Instructions: Do one problem from each of the following 8 sets of problems. Each problem is worth 25 pts. You may do additional problems for 5 pts each. Start the additional problems on a separate sheet and label them "Additional."

1) Suppose that \( f \) is a function of bounded variation on \( [0,1] \) with variation \( V_f \) and that \( C \) is the curve given by \( \vec{r}(t) = (t, f(t)), 0 \leq t \leq 1 \). Prove that \( C \) is rectifiable and show that its length \( L_C \) satisfies \( L_C \leq 1 + V_f \). (Use definition of rectifiable).

2) i) Show that \( g(t) = t^3 \sin(t) \) is continuously differentiable on the interval \( [0,1] \).

ii) Prove that the curve \( C \) given by \( \vec{r}(t) = (t, t^3 \sin(t)) \) \( 0 \leq t \leq 1 \) is rectifiable and express its arc length as a definite integral. (Use (i) and an appropriate theorem).

2) A) Give an \( \varepsilon/\delta \) proof of \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2 \sin(xy)}{x|x| + y|y|} = 0 \).

B) Let \( D \) be a compact set and \( A \) a fixed point not in \( D \). Define \( f: D \to \mathbb{R} \) by \( f(\vec{r}) = |\vec{r}^2 - \vec{A}^2| \).

i) Give an \( \varepsilon/\delta \) proof to show \( f \) is continuous on \( D \).

ii) Why does \( f \) attain a minimum value on \( D \)? What is the geometric interpretation of this minimum value?
A) Suppose that \( \hat{F} : S \rightarrow \mathbb{R}^m \) is a continuous function defined on the arc-connected subset \( S \) of \( \mathbb{R}^n \). Prove that \( \hat{F}(S) \) is arc-connected.

B) Suppose that \( \{ \hat{F}_n \} \) is a Cauchy sequence of points in a closed set \( S \) (in \( \mathbb{R}^n \)) and that \( f : S \rightarrow \mathbb{R} \) is a continuous function on \( S \). Prove that \( \{ f(\hat{F}_n) \} \) is a Cauchy sequence of real numbers.

A) Suppose that \( \hat{F} : D \rightarrow \mathbb{R}^m \) is differentiable at a point \( \hat{p} \in D \) with derivative \( \vec{D}_p \). Prove that for any unit vector \( \vec{u} \), \( \nabla_{\vec{u}} \hat{F}(\hat{p}) = \vec{D}_p (\vec{u}) \). (Use definitions).

B) Let \( f \) be a real valued function defined on \( \mathbb{R}^n \) such that
\[
f(t \hat{p}) = t^k f(\hat{p}) \quad \forall \hat{p} \in \mathbb{R}^n \text{ and } \forall t \in \mathbb{R},
\]
where \( k \) is a fixed positive integer.

i) Show that \( f(0) = 0 \).

ii) Let \( \vec{a} \) be any unit vector in \( \mathbb{R}^n \). Compute \( \nabla_{\vec{a}} f(0) \).

(Your answer will depend on \( k \)).
Let
\[ f(x, y) = \begin{cases} 
(x^2 + y^2) \sin \left( \frac{1}{x^2 + y^2} \right), & (x, y) \neq (0, 0) \\
0, & (x, y) = (0, 0) 
\end{cases} \]
Prove that \( f \) is differentiable on \( \mathbb{R}^2 \). (All of \( \mathbb{R}^3 \)).

Let \( D \) be an open connected set in \( \mathbb{R}^n \) and \( f: D \rightarrow \mathbb{R} \) be a differentiable function on \( D \) with \( f'(\mathbf{0}) = 0 \) (the zero transformation) \( \forall \mathbf{0} \in D \). Prove that \( f \) is constant on \( D \). (You may assume that any two points in \( D \) can be connected by a polygonal line segment).

Verify that \( (1, 1, 1) \) is a critical point of the function
\[ f(x, y, z) = -\frac{1}{4} (x^4 + y^4 + z^4) + yz - x - 2y - 2z \]
and classify it as a local max, min or saddle point.

Using a Taylor expansion about \((0, 0)\) we can write
\[ \sin(x^2 + y^2) = L(x, y) + E(x, y) \]
where \( L(x, y) \) is the first order approximation and \( E(x, y) \) is the error.

i) Find \( L(x, y) \) and \( E(x, y) \), explicitly, expressing \( E(x, y) \) in terms of a point \( \mathbf{c} \). (Where is \( \mathbf{c} \) located?)

ii) Find an upper bound for \( |E(x, y)| \) for \( |x| \leq 1, |y| \leq 1 \).
A) Let $f, g$ be continuously differentiable functions on $\mathbb{R}^3$ and $C$ be the curve defined by the intersection of the surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$. Assume that $(0, 0, 0) \in C$, that is, $f(0) = g(0) = 0$.

i) State sufficient conditions in order to be able to solve for $y$ and $z$ in terms of $x$ on a neighborhood of $(0, 0, 0)$. (That is, to give a parametric equation for $C$ in terms of $x$).

ii) Give a formula for the tangent vector to $C$ at $(0, 0, 0)$ in terms of $f_x, f_y, f_z, g_x, g_y, g_z$, each evaluated at $(0, 0, 0)$.

B) Consider the equation $3x + x^5 + yx + y^5 + y = 0$.

i) Why can we solve for $y$ as a continuously differentiable function $y = f(x)$ on a neighborhood of $(0, 0)$?

ii) Find the first order approximation of $f(x)$ expanded about $0$.

A) Suppose that $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$ such that $f_n \to f$ uniformly on $[0, 1]$. Prove that $f$ is continuous on $[0, 1]$.

B) Consider the sequence of functions $\left\{ \frac{n^2 x}{1 + n^3 x^2} \right\}$. 

i) Does this sequence converge uniformly on $[0, 1]$?

ii) Prove that the sequence converges to a continuous function on the interval $[1, \infty)$. 