BOUNDS ON FEW NOMIAL EXPONENTIAL SUMS OVER $\mathbb{Z}_p$

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Abstract. We obtain a number of new bounds for exponential sums of the type $S(\chi, f) = \sum_{x=1}^{p-1} \chi(x) e_p(f(x))$, with $p$ a prime, $f(x) = \sum_{i=1}^{r} a_i x^{k_i}$, $a_i, k_i \in \mathbb{Z}$, $1 \leq i \leq r$ and $\chi$ a multiplicative character (mod $p$). The bounds refine earlier Mordell-type estimates and are particularly effective for polynomials in which a certain number of the $k_i$ have a large gcd with $p-1$. For instance, if $f(x) = \sum_{i=1}^{m} a_i x^{k_i} + g(x^d)$ with $d(p-1)$ then $|S(\chi, f)| \leq p (k_1 \cdot \cdots \cdot k_m)^{1/d}/d^{1/2}$, if $f(x) = ax^k + h(x^d)$ with $d(p-1)$ and $(k, p-1) = 1$ then $|S(\chi, f)| \leq p/\sqrt{d}$, and if $f(x) = ax^k + bx^{-k} + h(x^d)$ with $d(p-1)$ and $(k, p-1) = 1$ then $|S(\chi, f)| \leq p/\sqrt{d} + \sqrt{2} p^{3/4}$.

1. Introduction

For a prime $p$, Laurent polynomial

$$f(x) = a_1 x^{k_1} + \cdots + a_r x^{k_r},$$

with $a_i, k_i \in \mathbb{Z}$, $1 \leq i \leq r$, and multiplicative character $\chi$ mod $p$ we consider the mixed exponential sum

$$S(\chi, f) := \sum_{x=1}^{p-1} \chi(x) e_p(f(x)),$$

where $e_p(\cdot)$ is the additive character $e_p(\cdot) = e^{2 \pi i \cdot /p}$ on the finite field $\mathbb{Z}_p$. (Unless specified, the $k_i$ need not be distinct, or nonzero.) We shall say that $f$ is in 'standard form' when $p \nmid a_1 \cdots a_r$, and the $k_i$ are distinct mod $p-1$. For such sums the classical Weil bound [10] (see [3] or [9] for Laurent $f$) yields,

$$|S(\chi, f)| \leq \max \{|k_1|, |k_i - k_j|\} p^{1/2},$$

nontrivial only if $\max \{|k_1|, |k_i - k_j|\} < \sqrt{p}$. Mordell [8] gave a different type of bound which depended rather on the product of all the exponents $k_i$. In [4] we obtained the following improvement in Mordell’s bound;

$$|S(\chi, f)| \leq \frac{1}{p^{1/2}} \left(\ell'_1 \ell'_2 \cdots \ell'_s\right)^{1/2} p^{1-\frac{1}{2r}},$$

for any nonconstant $f$ in standard form, where

$$\ell'_i = \begin{cases} k_i, & \text{if } k_i > 0, \\ r|k_i|, & \text{if } k_i < 0. \end{cases}$$

In [5] we showed that some of the larger $\ell'_i$ can in fact be omitted from the product (at the cost of a worse dependence on $p$), obtaining for any $m$ with $\frac{1}{2} r < m \leq r$,

$$|S(\chi, f)| \leq \frac{1}{p^{1/2}} \left(\ell_1 \cdots \ell_m\right)^{1/m} p^{1-\frac{1}{m} \left(m-\frac{1}{2}r\right)}.$$
provided \( k_1, \ldots, k_m \) are distinct and nonzero \((\text{mod } p-1)\) and \( p \nmid a_1 \cdots a_m \), where
\[
\ell_i = \begin{cases} 
  k_i, & \text{if } k_i > 0, \\
  m|k_i|, & \text{if } k_i < 0.
\end{cases}
\]

