GAPS BETWEEN INTEGERS WITH THE SAME PRIME FACTORS

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ABSTRACT. We give numerical and theoretical evidence in support of the conjecture of Dressler that between any two positive integers having the same prime factors there is a prime. In particular, it is shown that the abc conjecture implies that the gap between two consecutive such numbers \( a < c \) is \( \gg a^{1/2 + \epsilon} \). Dressler's conjecture is verified for values of \( a \) and \( c \) up to \( 7 \cdot 10^{13} \).

We start with the following conjecture of Dressler.

Conjecture 1. Between any two positive integers having the same prime factors there is a prime.

If the two integers have just one prime factor then the conjecture is a trivial consequence of Bertrand's Postulate. On the other hand, the validity of the conjecture for numbers composed of 2's and 3's implies Bertrand's Postulate. Indeed, for \( n \geq 5 \) one can always find positive integers \( i \) and \( j \) such that \( n \leq 2^i 3^j 2^i 3^j < 2n \). The primary reason for believing the conjecture is evidence, both numerical and theoretical, indicating that the gap between two integers with the same prime factors is relatively large.

Conjecture 2. For any \( \epsilon > 0 \) there exists a constant \( C(\epsilon) \) such that if \( a < c \) are positive integers having the same prime factors then

\[
c - a \geq C(\epsilon) a^{\frac{1}{2} - \epsilon}.
\]  

(1)

It is clear that Conjecture 1 is an easy consequence of Conjecture 2 modulo good information on \( C(\epsilon) \) and on the maximal gap between consecutive primes. In this paper we shall prove that Conjecture 2 in turn is an easy consequence of the the abc conjecture.

Theorem 1. The abc conjecture implies Conjecture 2.

We shall also deduce the following unconditional result as a consequence of a weaker version of the abc conjecture due to Stewart and Yu (1991).

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Theorem 2. If $a < c$ are positive integers having the same prime factors then,

$$c - a \geq C(e)(\log c)^{3-\epsilon}.$$  

Cramér (1937) conjectured that the gap between consecutive primes $p_n$ and $p_{n+1}$ is $O(\log^2 p_n)$, in fact he made the stronger conjecture that $\limsup_{n \to \infty} (p_{n+1} - p_n)/\log^2 (p_n) = 1$. Computer searches have shown that $p_{n+1} - p_n < \log^2 p_n$ for values of $p_n$ up to $7 \times 10^{13}$; see Shanks (1964), Lander and Parkin (1967), Brent (1980) and Young and Potler (1989). On the assumption of the Riemann Hypothesis, Cramér proved that there always exists a prime between $n$ and $n + O(n^{1/2} \log n)$. In order to deduce Conjecture 1 from Conjecture 2 one needs gaps of size $O(n^{1/2 - \epsilon})$, which is somewhere between what one obtains from the Riemann Hypothesis and what Cramér has conjectured. On the other hand, with just a "modest" improvement in Theorem 2, specifically obtaining $c - a \geq (\log c)^2$, Conjecture 1 is essentially a consequence of Cramér's conjecture.

On the assumption of Cramér's conjecture it follows that the exponent on the righthand side of (1) cannot be taken to be greater than $1/2$. For let $2^k$ be any power of 2, and $p$ be any prime with $0 < p - 2^k \ll \log^2 p$. Put $c = 2p^2$, $a = 2^{k+1} p$. Then $|c - a| \ll p \log^2 p \ll \sqrt{a} \log^2 a$. The infinitude of Mersenne primes (or Fermat primes) implies a slightly stronger statement. In this case if we let $p = 2^k - 1$ be a Mersenne prime and put $c = 2^{k+1} p$, $a = 2p^2$ then we have $c - a = 2p = \sqrt{2a^{1/2}}$. We do not know in general if the exponent in (1) can be taken as 1/2. However, if $a$ and $c$ have just two prime divisors then we can get 1/2.

Theorem 3. Suppose that $a < c$ are positive integers having the same two prime divisors. Then, on the assumption of the abc conjecture, $c - a \gg a^{1/3}$.

If the prime factors of $a$ and $c$ are restricted to a fixed finite set $S$ of primes then we have the much stronger lower bound of Tijdeman (1973),

$$c - a > \frac{a}{(\log a)^C},$$  

with the drawback being that the constant $C$ depends on the set $S$. In Section 3 we use results of de Weger (1987) to prove (Theorem 4) that the only positive integers $a < c$ composed of the same two primes $p, q$ with $p < q < 200$ and

$$c - a < \sqrt{a}$$  

are $(a, c) = (48, 54) = (2^4 \cdot 3, 2 \cdot 3^3)$, $(a, c) = (1250, 1280) = (2 \cdot 5^4, 2^8 \cdot 5)$ and $(a, c) = (11859482, 11862016) = (2 \cdot 181^3, 2^{16} \cdot 181)$. The following is an open question.

