof.** Checking that \( H \) is hereditary is straightforward. Suppose that \( v \) lies on a cycle, and let \( w \) be a regular vertex which only receives edges \( e_1, \ldots, e_n \) with source belonging to \( H \). Suppose that \( w \) were not in \( H \), so that there exists a path from \( v \) to \( w \). Unless the path were constant, it would have to contain one of the edges \( e_1, \ldots, e_n \), so that \( v \) could reach the source of such an edge, contradicting our assumptions about \( s(e_1), \ldots, s(e_n) \). Thus the only way that \( w \) could fail to lie in \( H \) is if \( w = v \). But \( v \in L(v) = E^0 \setminus H \) via the constant path of length 0. Thus \( w \) must lie in \( H \). So if \( v \) lies on a cycle, \( E^0 \setminus L(v) \) is saturated.

Now suppose that \( v \) is singular, and let \( w \) be a regular vertex receiving edges \( e_1, \ldots, e_n \) with \( s(e_i) \in H \) for \( i = 1, \ldots, n \). The only way that \( w \) could fail to belong to \( H \) is if \( w = v \); as \( v \) is regular and \( w \) and singular this is impossible. Thus \( w \) belongs to \( H \).

Now suppose that \( v \) is regular and lies on no cycle, receiving edges \( e_1, \ldots, e_n \). Each source \( s(e_i) \) for \( i = 1, \ldots, n \) must belong to \( H \), or else \( v \) would lie on a cycle. This shows that \( H \) is not saturated if \( v \) is regular and does not lie on a cycle. \( \square \)

**Definition 3.8.** Let \( E \) be a directed graph and let \( H \) be a saturated and hereditary subset of \( E^0 \). The quotient graph is defined as \( E \setminus H = (E^0 \setminus H, s^{-1}(E^0 \setminus H), r, s) \).

Quotient graphs allow us to realize certain quotients of \( C^* \)-algebras as graph algebras themselves.

**Proposition 3.9 ([11, Cor. 3.5]).** Suppose that \( H \subseteq E^0 \) is saturated and hereditary, and let \( I_H \) denote the ideal of \( C^*(E) \) generated by \( \{ p_v : v \in H \} \). Then \( C^*(E)/I_H \cong C^*(E \setminus H) \).

This shows that if \( H \) is a proper saturated and hereditary subset of \( E^0 \), then the quotient \( C^*(E)/I_H \) is nonzero.

**Lemma 3.10.** Let \( E \) be a directed graph and suppose that \( C^*(E) \) is stable. If \( v \in E^0 \) lies on a cycle or is singular, then \( v \) is left infinite.
Proof. Let \( v \) be such a vertex. By the preceding lemma \( H := E^0 \setminus L(v) \) is saturated and hereditary. The quotient graph \( E \setminus H \) has vertex set \( L(v) \). The quotient \( C^*(E)/I_H \cong C^*(E \setminus H) \) is a nonzero stable \( C^* \)-algebra and hence non-unital. The graph algebra \( C^*(E \setminus H) \) is unital exactly when \( L(v) \) is finite, so we see that \( L(v) \) must be infinite. \qed

Tracial states form another obstruction to stability.

**Definition 3.11.** A tracial state on a \( C^* \)-algebra \( A \) is a state \( \phi \in S(A) \) such that \( \phi(xy) = \phi(yx) \) for all \( x, y \in A \). The (possibly empty) set of tracial states on \( A \) is denoted by \( T(A) \subset S(A) \).

The following lemma follows from the description of stable \( C^* \)-algebras as inductive limits as in Lemma 3.2.

**Lemma 3.12.** A stable \( C^* \)-algebra has no tracial states.

**Proof.** Identify \( A \) and \( \bigcup_{n=1}^\infty M_n(A) \). Suppose that \( \phi \) were a tracial state on \( A \). Let \( a \in M_n(A) \) such that \( a \geq 0 \), \( ||a|| = 1 \). Note that \( \phi(a) \leq 1 \). Then considering the direct sum \( a \oplus a \in M_n(A) \), we have \( \phi(a \oplus a) = 2\phi(a) \). Thus \( 2\phi(a) \leq 1 \). Repeating the process yields \( \phi(a) \leq \frac{1}{2^k} \) for any \( k \geq 1 \). Thus \( \phi(a) = 0 \). Then density of \( \bigcup_{n=1}^\infty M_n(A) \) yields \( \phi \equiv 0 \), contradicting \( \phi \in S(A) \). \qed

**Definition 3.13.** A graph trace on a directed graph \( E \) is a function \( g : E^0 \to \mathbb{R}^+ \) such that

1. for any \( v \in E^0 \), we have \( g(v) \geq \sum_{e(v) = v} g(s(e)) \), and
2. for any regular \( v \in E^0 \), we have \( g(v) = \sum_{e(v) = v} g(s(e)) \).