A different type of bound was obtained by Akuliničev [1], showing for binomials that
\[
|S(\chi, ax^{k_1} + bx^{k_2})| \leq \frac{p}{\sqrt{k_1}} + \sqrt{(p-1) - 1p^{\frac{3}{2}}},
\]
when \( k_1 |(p-1) \) and \((k_1,k_2) = 1\), and for trinomials that
\[
|S(\chi, ax^{k_1} + bx^{k_2} + cx^{k_3})| \leq \sqrt{2p \ k_2^{-\frac{1}{4}}},
\]
if \((k_3,p-1) = 1\) and \((k_1,k_2)| (p-1) \) with \((k_1,k_2) = 1\) and \( k_2 < k_1 \). The condition \( k_2 < k_1 \) is certainly needed in view of the example
\[
|S(\chi, ax^{(p-1)/2} + bx^{(p+1)/2} - bx)| = \frac{1}{2}p + O(\sqrt{p}),
\]
when \( \chi \) is the principal character or the Legendre symbol. For trinomial polynomials the authors showed in [4] and [5] the Mordell [8] type bounds
\[
|S(\chi, ax^{k_1} + bx^{k_2} + cx^{k_3})| \leq \left(\frac{80}{9}\right) \left(\frac{1}{2}\right)^{\frac{1}{4}} \left(k_1 k_2 k_3\right)^{\frac{1}{4}} p^{\frac{3}{2}},
\]
and
\[
|S(\chi, ax^{k_1} + bx^{k_2} + cx^{k_3})| \leq \left(\frac{k_1 k_2 k_3 (p-1)}{k_2 k_3}\right)^{\frac{1}{4}} \left(k_2 k_3\right)^{\frac{1}{4}} p^{\frac{3}{2}},
\]
while in [6] they obtained
\[
|S(\chi, ax^{k_1} + bx^{k_2} + cx^{k_3})| \leq 3^{\frac{1}{4}} (k_1 k_2 k_3 (p-1))^{\frac{1}{4}} p^{\frac{3}{2}} + \sqrt{5} (k_1 k_2 k_3)^{\frac{1}{4}} p^{\frac{3}{2}}.
\]

In general, if all but one of the exponents (or all but a few small degree exponents) have a large common factor with \((p-1)\) then repeatedly applying Akuliničev’s averaging method gives the following bound:

**Theorem 1.1.** Suppose that \( f \) is of the form
\[
f(x) = g(x) + g_2(x^{k_2}) + \cdots + g_r(x^{k_r})
\]
with \( r \geq 2 \), where \( g, g_2, \ldots, g_r \) are Laurent polynomials over \( \mathbb{Z} \), and \( g \) (written in standard form) contains a monomial \( a_1 x^{k_1} \), \( k_1 \neq 0 \), with \( p \nmid a_1 \).

Then for any multiplicative character \( \chi \) \((\text{mod } p)\),
\[
|S(\chi, f)| \leq p \sum_{i=0}^{r-2} \left(\frac{(k_{r-i}, p-1)}{k_{r-i}, p-1}\right)^{\frac{1}{r-i}} + D \frac{1}{2-r},
\]
where
\[
D = \begin{cases} 
  \deg g - 1, & \text{if } g(x) \text{ is a polynomial}, \\
  (k_1, p-1) - 1, & \text{if } g(x) = a_1 x^{k_1} \text{ is a monomial},
\end{cases}
\]
and \( D \) is the maximum difference of the exponents when \( g \) contains exponents of both signs.

Replacing \( x \) by \( x^{-1} \) we can of course assume that \( g \) contains at least one positive exponent. The same bound also holds for the complete untwisted sum \( \sum_{k=0}^{p-1} e_p(f(x)) \) when \( f \) is a polynomial (no extra +1 is needed to account for the \( x = 0 \) term). The second term in (1.11) arises from using the Weil bound for exponential sums involving Laurent polynomials with the same set of exponents as \( g \). The more general version given in (3.2) can be used when there are better bounds for such sums. Notice that when \( g \) is a monomial \( a_1 x^{k_1} \) with \((k_1, p-1) = 1\) this second term vanishes. The bound of Akuliničev (1.6) is implied by the case \( r = 2 \) and (1.7) essentially from \( r = 3 \) (using the unsimplified form). The more general version given in (3.2) can be used when there are better bounds for such sums.
form (3.2) to obtain the same constant). The theorem follows from the reduction formula given in Lemma 3.1.

We observe here that combining the approach we used in [5] with features of the Akulinichev approach can lead to a variety of new fewnomial bounds appropriate when \( t \) of the exponents, \( k_{r-t+1}, \ldots, k_r \) say, share a large common factor with \((p-1)\):

