Advanced Calculus Test 2

1) Suppose that $\mathbf{F}, \mathbf{G} : D \rightarrow \mathbb{R}^m$ are differentiable at a point $P \in D$. Prove that $\mathbf{F} + \mathbf{G}$ is differentiable at $P$ and that $(\mathbf{F} + \mathbf{G})'(P) = \mathbf{F}'(P) + \mathbf{G}'(P)$. (use definition of diff.)

2) Let $f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Compute $\frac{\partial f}{\partial x}(0,0)$, $\frac{\partial f}{\partial y}(0,0)$ and $\nabla_x f(\mathbf{z})$ where $\mathbf{z} = \frac{1}{\sqrt{2}} (1, 1)$.

3) Let $\mathbf{F}(x,y) = (xy, x-y, x^2)$ and $\mathbf{G}$ be a function differentiable at $(0, -1, 0)$ with $\mathbf{T}_x \mathbf{G}(0,-1,0) = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 3 & 1 \end{bmatrix}$

Set $\mathbf{H} = \mathbf{G} \circ \mathbf{F}$.

a) Why is $\mathbf{F}$ differentiable at $(0,1)$?

b) Why is $\mathbf{H}$ diff. at $(0,1)$?

c) Find $\nabla_{\mathbf{z}} \mathbf{H}(0,1)$

4) Classify the critical points of $f(x,y,z) = x^2 + xz - 3\cos y + z^2$
5) Let \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a nonsingular linear transformation. Prove that for \( \delta > 0 \) such that 
\[ |L(\vec{u})| > \delta |\vec{u}| \quad \forall \vec{u} \neq \vec{0} \in \mathbb{R}^n. \]
(Hint: First consider the sphere of radius one \( \frac{1}{\delta} \).)

6) For any differentiable vector function \( \vec{F} = (f_1, f_2, f_3) \) we define curl \( \vec{F} \) by
\[
\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}.
\]
Suppose that \( f(x,y,z) \) has continuous 2nd partial derivatives on an open set \( D \). Prove that \( \text{Curl } \nabla f = \mathbf{0} \) on \( D \).

7) State and prove the Mean Value Theorem for a function \( f(x_1, x_2, \ldots, x_n) \) differentiable on a NEO \( N \) in \( \mathbb{R}^n \).