We define the norm of \( g \) to be \( ||g|| := \sum_{v \in E^0} g(v) \), and when \( g \) has finite norm we say that \( g \) is bounded. The (possibly empty) collection of graph traces on \( E \) with norm 1 is denoted by \( T(E) \).

**Remark.** Any bounded graph trace can be normalized to obtain a graph trace of norm 1.

**Example 3.14.** If \( E \) is a directed graph and \( \tau \) is a tracial state on \( C^*(E) \), then we can define a graph trace \( g_\tau \) on \( E \) via

\[ g_\tau(v) = \tau(p_v). \]

That is, any tracial state on \( C^*(E) \) induces a graph trace on \( E \).

The following theorem shows that every graph trace arises in this fashion. In other words, we can induce tracial states from graph traces.

**Theorem 3.15** ([9 Prop. 3.2]). Let \( E \) be a directed graph, and let \( g \in T(E) \) be a graph trace of norm 1. Then there is a tracial state \( \tau_g \in T(C^*(E)) \) such that \( \tau_g(p_v) = g(v) \) for every vertex \( v \in E^0 \).

**Corollary 3.16.** If \( E \) is a directed graph such that \( C^*(E) \) is stable, then \( E \) has no bounded graph traces.

Stability is easier to describe for \( C^* \)-algebras which have enough projections in a specific way.

**Definition 3.17.** A \( C^* \)-algebra \( A \) is called \( \sigma_p \)-unital if it has a countable approximate identity \( (p_n)_{n=1}^\infty \) with \( p_n \in \mathcal{P}(A) \) for all \( n \) and \( p_n \leq p_{n+1} \) for all \( n \).
Example 3.18. Let $E$ be a directed graph and let $C^*(E)$ be its graph algebra. If we enumerate the vertices as $E^0 = \{v_0, v_1, \ldots\}$ then defining $p_n = \sum_{i=0}^{\infty} P_{v_i}$ yields an approximate identity for $C^*(E)$ consisting of projections. This follows from the equation $C^*(E) = \operatorname{span}\{s_\lambda s_\mu^* : \lambda, \mu \in E^*\}$ (see [10, Corollary 1.16]).

Lemma 3.19 ([4, Lemma 2.3]). Let $A$ be a $C^*$-algebra with approximate identity $(p_n)_{n=1}^\infty$ consisting of projections. Then the following are equivalent.

1. $A$ is stable.
2. For every projection $p \in A$ there exists a projection $q \in A$ such that $p \sim q$ and $p \perp q$.
3. For every $n \in \mathbb{N}$ there exists $m > n$ such that $p_n \preceq p_m - p_n$.

We single out an application of this result to graph algebras.

Lemma 3.20 ([4, Lemma 2.3]). Let $E$ be a directed graph, and let $E^0 = \{v_0, v_1, \ldots\}$ be an enumeration of the vertices of $E$, with approximate identity of projections $(p_n)_{n=0}^\infty$. Then $C^*(E)$ is stable if and only if for any $F \subset E^0$, there exists a finite set $W \subset E^0 \setminus F$ such that $\sum_{v \in F} P_v \preceq \sum_{w \in W} P_w$.

Remark. By an induction argument $C^*(E)$ is stable as long as we can, for every $v \in E^0$ and finite $F \subset E^0$, find finite $W \subset E^0 \setminus F$ such that $p_v \preceq \sum_{w \in W} p_w$.

Lemma 3.21. Suppose that $E$ is a directed graph in which every cycle is left infinite and there are no bounded graph traces. Then every singular vertex is left infinite.

Proof. Suppose that $w \in E^0$ were a left finite singular vertex. Note that $w$ doesn’t lie on a cycle. Claim: there must be some singular vertex $v \in L(w)$ such that $\{\lambda : s(\lambda) = v\}$ is finite. If $w$ is the source of infinitely many paths, then there must be some vertex $v_1$ in $r(s^{-1}(w)) = \{r(e) : s(e) = w\}$ which lies on infinitely many of these, in particular which must receive infinitely many edges from $v$. Note that $v_1$ is a singular vertex in $L(v)$ with $|L(v_1)| < |L(v)|$. Repeating this operation we eventually obtain a singular vertex which is the source of finitely many paths, $v$. Now we can define a graph trace on $E$ by setting $g(z) = |\{\text{paths } z \leftarrow v\}|$. It is straightforward to confirm that this is a graph trace. It is bounded because the total number of paths emitted by $v$ is finite by construction of $v$. This contradicts the assumption that there are no bounded graph traces on $E$. \qed

4. Projection comparison and graph traces

In this section we show that if a certain type of projection comparison (such as those described in Lemma 3.20) fails to exist in $C^*(E)$, then a nonzero bounded graph trace can be defined on $E$. This is key to the characterization of stability for graph algebras.