**Theorem 1.2.** For positive integers \( m, r, t \) with \( 1 \leq t < r \) and \( \frac{1}{2} (r-t) < m \leq (r-t) \), any Laurent polynomial \( f(x) \) as in (1.1) with \( k_1, \ldots, k_{r-t} \) distinct and nonzero \((mod \ p-1), \ p \nmid a_1 \cdots a_{r-t} \), and any multiplicative character \( \chi \) \((mod \ p)\),

\[
|S(\chi, f)| \leq c_m (\ell_1 \cdots \ell_m)^{\frac{1}{2^m}} \frac{p^{(r-t-m)} (k_{r-t+1}, \ldots, k_r, p-1)}{(k_1, \ldots, k_m)^{m+1}} \left( \frac{1}{2^m} \right),
\]

where

\[
\mu_{m, t} = \frac{(k_1, \ldots, k_r, p-1)(k_1, \ldots, k_{r-t}, p-1)(k_1, \ldots, k_m, k_{r-t+1}, \ldots, k_r, p-1)^{m-1}}{(k_1, \ldots, k_m)^{2m}},
\]

and \( c_1 = 1, c_2 = (3/2)^{1/2}, c_m = (m+1)^{1/2} \) for \( m \geq 3 \). If \( k_1, \ldots, k_m \) are all positive then we can take \( c_m = 1 \) for any \( m \).

In particular, this bound is non-trivial if \((k_{r-t+1}, \ldots, k_r, p-1) \geq 4^{2/3} (\ell_1 \cdots \ell_m)^{2/3} p^{(r-t-m)/m}\) and improves upon (1.5) if \((k_{r-t+1}, \ldots, k_r, p-1) \geq p^{1/2} \). The factor \( \mu_{m, t} \) is at most one, and yields a bonus savings in certain situations. The bound comes from counting the number of solutions \( T_{m, t} \) in \( \mathbb{Z}_p^{2m} \) to the system

\[
\sum_{i=1}^{k_1} x_i^k + \cdots + x_m^k \equiv y_1^k + \cdots + y_m^k \pmod{p}, \quad 1 \leq i \leq r-t,
\]

and \( T_{m, t} \) the number of those solutions with \( x_i^k = y_i^k = 1, i = r-t+1, \ldots, r, j = 1, \ldots, m \).

Theorem follows at once from the estimates for \( T_{m, t} \) given in Lemma 2.1,

\[
T_{m, t} \leq C_m \frac{(k_1, \ldots, k_r, p-1)(k_1, \ldots, k_m, p-1)^{m-1}}{(k_1, \ldots, k_m)^{2m}} (\ell_1 \cdots \ell_m) (p-1)^m,
\]

and the following, (setting \( v = w = m \)), with \( c_m = C_m^{1/2} \).

**Lemma 1.1.** For any positive integers \( r, t, v, w \) with \( 1 \leq t < r \) and Laurent polynomial \( f(x) \) as in (1.1) with \( k_1, \ldots, k_{r-t} \) distinct and nonzero \((mod \ p-1), \ p \nmid a_1 \cdots a_{r-t} \), and multiplicative character \( \chi \) \((mod \ p)\),

\[
|S(\chi, f)| \leq (p-1)^{-1/2} \frac{p^{1/2} (T_{m, t})^{1/2}}{(r-t+1, \ldots, k_r, p-1)^{1/2}}.
\]

The case \( m = 1 \) of Theorem 1.2 may be stated as follows: For \( r \geq 2 \) and any \( f(x) \) as in (1.1) with \( k_1 \) nonzero \((mod \ p-1)\) and \( p \nmid a_1 \),

\[
|S(\chi, f)| \leq p \left( \frac{(k_1, p-1)(k_1, k_2, k_r, p-1)}{(k_1, k_2, k_r, p-1)} \right)^{1/2},
\]

a bound also discovered by Yu [12, Theorem 1] for the case of binomials. The case \( r-t = m \) gives for any \( m \) with \( 1 \leq m \leq r-1 \), and any \( f(x) \) as in (1.1) with \( k_1, \ldots, k_m \) distinct and nonzero \((mod \ p-1), \ p \nmid a_1 \cdots a_m \),

\[
|S(\chi, f)| \leq c_m p \frac{(\ell_1 \cdots \ell_m)^{1/2}}{(k_1, \ldots, k_m)^{2m}} \left( \frac{(k_1, k_r, p-1)(k_1, \ldots, k_m, p-1)}{(k_1, \ldots, k_m)^2} \right)^{1/2}.
\]

In particular for any \( f(x) \) in standard form of the type \( f(x) = a_1 x^{k_1} + \cdots + a_m x^{k_m} + h(x^d) \) with \( d(p-1), k_i > 0, 1 \leq i \leq m \) and \( h(x) \) any Laurent polynomial,

\[
|S(\chi, f)| \leq p \frac{(k_1 \cdots k_m)^{1/2}}{d^{1/2}}.
\]
For monomials we gain nothing new from Theorem 1.2 while for binomials we just have the bound in (1.13). For trinomials we can take \( m = 2, t = 1 \) to gain the new bound,

\[
|S(\chi, f)| \leq \left( \frac{3}{2} \right)^{1/4} \left( \frac{(k_1, k_2, k_3, p - 1)(k_1, k_2, p - 1)}{(k_1, k_2)^2} \right)^{1/4} \left( \ell_1 \ell_2 \right)^{\frac{1}{2}} (k_3, p - 1)^{-\frac{1}{p}}.
\]