Question 1. Are there infinitely many pairs $a < c$ having the same prime factors satisfying (3)?

Using the table of Young and Potler (1989) on first occurrences of prime gaps, we have been able to verify with a computer search that Conjecture 1 is valid for $a < c < 7 \cdot 10^{13}$. The only example in this range with $c - a$ less than the maximal gap between primes up
to \( c \) is \((a, c) = (2400, 2430)\). The largest gap between consecutive primes up to \( 7 \cdot 10^{13} \) is just 778, substantially smaller than the cube root of \( 7 \cdot 10^{13} \). Thus for \( n > 7 \cdot 10^{13} \) it is reasonable to believe that there is always a prime between \( n \) and \( n + n^{1/3} \). In this case, Conjecture 1 follows if one can establish that for any \( a < c \) having the same prime factors,

\[
c - a > a^{1/3}. \tag{4}
\]

We know of no example for which (4) fails, and so we ask

**Question 2.** Is there any pair \( a < c \) composed of the same prime factors with \( c - a < a^{1/3} \)?

From de Weger’s work we can obtain (Theorem 5) all solutions of (3) with \( a \) and \( c \) composed of the primes 2,3,5,7,11 and 13, and having the same prime factors. All of these solutions satisfy (4) as well. Thus (4) holds for all \( a, c \) composed of the same primes from the set 2,3,5,7,11, and 13. Further examples satisfying (3) may be gleaned from the tables of Nitaj (1993) and Browkin and Brzezinski (1994) on extremal examples for the abc conjecture. All of these examples satisfy (4) as well.

**Section 2. Proofs of Theorems 1 and 2.**

For any positive integer \( n \) let \( N_0(n) = \prod_{p|n} p \), the product being over the distinct prime factors of \( n \).

The abc conjecture. For any \( \epsilon > 0 \) there exists a constant \( C(\epsilon) \) such that for any nonzero relatively prime integers \( a, b \) and \( c \) with \( a + b = c \) we have

\[
\max(|a|, |b|, |c|) \leq C(\epsilon) N_0(abc)^{1+\epsilon}. \tag{5}
\]

Suppose now that \( a < c \) are positive integers having the same prime factors. Let \( b = c - a \). Put \( P = N_0(a) = N_0(c) \) and \( d = (a, b) = (a, c) = (b, c) \). Then \( \frac{a}{d}, \frac{b}{d}, \) and \( \frac{c}{d} \) are relatively prime. Now

\[
N_0(\frac{a}{d} \frac{b}{d} \frac{c}{d}) \leq N_0(ac) N_0(\frac{b}{d}) \leq P \frac{b}{d} \leq \frac{b^2}{d}, \tag{6}
\]

the last inequality following since \( P|b \). It follows from (5) that \( \frac{c}{d} \leq C(\epsilon)(\frac{b^2}{d})^{1+\epsilon} \), and so

\[
c \leq C(\epsilon)b^{2(1+\epsilon)},
\]

that is \( b \geq C'(\epsilon)c^{\frac{1}{3}-\epsilon} \). This establishes Theorem 1.

For the proof of Theorem 2 we proceed as above but instead of applying the abc conjecture we apply the following weaker, but proven, result of Stewart and Yu (1991): Under the same assumptions as in the abc conjecture above we have

\[
\max(\log |a|, \log |b|, \log |c|) \leq C(\epsilon) N_0(abc)^{\frac{4}{3}+\epsilon}.
\]

In our application we obtain

\[
\log(c/d) \ll (b^2/d)^{\frac{3}{2}+\epsilon},
\]
from which we deduce
\[ b^2 \gg d(\log(c/d))^{3-\epsilon} \gg (\log c)^{3-\epsilon}, \]
which completes the proof of Theorem 2. The latter inequality follows from the claim: For \(2 \leq d \leq c/2\), and \(0 < \epsilon < 3/2\) we have
\[ d(\log(c/d))^{3-\epsilon} \geq .7(\log c)^{3-\epsilon}. \]

The claim follows from observing that
\[ d\left(1 - \frac{\log d}{\log c}\right)^{3-\epsilon} \geq d\left(1 - \frac{\log d}{\log 2d}\right)^{3/2} = d\left(\frac{\log 2}{\log 2d}\right)^{3/2} \geq 2\left(\frac{\log 2}{\log 4}\right)^{3/2} > .7. \]