Lemma 4.1. Let $E$ be a directed graph and let $v \in E^0$ be left infinite. For any finite subset $F \subset E^0$ there is a finite subset $W \subset E^0 \setminus F$ such that $p_v \preceq \sum_{w \in W} p_w$.

Proof. If $w \in L(v) = \{r(\lambda) : \lambda \in E^* \land s(\lambda) = v\}$, then $p_v \preceq p_w$ as in Lemma 2.3. Therefore we can take any vertex $w \in L(v) \setminus F$ and set $W = \{w\}$. \qed

We are interested in subsets like the $W$ mentioned in the preceding lemma. If $v$ is a vertex and $F$ is a (finite) subset of $E^0$, then a cover for $v$ avoiding $F$ is a finite subset of $E^0 \setminus F$ such that $p_v \preceq \sum_{w \in W} p_w$. A path $\lambda$ is said to contain the path $\mu$ if $\lambda = \mu \nu$ for some path $\nu$. The following is a restatement of parts of Corollary 2.2.
Lemma 4.2. Let $\lambda$ and $\mu$ be two directed paths in $E$. Then $s_\lambda s_\lambda^* \perp s_\mu s_\mu^*$ if either of the two holds:
(1) $r(\lambda) \neq r(\mu)$, or
(2) neither $\lambda$ nor $\mu$ contains the other.

Lemma 4.3. Let $\lambda, \mu \in E^*$ be paths such that $s(\lambda) = s(\mu)$. Then $s_\mu s_\mu^* \lesssim p_{r(\lambda)}$.

Proof. To check this we apply the Cuntz-Krieger relations:

$$s_\mu s_\mu^* \sim s_\mu^* s_\mu = p_{s(\mu)} = p_{s(\lambda)} \sim s_\lambda s_\lambda^* \lesssim p_{r(\lambda)}.$$ 

Lemma 4.4. Let $p, p', q, q'$ be projections in $A$ with $p \perp q$ and $p' \perp q'$. If $p \lesssim p'$ and $q \lesssim q'$, then $p + q \lesssim p' + q'$.

Proof. If $s$ and $t$ are the partial isometries implementing the given comparisons, then $s + t$ implements the desired comparison.

The following technical graph theoretic lemma allows us to exhibit boundedness of certain graph traces. This lemma allows us to say that bounding the number of “nice” paths emitted by a vertex can be used to bound the total number of paths emitted by the vertex.

Lemma 4.5. Let $E$ be a directed graph in which every cycle and singular vertex is left infinite. Let $v \in E^0$ be left finite and denote the set of paths with source $v$ by $C$. Let $F \subset E^0$ be a finite set of vertices. Let $C_F$ denote the set of paths of the form $e_1 \ldots e_n$ with $r(e_i) \in F$ for $i = 1, \ldots, n$ and $s(e_n)$ left finite. Then $D_F$ is finite and

$$|C| \leq (|D_F| + 1)|C_F| + |D_F|.$$ 

Proof. Any cycle is left infinite by hypothesis, so $C$ is finite. Similarly no left finite vertex can reach a cycle in $F$, so the set $D_F$ is finite. A path starting at $v$ either belongs to $C_F$ (bounded by $|C_F|$), or is a composition of a path that lands outside of $F$ and a path in $D_F$ (bounded by $|C_F||D_F|$), or is a path in $D_F$ with source $v$ (bounded by $|D_F|$).

Lemma 4.6. Let $E$ be a directed graph in which every cycle and singular vertex is left infinite. Suppose that there exist $v \in E^0$ and finite $F \subset E^0$ such that there is no finite $W \subset E^0 \setminus F$ satisfying $p_v \lesssim \sum_{w \in W} p_w$. Then there is a nonzero bounded graph trace on $E$.

Proof. The proof proceeds by showing that the number of paths starting at certain vertices and ending in $E^0 \setminus F$ is bounded in a certain way by the number of paths of fixed length terminating at $v$. The basic idea is that more paths between vertices makes it harder to produce comparisons but easier to produce graph traces. In what follows we will assume that every vertex is regular, because singular vertices are left infinite and so we can always produce whatever covers we need for singular vertices.

The idea of the proof is to single out a chain of vertices and edges that looks like

$$\begin{array}{ccccccc}
N_1 & \xrightarrow{v_1} & N_2 & \xrightarrow{v_2} & N_3 & \xrightarrow{v_3} & N_4 & \cdots
\end{array}$$
Here $N_i$ denotes the number of edges from $v_i$ to $v_{i-1}$. (The vertex $v_i$ can receive vertices from other vertices, and

As we cannot construct a comparison for $v$ that avoids $F$, we know that $v$ is regular by Lemma 4.1 and the assumption that singular vertices are left infinite. We list the edges of $r^{-1}(v)$ as $e_1, \ldots, e_n$. The sources of these edges we list as $v_1^1, \ldots, v_{n_1}^1$, where $n_1$ is the number of vertices that emit edges into $v$. We define $N_k^1 = |\{\text{edges from } v_k^1 \text{ to } v\}|$. Here is an illustration in the case where $v$ receives edges from 3 vertices, with $N_k^1$ edges emitted by $v_k^1$ to $v$ for $k = 1, 2, 3$. Let $D_k^1$

![Figure 1. A regular vertex](image)

denote the set of $\lambda \in E^*$ such that $s(\lambda) = v_k^1, r(\lambda) \notin F$ and $\lambda$ does not overlap with any path with source in $\{v^1, \ldots, v_{k-1}^1, v_{k+1}^1, \ldots, v_{n_1}^1\}$. Define $d_k^1 = |D_k^1|$. This is illustrated in Figure 1.