**Example 1.1.** Let \( f(x) = ax^k + h(x^d) \) where \( d|(p - 1), k \) is nonzero  \((\text{mod } p - 1), p \not| a\), and \( h(x) \) is any Laurent polynomial. Then by (1.13),

\[
|S(\chi, f)| \leq p \sqrt{\frac{(k, p - 1)(k, d)}{d}} \leq p(k, p - 1)^{\frac{1}{2}},
\]

while from Theorem 1.1 (using \( r = 2 \), \( g(x) = ax^k \)),

\[
|S(\chi, f)| \leq p \sqrt{\frac{(k, d)}{d} + \sqrt{(k, p - 1) - 1}} p^{\frac{1}{2}}.
\]

**Example 1.2.** Let \( f(x) = ax^k + bx^{-k} + h(x^d) \) where \( d|(p - 1), k > 0 \) is nonzero \((\text{mod } p - 1), p \not| ab\), and \( h(x) \) is any Laurent polynomial. Then, by (1.14) with \( m = 2 \) we have

\[
|S(\chi, f)| \leq 3^{1/2} \frac{(k, p - 1)^{1/4}(k, d)^{1/4}p}{d^{1/4}},
\]

while from Theorem 1.1 (using \( r = 2 \), \( g(x) = ax^k + bx^{-k} \)),

\[
|S(\chi, f)| \leq p \sqrt{\frac{(k, d)}{d} + \sqrt{(k, p - 1)p^{\frac{1}{2}}}}.
\]

**Example 1.3.** Let \( f(x) = ax^k + bx^{-k} + cx^{-\ell} + h(x^d) \) with \( d|(p - 1), k, -k, \ell \) distinct and nonzero \((\text{mod } p - 1), k > 0, p \not| abc\), and \( h(x) \) is any Laurent polynomial. Then using Theorem 1.2 with \( m = 2, t = r - 3 \), we have

\[
|S(\chi, f)| \leq 3^{1/2} \frac{[k, d, (k, p - 1)(k, l, d)(k, l, p - 1)]^{1/2}p^{9/8}}{d^{1/4}} \leq 3^{1/2} (k, p - 1)^{1/2}p^{9/8}d^{1/4},
\]

while from Theorem 1.1 (using \( r = 2 \), \( g(x) = ax^k + bx^{-k} + cx^{-\ell} \)),

\[
|S(\chi, f)| \leq p \sqrt{\frac{(k, d)}{d} + \sqrt{\max\{2k, k + |\ell|\}}} p^{\frac{1}{2}}.
\]

Another class of polynomials for which nontrivial estimates are available was introduced by Bourgain [2]:

**Theorem 1.3.** [2, Theorem 7] For \( r \in \mathbb{Z}_+, \epsilon > 0 \), there is a \( \delta = \delta(r, \epsilon) \) such that if \( \{k_1, \ldots, k_r\} \) are distinct positive integers with \( k_i < p^{\frac{1}{2}} - \epsilon \), \( (k_i - k_1, p - 1) < p^{1-\epsilon} \), \( 2 \leq i \leq r \), and \( f(x) = g(x) + a_2x^{k_2} + \cdots + a_r x^{k_r} \in \mathbb{Z}[x] \) with and \( g(x) \) is a polynomial of degree \( d < p^{\frac{1}{2}} - \epsilon \) involving \((\text{written in standard form})\) a monomial \( a_i x^{k_i}, p \not| a_i \), then \( \sum_{p-1}^p e_p(f(x)) < p^{1-\delta}. \)

In this direction we offer the following corollary of Theorem 1.1

**Corollary 1.1.** Let \( \epsilon > 0, r \geq 2 \) and \( \{k_1, \ldots, k_r\} \) be integers such that \( 1 \leq k_i < p^{\frac{1}{2}} - \epsilon \) and \( (k_i, k_i, p - 1) < (k_i, p - 1)^{-\epsilon} \), \( 2 \leq i \leq r \).

