1 \( C^\ast\)-algebras

I am an operator algebraist studying the \( C^\ast\)-algebras of directed graphs and étale groupoids. My thesis work focused on determining how the structure of a directed graph \( E \) determined the behavior of its \( C^\ast\)-algebra \( C^\ast(E) \). I found characterizations for when \( E \) yielded \( C^\ast(E) \) that had \textit{continuous trace}; I also repaired an existing proof in the literature characterizing when \( C^\ast(E) \) is \textit{stable} under a certain tensor product operation. I also found partial generalizations of both of these results to the realm of \( k \)-graphs or higher-rank graphs, generalizations of directed graphs that tend to have much more combinatorial complexity. Much of this work can be framed in terms of \textit{groupoid \( C^\ast\)-algebras}: to a “nice” topological groupoid \( G \) we can assign a \( C^\ast\)-algebra \( C^\ast(G) \) that mirrors the properties of \( G \). As a postdoc at Kansas State, I worked with Gabriel Nagy to develop a flexible method for producing \textit{tracial states} on \( C^\ast\)-algebras of graphs and groupoids. I’m currently working to characterize the simplicity of groupoid \( C^\ast\)-algebras in terms of the underlying groupoid (joint work with Gabriel Nagy and Sarah Reznikoff).

My research area is part of \textit{operator algebras}, a sub-field of functional analysis with significant algebraic and topological aspects. More specifically, I study how discrete objects like directed graphs or groups determine the structure of related analytic objects called \( C^\ast\)-algebras. A \( C^\ast\)-algebra is a (complex) Banach algebra \( A \) with an antilinear involution \( * : A \to A \) such that \((ab)^* = b^*a^*\) and \( \|a^*a\| = \|a\|^2 \) for all \( a, b \in A \). Some basic examples are:

1. \textit{(Matrix algebras)} \( A = M_n(\mathbb{C}) \) with operator norm, standard algebra operations, and the adjoint as \( * \)-operation

2. \textit{(Abelian \( C^\ast\)-algebras)} \( A = C(X) \), the continuous \( \mathbb{C} \)-valued functions on a compact Hausdorff space \( X \) with supremum norm, pointwise operations, and pointwise conjugation as \( * \)-operation.

The study of operator algebras blends functional analysis and algebra with the study of various topological dynamical systems and related concepts. Most of my results center around constructions of the form

\[
\begin{array}{rcl}
G: \text{graph, groupoid, etc.} & \mapsto & C^\ast(G)
\end{array}
\]

The questions I study have the following general flavor: \textit{what combinatorial or topological properties of} \( G \) \textit{are equivalent to desired algebraic or analytic properties of} \( C^\ast(G) \)? (‘Equivalent’ is often out of reach, so we settle for ‘necessary’ or ‘sufficient.’)
2 Graphs, groupoids, and their $C^*$-algebras

Many interesting examples of $C^*$-algebras come from groups, graphs, and groupoids. Given a discrete group $G$, one can consider the complex $*$-algebra $C_r(G)$, the set of complex-valued functions of finite support on $G$ equipped with pointwise addition and convolution product, which is essentially the complex group ring $\mathbb{C}G$ with $g^* = g^{-1}$ for $g \in G$. There are usually two interesting norms one can use to make $C_r(G)$ into a normed algebra: the universal norm $||\cdot||$, corresponding to the universal representation of $G$, and the reduced norm, $|| \cdot ||_r$, corresponding to the left-regular representation. Completing $C_r(G)$ with respect to these norms we obtain the full group $C^*$-algebra $C^*(G)$ and the reduced group $C^*$-algebra $C^*_r(G)$ respectively. Much work has been done to understand how the group-theoretic and dynamical properties of $G$ correspond with the operator algebraic properties of the related $C^*$-algebras. For example, it is known that $C^*(G) = C^*_r(G)$ exactly when $G$ is an amenable group. A basic question to ask of a $C^*$-algebra is whether or not it is simple (no closed two-sided ideals), and while $C^*(G)$ is never simple unless $G$ is the trivial group, it is possible to find nontrivial discrete groups $G$ such that $C^*_r(G)$ is simple (Powers showed that the free group on two generators is an example $[14]$).