We claim that if $N_k^1 \leq d_k^1$ for $k = 1, \ldots, n$ then we can produce a cover for $v$ that avoids $F$. To see this note that for each edge $e$ from $v_k$ to $v$ we can take a path $\lambda_e$ with $s(\lambda) = v_k$ and $r(\lambda) \notin F$. By definition of $D_k^1$ the paths terminating at the same vertex don’t overlap, so that the range projections $s_{\lambda_e}^* s_{\lambda_e}^*$ are mutually orthogonal. In this case we have

$$p_v = \sum_{r(e) = v} s_{e}^* s_{e}^* \lesssim \sum_{k=1}^{n_1} \sum_{\lambda \in D_k^1} s_{\lambda}^* s_{\lambda}^* \perp \sum_{f \in F} p_f.$$

Now we can group the $\lambda$ together by common range and we see that

$$p_v = \sum_{r(e) = v} s_{e}^* s_{e}^* \lesssim \sum_{w \in \{r(\lambda) : \lambda \in D_k^1\}} p_w.$$

As none of the $w = r(\lambda)$ lie in $F$ by construction this yields a cover for $p_v$ that avoids $F$. This contradicts our assumption that no cover for $v$ avoiding $F$ exists. So there must be some $k = k_1$ with $d_{k_1}^1 < N_{k_1}^1$.

Now we use the fact that we cannot construct a comparison again at $v_k^1$. Let $N_j^2$ be the number of edges from $v_j^2$ to $v_j^2$ for the (finitely many) vertices $v_j^2$ emitting edges to the regular vertex $v_{k_1}^1$. We covered $d_{k_1}^1$ of the range projections affiliated to edges from $v_{k_1}^1$ to $v$ in the previous part, so there are $N_{k_1}^1 - d_{k_1}^1$ left. Now we split each of these at $v_{k_1}^1$, obtaining $(N_{k_1}^1 - d_{k_1}^1)N_j^2$ range projections affiliated to...
paths from $v^2_j$ to $v_{k_1}$ to $v$ that do not intersect any of the other paths among the vertices $v^2_j$ in $s(r^{-1}(v^1_{k_1}))$. Let $d^2_j$ be the number of paths that

1. start at $v^2_j$,
2. do not overlap with any path that has source in \{v^2_1, \ldots, v^2_{k_2}\},
3. do not include the other vertices in $s(r^{-1}(v^1_{k_1}))$, and
4. terminate at vertices not in $F$.

By the same argument, we see that there must be some $j$ with $(N^1_{k_1} - d^1_{k_1})N^2_j > d^2_j$. Repeating this process inductively we obtain a chain of vertices as in Figure 2.2.

Note that we cannot have the same vertex repeated along the highlighted path, because that would give us a (necessarily left infinite) cycle and we are ignoring the left infinite vertices. If we single out the vertices where the comparison fails together with the multiple edges that they emit to one another, we obtain a chain, where we have reduced the subscripts and singled out the edges and vertices on the dashed path.

Figure 2. Finding the chain
We have chosen $v_k$ and $N_k$ such that if $d_k$ is the number of paths which do not touch the chain before $v_k$ and do not terminate at a vertex of $F$, we have

$$
d_1 < N_1
$$
$$
d_2 < N_2(N_1 - d_1)
$$
$$
d_3 < N_3(N_2(N_1 - d_1) - d_2) - d_3
$$
$$
\vdots
$$
$$
d_k < N_k(N_{k-1}(\ldots N_2(N_1 - d_1) - d_2)\ldots) - d_{k-1})
$$

We can rewrite these inequalities as

$$
\frac{d_1}{N_1} < 1
$$
$$
\frac{d_1}{N_1} + \frac{d_2}{N_1N_2} < 1
$$
$$
\vdots
$$
$$
\frac{d_1}{N_1} + \frac{d_2}{N_1N_2} + \ldots + \frac{d_k}{N_1N_2\ldots N_k} < 1
$$
$$
\vdots
$$
$$
\sum_{i=1}^{\infty} \frac{d_i}{N_1N_2\ldots N_i} < 1
$$