Suppose that \( f \) is of the form

\[
f(x) = g(x) + g_2(x^{k_2}) + \cdots + g_r(x^{k_r})
\]

where \( g, g_2, \ldots, g_r \) are Laurent polynomials over \( \mathbb{Z} \) and \( g(x) \) (written in standard form) contains a monomial \( a_i x^{k_i}, p \not| a_i \), and has degree \((\text{maximum difference of exponents if } g \text{ has negative exponents}) \)

\[
d < p^{\frac{1}{2}} - \epsilon.
\]
Then
\[ |S(\chi, f)| \leq 3 \, p \, \frac{1}{3 - r}. \]

2. Proofs of Lemma 1.1 and Lemma 1.2

Let \( r, t, v, w \) be positive integers with \( 1 \leq t < r \) and \( k_i, a_i \) be integers with \( p \mid a_1 \cdots a_{r-t} \).

The proof of Lemma 1.1 is similar to that of Lemma 1.1 of [5] except we average only over those \( y \) from the set
\[ Y = \{ y \in \mathbb{Z}_p^* : y^{k_i} = 1, \ i = r - t + 1, \ldots, r \}, \quad |Y| = (k_{r-t+1}, \ldots, k_r, p - 1). \]

For \( \vec{u} = (u_1, \ldots, u_{r-t}) \in \mathbb{Z}_p^{r-t} \) and positive integer \( w \), put
\[ N_w(\vec{u}) = \# \{ (x_1, \ldots, x_w) \in \mathbb{Z}_p^* : \sum_{i=1}^w x_i^{k_j} = u_j, \ j = 1, \ldots, r - t \}, \]
so that
\[ (2.1) \quad \sum_{\vec{u} \in \mathbb{Z}_p^{r-t}} N_w(\vec{u}) = (p - 1)^w, \quad \sum_{\vec{u} \in \mathbb{Z}_p^{r-t}} N_w^2(\vec{u}) = T_{w,t}. \]

For any multiplicative character \( \chi \) and positive integer \( v \), the simple observation that \( \sum_{u \in \mathbb{Z}_p} c_p(u) = p \) if \( a \equiv 0 \) (mod \( p \)) and zero otherwise, gives
\[ \sum_{\vec{u} \in \mathbb{Z}_p^{r-t}} \left| \sum_{y \in Y} \chi(y) c_p(a_1 y^{k_1} + \cdots + a_{r-t} y^{k_{r-t}}) \right|^{2w} \]
\[ = \sum_{y_1, \ldots, y_v \in Y} \chi(x_1 \cdots x_v y_1^{-1} \cdots y_v^{-1}) \sum_{\vec{u} \in \mathbb{Z}_p^{r-t}} c_p \left( \sum_{j=1}^{r-t} a_j u_j (x_1^j + \cdots + x_v^j - y_j - \cdots - y_v^j) \right) \]
\[ = \sum_{y_1, \ldots, y_v \in Y} \chi(x_1 \cdots x_v y_1^{-1} \cdots y_v^{-1}) \leq p^{r-t} T_{v,t}, \]
where \( \sum^\star \) denotes a sum over the \( x_1, \ldots, x_v, y_1, \ldots, y_v \) in \( Y \) satisfying \( \sum_{j=1}^v x_j^k \equiv \sum_{j=1}^v y_j^k \) (mod \( p \)) for \( 1 \leq i \leq r - t \).

Writing \( S = S(\chi, f) \), we have for any positive integer \( w \),
\[ |Y|^w |S|^w \leq \sum_{y \in Y} \left( \sum_{x=1}^{p-1} \chi(y x) c_p(a_1 (y x)^{k_1} + \cdots + a_r (y x)^{k_r}) \right)^w \]
\[ = \sum_{y \in Y} \chi^w(y) \sum_{x_1, \ldots, x_w \in \mathbb{Z}_p^*} \chi(x_1 \cdots x_w) \sum_{j=1}^{r-t} a_j y^{k_j} (x_1^j + \cdots + x_w^j) \]
\[ = \sum_{x_1, \ldots, x_w \in \mathbb{Z}_p^*} \chi(x_1 \cdots x_w) c_p \left( \sum_{j=1}^{r-t} a_j (x_1^j + \cdots + x_w^j) \right) \cdot \sum_{y \in Y} \chi^w(y) c_p \left( \sum_{j=1}^{r-t} a_j y^{k_j} (x_1^j + \cdots + x_w^j) \right), \]
and so
\[ (2.3) \quad |Y|^w |S|^w \leq \sum_{\vec{u} \in \mathbb{Z}_p^{r-t}} N_w(\vec{u}) \left| \sum_{y \in Y} \chi^w(y) c_p \left( \sum_{j=1}^{r-t} a_j u_j y^{k_j} \right) \right|. \]

