Write solutions neatly on a separate sheet of paper. Always start by restating the problem or writing a self explanatory sentence.

(2) 1. a) Prove that for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2(|\vec{u}|^2 + |\vec{v}|^2)$.

b) Give a geometric interpretation of the identity in (a) for a parallelogram.

(2) 2. Prove that the diagonals of a Rhombus are orthogonal using vectors.

(2) 3. a) Prove that for any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $|\vec{v} - \vec{u}|^2 = |\vec{v}|^2 + |\vec{u}|^2 - 2\vec{u} \cdot \vec{v}$.

b) Deduce from part (a) the law of cosines relating the lengths of three sides of a triangle. (Hint: Let $\vec{u}, \vec{v}, \vec{v} - \vec{u}$ represent the three sides.)

4. (For simplicity we will drop the overline “arrow” for vectors in a general vector space.) An inner product on a real vector space $V$ is a mapping $\langle , \rangle: V \times V \to \mathbb{R}$ such that for any $u, v, w \in V$ and $\lambda \in \mathbb{R}$

(i) (symmetry) $\langle u, v \rangle = \langle v, u \rangle$,

(ii) (linearity) $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$,

(iii) (linearity) $\langle \lambda u, w \rangle = \lambda \langle u, w \rangle$,

(iv) (positivity) $\langle u, u \rangle \geq 0$ with equality iff $u = 0$.

The length or norm of a vector $u$ wrt the inner product $\langle , \rangle$ is defined by $|u| = \sqrt{\langle u, u \rangle}$.

(2) a) Prove that the CBS inequality holds for any inner product: $|\langle u, v \rangle| \leq |u| \cdot |v|$ for any $u, v \in V$.

(2) b) Prove that the triangle inequality holds for any norm defined by an inner product: $|u + v| \leq |u| + |v|$, for any $u, v \in V$.

5. Let $V$ be the vector space of continuous real valued functions on $[0, 1]$, and define an inner product on $V$ by $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$.

(2) a) Verify that $\langle , \rangle$ is an inner product. (That is, verify the four properties given in problem 4.)

(2) b) Two functions $f, g \in V$ are said to be orthogonal if $\langle f, g \rangle = 0$. Prove that for any integers $m \neq n$ the functions $\sin(\frac{2\pi mx}{2})$ and $\sin(\frac{2\pi nx}{2})$ are orthogonal. (This plays an important role in the theory of Fourier series.)

(2) 6. Let $S$ be the plane in $\mathbb{R}^3$ given by $ax + by + cz = d$.

(a) Prove that the vector $(a, b, c)$ is perpendicular to the plane $S$, that is, perpendicular to any vector $\vec{v} = \vec{P}_1 - \vec{P}_2$ with $\vec{P}_1, \vec{P}_2 \in S$.

(b) Show that the minimum distance from the plane to the origin is given by $|d|/\|(a, b, c)\|$. (Find the point $\vec{P}$ on the plane such that, viewed as a vector, $\vec{P}$ is perpendicular to the plane.)