The bound on this sum implies that a certain graph trace is bounded. Note that $d_i$ is eventually greater than or equal to 1, because the vertices on the chain eventually leave $F$ and then the constant paths at those vertices provide paths which start on the chain and do not terminate in $F$. Thus as we have the convergent sum

$$
\sum_{i=1}^{\infty} \frac{d_i}{N_1\ldots N_i} < \infty,
$$

we also have the convergent sum

$$
\sum_{i=1}^{\infty} \frac{1}{N_1\ldots N_i} < \infty.
$$

Now we count the paths which start at vertices on the chain and end in $F$, which we had previously ignored. (We have to do this in order to bound our graph trace.) For the infinite chain of vertices we constructed in the previous example, define

$$
C_i = \{\lambda \in E^*: s(\lambda) = v_i, \lambda \text{ does not factor through } v_{i-1} \overset{N_i}{\leftarrow} v_i\}.
$$

Set $c_i := |C_i|$. By Lemma 34, we have that

$$
c_i \leq (|d(F)| + 1)d_i + |d(F)|
$$

for all $i = 1, \ldots$. Thus we have

$$
\sum_{i=1}^{\infty} \frac{c_i}{N_1N_2\ldots N_i} \leq \sum_{i=1}^{\infty} \frac{(|d(F)| + 1)d_i + |d(F)|}{N_1\ldots N_i}
$$
To show that $\sum_{i=1}^{\infty} \frac{c_i}{N_1 \cdots N_i}$ converges we need only verify that $\sum_{i=1}^{\infty} \frac{|d(F)|}{N_1 \cdots N_i} < \infty$, where $d(F)$ denotes the finite set of acyclic paths among the finite set $F \subset E^0$. But we already showed that $\sum_{i=1}^{\infty} \frac{1}{N_1 \cdots N_i} < \infty$. Thus $\sum_{i=1}^{\infty} \frac{c_i}{N_1 \cdots N_i} < \infty$.

Now we define a graph trace on $E$. Now define a graph trace on $w \in E^0$ by

$$g(w) = \sum_{i=1}^{\infty} \left\{ \text{paths } w \leftarrow v_i \text{ that do not factor through } v_{i-1} \right\}.$$

We have not yet shown that $g(v) < \infty$ for $v \in E^0$. We do this by showing $\|g\| < \infty$.

$$\|g\| = \sum_{w \in E^0} \sum_{i=1}^{\infty} \left\{ \text{paths } w \leftarrow v_i \text{ that do not include any of the edges between } v_{i-1} \text{ and } v_i \right\} \frac{c_i}{N_1 \cdots N_i}$$

$$= \sum_{i=1}^{\infty} \frac{c_i}{N_1 \cdots N_i} < \infty$$

Now we have shown that $\|g\| < \infty$. It is straightforward to confirm that $g$ satisfies the requirements to be a graph trace. In fact, you only need to verify the relations at regular vertices because $g$ vanishes on infinite receivers as well as vertices emitting edges to singular vertices (both types of vertex are left infinite). Thus a failure of projection comparison induces a graph trace. \qed

5. Stability of graph algebras

The following characterization of stable graph algebras is given in [11] Theorem 3.2.

**Theorem 5.1.** Let $E$ be a directed graph. Then the following are equivalent.

(a) $C^*(E)$ is stable.
(b) $C^*(E)$ has no nonzero unital quotients and no tracial states.
(c) $E$ has no left finite cycles and $T(E) = \emptyset$.
(d) $E$ has no left finite cycles and no nonzero bounded graph traces.
(e) For any $v \in E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.
(f) For any finite $V \subset E^0$ there exists finite $W \subset E^0 \setminus V$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.

The proof of this theorem in [11] contains a subtle flaw. Let $H = H(E) \subset E^0$ be the set of left infinite vertices and let $\overline{H}$ denote its saturation. There is a small error in the proof that for any $v \in \overline{H}$ and any finite $F \subset E^0$ there exists finite $W \subset E^0 \setminus F$ with $p_v \lesssim \sum_{w \in W} p_w$. Specifically the comparison constructed is backwards: $xp^{*}x = q$ means that $q \lesssim p$ as projections, not vice versa. This doesn’t present much trouble as one can construct the comparison needed without too much difficulty and without the use of Cuntz comparison by adding appropriately orthogonal partial isometries. The real problem is that [11] Lemma 3.8 seems to use the fact that the ideal $I_H$ in question is stable, an additional hypothesis not included in the statement of the lemma. The lemma is repaired if we include this as a hypothesis, but this prevents us from applying it in the proof of [11] Theorem 3.2. While it is true that for any vertex $v \in \overline{H}$ and any finite set $F \subset E^0$, a comparison $p_v \lesssim \sum_{w \in W} p_w$ with $W \cap F = \emptyset$ exists, there is no guarantee that
$W \subset H$. In fact, it is not difficult to find graphs for which the saturation $\mathcal{P}$ of the set of left infinite vertices consists of a single point, in which case the necessary comparisons could not be constructed using other elements in $\mathcal{P}$. Since the basic idea of the proof is to create the part of a comparison needed in the ideal, then create the part of the comparison needed in the quotient, and lift the comparison from the quotient, this seems to cause an issue.