Applying Hölder’s inequality twice, the second time splitting
\[ (2.4) \quad N_w(\vec{u})^{\frac{2r}{3-r}} = N_w(\vec{u})^{\frac{2r-2}{3-r}} N_w(\vec{u})^{\frac{2}{3-r}}, \]
and using (2.1) and (2.2) gives

\[
|Y|^n |S|^w \leq \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( T_{v,t}^{*} p_{r-t}^{-1} \right)^{\frac{1}{r}}
\]

\[
\leq \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( \sum_{\alpha} \frac{N_{w}(\alpha)}{2^{w-1}} \right)^{2w-1} \left( T_{v,t}^{*} p_{r-t}^{-1} \right)^{\frac{1}{r}}
\]

\[
= (p-1)^{\frac{w-1}{w}} (T_{v,t}^{*} p_{r-t}^{-1})^{\frac{1}{r}} = (p-1)^{\frac{w-1}{w}} (T_{v,t}^{*} T_{w,t})^{\frac{1}{w}}
\]

Hence as claimed

\[
|S| < (p-1)^{\frac{1}{w}} p_{r-t}^{\frac{1}{r}} (T_{v,t}^{*} T_{w,t})^{\frac{1}{w}} |Y|^{\frac{1}{w}}.
\]

Lemma 2.1. Let \( m \in \mathbb{N}, k_1, \ldots, k_r \in \mathbb{Z} \) such that \( k_1, \ldots, k_m \) are distinct and nonzero (mod \( p-1 \)). Then we have the following estimates for \( T_{m,t} \) and \( T_{m,t}^{*} \).

(i) \( m = 1 \): For \( r \geq 2 \), \( 1 \leq t \leq r-1 \),

\[
T_{1,t} = (k_1, \ldots, k_{r-t}, p-1)(p-1), \quad T_{1,t}^{*} = (k_1, \ldots, k_r, p-1)(k_{r-t+1}, \ldots, k_r, p-1).
\]

(ii) \( m \geq 2 \): For \( r \geq 3 \), \( t \geq 1 \) and \( 2 \leq m \leq r-t \),

\[
T_{m,t} \leq C_m \frac{(k_1, \ldots, k_{r-t}, p-1)(k_1, \ldots, k_m, p-1)^{m-1}}{(k_1, \ldots, k_m)^m}((\ell_1, \ldots, \ell_m)(p-1)^m),
\]

\[
T_{m,t}^{*} \leq C_m \frac{(k_1, \ldots, k_{r-t}, p-1)(k_1, \ldots, k_m, k_{r-t+1}, \ldots, k_r, p-1)^{m-1}}{(k_1, \ldots, k_m)^m}((\ell_1, \ldots, \ell_m)(k_{r-t+1}, \ldots, k_r, p-1)^m),
\]

where \( C_m = 1 \) if \( k_1, \ldots, k_m \) are all positive, \( C_m = 1/m^m \) if \( k_1, \ldots, k_m \) are all negative, and \( C_2 = \frac{2}{\pi} \) and \( C_m = m+1, m \geq 3 \), if \( k_1, \ldots, k_m \) have mixed signs.

The Lemma refines estimates of Lemma 2.1 and 2.2 in [5].

Proof. The \( m = 1 \) bounds are immediate. For \( m \geq 2 \) we set

\[
d^* = (k_1, \ldots, k_m), \quad d = (k_1, \ldots, k_m, p-1), \quad d_1 = (k_1, \ldots, k_{r-t}, p-1)
\]

and observe that

\[
T_{m,t} = \# \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Z}_{p}^{2m} : \sum_{i=1}^{m} x_i^k = \sum_{i=1}^{m} y_i^j, j = 1, \ldots, r-t \right\}
\]

\[
= \# \left\{ (z, x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Z}_{p}^{m+1} : z^k \cdot x_i^j = \sum_{i=1}^{m} x_i^k = \sum_{i=1}^{m} y_i^j, j = 1, \ldots, m, \right\}
\]

\[
x_1^d = z, \quad x_1^d = \sum_{i=1}^{m} y_i^j = \sum_{i=2}^{m} x_i^k, j = m+1, \ldots, r-t \}
\]

\[
\leq d_1 \# \left\{ (z, x_2, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Z}_{p}^{2m} : z^k \cdot x_i^j = \sum_{i=1}^{m} x_i^k = \sum_{i=1}^{m} y_i^j, j = 1, \ldots, m, \right\}
\]

\[
= d_1 \# N
\]