**Theorem 2.1** (Powers, 1976). Let $G = \mathbb{F}_2$ be the non-abelian free group on two generators. Then $C^*_r(G)$ is simple as a $C^*$-algebra.

Another example comes from directed graphs, which were defined by Kumjian and Pask $[12]$, building off of work of Cuntz and others ($[12]$). Any directed graph $E$ defines a canonical $C^*$-algebra, denoted $C^*(E)$, which is generated by partial isometries $s_e$, representing the edges of $E$, and projections $p_v$, representing the vertices of $E$. A set of equations called the Cuntz-Krieger relations define the algebraic structure of $C^*(E)$, and as a result the structure of $C^*(E)$ is closely bound to the dynamics of directed paths in $E$. (For details see $[15]$.) For example, the graph $E$ has no directed cycles if and only if $C^*(E)$ can be written as an inductive limit of direct sums of matrix algebras (that is, $C^*(E)$ is an AF algebra). Many other operator algebraic properties of $C^*(E)$ can be “read” from the parent graph $E$—hence the nickname of graph algebras as “$C^*$-algebras we can see.” There are generalizations of graphs called $k$-graphs, which correspond to certain quotient spaces of path categories for colored graphs, and these also give rise to $C^*$-algebras called $k$-graph $C^*$-algebras. Some of my work has centered around determining properties of a $k$-graph $C^*$-algebra $C^*(\Lambda)$ based on its parent $k$-graph $\Lambda$.

Discrete group $C^*$-algebras and graph $C^*$-algebras are both examples of groupoid $C^*$-algebras. A groupoid is a set $G$ along with a partially defined multiplication $\cdot : G^{(2)} \to G$, $G^{(2)} \subset G \times G$, written $(\alpha, \beta) \mapsto \alpha \beta$, and an inversion operation $^{-1} : G \to G$, such that the multiplication $\cdot$ is associative and $\alpha^{-1}(\alpha \beta) = \beta$ and $(\alpha \beta)^{-1} = \alpha^{-1} \beta^{-1}$ for all composable $(\alpha, \beta)$. Thus groupoids are “partial groups” and the failure of the multiplication to be everywhere defined corresponds with the size of the unit space $G^{(0)} = \{u \in G : u = u^2\}$. Range and source maps $r, s : G \to G^{(0)}$ given by $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$ allow us to orient a groupoid as a category of morphisms.
with object set equal to the unit space. Topological groupoids are groupoids equipped
with a suitable topology so that the multiplication and inversion become continuous.
In my research, I focus mainly on étale groupoids, in which the topology is locally
compact and the range map $r : G \rightarrow G$ is a local homeomorphism. Groups are
groupoids with a single unit; the étale groupoids are the correct generalization of
discrete groups. A directed graph $E$ gives rise to an associated groupoid $G_E$ called
path groupoid of $E$, which is defined using infinite paths in $E$. There are constructions
for reduced and full groupoid $C^*$-algebras which suitably generalize the corresponding
group constructions; in the case of the path groupoid $G_E$ of a directed graph we have
$C^*(G_E) \cong C^*(E)$.

3 Continuous-trace graph and $k$-graph $C^*$-algebras

A $C^*$-algebra $A$ has continuous trace (or is continuous-trace) if its spectrum $\hat{A}$ of
*-representations is Hausdorff and if it is always possible to find certain projections
in any irreducible representation (for a more precise definition, see [16]). Continuous-
trace $C^*$-algebras are particularly well-behaved and enjoy a nice classification theory
by cohomology of $\hat{A}$ (again see [16]). Giving the characterization of continuous-trace
$C^*$-algebras requires some graph terminology: An entrance to a cycle is an
edge that meets a vertex of a cycle without belonging to the cycle. An ancestry pair
for vertices $v, w$ is a pair of directed paths $(\lambda, \mu)$, neither containing a cycle, such that
$r(\lambda) = v$ and $r(\mu) = w$ and $s(\lambda) = s(\mu)$. One ancestry pair can contain another as
$(\lambda \nu, \mu \nu) \supset (\lambda, \mu)$. In [2] I showed the following:

Theorem 3.1 ([2]). Let $E$ be a directed graph. Then $C^*(E)$ has continuous trace if
and only if both of the following conditions hold:

1. no cycle of $E$ has an entrance;
2. any pair of vertices $(v, w)$ has at most finitely many minimal ancestry pairs
   $(\alpha, \beta)$.