Giving a proof that avoids this issue requires us to consider singular vertices. This lead to a strengthening of the result, giving another condition that a graph must satisfy in order to yield a stable $C^*$-algebra: all singular vertices must be left infinite.

**Theorem 5.2.** Let $E$ be a directed graph. The following are equivalent.

1. $C^*(E)$ is stable.
2. $C^*(E)$ has no nonzero unital quotients and no tracial states.
3. $C^*(E)$ has no left finite cycles and no bounded graph traces.
4. $E$ has no left finite cycles, no left finite singular vertices, and no bounded graph traces.
5. For any vertex $v \in E^0$ and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$.
6. For any finite $V \subset E^0$, and any finite $F \subset E^0$, there exists finite $W \subset E^0 \setminus F$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.

**Proof.** (1) implies (2): A stable $C^*$-algebra has no nonzero unital quotients (Corollary 3.4) and no tracial states (Lemma 3.8).

(2) implies (3): Combine Lemma 3.10 and Theorem 3.15.

(3) implies (4): Apply Lemma 3.22.

(4) implies (5): Apply Lemma 4.5.

(5) implies (6): This follows using the same argument as in [11]. Let $V = \{v_1, \ldots, v_n\}$. Take a cover $W_1$ for $v_1 \in V$ which avoids $F$. Then find a cover $W_2$ for $v_2$ which avoids $F \cup W_1$. Let $W = W_1 \cup W_2 \cup \ldots \cup W_n$. The comparisons add because all the projections are orthogonal as in Lemma 4.4.

(6) implies (1): This follows from Lemma 3.21. $\square$

### 6. Stability of Balanced $k$-Graph Algebras

In this section we extend our result to a certain class of $k$-graphs. Higher-rank graphs were introduced by Kumjian and Pask in [6] as generalizations of directed graphs. The semigroup $\mathbb{N}^k$ is treated as a category with one object and composition given by coordinate-wise addition. For an element $m \in \mathbb{N}^k$ we denote the coordinates by $m_i$ for $i = 1, \ldots, k$. The standard basis elements in $\mathbb{N}^k$ are denoted by $e_1, e_2, \ldots, e_k$.

**Definition 6.1.** A $k$-graph consists of a countable category $\Lambda$ along with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the factorization property: if $\lambda \in \Lambda$ and $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\mu \nu = \lambda$.

In analogy with directed graphs the morphisms in a $k$-graph are often called paths. For $n \in \mathbb{N}^k$ we write $\Lambda^n := d^{-1}(n) = \{ \lambda \in \Lambda : d(\lambda) = n \}$ for the paths of degree $n$ in $\Lambda$. The set of objects of $\Lambda$ is precisely $\Lambda^0 = d^{-1}(0)$, which we refer to as the set of vertices of $\Lambda$. The range and source maps $r, s : \Lambda \to \Lambda^0$ satisfy $r(\lambda)\lambda = \lambda = \lambda s(\lambda)$ for all $\lambda \in \Lambda$.  

Definition 6.2. The 1-skeleton of $\Lambda$ is the colored directed graph $E_\Lambda = (\Lambda^0, \cup_{i=1}^k \Lambda^{e_i}, r, s, c)$, where $c(\Lambda^{e_i}) = \{i\}$. We say that $\Lambda$ is row-finite if $E_\Lambda$ is row-finite (considered as a directed graph). We say that $\Lambda$ has no sources if every vertex of $E_\Lambda$ receives an edge of each color.

Definition 6.3. If $\Lambda$ is a row-finite $k$-graph with no sources and $B$ is a $C^*$-algebra then a Cuntz-Krieger $\Lambda$-family in $B$ is a collection of partial isometries $S = \{S_\lambda : \lambda \in \Lambda\} \subset B$ satisfying:

(i) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(ii) $S_\lambda S_\mu = S_\mu$ if $s(\lambda) = r(\mu)$;
(iii) $S_\lambda^* S_\lambda = S_\mu$ for every $\lambda \in \Lambda$;
(iv) $S_v = \sum_{\lambda \in \Lambda^0 : r(\lambda) = v} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and every $n \in \mathbb{N}^k$.

The universal $C^*$-algebra generated by a $\Lambda$-family is called the $k$-graph algebra of $\Lambda$ denoted by $C^*(\Lambda) = C^*((S_\lambda : \lambda \in \Lambda))$.

A $k$-graph yields a unital $C^*$-algebra if and only if $\Lambda^0$ is finite; otherwise the sequence $\{\sum_{i=1}^n p_v\}_{n=1}^\infty$ forms an approximate identity consisting of projections.