where

\[
N = \# \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Z}_{p}^{2m} : \sum_{i=1}^{m} x_i^k = \sum_{i=1}^{m} y_i^j, j = 1, \ldots, m \right\}.
\]
Further
\[ N = d^{2m} \# \left\{ (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{Z}_{p}^{2m} : \sum_{i=1}^{m} u_i k_i/d^* = \sum_{i=1}^{m} v_i k_i/d^* , \ j = 1, \ldots, m, \ u_i, v_i \ are \ d^* \ \text{th \ powers} \right\} \]
\[ = d^{2m} \# \left\{ (u_1, \ldots, u_m, v_1, \ldots, v_m) \in \mathbb{Z}_{p}^{2m} : \sum_{i=1}^{m} u_i k_i/d^* = \sum_{i=1}^{m} v_i k_i/d^* , \ j = 1, \ldots, m, \ u_i, v_i \ are \ d^* \ \text{th \ powers} \right\} \]
\[ = \# \left\{ (x_1, \ldots, x_m, y_1, \ldots, y_m) \in \mathbb{Z}_{p}^{2m} : \sum_{i=1}^{m} x_i k_i/d^* = \sum_{i=1}^{m} y_i k_i/d^* , \ j = 1, \ldots, m \right\} . \]
Applying Wooley’s [11, Theorem 1.2] for \( m \geq 3 \) and Lemma 3.2 of [4] for \( m = 2 \), we have
\[ N \leq C_m \left( \frac{\ell_1 d}{d^*} \cdots \frac{\ell_m d}{d^*} \right) (p-1)^m , \]
with the stated values of \( C_m \) and
\[ T_{m,t} \leq C_m d_1 \left( d \right)^{m-1} \left( \frac{\ell_1 \cdots \ell_m}{d^*} \right) (p-1)^m . \]
Observing that \( (p-1)/(k_{r-t+1}, \ldots, k_r, p-1) \) equals \( T_{m,t} \) with exponents \( p-1 \), then gives the bound for \( T_{m,t} \).

3. Proof of Theorem 1.1

For any integers \( k_1, \ldots, k_r \) we define
\[ B(k_1, \ldots, k_r) = \max \sum_{x=1}^{p-1} e_p(f(x)) , \]
where the max is taken over all Laurent polynomials over \( \mathbb{Z} \) of the form
\[ f = a_1 x^{k_1} + \cdots + a_r x^{k_r} , \quad p \nmid a_1 . \]
When none of the \( k_i \) are negative we similarly define
\[ B^*(k_1, \ldots, k_r) = \max \sum_{x=0}^{p-1} e_p(f(x)) . \]

Lemma 3.1. For \( r \geq 2 \), if \( d \mid k_{t+1}, \ldots, k_r \) for some \( 1 \leq t < r \), and \( p \nmid a_1 \) then
\[ S(\chi, a_1 x^{k_1} + \cdots + a_r x^{k_r}) \leq p \left( \frac{(k_1, d, p-1)}{(d, p-1)} + \frac{B(k_1, \ldots, k_t)}{p} \right)^{\frac{1}{2}} . \]
If the \( k_1, \ldots, k_r \) are all non-negative then \( B(k_1, \ldots, k_t) \) may be replaced by \( B^*(k_1, \ldots, k_t) \).

If the \( k_1, \ldots, k_r \) are all non-negative then similarly
\[ B^*(k_1, \ldots, k_r) \leq p \left( \frac{(k_1, d, p-1)}{(d, p-1)} + \frac{B^*(k_1, \ldots, k_t)}{p} \right)^{\frac{1}{2}} . \]
For
\[ f(x) = g(x) + g_2(x^{k_2}) + \cdots + g_r(x^{k_r}) \]
where \( g \) has distinct exponents \( k_1, t_1, \ldots, t_s \) say, repeated application of the Lemma gives
\[ |S(\chi, f)| \leq p \left( \frac{(k_1, k_r, p-1)}{(k_r, p-1)} + \sqrt{\cdots} \left( \frac{(k_1, k_2, p-1)}{(k_2, p-1)} + \frac{B(k_1, t_1, \ldots, t_s)}{p} \right)^{\frac{1}{2}} \right) \leq p \sum_{i=0}^{r-2} \left( \frac{(k_1, k_{r-i}, p-1)}{(k_{r-i}, p-1)} \right)^{\frac{1}{2(i+1)}} + B(k_1, t_1, \ldots, t_s) \frac{1}{\sigma-1} p^{1-\frac{1}{\sigma}} , \]
where $B(k_1, t_1, \ldots, t_s)$ may be replaced by $B^*(k_1, t_1, \ldots, t_s)$ if $g$ is a polynomial. The Weil bound $B^*(k_1, t_1, \ldots, t_s) \leq D \sqrt{p}$ when $g$ is a polynomial, and $B(k_1, t_1, \ldots, t_s) \leq D \sqrt{p}$ when $g$ contains exponents of both signs, completes the proof of Theorem 1.1.