**Section 3. The case of two prime factors. Proof of Theorem 3:**

Suppose that \(a < c\) are positive integers composed of the same two prime divisors \(p, q\). Let \((a, c) = p^e q^f\) and write
\[ c = p^e q^f, \quad a = p^e q^{f+h}, \quad b = c - a = p^e q^f (p^g - q^h). \]

We start by observing that in this case a large gap between \(a\) and \(c\) is tantamount to a large gap between the prime powers \(p^g\) and \(q^h\). To be precise, the inequality
\[ c - a \gg a^{1/2} \]

is equivalent to the inequality
\[ p^g - q^h \gg p^{3/2 (1-\frac{f}{g} - \frac{h}{g})}. \]

To see this we consider two cases. If \(q^h < \frac{1}{2}p^g\) then (8) and (9) are both trivially true, and so we may assume that \(\frac{3}{2}p^g \leq q^h < p^g\). Now (8) is equivalent to
\[ p^g - q^h \gg p^{\frac{3}{2} - \frac{f+h}{2}}. \]

Substituting \(q \approx p^g/h\) into the righthand side yields (9).

We conclude the proof of Theorem 3 by showing that (9) holds true under the assumption of the abc conjecture.

It suffices to consider the case \(e = f = 1\) whence (9) becomes
\[ p^g - q^h \gg p^{\frac{3}{2} (1 - \frac{1}{h} - \frac{1}{g})}. \]

If \(h = 1\) or \(g = 1\) or \((h, g) = (2, 2)\) then (10) is trivial. Thus we may assume that \(h \geq 2\), \(g \geq 2\), and that either \(h\) or \(g\) is \(\geq 3\). Now, the abc conjecture, applied to the sum \(p^g - q^h = (p^g - q^h)\), implies that
\[ p^g \ll (pq|p^g - q^h|)^{1+\epsilon}, \]
or equivalently
\[ |p^g - q^h| \gg p^g(1 - \frac{1}{h} - \frac{1}{h} - \frac{1}{h}) \tag{11} \]
the constants depending on \( \varepsilon \). Since \( 1/h + 1/g \leq 5/6 \), one obtains (10) from (11) on taking \( \varepsilon < 1/12 \). This completes the proof of Theorem 3.

**Remark.** The argument above applies just as well to any relatively prime integers \( p \) and \( q \) (not necessarily primes). Thus Theorem 3 is valid for any \( a, c \) as in (7) with \( p, q \) relatively prime positive integers.

As one can see by the equivalence of (8) and (9), finding pairs \( a, c \) with \( c - a \) small amounts to finding two prime powers close together. Cijswouw, Korlaar and Tijdeman (1982) found all solutions of the inequality
\[ |p^g - q^h| < p^{g/2}, \tag{12} \]
in positive integers \( g, h \) and primes \( p < q < 20 \). Their work was extended by de Weger (1987, Theorem 4.3) to the range \( p < q < 200 \); see also Deze and Tijdeman (1992, Lemma 1). Now any solution of (3) with \( a, b, c \) as in (7) satisfies
\[ p^g - q^h < p^{\frac{g}{2}} q^{\frac{h-1}{2}} < p^{\frac{g-1}{2}} q^{\frac{h-1}{2}}. \]
If \( p < q \) then using the fact that \( q < p^{g/h} \) we obtain
\[ p^g - q^h < p^g(1 - \frac{1}{h} - \frac{1}{h}), \tag{13} \]
which is a stronger inequality than (12). If \( q < p \) then \( p^{-1/2} < q^{-1/2} \) and so we obtain
\[ p^g - q^h < q^{\frac{h-1}{2}} \]
which again is stronger than (12) with the roles of \( p \) and \( q \) reversed. Thus all solutions of (3) with \( p, q < 200 \) may be found by testing the solutions of (12) given in de Weger’s paper. By doing so we obtain,

**Theorem 4.** Suppose that \( a < c \) are positive integers as in (7) with \( p, q < 200 \) and \( c - a < a^{1/2} \). Then \( (a, c) = (48, 54) = (2^4 \cdot 3, 2 \cdot 3^3), (1250, 1280) = (2 \cdot 5^4, 2^8 \cdot 5) \) or \( (11859482, 11862016) = (2 \cdot 181^3, 2^{16} \cdot 181) \).