Definition 6.4. For $m, n \in \mathbb{N}^k$ denote $m \vee n \in \mathbb{N}^k$ by $(m \vee n)_i = \max(m_i, n_i)$ for $i = 1, \ldots, k$. If $\lambda, \mu \in \Lambda$ then let $MCE(\lambda, \mu) = \{v \in \Lambda : v = \lambda \alpha = \mu \beta \text{ for some } \alpha, \beta \in \Lambda, d(v) = d(\lambda) \vee d(\mu)\}$ denotes the set of minimal common extensions of $\lambda$ and $\mu$. We define $\Lambda^\text{min}(\lambda, \mu) = \{\alpha, \beta : s(\lambda) = r(\alpha), s(\mu) = r(\beta), \lambda \alpha = \mu \beta \in MCE(\lambda, \mu)\}$.

The *-algebraic structure of $C^*(\Lambda) = \mathfrak{sp}(\{S_\lambda S_\mu^* : \lambda, \mu \in \Lambda\})$ is described by the following lemma.

Lemma 6.5 (\cite{[17]} Lemma 10.6). Let $\Lambda$ be a row-finite $k$-graph with no sources, and let $S = \{S_\lambda\}$ be a Cuntz-Krieger $\Lambda$-family. Then for every $\lambda, \mu \in \Lambda$ and $q \geq d(\lambda) \vee d(\mu)$, we have

$$S_\lambda S_\mu = \sum_{(\alpha, \beta) \in \Lambda^{\text{min}}(\lambda, \mu)} S_\alpha S_\beta^*.$$

Definition 6.6. Let $\Lambda$ be a row-finite $k$-graph with no sources. A subset $H \subset \Lambda^0$ is said to be hereditary if $r(\lambda) \in H$ implies $s(\lambda) \in H$. A subset $H \subset \Lambda^0$ is said to be saturated if $s(v \Lambda^a) \subset H$ implies $v \in H$ for any $v \in \Lambda^0$ and $a \in \mathbb{N}^k$. If $H \subset \Lambda^0$ then we let $I_H$ denote the ideal of $C^*(\Lambda)$ generated by $\{p_v : v \in H\}$.

Theorem 6.7. If $\Lambda$ is a $k$-graph and $H \subset \Lambda^0$ is saturated and hereditary then $\Lambda \setminus H := (\Lambda^0 \setminus H, s^{-1}(\Lambda^0 \setminus H), r, s)$ is a $k$-graph and $C^*(\Lambda \setminus H) \cong C^*(\Lambda) / I_H$.

Definition 6.8. Let $\Lambda$ be a row-finite $k$-graph with no sources. A $k$-graph trace on $\Lambda$ is a function $g : \Lambda^0 \to [0, \infty)$ such that $g(\lambda) = \sum_{\alpha \in \Lambda^a} g(s(\lambda))$ for all $\lambda \in \Lambda^0$ and all $a \in \mathbb{N}^k$. A $k$-graph trace $g$ is bounded if $\sum_{\lambda \in \Lambda^0} g(\lambda) < \infty$, in which case the sum $\sum g(\lambda)$ is referred to as the norm of $g$. The (possibly empty) collection of $k$-graph traces on $\Lambda$ is denoted by $T(\Lambda)$.

Producing tracial states on $k$-graph algebras requires a different approach from that used by Tomforde in \cite{[10]}. In \cite{[3]} Lemma 2.1 it is shown that any $g \in T(\Lambda)$ which is faithful (in the sense that $g(v) > 0$ for all $v$) lifts to a faithful tracial state $\tau_g$ on $C^*(\Lambda)$ such that $\tau_g(p_v) = g(v)$ for all $v \in \Lambda^0$.

The following lemma follows directly from the definition of a $k$-graph trace. It is essentially that same as \cite{[10]} Lemma 3.7.
Lemma 6.9. Let $\Lambda$ be a $k$-graph and let $g$ be a $k$-graph trace on $\Lambda$. Then $H = \{v \in \Lambda^0 : g(v) = 0\}$ is a saturated and hereditary subset of $H$.

Lemma 6.10. Let $\Lambda$ be a $k$-graph and let $g$ be a $k$-graph trace on $\Lambda$ with norm 1. Then there is a tracial state $\tau_g$ on $C^*(\Lambda)$ such that $\tau_g(p_v) = g(v)$ for each $v \in \Lambda^0$.

Proof. Let $H$ be the zero set of $g$. By the preceding lemma $H$ is a saturated and hereditary subset of $\Lambda^0$. The restriction of $g$ to $\Gamma(\Lambda \setminus \Lambda H)$ is faithful by construction of $H$. Thus by \(\mathbb{[3]}\) Lemma 2.1 it lifts to a faithful tracial state $\tau$ on the quotient $C^*(\Lambda)/IH$. If $q$ is the quotient map $C^*(\Lambda) \to C^*(\Lambda)/IH$, then $\tau \circ q$ is the desired tracial state on $C^*(\Lambda)$. \qed

The graph trace constructions of the previous sections only readily generalize to a limited class of $k$-graphs.