(If only the first condition holds, then $C^*(E)$ merely has Hausdorff spectrum, a fact
that Goehle pointed out in [5] and which I used in my proof.)

In proving this I used a technique of Drinen and Tomforde called desingularization,
which replaces a directed graph $E$ with a graph $\tilde{E}$ which satisfies certain existence and
finiteness conditions in such a way that the affiliated $C^*$-algebras $C^*(E)$ and $C^*(\tilde{E})$
are Morita equivalent. Morita equivalence is an equivalence relation on $C^*$-algebras
weaker than isomorphism that preserves many useful structural properties. A lot of
the work in [2, Sec. 4] goes into characterizing structure of the desingularization $\tilde{E}$
in terms of the parent graph $E$.

This theorem holds in a wider context than graph $C^*$-algebras, although not yet in
full generality. A $k$-graph (or higher-rank graph) is a countable category $\Lambda$ equipped
with a degree functor $d : \Lambda \to \mathbb{N}^k$ (where the range is a category under addition with a single object) that satisfies the factorization property: if $d(\lambda) = m + n$ for $m, n \in \mathbb{N}^k$, then there are unique $\mu, \nu \in \Lambda$ satisfying $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu \nu$. The simplest example of such an object is a 1-graph, which is always equal to the category of paths of some graph $E$, with length function serving as degree functor.

Each $k$-graph $\Lambda$ defines a $C^*$-algebra $C^*(\Lambda)$ (actually, $\Lambda$ must satisfy a mild finiteness condition for the $C^*$-algebra to be defined). The higher-rank graph $C^*$-algebra construction includes all graph $C^*$-algebras as well as many others. The aforementioned dichotomy for simple graph algebras breaks down for higher-rank graph $C^*$-algebras.

The techniques I used to characterize graph $C^*$-algebras with continuous trace don’t work for all $k$-graph $C^*$-algebras: I restricted to the strictly aperiodic $k$-graphs, which are the $k$-graph analog of graphs with no cycles. Row-finite $k$-graphs satisfy a strict finiteness condition, and ancestry pairs are defined for $k$-graphs in much the same way as for graphs. Under these requirements, I proved the following theorem:

**Theorem 3.2** ([2]). Let $\Lambda$ be a row-finite strictly aperiodic $k$-graph. Then $C^*(\Lambda)$ has continuous trace if and only if any two vertices of $\Lambda$ have at most finitely many minimal ancestry pairs.

## 4 Stability

A $C^*$-algebra is called stable if it satisfies $A \otimes K \cong A$, where $K$ denotes the algebra of compact operators on Hilbert space. The tensor product of two $C^*$-algebras is sometimes a subtle object to define, but here $A \otimes K \cong \bigcup M_n(A)$, where the embedding $M_n(A) \hookrightarrow M_{n+1}(A)$ takes the domain into upper left $n \times n$ corner. If one knows that $C^*$-algebra is stable then a good deal of structural information becomes immediately available, so it is of interest to determine which members of given classes of $C^*$-algebras are stable. Stability of $A$ is closely connected to non-existence of tracial states on $A$: a tracial state on $A$ is a positive linear function $\phi : A \to \mathbb{C}$ of norm 1 such that $\phi(ab) = \phi(ba)$ for all $a, b \in A$.

Tomforde [17], building on work of Hjelmborg and Rørdam [6], gave a characterization of those graphs which have stable $C^*$-algebras, showing that a graph algebra is stable if and only if it has no unital quotients and no tracial states. Unital quotients are (in a certain sense) parametrized by left-finite vertices (vertices that can only reach finitely many others following directed paths) and tracial states on $C^*(E)$ are parametrized by graph traces (positive $\ell^1$ functions on vertices that satisfy Cuntz-Krieger relations). While I was trying to generalize the theorem I found a gap in the published proof in [17] and repaired it. The work of Hjelmborg and Rórdam is based on the notion of subequivalence of projection: $p \preceq q$ if there is a partial isometry $v$ with $v^* v = p$ and $v v^* \leq q$ (for bounded linear operators on a Hilbert space, this is equivalent to the rank of $p$ being bounded by the rank of $q$). I re-organized Tomforde’s analysis of stability for graph algebras, and generalized parts of it to the context of $k$-graph algebras. For $k$-graphs it is harder to find graph traces, so I restricted my analysis to
Research Statement