**Proof of Lemma 3.1.** Suppose that $f(x) = a_1 x^{k_1} + \cdots + a_r x^{k_r}$ is a Laurent polynomial over $\mathbb{Z}$ with $p \nmid a_1$ and $d \mid k_{t+1}, \ldots, k_r$ and define

$$Y = \left\{ y \in \mathbb{Z}_p^* : y^d = 1 \right\}.$$ 

Then, with $\chi$ a multiplicative character,

$$(d, p-1) \sum_{x=1}^{p-1} \chi(x)e_p(f(xy)) = \sum_{y \in Y} \sum_{x=1}^{p-1} \chi(xy)e_p(f(xy)) = \sum_{y \in Y} \chi(y)e_p(a_{t+1} x^{k_{t+1}} + \cdots + a_r x^{k_r}) \sum_{x=1}^{p-1} \chi(y)e_p \left( a_1 y^{k_1} x^{k_1} + \cdots + a_r y^{k_r} x^{k_r} \right).$$

Applying the Cauchy-Schwarz inequality,

$$\left| \sum_{x=1}^{p-1} \chi(x)e_p(f(xy)) \right| \leq \frac{1}{(d, p-1)} \sum_{x=1}^{p-1} \chi(y)e_p \left( a_1 y^{k_1} x^{k_1} + \cdots + a_r y^{k_r} x^{k_r} \right)$$

$$\leq \frac{1}{(d, p-1)} \left( \sum_{x=1}^{p-1} \chi(y_1 y_2 x^{k_1}) \sum_{x=1}^{p-1} \chi(y_1 y_2 x^{k_1}) \right)^{\frac{1}{2}}$$

$$= \frac{1}{(d, p-1)} \left( \sum_{y \in Y} \chi(y_1 y_2) \sum_{x=1}^{p-1} e_p \left( a_1 (y_1 y_2)^{k_1} + \cdots + a_r (y_1 y_2)^{k_r} \right) \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{(d, p-1)} \left( \sum_{y \in Y} B(k_1, \ldots, k_t) \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{(d, p-1)} \left( (d, p-1)(k_1, d, p-1) + (d, p-1)^2 B(k_1, \ldots, k_t) \right)^{\frac{1}{2}},$$

and the claimed result follows. In these last inequalities we can plainly add an extra $x = 0$ term on the right to obtain $B^*$ in place of $B$ when the $k_1, \ldots, k_t$ are all non-negative. When $f$ is a polynomial, starting with $\sum_{x=0}^{p-1} e_p(f(x))$ we obtain (3.1) in the same way. \hfill \Box

### 4. Proof of Corollary 1.1

Suppose that $f$ is of the form

$$f(x) = g(x) + g_2(x^{k_2}) + \cdots + g_r(x^{k_r}),$$

with $g(x)$ containing a monomial $a_1 x^{k_1}$, satisfying the hypotheses of Corollary 1.1. Then $(k_1, k_2, p-1) \leq p^{-\epsilon}$, $2 \leq i \leq r$, and the value $D$ in Theorem 1.1 satisfies $D < p^{2^{-\epsilon}}$. We get from Theorem 1.1,

$$|S(\chi, f)| \leq p \sum_{i=0}^{r-2} p^{-\epsilon/2^{i+1}} + \left( 2^\frac{1}{2} \right)^{1-\epsilon} p^1 \epsilon = p \sum_{i=1}^{r-1} p^{-\epsilon/2^i} + p^{1-\epsilon/2^{i-1}} < 3p^{1-\epsilon/2^{i-1}}.$$

To obtain the constant 3 in the last inequality observe that trivially $|S(\chi, f)| \leq p < 2p^{1-\epsilon/2^{i-1}}$ if $p^{1-\epsilon/2^{i-1}}$ so we may assume $p^{1-\epsilon/2^{i-1}}$. Set $\rho = p^{-\epsilon/2^{i-1}}$ so that $\rho \leq \frac{1}{2}$. Then

$$\sum_{i=1}^{r-1} p^{-\epsilon/2^i} < \rho + \rho^2 + \rho^4 + \rho^8 + \cdots < 2\rho.$$
References


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