Definition 6.11. Let $\Lambda$ be a row-finite $k$-graph, and let $M_1, M_2, \ldots, M_k$ be its vertex matrices (each of size $\Lambda^0 \times \Lambda^0$), so that $M_1(v, w) = |\{e \in \Lambda^* : r(e_i) = v, s(e_i) = w\}|$. Then $\Lambda$ is said to be balanced if $M_1 = M_2 = \ldots = M_k$.

Definition 6.12. A cycle in a $k$-graph is a path $\lambda$ with $d(\lambda) \neq 0$ and $r(\lambda) = s(\lambda)$. A vertex $v$ in $\Lambda$ is left finite (resp. left infinite) if $v$ is left finite (resp. left infinite) considered as a vertex of the 1-skeleton of $\Lambda$. A cycle is left finite (resp. left infinite) if its source vertex is left finite (resp. left infinite). A red path $\lambda \in \Lambda$ is a path with degree $d(\lambda) = (n_1, 0, \ldots, 0)$.

Remark. If $\lambda$ and $\mu$ are two red paths with $r(\lambda) = r(\mu)$ and $s(\lambda) = s(\mu)$, then $ss^*_\lambda \perp ss^*_\mu$. For this reason we compare the number of path partial isometries created by iterating the Cuntz-Krieger relations with the number of red paths emitted by the vertices on the chain. Otherwise two paths emitted from a fixed vertex $v_i$ to a fixed vertex $w$ might not yield partial isometries with orthogonal range projections, which we need to create comparisons between vertices.

Theorem 6.13. Let $\Lambda$ be a balanced row-finite $k$-graph with no sources. The following are equivalent:

1. $C^*(\Lambda)$ is stable.
2. $C^*(\Lambda)$ has no unital quotients and no tracial states.
3. $\Lambda$ has no left finite cycles and no bounded $k$-graph traces.
4. For any $v \in \Lambda^0$ and any finite $F \subset \Lambda^0$ there exists finite $W \subset \Lambda^0 \setminus F$ with $p_v \lesssim \sum_{w \in W} p_w$.
5. For any finite $V \subset G^0$, and any finite $F \subset G^0$, there exists finite $W \subset G^0 \setminus F$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.

Proof. (1) implies (2): We have already seen this in the case of graph algebras.

(2) implies (3): A left finite cycle yields a nonzero quotient of $C^*(\Lambda)$ as in the graph case. A non-zero bounded graph trace can be normalized to obtain a graph trace of norm 1, which induces a tracial state on $C^*(\Lambda)$ as in Lemma 6.9.

(3) implies (4): The proof is similar to the graph case. Assume that (4) fails, so that there is a vertex $v$ and a finite set $F \subset \Lambda^0$ such that there does not exist a finite $W \subset \Lambda^0 \setminus F$ with $p_v \lesssim \sum_{w \in W} p_w$. Then for every $e_i$, $i = 1, \ldots, k$, we cannot produce a comparison using the Cuntz-Krieger relations for the degree $e_i$. For some vertex $v_1$ emitting edges to $v$ it must be the case that, for every $i = 1, \ldots, k$, there are fewer red paths out of $v$ not terminating in $F$ is less than the number of edges of degree $e_i$ that $v_1$ emits to $v$. (Here we are using the fact that $\Lambda$ is balanced to
guarantee that the same vertex fails for each degree simultaneously.) At each step we are comparing the number of red paths emitted by \( v_i \) whose range vertex does not belong to \( F \) with the number of partial isometries obtained by iterating the Cuntz-Krieger relations back from \( v \) to \( v_i \) (any choice of degrees yields the same number). Therefore we can extract a similar chain of vertices \( v, v_1, v_2, \ldots \) as in the proof of Lemma 4.6. Define

\[
g(w) = \sum_{i=1}^{\infty} \left| \{ \text{red paths } w \leftarrow v_i \text{ that do not factor through } v_{i-1} \xleftarrow{N_i} v_i \} \right|.
\]

Confirming that this yields a bounded \( k \)-graph trace on \( \Lambda \) is similar to the proof of Lemma 4.6.

(4) implies (5): Same as graph case.

(5) implies (1): Same as graph case. \( \square \)

7. Conclusion

Requiring that a \( k \)-graph be balanced is fairly strict. Hopefully this approach can be extended to characterize stability for other \( k \)-graphs. One possible direction in which to generalize is to consider \( k \)-graphs for which \( |v^e_i| = |v^e_j| \) for all \( i, j \); that is, the number of paths of degree \( n \) terminating at vertex \( v \) depends only on \( v \) and \( |n| \). The arguments used here become tricky in this case.

References