**Section 4.** \( a, c \) restricted to the primes 2, 3, 5, 7, 11 and 13.

De Weger (1987, Theorem 4.6) solved the diophantine inequality
\[ 0 < c - a < a^{1/2} \tag{14} \]
with
\[ a, c \in \{2^{x_1} \ldots 13^{x_6} : x_i \in \mathbb{Z}, x_i \geq 0, (1 \leq i \leq 6) \}, \]
and \( (a, c) = 1 \). He found exactly 605 solutions, and all of them satisfy \( \nu_2(ac) \leq 26, \nu_3(ac) \leq 19, \nu_5(ac) \leq 13, \nu_7(ac) \leq 13, \nu_{11}(ac) \leq 7 \) and \( \nu_{13}(ac) \leq 8 \). Here, \( \nu_p(n) \) denotes the multiplicity of \( p \) dividing \( n \). We ran a program in UBASIC to test which of these satisfy the stronger inequality
\[ 0 < P(c - a) < (Pa)^{1/2} \]
where \( P \) is the product of the primes appearing in \( ac \). In this manner we were able to establish
Theorem 5. There are 58 pairs of positive integers \( a < c \) having the same prime factors, with the primes selected from the set \( \{2, 3, 5, 7, 11, 13\} \), such that \( c - a < a^{1/2} \). In every such pair we have \( c < 15 \cdot 10^9 \), and \( c - a > a^{1/3} \). Of these pairs, 19 are primitive, \((a, c) = 1\).

A complete listing of the pairs in Theorem 5 is available upon request.

Section 5. Small gaps with \( a < c < 7 \cdot 10^{13} \).

In the chart below we list all pairs \( 0 < a < c < 7 \cdot 10^{13} \), having the same prime factors, with \( c - a \) less than twice the maximal gap between primes up to \( c \).

<table>
<thead>
<tr>
<th>( a )</th>
<th>( c )</th>
<th>( c - a )</th>
<th>max prime gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 48 = 2^4 \cdot 3 )</td>
<td>( 54 = 2 \cdot 3^3 )</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>1250 = ( 2 \cdot 5^4 )</td>
<td>1280 = ( 2^8 \cdot 5 )</td>
<td>30</td>
<td>22</td>
</tr>
<tr>
<td>2016 = ( 2^5 \cdot 3^2 \cdot 7 )</td>
<td>2058 = ( 2 \cdot 3 \cdot 7^3 )</td>
<td>42</td>
<td>34</td>
</tr>
<tr>
<td>2400 = ( 2^5 \cdot 3 \cdot 5^2 )</td>
<td>2430 = ( 2 \cdot 3^5 \cdot 5 )</td>
<td>30</td>
<td>34</td>
</tr>
<tr>
<td>2646 = ( 2 \cdot 3^3 \cdot 7^2 )</td>
<td>2688 = ( 2^7 \cdot 3 \cdot 7 )</td>
<td>42</td>
<td>34</td>
</tr>
<tr>
<td>15972 = ( 2^2 \cdot 3 \cdot 11^3 )</td>
<td>16038 = ( 2 \cdot 3^6 \cdot 11 )</td>
<td>66</td>
<td>44</td>
</tr>
<tr>
<td>29376 = ( 2^6 \cdot 3^3 \cdot 17 )</td>
<td>29478 = ( 2 \cdot 3 \cdot 17^2 )</td>
<td>102</td>
<td>52</td>
</tr>
<tr>
<td>58368 = ( 2^{10} \cdot 3 \cdot 19 )</td>
<td>58482 = ( 2 \cdot 3^4 \cdot 19^2 )</td>
<td>114</td>
<td>72</td>
</tr>
<tr>
<td>504000 = ( 2^6 \cdot 3^2 \cdot 5^3 \cdot 7 )</td>
<td>504210 = ( 2 \cdot 3 \cdot 5 \cdot 7^5 )</td>
<td>210</td>
<td>114</td>
</tr>
<tr>
<td>918540 = ( 2^2 \cdot 3^3 \cdot 5 \cdot 7 )</td>
<td>918750 = ( 2 \cdot 3 \cdot 5^5 \cdot 7^2 )</td>
<td>210</td>
<td>114</td>
</tr>
</tbody>
</table>

The table above was obtained by a direct search on a PC using UBASIC. The idea of the program is very simple, and it runs extremely fast. For example if \( a, c \) have three odd primes in common say \( p_1, p_2, p_3 \), then we know \( p_1 p_2 p_3 < 778/2 \), half the maximal gap between consecutive primes up to \( 7 \cdot 10^{13} \), and so the choices for \( p_1, p_2 \) and \( p_3 \) are very restricted, etc..

References