Danny Crytser
October 2016

$k$-graphs which satisfy a balancing condition: the number of edges of fixed color $e_i$ between two vertices is the same for all $i = 1, \ldots, k$. Under this condition I proved the following ([3]):

**Theorem 4.1 ([3]).** Let $E$ be a balanced row-finite $k$-graph with no sources. Then the following are equivalent.

(i) $C^*(\Lambda)$ is stable;

(ii) $C^*(\Lambda)$ has no unital quotients and no tracial states;

(iii) $\Lambda$ has no left finite cycles and no bounded nonzero $k$-graph traces;

(iv) for every $v \in \Lambda^0$ and every finite $F \subset \Lambda^0$ there is finite $W \subset \Lambda^0 \setminus F$ such that $p_v \lesssim \sum_{w \in W} p_w$;

(v) for every finite $V \subset \Lambda^0$ there exists finite $W \subset \Lambda^0 \setminus F$ such that $\sum_{v \in V} p_v \lesssim \sum_{w \in W} p_w$.

5 Tracial states

Gabriel Nagy and I have come up with a flexible method for constructing tracial states on many $C^*$-algebras. Instead of looking at the entire $C^*$-algebra $A$, the idea is to look at a certain abelian $C^*$-subalgebra $B \subset A$, and then try to find states on $B$ which extend suitably to tracial states on $A$. A normalizer of $B$ ([11]) is an element $n \in A$ such that $nBn^* \cup n^*Bn \subset A$; the collection of such elements is denoted $N(B)$. If $B$ contains an approximate identity for $A$, then $n^*n$ and $nn^*$ belong to $B$ for each $n \in N(B)$. A state on $B$ is fully invariant if $\phi(nbn^*) = \phi(n^*nb)$ for all $b \in B$ and $n \in N(B)$. Note that if $\phi$ is the restriction of a tracial state to $B$, then it is fully invariant. Our work centered around finding contexts in which the converse to this hold, that is, situations in which every fully invariant state on the $C^*$-subalgebra extends to a tracial state on the larger $C^*$-algebra.

In order to do this we need some convenient way to extend states to the rest of the algebra. (At least one extension always exists by the Hahn-Banach theorem but it might be hard to describe concretely.) A conditional expectation $\mathbb{E} : A \to B$ is a completely positive linear projection onto $B$ (typically not a $C^*$-homomorphism). A conditional expectation $\mathbb{E} : A \to B$ is normalized by $n \in N(B)$ if $n\mathbb{E}(a)n^* = \mathbb{E}(nan^*)$ for all $a \in A$. (Typically the conditional expectations one encounters in the wild are normalized by sufficiently many normalizers.)

**Theorem 5.1 ([4]).** Let $B \subset A$ be a non-degenerate abelian $C^*$-subalgebra with a conditional expectation $\mathbb{E} : A \to B$ which is normalized by $N_0 \subset N(B)$. For a state $\phi \in S(B)$, the following are equivalent:

(i) $\phi$ is $n$-invariant for all $n \in N_0$;

(ii) $\phi \circ \mathbb{E}(na) = \phi \circ \mathbb{E}(an)$ for all $a \in A$ and $n \in N_0$;
Corollary 5.2. If $\phi$, $E$, and $N_0 \subset N(B)$ satisfy the conditions of the above theorem, and $N_0 \cup B$ generates $A$ as a $C^*$-algebra, then $\phi \circ E$ is a tracial state on $A$.

This analysis has a natural interpretation in the context of étale groupoids. A Radon probability measure $\mu$ on the unit space $G^{(0)}$ of an étale groupoid $G$ is called totally balanced if for any Borel $X \subset G^{(0)}$ and any open bisection $B \subset G$, we have $\mu(BXB^{-1}) = \mu(B^{-1}BX)$. (This is the groupoid analogue of a fully invariant state on $C_0(G^{(0)})$.) Étale groupoid $C^*$-algebras always have a conditional expectation $P : C^*_{r}(G) \to C_0(G^{(0)})$, given by extending the restriction map $C_c(G) \to C_c(G^{(0)})$.

Theorem 5.3 ([4]). Let $G$ be a second countable locally compact Hausdorff étale groupoid.

(i) Any totally balanced measure on $G^{(0)}$ induces a tracial state on $C^*_{r}(G)$ via $\tau_{\mu}(f) = \int P(f) \, d\mu$ for $f \in C_c(G)$.

(ii) If $G$ is principal (in the sense that $r(g) = s(g)$ implies $g \in G^{(0)}$), then every tracial state on $C^*_{r}(G)$ is formed from a totally balanced measure as in (i).

6 $C^*$-simplicity for étale groupoid $C^*$-algebras

A discrete group $\Gamma$ is $C^*$-simple if its reduced group $C^*$-algebra is simple. (The full group $C^*$-algebra is never simple for $\Gamma \neq 1$ because it projects onto the one-dimensional trivial representation.) The first non-trivial example of a $C^*$-simple group is the non-abelian free group on two generators, a result of Powers ([14]). Recent work ([10]) has powerfully extended the study of $C^*$-simplicity. Specifically, a discrete group is $C^*$-simple if and only if it acts freely on a related compact space called its Furstenberg boundary. (This is a universal construction that mirrors the boundary of the free group.) The groupoid analogue of discrete groups are the étale groupoids, in which the range map is a local homeomorphism–these encompass all crossed product groupoids by discrete groups, as well as many others.

Étale groupoids define $C^*$-algebras, so I’ve been looking to extend the characterization of [10] to étale groupoids. This requires a variety of technical modifications to the approach of [10]. First, the appropriate analogue of a group $\Gamma$ acting on a compact space $X$ is a groupoid $G$ acting on a locally compact space $X$ which is fibered via a continuous and proper map $X \to G^{(0)}$. The first step is correctly extending the definition of a “$G$-boundary” to the groupoid context, which I believe I’ve completed.

The work of [1] points the way as to how such constructions as the injective and $G$-injective envelope of an operator system are constructed, but they only treat the case of discrete groupoids (groupoids without interesting topology). The remaining work will consist of extending the arguments of [10] to the locally compact étale groupoid context. Its interesting to note that a nontrivial étale groupoid can yield a simple full $C^*$-algebra, as opposed to the group context where this is not possible. This ties in to the theory of amenable groupoids, which is rich and complicated.
7 Research opportunities with undergraduates

In summer 2016 I mentored an REU group of three undergraduates on a project relating to operator algebras. Specifically, my group explored extreme traces on products of higher-rank graphs. If Λ₁ is a k-graph and Λ₂ is an ℓ-graph, then Λ₁ × Λ₂ is a (k + ℓ)-graph whose C*-algebra is isomorphic to the tensor product of C*(Λ₁) and C*(Λ₂). (For this and other product constructions for graphs and k-graphs, see [9].) Of particular interest are the extreme traces, those traces on a higher-rank graph which are extreme points in the compact convex set of all traces on the graph. We were able to obtain two results, one of which extended [8] and one of which resembled a result from [13].

Theorem 7.1. (cf. [8]) Let Λ be a locally convex finite k-graph. Then there is a bijection from the set of sources S_Λ onto the extreme traces ext T(Λ) given by v ↦→ g_v, where

\[ g_v(w) = \frac{|\{λ ∈ Λ : s(λ) = v, r(λ) = w\}|}{s^{-1}(v)} \]

Theorem 7.2. Let Λ be a locally convex ℓ-graph and let M be a locally convex m-graph. Then Λ × M is a locally convex (ℓ + m)-graph, and furthermore if g₁ is an extreme trace on Λ and g₂ is an extreme trace on M, then g₁ ⊗ g₂(v, w) = g₁(v)g₂(w) defines an extreme trace on Λ × M.

The map (g₁, g₂) ↦ g₁ ⊗ g₂ may not always be a surjection from ext T(Λ) × ext T(M) onto ext T(Λ × M). (My group tried to establish a result along these lines that did not use the heavy-duty functional analysis of [13].) We are currently trying to get this paper ready for submission to a journal [7].

I would like to explore further possibilities for undergraduate research, possibly centering around the groupoid model of graph or k-graph C*-algebras.
References


